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ON THE THEORY OF STABILITY OF A PROLATE SPHEROIDAL SHELL UNDER UNIFORM EXTERNAL PRESSURE

(K Teorii Ustoichivosti Vytyanutoi Ellipsoidal'noi Obolochki Vrashcheniya Pri Vneshnem Ravnomernom Davlenii)

by

R. G. Surkin

Kazanskii Filial Akademii Nauk SSSR,
Seriya Fiziko- Matematicheskikh i
Tekhnicheskikh Nauk, No. 7, 1955

STRUCTURAL MECHANICS LABORATORY

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ON THE THEORY OF STABILITY OF A PROLATE SPHEROIDAL SHELL UNDER UNIFORM EXTERNAL PRESSURE.

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PREFACE

This paper is the only known published work on the nonlinear theory of stability of prolate spheroids under external pressure. An extensive search by the translator revealed that the particular issue of the journal (Kazanskii Filial Akademii Nauk SSSR, Seriya Fiziko- Matematicheskikh i Tekhnicheskikh Nauk, No. 7, 1955) in which this paper appeared was not available in this country. A copy of the article as it appeared in the journal was obtained after direct correspondence with the author; and this translation serves to make this work available on a wide scale.

The translator wishes to acknowledge his indebtedness to Mrs. P. Hale of Virginia Polytechnic Institute and to Dr. B. Nakonechny of the David Taylor Model Basin for their valuable assistance in the translation of this paper.

NOTATION

D	Bending rigidity, $\frac{Et^3}{12(1-\nu^2)}$
E	Young's modulus
K	Tensile rigidity, $\frac{Et}{1-\nu^2}$
p_e	Critical pressure of the shell according to the linear theory
p_k	Pressure at which the stable and unstable states of equilibrium coincide, i.e., at which the first and second variations of the energy functional Φ are equal to zero
p_m	Critical pressure of the shell according to the nonlinear theory, i.e., the lower limit of all values of the pressure p for which the energy of the "nonlinear" state is smaller than the energy of the "zero" state
T_1, T_2	Additional stresses in the middle surface (after snapping)
T_{01}, T_{02}	Stresses in the middle surface of the shell for the membrane state (prior to snapping)
t	Thickness of shell

u, v Projections of the displacement of a point of the middle surface along the lines α and β

W Specific work of the external load

w Projection of the displacement on the inward normal to the middle surface

$u_x = \frac{\partial u}{\partial x}, v_y = \frac{\partial v}{\partial y}, w_x = \frac{\partial w}{\partial x}, \ldots$ Corresponding partial derivatives of the displacements

$\varkappa_1, \varkappa_2, \varkappa_{12}$ Curvature parameters

α and β Gaussian coordinates of the middle surface of a shell of revolution along the meridians and parallels; R_1 and R_2 are their radii of curvature

δ $\frac{R_1}{R_2}$

ϵ_1, ϵ_2 Relative elongations in the directions of the coordinate lines α and β

ν Poisson's ratio

τ $\frac{1}{\sqrt{12}} \frac{t}{R_1}$

ω Angle of displacement between the coordinate lines α and β.

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ABSTRACT

The Rayleigh-Ritz method is used to solve the problem of stability of prolate spheroidal shells under uniform external pressure. Nonlinear terms are retained in the analysis. The "equal energy" load and the minimum post-buckling load are determined for several cases that demonstrate the effect of varying the eccentricity of the generating ellipse.

INTRODUCTION

This paper deals with the possibility of the local loss of stability of a prolate spheroid, under the influence of uniform external normal pressure distributed over the entire shell. Large displacement theory, which allows for snap-through buckling, is used.

The present work represents a generalization of the well-known theory of snapping of shells[1] for the case of prolate spheroids.

Here the critical pressures p_m and p_k are determined by the energy method, as was done for the spherical shell. This means that the pressure p_m corresponds to equal levels of total energy of the shell in the "zero" and "nonlinear" states; the pressure p_k corresponds to the case where the stable and unstable states of equilibrium coincide, at which point the energy function Φ has a parabolic point, i.e., the first and second variations of Φ are equal to zero.

The problem under discussion is solved in a general form, and in addition, certain numerical examples are investigated.

1. DETERMINATION OF p_m

The solution of the problem is carried out for the assumption that the center of the snap lies on the equator of the shell and in a plan view the region of the snap resembles the form of an ellipse. This assumption is reasonable, since in the investigation of the local loss of stability of a geometrically perfect spheroid shell the weakest part is in the region of the equator. In the regions remote from the equator the curvature of the shell is greater; consequently, the stiffness of the shell will also be greater.

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[1] References are listed on page 16.

If, in addition to the fundamental "zero" state of equilibrium of the shell, it is possible that for the same loading there is a stable position of equilibrium after snapping, then the total energy of the shell must also be at a minimum in this final state. Thus, the problem is reduced to a minimization of the functional:

$$\Phi = \iint\limits_{(s)} \Big\{ \frac{K}{2}\Big[(\varepsilon_1^0+\varepsilon_1)^2+(\varepsilon_2^0+\varepsilon_2)^2+2\nu(\varepsilon_1^0+\varepsilon_1)(\varepsilon_2^0+\varepsilon_2)+\frac{(1-\nu)}{2}\omega^2\Big]+ \qquad [1.1]$$

$$+\frac{D}{2}[\varkappa_1^2+\varkappa_2^2+2\nu\varkappa_1\varkappa_2+2(1-\nu)\varkappa_{12}^2]-W\Big\}\,dxdy,$$

where

$$R_1 d\alpha = dx, \quad R_2 d\beta = dy;$$

and

$$\varepsilon_1^0 = -\frac{w_0}{R_1}, \quad \varepsilon_2^0 = -\frac{w_0}{R_2}$$

are the strains in the middle surface of the shell before snapping for the assumption that prior to the local loss of stability the shell is in a membrane state; w_0 is the original deflection in the membrane state; [1.1] is integrated over the entire middle surface of the shell; and the normal to the shell is considered to be directed inward.

The relative displacements and curvature parameters can be written in the form:

$$\varepsilon_1 = u_x + \frac{1}{2}w_x^2 - w/R_1, \quad \varepsilon_2 = v_y + \frac{1}{2}w_y^2 - w/R_2;$$

$$\omega = v_x + u_y + w_x w_y; \quad \varkappa_1 = w_{xx}, \quad \varkappa_2 = w_{yy}, \quad \varkappa_{12} = w_{xy}. \qquad [1.2]$$

Taking into account the condition for equilibrium of an element of the shell in the direction normal to the middle surface in the presence of external uniform pressure on the shell

$$\frac{T_{01}}{R_1} + \frac{T_{02}}{R_2} = -p, \qquad [1.3]$$

where

$$T_{01} = K(\varepsilon_1^0 + \nu\varepsilon_2^0), \quad T_{02} = K(\varepsilon_2^0 + \nu\varepsilon_1^0), \qquad [1.4]$$

we present the work per unit area of the external forces in the form:

$$W = p(w + w_0) = -\Big(\frac{T_{01}}{R_1} + \frac{T_{02}}{R_2}\Big)(w + w_0) = \qquad [1.5]$$

$$= -K\Big[\varepsilon_1^0\Big(-\varepsilon_1^0 + \frac{w}{R_1} + \nu\frac{w}{R_2}\Big) + \varepsilon_2^0\Big(-\varepsilon_2^0 + \nu\frac{w}{R_1} + \frac{w}{R_2}\Big) - 2\nu\varepsilon_1^0\varepsilon_2^0\Big],$$

here we include in the energy functional only the work of the normal pressure on the shell since, by virtue of the boundary conditions for all the cases which are considered further, the work of the reaction of the remaining part of the shell on the boundary of the snap region is equal to zero.

Considering expressions [1.2] - [1.5], we can reduce the functional [1.1] to the form:

$$\Phi = \iint\limits_{(s)} \left\{ \frac{1}{2K(1-\nu^2)} \left[-(T_{01}^2 + T_{02}^2) + 2\nu T_{01} T_{02} \right] + u_x T_{01} + v_y T_{02} \right\} dxdy + \\ + \iint\limits_{(s')} \left\{ \frac{K}{2} \left[\varepsilon_1^2 + \varepsilon_2^2 + 2\nu\varepsilon_1\varepsilon_2 + w_x^2 \frac{T_{01}}{K} + w_y^2 \frac{T_{02}}{K} + \frac{(1-\nu)}{2}\omega^2 \right] + \\ + \frac{D}{2} \left[\varkappa_1^2 + \varkappa_2^2 + 2\nu\varkappa_1\varkappa_2 + 2(1-\nu)\varkappa_{12}^2 \right] \right\} dxdy. \quad [1.6]^{*}$$

At the same time, it is assumed that one can neglect the change in R_1 and R_2 in the region of the snap, since the size of the snap region is small in comparison to the size of the shell.

Obviously, the total energy in the first form of equilibrium (before snapping) is equal to

$$\Phi_0 = \iint\limits_{(s)} \frac{1}{2K(1-\nu^2)} \left[-(T_{01}^2 + T_{02}^2) + 2\nu T_{01} T_{02} \right] dxdy,$$

where the integral is taken over the entire shell. Then, the problem is reduced to the minimization of the functional

$$\Phi' = \Phi - \Phi_0.$$

On the assumption that, in the snap region, T_{01} and T_{02} are constant, the expression

$$\varphi = \iint\limits_{(s)} (u_x T_{01} + v_y T_{02})\, dxdy,$$

which enters into [1.6], is equal to zero in virtue of the boundary conditions, since we put

$$u = 0,\ v = 0,\ w_x = 0,\ w_y = 0 \text{ for } \alpha = \alpha_0 \text{ and } \beta = \beta_0.$$

Thus, we have

$$\Phi' = \frac{K}{2} \iint\limits_{(s')} \left\{ \varepsilon_1^2 + \varepsilon_2^2 + 2\nu\varepsilon_1\varepsilon_2 + \frac{(1-\nu)}{2}\omega^2 + \tau^2 R_1^2 \left[\varkappa_1^2 + \varkappa_2^2 + \\ + 2\nu\varkappa_1\varkappa_2 + 2(1-\nu)\varkappa_{12}^2 \right] + w_x^2 \frac{T_{01}}{K} + w_y^2 \frac{T_{02}}{K} \right\} dxdy. \quad [1.7]$$

Here the integration is carried out only over the region of buckling S', since the quantities characterizing the snap may be different from zero

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*The terms w_x^2 and w_y^2 were incorrectly printed as e_{13}^2 and e_{23}^2 in the original.

only in the region $0 \leqslant \alpha \leqslant \alpha_0$ and $0 \leqslant \beta \leqslant \beta_0$.

We introduce the new variables

$$\frac{x^2}{x_0^2} = \xi, \quad \frac{y^2}{y_0^2} = \eta, \quad \begin{matrix} 0 \leqslant \xi \leqslant 1, \\ 0 \leqslant \eta \leqslant 1, \end{matrix} \qquad [1.8]$$

and the notation

$$x_0 = R_1 \alpha_0, \quad y_0 = R_2 \beta_0, \quad \delta = \frac{R_1}{R_2}, \qquad [1.9]$$

where x_0 and y_0 are the linear dimensions of the snap region in the directions of the meridian and equator of the shell. In the following, we will assume that the contour of the snap region is determined by the ellipse

$$\xi + \eta = 1.$$

We choose the displacements in the general form:

$$u = \rho_1 \lambda \alpha_0^3 R_1 h(\xi, \eta), \quad v = \rho_2 \lambda \alpha_0^2 \beta_0 R_1 j(\xi, \eta), \qquad [1.10]$$
$$w = \lambda \alpha_0^2 R_1 g(\xi, \eta),$$

where

$$h(\xi, \eta), \ j(\xi, \eta) \text{ and } g(\xi, \eta)$$

are some functions of ξ, η, characterizing the displacements and which should satisfy the boundary conditions, i.e.,

$$h(\xi, \eta) = j(\xi, \eta) = 0 \text{ for } \xi = \eta = 0 \text{ and } \xi + \eta = 1, \qquad [1.11]$$
$$g(\xi, \eta) = 1 \text{ for } \xi = \eta = 0 \text{ and } g(\xi, \eta) = 0 \text{ for } \xi + \eta = 1,$$

where ρ_1, ρ_2, and λ are unknown parameters. The magnitude of the angles α_0 and β_0, which determine the extent of the buckled region, are also unknown. However, a simple relation exists between these angles.

In fact, according to our assumption, the contour of the buckled region projected onto a plane tangent to the spheroid at the equator is the ellipse (see Figure 1)

$$\frac{x^2}{x_0^2} + \frac{y^2}{y_0^2} = 1. \qquad [1.12]$$

The equation of the ellipsoid formed by rotating an ellipse with semiaxes a and b about the axis 0_1x_1 has the form

$$\frac{x_1^2}{b^2} + \frac{y_1^2 + z_1^2}{a^2} = 1. \qquad [1.13]$$

If the contour of the snap region lies in the π-plane parallel to the plane x_1Oy_1, and is separated from it by a distance $z_1 = a-d$, then

$$z_1^2 = a^2 - 2ad + d^2, \qquad [1.14]$$

where the distance $d \ll a$.

Substituting [1.14] into [1.13], we obtain the equation of an ellipse lying in the π-plane:

$$\frac{x_1^2}{2\frac{db^2}{a}\left(1-\frac{d}{2a}\right)} + \frac{y_1^2}{2ad\left(1-\frac{d}{2a}\right)} = 1.$$

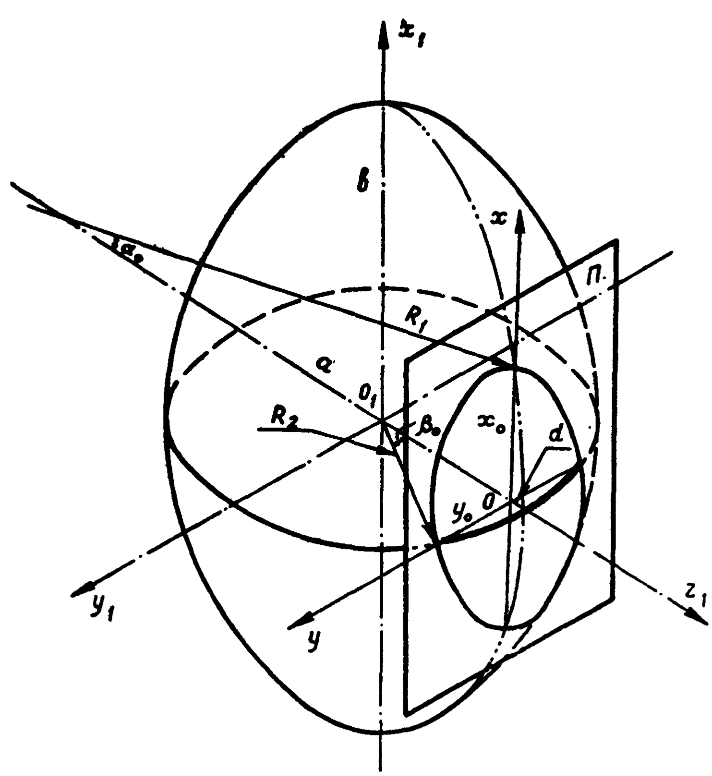

Figure 1

Comparing this equation with [1.12], we find:

$$x_0^2 : y_0^2 = b^2 : a^2 = R_1 : R_2 = \delta.$$

On the other hand

$x_0 = R_1\alpha_0,\ y_0 = R_2\beta_0$, where α_0 and β_0 are small angles. Therefore

$$R_1^2\alpha_0^2 : R_2^2\beta_0^2 = \delta,\ \alpha_0^2 : \beta_0^2 = 1 : \delta. \qquad [1.15]$$

Thus, in place of the quantities α_0 and β_0, we may introduce a single unknown parameter, namely

$$r = \alpha_0\beta_0 . \qquad [1.16]$$

We take into account that

$$T_{02} = T_{01}\left(2 - \frac{1}{\delta}\right), \qquad [1.17]$$

and introduce the new symbols

$$e_0 = -T_{02} : K = |\sigma_{02}|(1-\nu^2) : E, \qquad [1.18]$$

$$A_1 = \frac{1}{\delta^2}\left[A_{11} + \frac{(1-\nu)}{2}\delta A_{12}\right],\ A_2 = A_{21} + \frac{1-\nu}{2\delta}A_{22},$$

$$A_3 = -\frac{1+\nu\delta}{\delta^2}A_{31},\ A_4 = \frac{1+2\nu\delta+\delta^2}{4\delta^2}\int_0^1\int_0^1 \frac{1}{\sqrt{\xi\eta}}g^2 d\xi d\eta,$$

$$A_5 = \frac{1}{\delta}\int_0^1\int_0^1 [2\nu h_\xi j_\eta + (1-\nu)h_\eta j_\xi]\, d\xi d\eta,\ A_6 = -\frac{\delta+\nu}{\delta}A_{32},$$

$$B_1 = \frac{2}{\delta^2}[B_{11} + \nu\delta B_{12} + (1-\nu)\delta B_{13}],\ B_2 = 2\left(B_{21} + \frac{\nu}{\delta}B_{22} + \frac{1-\nu}{\delta}B_{23}\right),$$

$$B_3 = -\frac{1}{\delta^2}\int_0^1\int_0^1\left[(1+\nu\delta)\sqrt{\frac{\xi}{\eta}}gg_\xi^2 + (\delta+\nu)\delta\sqrt{\frac{\eta}{\xi}}gg_\eta^2\right]d\xi d\eta,$$

$$D_1 = \int_0^1\int_0^1\left[\frac{1}{\delta^2}\frac{\xi^{3/2}}{\sqrt{\eta}}g_\xi^4 + \frac{\eta^{3/2}}{\sqrt{\xi}}g_\eta^4 + \frac{2}{\delta}\sqrt{\xi\eta}g_\xi^2 g_\eta^2\right]d\xi d\eta,$$

$$C = \int_0^1\int_0^1\left[\frac{1}{\delta}\frac{1}{\sqrt{\xi\eta}}(2\xi g_{\xi\xi} + g_\xi)^2 + \frac{\delta}{\sqrt{\xi\eta}}(2\eta g_{\eta\eta} + g_\eta)^2 + \right.$$

$$\left. + \frac{2\nu}{\sqrt{\xi\eta}}(2\xi g_{\xi\xi} + g_\xi)(2\eta g_{\eta\eta} + g_\eta) + 8(1-\nu)\sqrt{\xi\eta}\, g_{\xi\eta}^2\right]d\xi d\eta,$$

$$A = \frac{1}{\sqrt{\delta}}\int_0^1\int_0^1\left[\frac{1}{2\delta-1}\sqrt{\frac{\xi}{\eta}}g_\xi^2 + \sqrt{\frac{\eta}{\xi}}g_\eta^2\right]d\xi d\eta.$$

$$A_{11}=\int_0^1\int_0^1\sqrt{\frac{\xi}{\eta}}\,h_\xi^2 d\xi d\eta,\quad A_{12}=\int_0^1\int_0^1\sqrt{\frac{\eta}{\xi}}\,h_\eta^2 d\xi d\eta,$$

$$A_{31}=\int_0^1\int_0^1\frac{1}{\sqrt{\eta}}\,gh_\xi d\xi d\eta,\quad B_{11}=\int_0^1\int_0^1\frac{\xi}{\sqrt{\eta}}\,h_\xi g_\xi^2 d\xi d\eta,$$

$$B_{12}=\int_0^1\int_0^1\sqrt{\eta}\,h_\xi g_\eta^2 d\xi d\eta,\quad B_{13}=\int_0^1\int_0^1\sqrt{\eta}\,h_\eta g_\xi g_\eta d\xi d\eta, \qquad [1.19]$$

where A_{21}, A_{22}, A_{32}, B_{21}, B_{22}, B_{23} are obtained respectively from A_{11}, A_{12}, A_{31}, B_{11}, B_{12}, B_{13} by replacing h by j and ξ by η.

Using the symbols just introduced, after lengthy but in reality simple calculations, we can represent the functional [1.7] in the following form:

$$\Phi^*=\frac{2\Phi'}{KR_1^2}=r^3\lambda^2[\rho_1^2A_1+\rho_2^2A_2+\rho_1A_3+A_4+\rho_1\rho_2A_5+\rho_2A_6+$$
$$+\lambda\rho_1B_1+\lambda\rho_2B_2+\lambda B_3+\lambda^2D_1]+\tau^2\lambda^2rC-e_0\lambda^2r^3A. \qquad [1.20]$$

Here λ, ρ_1, ρ_2, and r are unknown parameters characterizing the snap region and e_o. To minimize the functional [1.20] by the Ritz-Timoshenko method for the determination of p_m, which is the lowest limit of all values of p for which the energy of the "nonlinear" state is less than the energy of the "zero" state, it is necessary to fulfill the following conditions:

$$\Phi^*=0,\ \Phi^*_\lambda=0,\ \Phi^*_{\rho_1}=0,\ \Phi^*_{\rho_2}=0,\ \Phi^*_r=0. \qquad [1.21]$$

Hence, we obtain equations for the determination of ρ_1, ρ_2, λ, r, and e_{om}:

$$\left.\begin{aligned}2\rho_1A_1+\rho_2A_5+\lambda B_1+A_3=0,\\ \rho_1A_5+2\rho_2A_2+\lambda B_2+A_6=0,\\ \rho_1B_1+\rho_2B_2+2\lambda D_1+B_3=0,\end{aligned}\right\} \qquad [1.22]$$

$$r^2[A_4-\rho_1^2A_1+\rho_2^2A_2-\rho_1\lambda B_1+\rho_2A_6-\lambda^2D_1]=C\tau^2 \qquad [1.23]$$

$$e_{0m}=\frac{2C}{rA}\tau^2, \qquad [1.24]$$

where

$$\tau=\frac{1}{\sqrt{12}}\frac{t}{R_1}=\frac{1}{\sqrt{12}}\frac{t}{\delta R_2}.$$

As is evident for the numerical solution of a particular problem, it is necessary to determine the values of the functionals A_1, A_2, ... , B_1, ... , and A. The latter (according to [1.19]) depend only on the form of the functions for the displacements h, j, and g. Proper selection of the displacement functions satisfying the boundary conditions [1.11] obviously guarantees a more dependable solution to the problem.

From our investigation of six alternate forms for the displacements, we retained the one that, in the final analysis, gave the minimum value for the pressure p_m at the values $\delta = 1, 2, 3, 4$:

$$h(\xi, \eta) = e^{-n(\xi+\eta)}[1 - k(\xi+\eta) - k_3(\xi^2+\eta^2)],$$
$$j(\xi, \eta) = e^{-n(\xi+\eta)}[1 - k_1(\xi+\eta) - k_4(\xi^2+\eta^2)],$$
$$g(\xi, \eta) = e^{-n(\xi+\eta)}[1 - k_2(\xi+\eta)]. \qquad [1.25]$$

Here k, k_1, k_2, k_3, k_4, and n are quantities to be determined, where we will assume that n will be chosen a number such that on the boundary of the buckled region ($\xi + \eta = 1$), the deflection becomes negligible.

Further, using formulas [1.19], we compute the coefficients A_1, A_2, ..., B_1, ..., and A of the energy functional [1.20]. In addition, in formulas [1.19], the limits of integration are taken from 0 to ∞ since, for the assumed form of the displacements [1.25], the displacements and stresses are negligible on the boundary of the snap region ($\xi + \eta = 1$).

Omitting the detailed calculations, we can write the coefficients of the energy functional in their final form:

$$A_{11} = \frac{6}{64}\cdot\frac{\pi}{n}\left(1 - \frac{k}{n} + \frac{k^2}{n^2} + \frac{1}{4}\frac{k_3}{n^2} + \frac{17}{8}\frac{kk_3}{n^3} + \frac{185}{64}\frac{k_3^2}{n^4}\right),$$
$$A_{12} = \frac{1}{32}\cdot\frac{\pi}{n}\left(1 - \frac{k}{n} + \frac{k^2}{n^2} - \frac{3}{4}\frac{k_3}{n^2} + \frac{21}{8}\frac{kk_3}{n^3} + \frac{225}{64}\frac{k_3^2}{n^4}\right),$$
$$A_{31} = -\frac{\pi}{16n}\left(2 - 2\frac{k}{n} + \frac{kk_2}{n^2} - \frac{9}{4}\frac{k_3}{n^2} + \frac{9}{4}\frac{k_2k_3}{n^3}\right),$$
$$A_4 = \frac{(1 + 2\nu\delta + \delta^2)}{16\delta^2}\,\frac{\pi}{n}\left(2 - 2\frac{k_2}{n} + \frac{k_2^2}{n^2}\right),$$
$$A_5 = \frac{\pi(1+\nu)}{16\delta n}\left(2 - \frac{k}{n} - \frac{k_1}{n} + 2\frac{kk_1}{n^2} - \frac{3}{4}\frac{k_3}{n^2} - \frac{3}{4}\frac{k_4}{n^2} + \frac{21}{8}\frac{kk_3}{n^3} + \frac{21}{8}\frac{kk_4}{n^3}\right)$$
$$B_{11} = \frac{\pi}{54}\left(\frac{k_2}{n} - \frac{k}{n} - \frac{kk_2}{n^2} - \frac{5}{3}\frac{k_3}{n^2} + \frac{1}{18}\frac{k_2k_3}{n^3} - \frac{2}{9}\frac{k_2^2k_3}{n^4}\right),$$
$$B_{12} = \frac{\pi}{9\cdot 54}\left[9 - 9\frac{k}{n} + 9\frac{k_2}{n} - 3\frac{kk_2}{n^2} + 3\frac{k_2^2}{n^2} - 2\frac{kk_2^2}{n^3} - \right.$$
$$\left. - \frac{5}{2}\left(3\frac{k_3}{n^2} - \frac{k_2k_3}{n^3} + \frac{k_2^2k_3}{n^4}\right)\right],$$
$$B_{13} = -\frac{\pi}{18\cdot 54}\left(9 + 6\frac{kk_2}{n^2} + 3\frac{k_2^2}{n^2} - 2\frac{kk_2^2}{n^3} + \frac{3}{2}\frac{k_3}{n^2} + 4\frac{k_2k_3}{n^3} - \frac{5}{2}\frac{k_2^2k_3}{n^4}\right),$$
$$B_3 = \frac{\pi}{162}\,\frac{(1 + 2\nu\delta + \delta^2)}{\delta^2}\left(2\frac{k_2^3}{n^3} - 3\frac{k_2^2}{n^2} - 9\right),$$

$$D_1 = \frac{\pi n}{8192\delta^2}(3\delta^2 + 2\delta + 3)\left(32 + 32\frac{k_2}{n} + 48\frac{k_2^2}{n^2} + 8\frac{k_2^3}{n^3} + 5\frac{k_2^4}{n^4}\right),$$

$$C = \frac{\pi n}{8\delta}(3\delta^2 + 2\delta + 3)\left(1 + \frac{k_2}{n} + \frac{k_2^2}{n^2}\right),$$

$$A \doteq \frac{\pi}{8}\frac{\sqrt{\delta}}{2\delta - 1}\left(2 + \frac{k_2^2}{n^2}\right). \qquad [1.26]$$

The coefficients A_{21}, A_{22}, A_{32}, B_{21}, B_{22}, and B_{23} are obtained from A_{11}, A_{12}, A_{31}, B_{11}, B_{12}, and B_{13} by replacing k and k_3 respectively by k_1 and k_4.

These coefficients are functions of δ and of the unknown parameters $\frac{k}{n}, \frac{k_1}{n}, \frac{k_2}{n}, \frac{k_3}{n^2}$, and $\frac{k_4}{n^2}$. The latter in our case are determined by means of successive selection and, for $\delta = 1$, proved to be equal to

$$\frac{k}{n} = \frac{k_1}{n} = 0.150; \ \frac{k_2}{n} = -0.545; \ \frac{k_3}{n^2} = \frac{k_4}{n^2} = -0.055. \qquad [1.27]$$

To simplify the computations we also used [1.27] for $\delta \neq 1$. The numerical determination of p_m was carried out in the following manner:

(a) For a given δ ($\delta = 1, 2, 3, 4$) and the values of the parameters $\frac{k}{n}, \frac{k_1}{n}, \cdots$ from [1.27], the coefficients of the energy functional $A_1, A_2, \cdots$, $B_1, \cdots$, A are calculated according to [1.19] and [1.26].

(b) The values ρ_1, ρ_2, and λ are determined from Equations [1.22]. In this case, ρ_1 and ρ_2 are not dependent on the order of the decay n in the displacement functions; however, λ does depend on n.

(c) The obtained values of ρ_1, ρ_2, and λ are substituted in Equation [1.23], and we calculate the parameter $r = \alpha_o \beta_o$, which depends on n and $\frac{t}{R_2}$. Knowing r and taking into account [1.15], we determine without difficulty the values of the small solid angles of the buckle

$$\alpha_0 = \sqrt{\frac{r}{\sqrt{\delta}}}, \quad \beta_0 = \sqrt{r\sqrt{\delta}}. \qquad [1.28]$$

(d) For a known r, we compute e_{om} according to formula [1.24]. Then σ^m_{01}, σ^m_{02}, and p_m are determined. Considering [1.3], [1.17], and [1.18] and assuming $r = r^*\tau$ (where r^* is a numerical coefficient), we write these in the general form

$$\sigma_{01}^{m} = -\frac{1}{\sqrt{3}(2\delta-1)}\frac{C}{r^{*}A}\frac{E}{1-\nu^{2}}\cdot\frac{t}{R_{2}},\qquad [1.29]$$

$$\sigma_{02}^{m} = -\frac{1}{\delta\sqrt{3}}\frac{C}{r^{*}A}\frac{E}{1-\nu^{2}}\cdot\frac{t}{R_{2}},$$

$$p_{m} = \frac{2}{\sqrt{3}(2\delta-1)}\frac{C}{r^{*}A}\cdot\frac{E}{1-\nu^{2}}\cdot\frac{t^{2}}{R_{2}^{2}}.\qquad [1.30]$$

(e) The maximum deflection in the center of the buckled region is determined from [1.10] and [1.25] for $\xi = \eta = 0$:

$$w_{max} = \lambda\alpha_{0}^{2}R_{1}.$$

Since

$$\alpha_{0}^{2} = \frac{r}{\sqrt{\delta}} \text{ and } r = r^{*}n\frac{t}{\sqrt{12}R_{1}},\quad \lambda = \lambda^{*}\frac{1}{n},$$

(where r^{*} and λ^{*} are numerical coefficients), we obtain

$$\frac{w_{max}}{t} = \frac{\lambda^{*}r^{*}}{\sqrt{12\delta}}.\qquad [1.31]$$

Thus the maximum relative displacement in the center of the buckled region does not depend on the relative thickness of the shell t/R_2 but on the order of the decay n.

In Table 1 the values of the critical pressure p_m and the dimensions of the buckled region for different values of δ are given for

$$\frac{k}{n} = \frac{k_1}{n} = 0.150,\ \frac{k_2}{n} = -0.545,\ \frac{k_3}{n^2} = \frac{k_4}{n^2} = -0.055 \text{ and } \nu = 0.3.$$

Table 1 shows that the solid angles of the buckled region α_o and β_o depend on n and $\frac{t}{R_2}$. Supposing that $\frac{t}{R_2} = \frac{1}{900}$ and assuming that for n=5 (or also for n=4) the buckle is very small on the boundary of the snap region, we calculate the values of the small angles α_o and β_o; see Table 2.

2. DETERMINATION OF p_k

For the pressure equal to p_k we have a parabolic point for Φ^{*} on the energy-deflection graph, i.e., the first and second variations of Φ^{*} [1.20] are equal to zero.

TABLE 1

$\delta = \frac{R_1}{R_2}$	1	2	3	4
ρ_1	1.2225	1.2243	1.2257	1.2267
ρ_2	1.2225	1.2216	1.2207	1.2208
λn	4.2619	4.2581	4.2563	4.2558
$r \cdot \frac{1}{n\tau}$	5.9933	7.7872	10.086	11.640
$\frac{x_o}{y_o}$	1	$\sqrt{2}$	$\sqrt{3}$	2
$\frac{\alpha_o}{\beta_o}$	1	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{3}$	$\frac{1}{2}$
$\alpha_o \sqrt{\frac{R_2}{nt}}$	1.3153	1.2971	1.2960	1.2610
$\beta_o \sqrt{\frac{R_2}{nt}}$	1.3153	1.7830	2.2457	2.5220
$e_{om} \frac{R_2}{t}$	0.2205	0.2135	0.1878	0.1816
$\lvert \sigma^m_{01} \rvert \frac{R_2}{Et}$	0.2423	0.1564	0.1238	0.1140
$\lvert \sigma^m_{02} \rvert \frac{R_2}{Et}$	0.2423	0.2346	0.2064	0.1996
$\frac{p_m R_2^2}{Et^2}$	0.4446	0.3128	0.2476	0.2280
$\frac{w_{max}}{t}$	7.37	7.25	7.15	7.14

TABLE 2

δ		1	2	3	4
α_o	n = 4	5°02'	4°58'	4°57'	4°48'
	n = 5	5°40'	6°51'	8°35'	9°55'
β_o	n = 4	5°02'	5°33'	5°30'	5°23'
	n = 5	5°40'	7°55'	9°36'	11°04'

Therefore, p_k is determined from the equations

$$\Phi^*_\lambda = 0,\ \Phi^*_{\rho_1} = 0,\ \Phi^*_{\rho_2} = 0,\ \Phi^*_r = 0,$$

$$\begin{vmatrix} \Phi^*_{\lambda\lambda} & \Phi^*_{\lambda\rho_1} & \Phi^*_{\lambda\rho_2} & \Phi^*_{\lambda r} \\ \Phi^*_{\rho_1\lambda} & \Phi^*_{\rho_1\rho_1} & \Phi^*_{\rho_1\rho_2} & \Phi^*_{\rho_1 r} \\ \Phi^*_{\rho_2\lambda} & \Phi^*_{\rho_2\rho_1} & \Phi^*_{\rho_2\rho_2} & \Phi^*_{\rho_2 r} \\ \Phi^*_{r\lambda} & \Phi^*_{r\rho_1} & \Phi^*_{r\rho_2} & \Phi^*_{rr} \end{vmatrix} = 0 \qquad [2.1]$$

where $\Phi^*_\lambda,\ \Phi^*_{\rho_1},\ \ldots,\ \Phi^*_r,\ \Phi^*_{\lambda\lambda},\ \ldots,\ \Phi^*_{rr}$ are the corresponding partial derivatives with respect to the parameters λ, ρ_1, ρ_2, and r. After certain transformations in the first four equations [2.1], we obtain:

$$\begin{gathered} 2\rho_1 A_1 + \rho_2 A_5 + \lambda B_1 + A_3 = 0 \\ \rho_1 A_5 + 2\rho_2 A_2 + \lambda B_2 + A_6 = 0 \\ \rho_1 B_1 + \rho_2 B_2 + 2\lambda D_1 + B_3 + \frac{2}{3}\,\frac{(2\tau^2 C - e_{0k} r A)}{r^2\lambda} = 0. \end{gathered} \qquad [2.2]$$

$$r^2 [\rho_1^2 A_1 + \rho_2^2 A_2 + \rho_1 A_3 + A_4 + \rho_1\rho_2 A_5 + \rho_2 A_6 - \lambda^2 D_1] = \tau^2 C. \qquad [2.3]$$

The calculation of the fourth order determinant in [2.1] does not present particular difficulty since $\Phi^*_{\rho_1 r}$ and $\Phi^*_{\rho_2 r}$ are equal to zero. Calculation of the fourth order determinant in [2.1] gives us:

$$\begin{gathered} [(2\tau^2 C - e_{0k} r A)^2 + (2\tau^2 C - e_{0k} r A)(\tau^2 C - e_{0k} r A)](A_5^2 - 4A_1A_2) - \\ - r^2\lambda^2(\tau^2 C - e_{0k} r A)\,[B_1 B_2 A_5 - A_1 B_2^2 - A_2 B_1^2 - D_1 (A_5^2 - 4A_1A_2)] = 0. \end{gathered} \qquad [2.4]$$

Equations [2.2], [2.3], and [2.4] are completely sufficient for determining the five unknowns λ, ρ_1, ρ_2, r, and e_{ok}. For the solution of the problem we will make certain transformations.

In the third equation of [2.2] we introduce the notation

$$\frac{2}{3}\,\frac{(2\tau^2 C - e_{0k} r A)}{r^2\lambda} = \varepsilon. \qquad [2.5]$$

Then ρ_1, ρ_2, and λ are determined from Equations [2.2] and are linear functions of ϵ. Further, for known ρ_1, ρ_2, and λ, from Equation [2.3] we determine $\frac{\tau^2 C}{r^2}$, which will be a quadratic function of ϵ. Using the notation [2.5], we transform Equation [2.4]:

$$\lambda\varepsilon^2 - \frac{1}{3}\frac{\tau^2 C}{r^2}\varepsilon + \left[\frac{2}{9}\frac{\tau^2 C}{r^2}\lambda - \frac{1}{3}\lambda^2\varepsilon\right]\frac{[B_1B_2A_5 - A_1B_2^2 - A_2B_1 - D_1(A_5^2 - 4A_1A_2)]}{A_5^2 - 4A_1A_2} = 0. \qquad [2.6]$$

After we substitute the values λ and $\frac{\tau^2 C}{r^2}$ obtained for the particular values of $\delta = 1, 2, 3, 4$, Equation [2.6] becomes a cubic equation involving ϵ. A cubic equation is solvable by well-known methods, and all three of its roots can be determined. Computations showed that for a given δ, only the smallest root of Equation [2.6] was applicable to the determination of p_k.

For known ϵ, we easily calculate e_{ok} from Equation [2.5]. Knowing e_{ok}, we obtain p_k from formulas [1.18], [1.17], and [1.3]. We have calculated the value of p_k for spheroids with different elongations, i.e., for the particular cases $\delta = 1, 2, 3, 4$. The results of the computations are given in Table 3.

Here we do not show the computations for the values α_0, β_0, $|\sigma_{01}^k|$, $|\sigma_{02}^k|$ and $\frac{w_{max}}{t}$, which are easily determined for the known quantities of ρ_1, ρ_2, λ, r, and e_{ok}. Further, we compare the values of the critical pressures p_m and p_k, which we derived for different δ, with results provided by the linear theory by constructing the graph of the dependence of the value $\frac{p R_2^2}{Et^2}$ on δ; see Figure 2. The formula for the determination of the value of the critical external pressure on the shell according to linear theory for $\delta \geq 1$ is easily obtained from Reference 2. It has the form

$$p_e = \frac{2E}{\sqrt{3(1-\nu^2)}}\,\frac{1}{(2\delta - 1)}\cdot\frac{t^2}{R_2^2}. \qquad [2.7]$$

TABLE 3

δ	1	2	3	4
ρ_1	$1.2225 - 2.6834\epsilon$	$1.2243 - 4.7537\epsilon$	$1.2257 - 5.8618\epsilon$	$1.2267 - 6.5650\epsilon$
ρ_2	$1.2225 - 2.6834\epsilon$	$1.2216 - 4.3668\epsilon$	$1.2203 - 5.1142\epsilon$	$1.2208 - 5.5090\epsilon$
λn	$4.2590 - 21.165\epsilon$	$4.2581 - 35.559\epsilon$	$4.2558 - 42.089\epsilon$	$4.2562 - 45.572\epsilon$
$\frac{\tau^2 C}{r^2} \cdot \frac{n}{\pi}$	$0.0215 + 4.259\epsilon - 10.583\epsilon^2$	$0.0129 + 4.259\epsilon - 17.685\epsilon^2$	$0.0111 + 4.257\epsilon - 21.048\epsilon^2$	$0.0102 + 4.256\epsilon - 22.788\epsilon^2$
$\epsilon \frac{1}{\pi}$	0.0040	0.0039	0.0042	0.0038
$e_{ok} \frac{R_2}{t}$	0.2002	0.1942	0.1758	0.1673
$p_k \frac{R_2^2}{Et^2}$	0.4138	0.2840	0.2220	0.2069

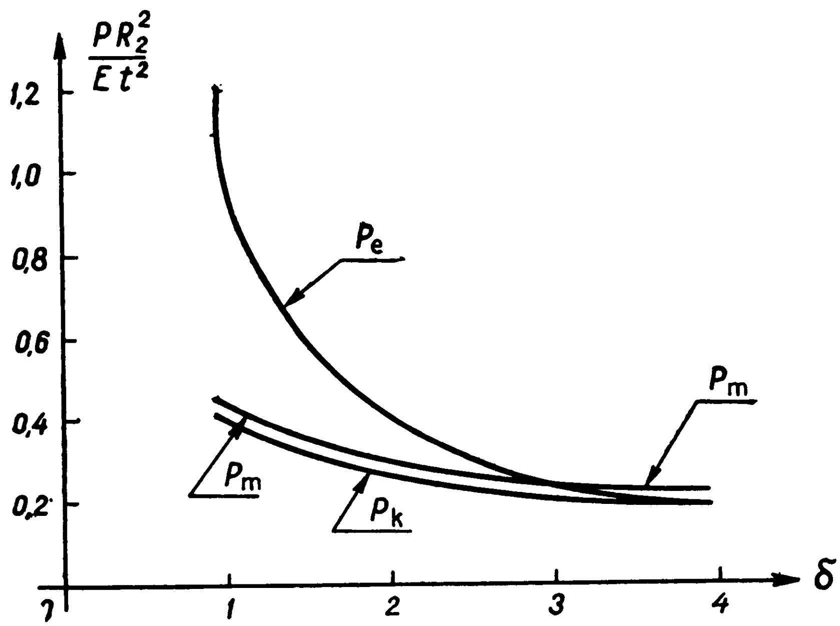

Figure 2

The graph shows that, for the chosen form of displacements, the solution of the problem of local loss of stability of prolate spheroids under external uniform pressure on the shell for $\delta = \frac{R_1}{R_2} > 3$ cannot be considered as satisfactory since, beginning with $\delta > 3$, the magnitude of the critical pressure for which the shell loses its stability, as found from the nonlinear theory, p_k becomes greater than the value of the upper limit of the critical pressure p_e as obtained from formula [2.7].

This discrepancy between p_k and p_e is explained primarily by the fact that in the choice of the displacement functions we limited ourselves, because of the complexity of the problem, to satisfying only the geometric boundary conditions.

Also, the solution of the problem is influenced by the proper determination of the unknown parameters $\frac{k}{n}, \frac{k_1}{n}, \frac{k_2}{n}, \frac{k_3}{n^2}, \frac{k_4}{n^2}$ which in our case were determined by successive selection only for the case of a sphere and were used for the other particular cases ($\delta = 2, 3, 4$). Finally, we found that the restriction we imposed on the region of buckling, assuming it to be elliptical, apparently had an effect on the solution.

It is necessary to point out that, in the particular case when $\delta = 1$, we obtain a fully satisfactory solution to the problem of the local loss of stability of a spherical shell. We refrain from investigating this case, which was satisfactorily discussed at length in Reference 1.

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Physico-Technical Institute of
Kazan Affiliate of the USSR Academy
of Science

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