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ON THE BUCKLING OF PLATES

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The stressing of plates by compressive and tensile forces in the longitudinal and transverse directions is first shown. Then, by Timoshenko's method, the buckling load under various conditions of stress is derived and the criterion for the shape and length of the buckles is determined as a function of the ratio of the normal longitudinal stresses to the normal transverse stresses.

It is further demonstrated that the customary method of computing the buckling strength of plates, although it does indicate the inception of buckling, does not show the actual resistance of the plate up to the limit of its strength. By means of a new formula it is shown how the buckling load of plates increases with increasing deflection. This plate effect determines the ability of the plating of ships to resist high compressive stresses.

1. GENERAL.

The problem of calculating the tensile flange of ship hulls under bending may today be regarded as solved. In the present instance, the plating of completely decked ships and ships with long superstructures is assumed to be fully effective, and thus the conclusions are acknowledged as valid, which I have here presented and subsequently extended on the basis of tests and theoretical reasoning. The conclusions were borne out by analysis of Biles' tests with the WOLF just as impressively as by the measurements [1] variously undertaken aboard vessels under operation.

In recent years attempts have also been made to compute the compressive flange. Hoffmann reported on this in a paper before the INA [2]. He there makes the following assumptions:

1. The plating carries the buckling load of a column which has the thickness of the plate.

2. At the stiffeners the effective thickness is fifty times that of the plates.

3. The plates are completely fixed (clamped) over the transverse frames.

This last assumption has already been the cause of criticism in the discussion of that paper. Attention was directed to the fact that observation had shown that there is no fixation.

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1) See Siemann, Schiffbau 1909/10.

2) See INA, 1924 and 1927.

Now, if there is no fixation, the paper has only a limited value, since conditions in various types of vessels may vary radically.

It is intended to show in the following how it is possible to compute the compression flange, taking into consideration the plate effect without assumption of fixation. Unfortunately, the difficulties are considerably increased by the stressing of the plates in two directions. For instance, if we consider the deck plating, we find it to be stressed longitudinally by tension or compression, and transversely likewise by compression and tension, due to the fact that it acts in conjunction with the flange of the deck beams. The latter stresses, however, are comparatively slight due to the powerful flange effect.

In the bottom, on the other hand, the transverse stresses are very great. If the midship section of the vessel lies above a trough, the inner and outer bottoms are under tensile stress in the longitudinal direction. In the transverse direction the outer bottom is under tensile stress, the inner bottom under compressive stress when the weight of cargo is greater than the pressure of the water. The reverse applies when the ship is floating on a wave crest.

Thus the plates are stressed as indicated in Figs. 1 and 2. In addition there is the case of tension on all sides, which, however, will not be treated here, since the strength in this case differs only slightly from that of a bar under simple stress.

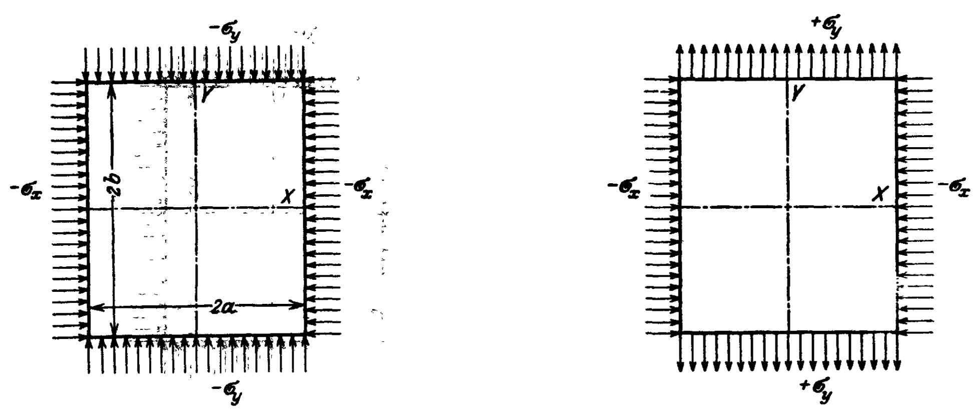

Fig. 1 Compression on all sides of a plate.

Fig. 2 Tension and compression in a plate.

In the recent discussions it has been stated that, even in ships with transverse frames, a plate effect is in evidence, whereby the longitudinal seams (whose moment of inertia is eight times as great as that of the un-reenforced plating because its thickness is twice as great) act as weak longitudinal frames[1].

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1) See Dahlmann's discussion, S.T.G. 1928.

Of course it must be noted in this connection that in the nature of things the longitudinals are able to stand only a relatively slight load before the strakes buckle. But even after this occurs they continue to support a load equivalent to the buckling load. This conclusion has also been reached by Müller-Breslau and H. Lorenz. The buckling load of the longitudinal sections will accordingly be the limiting load to which the plating may be stressed.

Using this theory as a basis for further calculation, the objection might also be raised that in all larger ships the operating stress is less than the buckling load of the plates, and that therefore no further investigation is necessary. However, it is of greatest importance that not only the maximum stress but also the safety factor of the construction be determined. This is only possible, however, if we are able to follow the stress curve above the maximum operating load up to the point of collapse.

2. SHAPE OF THE BUCKLES.

Hitherto, several successful attempts have been made in a series of simple cases to compute the buckling load of such plates, i.e. the load at which the plates begin to buckle.

Reissner, Bryan, and Timoshenko simultaneously treated the fundamental cases.

In discussing the somewhat extended questions, we shall apply the method of the last-named investigator because it renders it possible to treat the more difficult cases also. This method is based upon the law of minimum strain energy. As a first approximation, the buckling load is derived by equating the external and internal work.

If we consider a plate having the dimensions $2a \times 2b$, whose deflection in buckling is designated by w then in each element of the length dx the external force acting upon it, $\sigma_x \cdot \delta \cdot dy$, will advance a distance $\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \cdot dx$, i.e. the difference between the length of the chord and that of the arc. Here δ is the thickness of the element.

Similarly it can be demonstrated that the forces $\sigma_y \cdot \delta \cdot dx$ advance $\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \cdot dy$ and the shear forces $\tau \cdot \delta \cdot dx$ rotate by $\frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \cdot dy$

From this the external work is found to be

$$L_a = 4\int_0^a\int_0^b\left(\frac{\delta}{2} \cdot \sigma_x \left(\frac{\partial w}{\partial x}\right)^2 \cdot dx \cdot dy + \frac{\delta}{2}\sigma_y \cdot \left(\frac{\partial w}{\partial y}\right)^2 \cdot dx \cdot dy + \delta \cdot \tau \cdot \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} \cdot dx \cdot dy\right). \quad (1)$$

In the same way it is possible to represent the strain energy by a slight variation from 0 of the deflection w, by the expression

$$L_i = 4N\int_0^a\int_0^b \frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - (1-\mu)\left[\frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \cdot \partial y}\right)^2\right] dx \cdot dy. \quad (2)$$

By expressing the deflection w by a double series for a plate supported on all sides, in the form

$$w = \sum\sum f_{mn} \cos\frac{m\pi x}{2a} \cdot \cos\frac{n\pi y}{2b} \tag{3}$$

we get

$$\left(\frac{\partial w}{\partial x}\right)^2 = \left(\sum\sum \frac{m\pi}{2a} f_{mn} \cdot \sin\frac{m\pi x}{2a} \cos\frac{n\pi y}{2b}\right)^2,$$

$$\left(\frac{\partial w}{\partial y}\right)^2 = \left(\sum\sum \frac{n\pi}{2b} f_{mn} \cos\frac{m\pi x}{2a} \cdot \sin\frac{n\pi y}{2b}\right)^2,$$

$$\frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} = \left(\sum\sum \frac{m\pi}{2a} \cdot f_{mn} \sin\frac{m\pi x}{2a} \cos\frac{n\pi y}{2b}\right)\left(\sum\sum \frac{n\pi}{2b} f_{mn} \cdot \cos\frac{m\pi x}{2a} \cdot \sin\frac{n\pi y}{2b}\right),$$

$$\frac{\partial^2 w}{\partial x^2} = -\sum\sum \left(\frac{m\pi}{2a}\right)^2 f_{mn} \cdot \cos\frac{m\pi x}{2a} \cdot \cos\frac{n\pi y}{2b},$$

$$\frac{\partial^2 w}{\partial y^2} = -\sum\sum \left(\frac{n\pi}{2b}\right)^2 f_{mn} \cdot \cos\frac{m\pi x}{2a} \cdot \cos\frac{n\pi y}{2b}.$$

In former solutions it is assumed that σ_x, σ_y and τ are constant and do not change during deflection. Under this assumption it is possible to calculate the buckling loads. The assumption obviously ceases to be applicable when there is a large deflection. Let us consider a grid of stiffened plate panels as shown in Fig 3. Here we can immediately see that for reasons

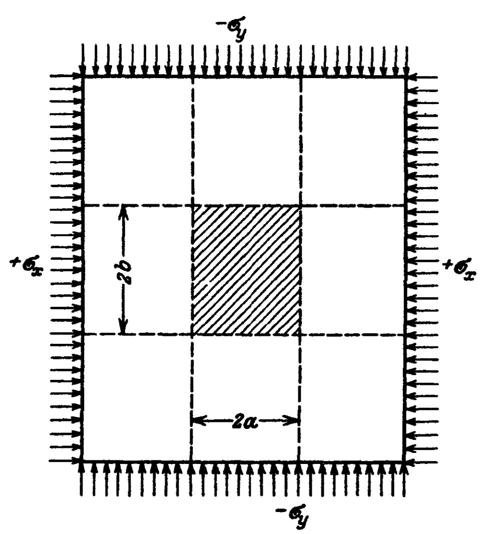

Fig. 3 Grid of stiffened plate panels.

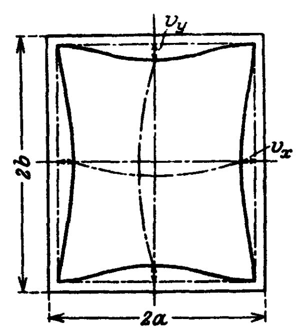

Fig. 4 Displacement of edges for cut out plates.

of symmetry the panels must retain their rectangular form. But as the equations show, the edges, originally straight, are forced by the deflections to curve inward (see Fig. 4). This deformation obviously can only occur when we release one panel from its connection with the others. On the other hand, it is vanishingly small if we investigate only the inception of buckling. When the deflection is very small, then the deformation of the edges is small of the second order and may be neglected.

It is important that the inception of buckling and the buckling deformation be first determined, and then to take up the variable stresses occurring when there are appreciable bulges.

Now let us consider the case represented in Figs. 1 and 2, i.e., a plate acted upon by compressive forces or compressive and tensile forces. Then, with σ_x and σ_y constant, we find

$$L_{a_1} = 4\delta \cdot \frac{\sigma_x}{2} \int_0^a \int_0^b \left(\sum \sum f_{mn} \left(\frac{m\pi}{2a}\right) \sin \frac{m\pi x}{2a} \cdot \cos \frac{n\pi y}{2b} \right)^2 \cdot dx \cdot dy$$

$$\underset{(-)}{+} 4\delta \cdot \frac{\sigma_y}{2} \int_0^a \int_0^b \left(\sum \sum f_{mn} \left(\frac{n\pi}{2b}\right) \cos \frac{m\pi x}{2a} \sin \frac{n\pi y}{2b} \right)^2 \cdot dx \cdot dy$$

$$= 2\delta \cdot \sigma_x \sum \sum \left(f_{mn}^2 \left(\frac{m\pi}{2a}\right)^2 \cdot \frac{a}{2} \cdot \frac{b}{2} \underset{(-)}{+} 2\delta \cdot \sigma_y f_{mn}^2 \left(\frac{n\pi}{2b}\right)^2 \frac{a}{2} \cdot \frac{b}{2} \right),$$

$$L_a = 2\delta \frac{ab}{4} \sigma_x \sum \sum f_{mn}^2 \cdot \left(\left(\frac{m\pi}{2a}\right)^2 \underset{(-)}{+} \left(\frac{\sigma_y}{\sigma_x}\right) \cdot \left(\frac{n\pi}{2b}\right)^2 \right). \tag{4}$$

Similarly the strain energy is found to be

$$L_i = \frac{4}{2} N \int_0^a \int_0^b \left[\sum \sum f_{mn}^2 \cdot \left(\left(\frac{m\pi}{2a}\right)^2 \cdot \cos \frac{m\pi x}{2a} \cdot \cos \frac{n\pi y}{2b} + \left(\frac{n\pi}{2a}\right)^2 \cos \frac{m\pi x}{2a} \cos \cdot \frac{n\pi y}{2b} \right)^2 \right] dx\, dy,$$

$$L_i = 4N \sum \sum \frac{f_{mn}^2}{2} \left(\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{2b}\right)^2 \right)^2 \frac{a}{2} \cdot \frac{b}{2},$$

$$Li = 2N \frac{ab}{4} \sum \sum f_{mn}^2 \left[\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{2b}\right)^2 \right]^2, \quad \text{where} \quad N = \frac{E\delta^3}{12(1-\mu^2)} \tag{4a}$$

is the flexural rigidity of the plates.

By equating the internal and external work the critical stress

$$\sigma_{x_k} = \frac{N\pi^2}{\delta} \cdot \frac{\sum \sum f_{mn}^2 \left[\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2 \right]^2}{\sum \sum f_{mn}^2 \left[\left(\frac{m}{2a}\right)^2 \underset{(-)}{+} \left(\frac{\sigma_y}{\sigma_x}\right) \left(\frac{n}{2b}\right)^2 \right]}. \tag{5a}$$

is derived. Here the minus sign in the denominator includes the case where σ_y represents a tensile stress.

In order to determine the form of the buckles we first assume that σ_x will be the determining factor in buckle formation. Then the plate will buckle in only one wave in the direction of y; i.e. n will equal unity. In determining the minimum by selecting a suitable f_2/f_1 ratio, we find that the ratio disappears. Therefore, with uniform stress, we may content ourselves with a single term of the double sums, so that we will have

$$\sigma_{xk} = \frac{N \cdot \pi^2}{\delta} \frac{\left[\left(\frac{m}{2a}\right)^2 + \left(\frac{1}{2b}\right)^2\right]^2}{\left(\frac{m}{2a}\right)^2 \underset{(-)}{+} \frac{\sigma_y}{\sigma_x}\left(\frac{1}{2b}\right)^2} \tag{5b}$$

If we let $\frac{\sigma_y}{\sigma_x} = k$, and $\frac{m}{2a} = x$, we obviously obtain the minimum by a definite ratio of m/2a to 1/2b.

$$\frac{\delta \sigma_k}{\delta\left(\frac{m}{2a}\right)} = 0 = \left[x^2 \underset{(-)}{+} k\left(\frac{1}{2b}\right)^2\right] \cdot 2\left[x^2 + \left(\frac{1}{2b}\right)^2\right] 2x - \left[x^2 + \left(\frac{1}{2b}\right)^2\right]^2 2x = 0$$

or

$$\left[x^2 \underset{(-)}{+} k \cdot \left(\frac{1}{2b}\right)^2\right] \cdot 2 = x^2 + \left(\frac{1}{2b}\right)^2,$$

$$x^2 = \left(\frac{m}{2a}\right)^2 = \left(\frac{1}{2b}\right)^2 \cdot \left(1 \underset{(+)}{-} 2k\right),$$

$$\left(\frac{m}{2a}\right) = \frac{1}{2b}\sqrt{1 \underset{(+)}{-} 2k}. \tag{6}$$

(See Table)

ratio $\frac{\sigma_y}{\sigma_x} = k$	$\frac{1}{4}$	0	$-0{,}6$	-1	$-1{,}5$	-2	$-7{,}5$
$\frac{m}{2a}$	$\frac{\sqrt{2}}{4b}$	$\frac{1}{2b}$	$\frac{\sqrt{2{,}2}}{2b}$	$\frac{\sqrt{3}}{2b}$	$\frac{2}{2b}$	$\frac{\sqrt{5}}{2b}$	$\frac{4}{2b}$
wave number m*	$\frac{a}{2b}\sqrt{2}$	$\frac{a}{b}$	$\frac{a}{b}\sqrt{2{,}2}$	$\frac{a}{b}\sqrt{3}$	$\frac{2a}{b}$	$\frac{a}{b}\sqrt{5}$	$4\frac{a}{b}$
half wave length λ	$2b\sqrt{2}$	$2b$	$0{,}67(2b)$	$2b\frac{\sqrt{3}}{3}$	$\frac{2b}{2}$	$2b\frac{\sqrt{5}}{5}$	$\frac{2b}{4}$

When k $>$ ½, the value m/2a becomes imaginary, i.e., the buckling in the case of a long plate no longer depends upon σ_x. When k = ½, m/2a = 0.

Actually a plate whose greatest dimension is in the direction of x and in which σ_y has half the magnitude of σ_x, will buckle in a single bulge. When σ_y = 0 the plate will have a tendency to buckle in squares.

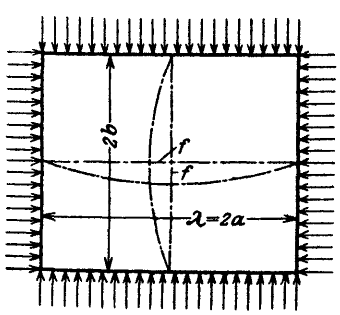

Fig. 5
Lobe form for $\frac{\sigma_y}{\sigma_x} \geqq \frac{1}{2}$

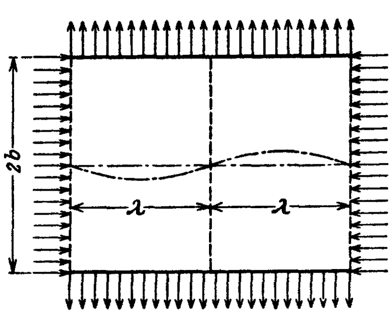

Fig. 6
Lobe form for $\frac{\sigma_y}{\sigma_x}$ =−0.6

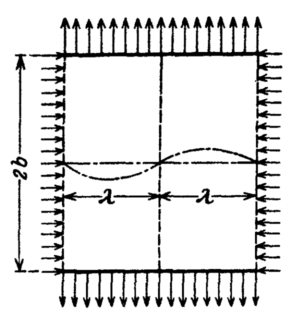

Fig. 7
Lobe form for $\frac{\sigma_y}{\sigma_x}$ = − 2

*The ratio a/b must be so selected that integral numbers will result.

Whenever 2a is smaller than the length λ, given in the table, the plate will buckle only in a half wave.

The wave form for several $k = \frac{\sigma_y}{\sigma_x}$ ratios is given in Figs. 5, 6, and 7.

If the tensile stresses are very high with respect to the compressive stresses, it is obvious that very narrow buckles will form. These buckle formations must be given special attention in very thin plates such as are used in aeroplane construction (See Wagner's paper at the W.G.L. meeting, 1928).

After the shape of the waves has been determined, determination of the buckling load will obviously be very simple.

When $0 \leqq k < \frac{1}{2}$

$$\left(\frac{m}{2a}\right)^2 = \frac{1-2k}{(2b)^2},$$

therefore

$$\sigma_{xk} = \frac{N}{\delta}\left(\frac{\pi}{2b}\right)^2 4(1-k) \tag{7}$$

and when k = 0

$$\sigma_{xk} = 4\frac{N}{\delta}\left(\frac{\pi}{2b}\right)^2.$$

When $k \gtreqless \frac{1}{2}$

$$\sigma_{xk} = \frac{N}{\delta}\left(\frac{\pi}{2b}\right)^2 \frac{\left[1+\left(\frac{b}{a}\right)^2\right]^2}{k+\left(\frac{b}{a}\right)}, \tag{8}$$

therefore when k = 1

$$\sigma_{xk} = \frac{N}{\delta}\left(\frac{\pi}{2b}\right)^2\left[1+\left(\frac{b}{a}\right)^2\right].$$

When k < 0 we get

$$\sigma_{xk} = \frac{N}{\delta}\cdot\left(\frac{\pi}{2b}\right)^2 4(1+k), \tag{9}$$

and therefore when k = –1.5

$$\sigma_{xk} = 10\frac{N}{\delta}\left(\frac{\pi}{2b}\right)^2$$

and when k = –7.5

$$\sigma_{xk} = 34\frac{N}{\delta}\left(\frac{\pi}{2b}\right)^2.$$

3. SUPPLEMENTARY STRESSES AFTER EXCEEDING THE BUCKLING LIMIT

As shown by experience, the calculated form of the buckles changes only slightly after the buckling limit has been exceeded. Therefore in our subsequent calculations, we can start with this form only in making the first approximation.

If the plate is put under continued compression after buckling, the parts at the edges obviously cannot deflect further and the compression there will continue to increase uniformly. In the middle, where the plate buckles, it seeks to avoid further stress. If we imagine the plate to be cut out, a deformation may occur which will be equal to the change in the lengths of the chords in the individual cuts. Thus the displacement at the edge x = a for one term will be

$$v_{xa} = \frac{1}{2}\int_0^a\left(\frac{\partial w}{\partial x}\right)^2 dx = \frac{m^2\pi^2}{32}\,\frac{f_{mn}^2}{a}\cdot\left(1+\cos\frac{n\pi y}{b}\right) \tag{10a}$$

and the displacement of the edge y = b

$$v_{yb} = \frac{1}{2}\int_0^b \left(\frac{\partial w}{\partial y}\right)^2 \cdot dy = \frac{n^2\pi^2}{32} \cdot \frac{f_{mn}^2}{b}\left(1 + \cos\frac{m\pi x}{a}\right). \tag{10b}$$

If several members are to be taken into account, we get

$$v_{xa} = \sum \frac{m^2\pi^2}{32a} \cdot \left(f_{mn}^2\left(1 + \cos\frac{n\pi y}{b}\right) + 2 f_{mn} f_{mr}\left(\cos\frac{(r-n)\pi y}{2b} + \cos\frac{(r+n)\pi y}{2b}\right)\right)$$

and correspondingly v_{yb}. Here $r \neq n$.

These deformations are plotted in Fig. 4. They are due solely to the bulges. Superimposed upon them is a uniform compression due to the increase in load. Since the field of plating is composite, however, this deformation can not set in. Rather, it is neutralized by a plane stress system superimposed on the above stresses.

Obviously the displacement can now be resolved into two parts, one which (for x = a and y = b) causes uniform displacements of v_{x_1} and v_{y_1}, and another which causes a cosine shaped displacement (see Fig. 8). The deflections v_{x_1} and v_{y_1} obviously produce a constant stress distribution.

$$\sigma_{x_1} = \frac{E}{1-\mu^2}\left(\frac{v_{x_1}}{a} + \mu \cdot \frac{v_{y_1}}{b}\right) = \frac{E}{1-\mu^2}\,\frac{\pi^2}{32}\left(\frac{m^2}{a^2} + \frac{n^2\mu}{b^2}\right) f_{mn}^2, \tag{11a}$$

and

$$\sigma_{y_1} = \frac{E}{1-\mu^2} \cdot \left(\frac{v_{y_1}}{b} + \frac{\mu \cdot v_{x_1}}{a}\right) = \frac{E}{1-\mu^2}\,\frac{\pi^2}{32}\left(\frac{n^2}{b^2} + \frac{m^2\mu}{a^2}\right) f_{mn}^2. \tag{11b}$$

For a = b and m = n = 1, v_{x_1} will be equal to v_{y_1} and

$$\sigma_{x_1} = \sigma_{y_1} = \frac{1+\mu}{1-\mu^2} \cdot \frac{\pi^2}{8}\left(\frac{f_n}{2a}\right)^2 E. \tag{11c}$$

In addition to this positive displacement there will naturally be other displacements arising from the increase of stresses at the edges of the plates. For the time being these do not interest us.

Computation of the second part of the displacement shown in Fig. 8 is considerably more difficult. The method followed will be only briefly indicated here.

We seek to determine the stress function in such a manner that these displacements

$$v_{y_2} = v_{y_1} \cdot \cos\frac{m\pi x}{a} \tag{10b}$$

and

$$v_{x_2} = v_{x_1} \cdot \cos\frac{n\pi y}{b} \tag{10a}$$

are obtained.

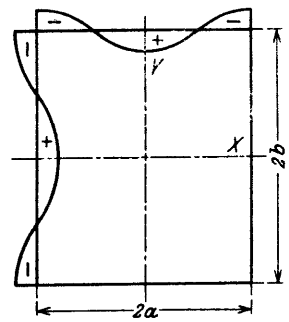

Fig. 8. Cosine shaped edge deflection.

This condition requires a stress distribution of the form:

$$\sigma_{x_2} = \sigma_{m_2} \cdot F''_{(y)} \cdot \left(\frac{a}{m\pi}\right)^2 \cdot \cos\frac{m\pi x}{a} - \sigma_{n_2} \cdot F_{(x)} \cdot \cos\frac{n\pi y}{b}$$

and

$$\sigma_{y_2} = -\sigma_{m_2} \cdot F_{(y)} \cos\frac{m\pi x}{a} + \sigma_{n_2} F''_{(x)} \cdot \left(\frac{b}{n\pi}\right)^2 \cos\frac{n\pi y}{b}$$

and

$$\tau = \sigma_{m_2}\left(\frac{a}{m\pi}\right) \cdot F'_{(y)} \cdot \sin\frac{m\pi x}{a} + \sigma_{n_2}\frac{b}{n\pi} F'_{(x)} \cos\frac{n\pi y}{b},$$

where the functions $F_{(x)}$ and $F_{(y)}$ are so selected that they correspond to a condition of plane stress.

Here we have $\sigma_{m_2} = E m^2 n^2 \frac{\pi^4}{64} \cdot \frac{f_{mn}^2}{a^2}$

and $\sigma_{n_2} = E \cdot m^2 n^2 \frac{\pi^4}{64} \cdot \frac{f_{mn}^2}{b^2}$; $\left.\begin{array}{l}\text{for } a = b \\ \text{and } m = n = 1\end{array}\right\}$

$$\sigma_{m_2} = \sigma_{n_2} = E\frac{\pi^2}{32} \cdot \frac{f^2}{b^2} \cdot \frac{\pi^2}{2} = \frac{\pi^2}{2} \cdot 0{,}75 \cdot \sigma_{x_1}.$$

The functions $F_{(x)}$ and $F_{(y)}$ are hyperbolic functions of the form

$$F_{(x)} = A_1\left(B_1 \mathfrak{Cof}\frac{n\pi x}{b} + \frac{x}{a}\mathfrak{Sin}\frac{n\pi x}{b}\right), \qquad F_{(y)} = A_2\left(B_2 \mathfrak{Cof}\frac{m\pi y}{a} + \frac{y}{b}\mathfrak{Sin}\frac{m\pi y}{a}\right).$$

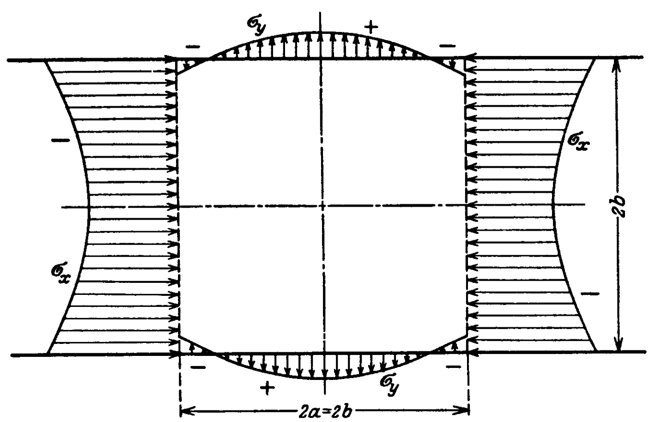

Fig. 9. Edge stresses corresponding to square bulge after buckling.

Here

$$A_1 = \frac{1}{\mathfrak{Sin}\frac{n\pi a}{b}}$$

and

$$A_2 = \frac{1}{\mathfrak{Sin}\frac{m\pi b}{a}}.$$

Furthermore

$$B_1 = -\left(\mathfrak{Ctg}\frac{n\pi a}{b} + \frac{b}{an\pi}\right)$$

and

$$B_2 = -\left(\mathfrak{Ctg}\frac{m\pi b}{a} + \frac{a}{bm\pi}\right).$$

These constants simultaneously satisfy the condition that the shear stresses vanish at the edges. It is further assumed that the frame does not relieve formation of a non-uniform stress at the edges. In view of the slight stiffness of the longitudinal seams in ships with transverse framing this condition is satisfied with sufficient accuracy.

If, for instance, we have a square bulge such as results when the stress $\sigma_y = 0$ and the edges parallel to the σ_x-stress are rigidly held a distribution of stress occurs at the edges such as is indicated in Fig. 9 for σ_y.

In the x-direction the stress is superimposed over the uniform edge stress σ_{xr}.

The absolute value of the edge stress is as yet unknown.

We will find it according to the principle of virtual displacement. We consider a certain deflection f. The stresses σ_{x_1} and σ_{y_1} as well as σ_{x_2} and σ_{y_2} are then known.

If we increase the deformation by a very small amount Δf, these stresses remain constant. Computing the virtual work of the internal and external forces, we find that the external work has increased.

$$\left.\begin{aligned}\frac{\Delta La}{\Delta f} = \delta = f_m ab \Bigg[\sigma_{xr}\left(\left(\frac{m\pi}{2a}\right)^2 + \frac{\sigma_{yr}}{\sigma_{xr}}\left(\frac{n\pi}{2b}\right)^2\right) - \sigma_{x1}\left(\frac{m\pi}{2a}\right)^2 \\ - \sigma_{y1}\left(\frac{n\pi}{2b}\right)^2 - \frac{\sigma_{n_2}}{(2a)^2}\left(\frac{bm}{an}\right)^2 - \frac{\sigma_{m_2}}{(2b)^2}\cdot\left(\frac{an}{bm}\right)^2\Bigg].\end{aligned}\right\} \tag{12}$$

Similarly the change in internal work is found to be

$$\frac{\Delta Li}{\Delta f} = Nab f_m\left[\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{2b}\right)^2\right]^2. \tag{13}$$

From this the mean stress at the edge is found to be

$$\sigma_{xr} = \frac{N}{\delta}\pi^2\frac{\left[\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2\right]^2}{\left(\frac{m}{2a}\right)^2 \pm \frac{\sigma_{yr}}{\sigma_{xr}}\left(\frac{n}{2b}\right)^2} + \frac{\sigma_{x_1}\left(\frac{m}{2a}\right)^2 + \sigma_{y_1}\left(\frac{n}{2b}\right)^2 + \sigma_{n_2}\left(\frac{m}{2a\pi}\right)^2\left(\frac{b}{an}\right)^2 + \sigma_{m_2}\left(\frac{n}{2b\pi}\right)^2\left(\frac{a}{bm}\right)^2}{\left(\frac{m}{2a}\right)^2 \pm \frac{\sigma_{yr}}{\sigma_{xr}}\left(\frac{n}{2b}\right)^2}. \tag{14}$$

We note at once from the formula that the stress σ_{xr} rises materially above the buckling stress, since the first term denotes the buckling stress.

We obtain a better view by writing $\sigma_{yr} = 0$. Then the form of the first bulge will be square and we get

$$\begin{gathered}\sigma_{x_1} = \sigma_{y_1}; \qquad \sigma_{m_1} = \sigma_{n_1}; \\ \sigma_{xr} = 4\frac{N}{\delta}\frac{\pi^2}{(2b)^2} + 2\sigma_{x_1} + \frac{2}{\pi^2}\cdot\sigma_{m_1}.\end{gathered} \tag{14a}$$

For a plate having a width 2b = 80 cm, for example, we get for f = 0.5 cm and δ = 1 cm thickness

$$\sigma_k = 4\frac{N}{\delta}\frac{\pi^2}{(2b)^2} = 1150\,\text{kg/cm}^2$$

and

$$\sigma_{x_1} = \frac{1+\mu}{1-\mu^2}\cdot\frac{\pi^2}{8}\left(\frac{f}{2b}\right)^2 = 158\,\text{kg/cm}^2,$$

$$\sigma_{m_2} = E\frac{\pi^4}{16}\cdot\left(\frac{f}{2b}\right)^2 = 585\,\text{kg/cm}^2.$$

Therefore

$$\sigma_{xr} = 1150 + 316 + 117 \approx \mathbf{1583\,kg/cm^2}.$$

The effectiveness of the plate, its "effective width", is obtained from the formula

$$b_m = \frac{\sigma_{xr} - \sigma_{x_1}}{\sigma_{xr}}\cdot b; \tag{15}$$

then $b_m = 0.9b$ and the mean stress is

$$\sigma_m = 1425\,\text{kg/cm}^2.$$

The stress distribution for this condition is shown in Fig. 9. With greater deflections this augmenting effect of tensile stress increases rapidly.

Beyond a certain limit, however, it is not possible to compute the edge stress simply by means of the formula. As soon as the tensile stresses become too high, the bulges begin to alter their shapes. These shapes can be determined by means of the minimum work of deformation, and the stresses may then be computed in the manner indicated. To determine the unknown terms, a sufficient number of equations is obtained, the solution of which will yield the desired edge stresses.

It is outside the range of this brief paper to apply this calculation to a concrete example. Its application, however, will cause no difficulty. In order to test the applicability of the formulas to ship members, Biles'[1] tests with the destroyer WOLF were checked by means of the new theory. Good agreement was obtained for the hogging as well as for the sagging trials, lying within the limits of test accuracy.[2]

For large vessels with thick plates the buckling modulus must be substituted for the modulus of elasticity E, in order to obtain the buckling loads.

In conclusion it must be stated that, according to this process, thin plates may also be computed which are simultaneously under tensile, compressive, and bending stress. As Pietzker has already stated, application of the usual plate theory which has been very successful in concrete construction is admissible to only a limited extent for thin ship plating. The method indicated in this paper will yield a sufficiently accurate result.

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1) See INA, 1905.

2) See Schiffbau 1928, Book 22.

APPENDIX.

(a) Stress Condition with Cosine Shaped Edge Displacement.

The stress function must satisfy the partial differential equation:

$$\frac{\partial^4 F}{\partial x^4} + \frac{2\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 .$$

Then

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} = \sigma_{m2} F''_{(y)} \left(\frac{a}{m\pi}\right)^2 \cos\frac{m\pi x}{a} - \sigma_{n2} F_{(x)} \cos\frac{n\pi y}{b},$$

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} = -\sigma_{m2} F_{(y)} \cos\frac{m\pi x}{a} + \sigma_{n2} F''_{(x)} \left(\frac{b}{n\pi}\right)^2 \cos\frac{n\pi y}{b},$$

$$\tau = -\frac{\partial^2 F}{\partial x \partial y} = \sigma_{m2}\left(\frac{a}{m\pi}\right) F'_{(y)} \sin\frac{m\pi x}{a} + \sigma_{n2}\left(\frac{b}{n\pi}\right) F'_{(x)} \sin\frac{n\pi y}{b} .$$

[Translator's Note: The Airy's stress function chosen is

$$F = \sigma_{m2}\, F_{(y)} \left(\frac{a}{m\pi}\right)^2 \cos\frac{m\pi x}{a} + \sigma_{n2}\, F_{(x)} \left(\frac{b}{n\pi}\right)^2 \cos\frac{n\pi y}{b} \Big].$$

Herein

$$F_{(x)} = A_1\left(B_1 \mathfrak{Cof}\frac{n\pi x}{b} + \frac{x}{a}\mathfrak{Sin}\frac{n\pi x}{b}\right),$$

$$F_{(y)} = A_2\left(B_2 \mathfrak{Cof}\frac{m\pi y}{a} + \frac{y}{b}\mathfrak{Sin}\frac{m\pi y}{a}\right).$$

Therefore

$$F'_{(x)} = \frac{n\pi}{b} A_1 \left[\left(B_1 + \frac{b}{an\pi}\right)\mathfrak{Sin}\frac{n\pi x}{b} + \frac{x}{a}\mathfrak{Cof}\frac{n\pi x}{b}\right],$$

$$F''_{(x)} = \left(\frac{n\pi}{b}\right)^2 A_1 \left[\left(B_1 + \frac{2b}{an\pi}\right)\mathfrak{Cof}\frac{n\pi x}{b} + \frac{x}{a}\mathfrak{Sin}\frac{n\pi x}{b}\right]$$

or

$$F''_{(x)} = \left(\frac{n\pi}{b}\right)^2 \left(F_{(x)} + \frac{2b}{an\pi} A_1 \mathfrak{Cof}\frac{n\pi x}{b}\right)$$

and correspondingly $F'_{(y)}$ and $F''_{(y)}$.

From the condition that $\tau = 0$ when $x = a$ and $y = b$, it follows that

$$[F'_{(x)}]_{x=a} = 0, \qquad [F'_{(y)}]_{y=b} = 0,$$

wherefor

$$\left(B_1 + \frac{b}{an\pi}\right)\mathfrak{Sin}\frac{n\pi a}{b} + \mathfrak{Cof}\frac{n\pi a}{b} = 0,$$

Then

$$B_1 = -\left(\mathfrak{Ctg}\frac{n\pi a}{b} + \frac{b}{an\pi}\right)$$

and

$$B_2 = -\left(\mathfrak{Ctg}\frac{m\pi b}{a} + \frac{a}{bm\pi}\right).$$

In order to bring about the edge displacement

$$v_x \cdot \cos\frac{n\pi y}{b} = \frac{m^2\pi^2}{32a} f_{mn}^2 \cos\frac{n\pi y}{b}$$

it is essential that

$$v_x \cdot \cos\frac{n\pi y}{b} = \frac{1}{E}\left[\int \sigma_x dx - \mu \int \sigma_y dy\right]$$

$$= \frac{\sigma_{m2}}{E}\int_0^a F''_{(y)}\left(\frac{a}{m\pi}\right)^2 \cos\frac{m\pi x}{a} dx - \frac{\sigma_{n2}}{E}\int_0^a F_{(x)} \cos\frac{n\pi y}{b} dx$$

$$+ \frac{\mu}{E}\sigma_{m2}\int_0^a F_{(y)} \cos\frac{m\pi x}{a} dy - \frac{\mu\sigma_{n2}}{E}\int_0^a F''_{(x)}\left(\frac{b}{n\pi}\right)^2 \cos\frac{n\pi y}{b} dy.$$

Here the first, third and fourth integrals will be zero. From the second integral it follows that

$$\frac{m^2\pi^2}{32a} f_{mn}^2 = -\frac{\sigma_{n2}}{E}\frac{b}{n\pi}A_1\left[-\left(\mathfrak{Ctg}\frac{n\pi a}{b} + \frac{2b}{an\pi}\right)\mathfrak{Sin}\frac{n\pi x}{b} + \frac{x}{a}\mathfrak{Cof}\frac{n\pi x}{b}\right]_0^a.$$

If we write

$$\sigma_{n2} = Em^2n^2\frac{\pi^4}{64}\frac{f_{mn}^2}{b^2}$$

and

$$\sigma_{m2} = Em^2n^2\frac{\pi^4}{64}\frac{f_{mn}^2}{a^2},$$

we will have

$$A_1 = \frac{1}{\mathfrak{Sin}\frac{n\pi a}{b}} \qquad \text{and} \qquad A_2 = \frac{1}{\mathfrak{Sin}\frac{m\pi b}{a}}.$$

The entire stress condition thus is composed of three parts

σ_{xo} and σ_{yo} , the mean external pressure,

σ_{x1} and σ_{y1} , the uniform tensile stresses, and

σ_{x2} , σ_{y2} , and τ , the constants of which have just been calculated.

(b) Equilibrium in Buckling.

Equilibrium for a deflection of

$$w = f_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$$

exists when with a very slight alteration in the amount of deflection by Δf_{mn} the internal and external energy are equal.

Calculation of the energy with the uniform stresses σ_{xo}, σ_{yo}, σ_{x1} and σ_{y1} has been demonstrated in Part I of the present paper.

The strain energy for σ_{x2}, σ_{y2}, and τ is calculated by means of the formulas

$$L_a = \frac{4\delta}{2}\int_0^a\int_0^b \sigma_x \left(\frac{\partial w}{\partial x}\right)^2 dx\,dy + \frac{4\delta}{2}\int_0^a\int_0^b \sigma_y \left(\frac{\partial w}{\partial y}\right)^2 dx\,dy + 4\delta\int_0^a\int_0^b \tau \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} dx\,dy,$$

where

$$\left(\frac{\partial w}{\partial x}\right)^2 = \frac{1}{4} f_{mn}^2 \left(\frac{m\pi}{2a}\right)^2 \left(1 - \cos\frac{m\pi x}{a}\right)\left(1 + \cos\frac{n\pi y}{b}\right),$$

$$\left(\frac{\partial w}{\partial y}\right)^2 = \frac{1}{4} f_{mn}^2 \left(\frac{n\pi}{2b}\right)^2 \left(1 + \cos\frac{m\pi x}{a}\right)\left(1 - \cos\frac{n\pi y}{b}\right),$$

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \frac{1}{4} f_{mn}^2 \left(\frac{m\pi}{2a}\right)\left(\frac{n\pi}{2b}\right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}.$$

As has just been shown

$$F''_{(y)} = \left(\frac{m\pi}{a}\right)^2 \left(F_{(y)} + \frac{2a}{bm\pi} \frac{\mathfrak{Cof}\,\frac{m\pi y}{a}}{\mathfrak{Sin}\,\frac{m\pi b}{a}} \right).$$

When

$$\sigma_{x2} = \sigma_{m2} F''_{(y)} \left(\frac{a}{m\pi}\right)^2 \cos\frac{m\pi x}{a}$$

we get as the energy of the normal stresses

$$L_{a2} = \sigma_{m2} f_{mn}^2 \frac{\delta}{8} \int_0^a\int_0^b F''_{(y)} \cos\frac{m\pi x}{a}\left(1 - \cos\frac{m\pi x}{a}\right)\left(1 + \cos\frac{n\pi y}{b}\right) dx\,dy$$

or

$$L_{a2} = -\sigma_{m2} f_{mn}^2 \frac{\delta}{16} a \int_0^b F''_{(y)} \left(1 + \cos\frac{n\pi y}{b}\right) dy.$$

Here

$$\int_0^b F''_{(y)}\,dy = [F'_{(y)}]_0^b = 0.$$

Then

$$L_{a2} = -\sigma_{m2} f_{mn}^2 \frac{\delta a}{16} \int_0^b F''_{(y)} \cos \frac{n\pi y}{b} dy .$$

By partial integration

$$L_{a2} = -\sigma_{m2} f_{mn}^2 \frac{\delta a}{16} \left(F'_{(y)} \cos \frac{n\pi y}{b} \right)_0^b + \int_0^b \left(\frac{n\pi}{b} \right) F'_{(y)} \sin \frac{n\pi y}{b} dy$$

$$= -\sigma_{m2} f_{mn}^2 \frac{\delta a}{16} \left(\frac{n\pi}{b} F_{(y)} \sin \frac{n\pi y}{b} \right)_0^b - \left(\frac{n\pi}{b} \right)^2 \int_0^b F_{(y)} \cos \frac{n\pi y}{b} dy$$

$$= -\sigma_{m2} f_{mn}^2 \frac{\delta a}{16} \left[- \left(\frac{n\pi}{b} \right)^2 \left(\frac{a}{m\pi} \right)^2 F''_{(y)} + \left(\frac{n\pi}{b} \right)^2 \frac{2a}{bm\pi} \frac{\mathfrak{Cof} \frac{m\pi y}{a}}{\mathfrak{Sin} \frac{m\pi b}{a}} \right].$$

Therefore

$$L_{a2} = -\sigma_{m2} f_{mn}^2 \frac{\delta a}{16} \frac{\left(\frac{n\pi}{b} \right)^2 \frac{2a}{bm\pi}}{1 + \left(\frac{na}{mb} \right)^2} \int \frac{\mathfrak{Cof} \frac{m\pi y}{a} dy}{\mathfrak{Sin} \frac{m\pi b}{a}} ,$$

$$L_{a2} = - \frac{\sigma_{m2} f_{mn}^2 \delta a}{16 \mathfrak{Sin} \frac{m\pi b}{a}} \frac{\frac{2a}{bm\pi} \left(\frac{n\pi}{b} \right)^2 \left(\frac{a}{m\pi} \right)^2}{\left[1 + \left(\frac{an}{bm} \right)^2 \right]^2} \left[\frac{m\pi}{a} \mathfrak{Sin} \frac{m\pi y}{a} \cos \frac{n\pi y}{b} + \frac{n\pi}{b} \mathfrak{Cof} \frac{m\pi y}{a} \sin \frac{n\pi y}{b} \right]_0^b ,$$

$$L_{a2} = +\sigma_{m2} f_{mn}^2 \frac{\left(\frac{an}{bm} \right)^2}{\left[1 + \left(\frac{an}{bm} \right)^2 \right]^2} .$$

For the second part of σ_{x2} we get

$$L_a = -\sigma_{n2} \frac{\delta}{8} \left(\frac{m\pi}{a} \right)^2 f_{mn}^2 \int_0^a \int_0^b F_{(x)} \cos \frac{n\pi y}{b} \left(1 - \cos \frac{m\pi x}{a} \right) \left(1 + \cos \frac{n\pi y}{b} \right) dx\, dy ,$$

$$L_a = -\sigma_{n2} \frac{\delta}{8} f_{mn}^2 \left(\frac{m\pi}{a} \right)^2 \frac{b}{2} \int_0^a F_{(x)} \left(1 - \cos \frac{m\pi x}{a} \right) dx .$$

The first part

$$L_{a3} = -\sigma_{n2} \frac{\delta}{8} f_{mn}^2 \frac{b}{2} \left(\frac{m\pi}{a} \right)^2 \int_0^a F_{(x)} dx$$

yields

$$L_{a3} = -\sigma_{n2}\frac{\delta}{8}f_{mn}^2\frac{b}{2}\left(\frac{m\pi}{a}\right)^2\frac{\frac{b}{n\pi}}{\mathfrak{Sin}\,\frac{n\pi a}{b}}\left[\left(-\mathfrak{Cotg}\,\frac{n\pi a}{b}-\frac{2b}{an\pi}\right)\mathfrak{Sin}\,\frac{n\pi x}{b}+\frac{x}{a}\,\mathfrak{Cof}\,\frac{n\pi x}{b}\right]_0^a.$$

$$L_{a3} = +\sigma_{n2}\frac{\delta}{8}f_{mn}^2\frac{b}{a}\left(\frac{bm}{an}\right)^2.$$

From

$$J = \left(\frac{m\pi}{a}\right)^2\int_0^a F_{(x)}\cos\frac{m\pi x}{a}\,dx$$

it follows by partial integration that

$$= -\left(\frac{m\pi}{a}\right)^2\left[F_{(x)}\left(\frac{a}{m\pi}\right)\sin\frac{m\pi x}{a}\right]_0^a+\left(\frac{m\pi}{a}\right)^2\left(\frac{a}{m\pi}\right)\int_0^a F'_{(x)}\sin\frac{m\pi x}{a}\,dx$$

$$= \left[F'_{(x)}\cos\frac{m\pi x}{a}\right]_0^a-\int_0^a F''_{(x)}\cos\frac{m\pi x}{a}\,dx$$

$$= -\int_0^a\left(\frac{n\pi}{b}\right)^2\left(F_{(x)}+\frac{2b}{an\pi}\cdot\frac{\mathfrak{Cof}\,\frac{n\pi x}{b}}{\mathfrak{Sin}\,\frac{n\pi b}{a}}\right)\cos\frac{m\pi x}{a}\,dx.$$

Then

$$J = -\frac{\left(\frac{n\pi}{b}\right)^2\cdot\frac{2b}{an\pi}}{\left[1+\left(\frac{an}{bm}\right)^2\right]}\cdot\int_0^a\frac{\mathfrak{Cof}\,\frac{n\pi x}{b}\cos\frac{m\pi x}{a}\,dx}{\mathfrak{Sin}\,\frac{n\pi a}{b}}$$

and

$$L_{a4} = \sigma_{n2}\frac{\delta}{8}f_{mn}^2\frac{b}{a}\frac{\left(\frac{bm}{an}\right)^2}{\left[1+\left(\frac{bm}{an}\right)^2\right]^2}.$$

Similarly we get for the stress σ_{y2}

$$L_{a5} = \sigma_{m2}\frac{\delta}{8}f_{mn}^2\left(\frac{a}{b}\right)\left(\frac{an}{bm}\right)^2,$$

$$L_{a6} = \sigma_{m2}\frac{\delta}{8}f_{mn}^2\left(\frac{a}{b}\right)\frac{\left(\frac{an}{bm}\right)^2}{\left[1+\left(\frac{an}{bm}\right)^2\right]^2},$$

$$L_{a7} = \sigma_{n2}\frac{\delta}{8}f_{mn}^2\frac{b}{a}\frac{\left(\frac{bm}{an}\right)^2}{\left[1+\left(\frac{bm}{an}\right)^2\right]^2}.$$

Energy of the Shear Stresses.

By double differentiation we find

$$F'''_{(y)} = \left(\frac{m\pi}{a}\right)^2\cdot\left(F'_{(y)}+\frac{2}{b}\frac{\mathfrak{Sin}\,\frac{m\pi y}{a}}{\mathfrak{Sin}\,\frac{m\pi b}{a}}\right).$$

The energy of the shear stresses τ_1 will be

$$L_{a8} = \sigma_{m2} \cdot \frac{\delta}{4} \frac{n\pi}{b} \int_0^a \int_0^b F'_{(y)} \sin^2 \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx\, dy$$

$$\int_0^b F'_{(y)} \sin \frac{n\pi y}{b} dy = \left(F'_{(y)} \frac{b}{n\pi} \cos \frac{n\pi y}{b} \right)_0^b + \frac{a}{2} \frac{b}{n\pi} \int F''_{(y)} \cos \frac{n\pi y}{b} dy\,.$$

$$J = \left(\frac{b}{n\pi}\right)^2 \left[F''_{(y)} \sin \frac{n\pi y}{b} \right]_0^b - \left(\frac{b}{n\pi}\right)^2 \int F'''_{(y)} \sin \frac{n\pi y}{b} dy,$$

$$J = -\left(\frac{b}{n\pi}\right)^2 \cdot \left(\frac{m\pi}{a}\right)^2 \cdot \left[\int_0^b F'_{(y)} \sin \frac{n\pi y}{b} dy + \frac{2}{b} \int_0^b \frac{\mathfrak{Sin}\, \frac{m\pi y}{a} \cdot \sin \frac{m\pi y}{b} dy}{\mathfrak{Sin}\, \frac{m\pi b}{a}} \right],$$

$$J = -\frac{\left(\frac{bm}{an}\right)^2 \frac{2}{b}}{\left(1 + \left(\frac{bm}{an}\right)^2\right)} \cdot \int_0^b \frac{\mathfrak{Sin}\, \frac{m\pi y}{a} \cdot \sin \frac{n\pi y}{b} dy}{\mathfrak{Sin}\, \frac{m\pi b}{a}},$$

$$J = \frac{\left(\frac{bm}{an}\right)^2 \frac{2}{b}}{\left(1 + \left(\frac{bm}{an}\right)^2\right)} \cdot \frac{1}{\mathfrak{Sin}\, \frac{m\pi b}{a}} \cdot \frac{\left(\frac{a}{m\pi}\right)^2}{\left(1 + \left(\frac{an}{bm}\right)^2\right)} \cdot \left[\frac{m\pi}{a} \mathfrak{Cof}\, \frac{m\pi y}{a} \cdot \sin \frac{n\pi y}{b} - \frac{n\pi}{b} \mathfrak{Sin}\, \frac{m\pi y}{a} \cos \frac{n\pi y}{b} \right]_0^b,$$

$$J = -\frac{2}{b} \frac{\left(\frac{a}{m\pi}\right)^2 \cdot \frac{n\pi}{b}}{\left[1 + \left(\frac{an}{bm}\right)^2\right]^2};$$

$$L_{a8} = -\sigma_{m2} \frac{\delta}{4} \frac{a}{b} \frac{\left(\frac{an}{bm}\right)^2}{\left[1 + \left(\frac{an}{bm}\right)^2\right]^2}.$$

Similarly we get for the shear stress τ_2

$$L_{a9} = -\sigma_{n2} \frac{\delta}{4} \frac{b}{a} \frac{\left(\frac{bm}{an}\right)^2}{\left[1 + \left(\frac{bm}{an}\right)^2\right]^2}.$$

If we compare the strain energy due to the stresses σ_{x2} , σ_{y2} and τ , we find that L_{a8} and L_{a9} cancel out L_{a4} and L_{a6} , and L_{a2} cancels out L_{a7}.

There remains as the energy of uniform stresses

$$L_a = -\frac{\delta}{2} \pi^2 ab f_{mn}^2 \left[\left(\frac{m}{2a}\right)^2 \sigma_{xr} - \left(\frac{m}{2a}\right)^2 \sigma_{x1} + \left(\frac{n}{2b}\right)^2 \sigma_{yr} - \left(\frac{n}{2b}\right)^2 \sigma_y \right].$$

The energy of the non-uniform stresses

$$L_a = \sigma_{m2} \frac{\delta}{8} f_{mn}^2 \left(\frac{a}{b}\right) \left(\frac{an}{bm}\right)^2 + \sigma_{n2} \frac{\delta}{8} f_{mn}^2 \left(\frac{b}{a}\right) \left(\frac{bm}{an}\right)^2.$$

Then the change in the total energy when the amount of deflection is changed by Δf_{mn} will be

$$\frac{\Delta L_a}{\Delta f_{mn}} = -\delta\pi^2 ab f_{mn}^2 \sigma_{xr}\left[\left(\frac{m}{2a}\right)^2 + \frac{\sigma_{yr}}{\sigma_{xr}}\left(\frac{n}{2b}\right)^2\right]$$

$$+ \delta\pi^2 ab f_{mn}^2\left[\sigma_{x1}\left(\frac{m}{2a}\right)^2 + \sigma_{y1}\left(\frac{n}{2b}\right)^2 + \frac{\sigma_{n2}}{\pi^2}\left(\frac{bm}{an}\right)^2\left(\frac{1}{2a}\right)^2 + \frac{\sigma_{m2}}{\pi^2}\left(\frac{an}{bm}\right)^2\left(\frac{1}{2b}\right)^2\right],$$

$$\sigma_{xr} = \frac{N}{\delta}\pi^2\cdot\frac{\left[\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2\right]^2}{\left(\frac{m}{2a}\right)^2 \pm \frac{\sigma_{yr}}{\sigma_{xr}}\left(\frac{n}{2b}\right)^2} + \frac{\sigma_{x1}\left(\frac{m}{2a}\right)^2 + \sigma_{y1}\left(\frac{n}{2b}\right)^2 + \sigma_{n2}\left(\frac{1}{2a\pi}\right)^2\left(\frac{bm}{an}\right)^2 + \sigma_{m2}\left(\frac{1}{2b\pi}\right)^2\left(\frac{an}{bm}\right)^2}{\left(\frac{m}{2a}\right)^2 \pm \frac{\sigma_{yr}}{\sigma_{xr}}\left(\frac{n}{2b}\right)^2}.$$

The change in the internal energy will be

$$\frac{\Delta L_i}{\Delta f_{mn}} = Nab f_{mn}\pi^4\left(\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2\right)^2.$$

From this follows the mean stress σ_{xr} ,

$$\Delta L_a + \Delta L_i = 0.$$

The effective width is easy to calculate. The mean of the stresses is

$$\sigma_{xr} - \sigma_{x1}, \qquad \text{since} \qquad \int_0^b [\sigma_{x2}]dy = 0.$$

Therefore

$$b_m = \frac{\sigma_{xr} - \sigma_{x1}}{\sigma_{xr}}.$$

When $x = 0$, the b_m value is more favorable, and when $x = a$ it is less favorable. For the buckling of the longitudinal, however, the mean value must be used.