Report 4259

ACCURACY AND PRECISION OF ALGORITHMS FOR BESSEL FUNCTIONS OF FIRST KIND AND INTEGER ORDER

L SHIP RESEARCH AND DEVELOPMENT CENTER

Bethesda, Md. 20034

ACCURACY AND PRECISION OF ALGORITHMS FOR BESSEL FUNCTIONS OF FIRST KIND AND INTEGER ORDER

by

T. Kooij

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SHIP ACOUSTICS DEPARTMENT

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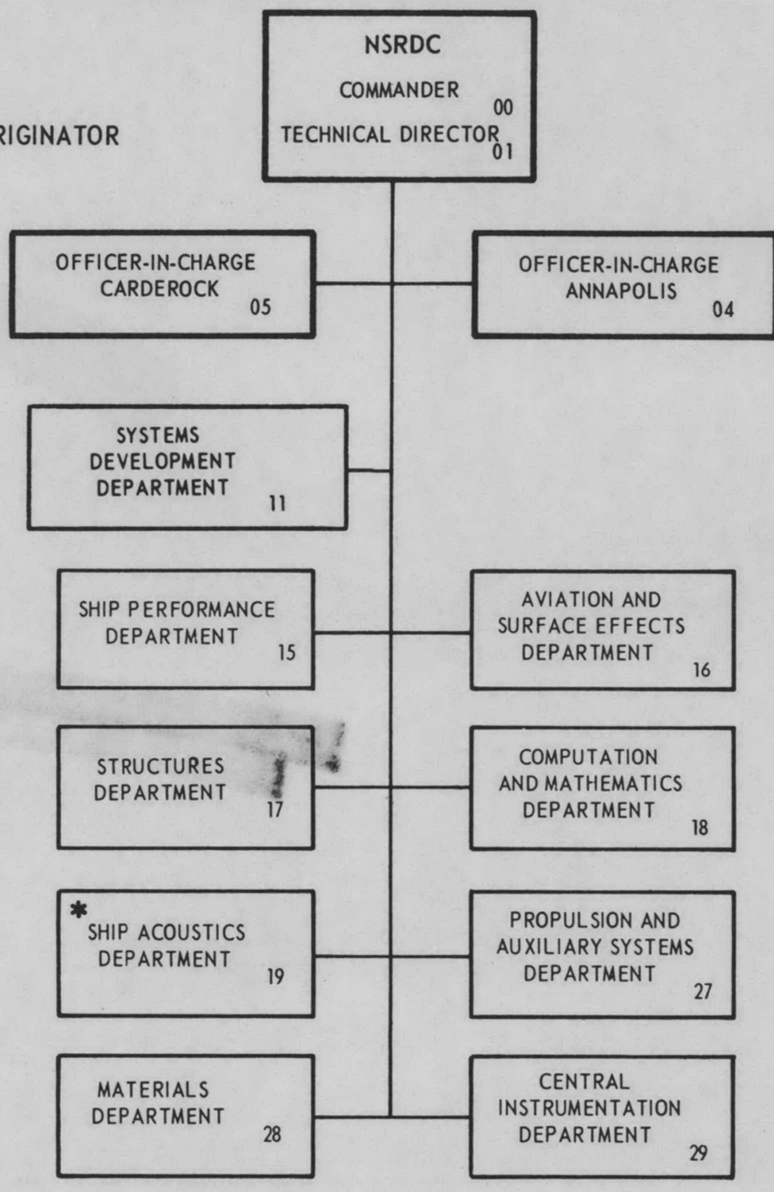

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DEPARTMENT OF THE NAVY
NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER
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ABSTRACT

Three different algorithms—the power series, the asymptotic series, and the recurrence relation method—are investigated with special attention to the single (10^{-14}) and double precision (10^{-29}) of Computer Data Corporation (CDC) 6000 Series computers. The final accuracy of each method depends partly on the magnitudes of the largest and smallest terms when floating point additions are involved. Another consideration is the number of terms required for each algorithm. Combination of all considerations leads to a partitioning of the order-argument domain into partially overlapping areas in which each algorithm is most appropriate. A wedged area not covered by any of the algorithms remains for large order and argument of approximately equal size.

Orders and arguments up to 1024 were investigated and checked where possible. A FORTRAN IV program in the form of an external function is included.

ADMINISTRATIVE INFORMATION

The particular work addressed herein was supported by the Naval Ship Systems Command (037) under Program Element 25684, Project S4628, Task Area S4628-019, Naval Ship Research and Development Center Work Unit 1932-010.

INTRODUCTION

Three well-known algorithms to determine the Bessel function $J_L(X)$ are:

(1) The power series expansion[1] for X/L not too large.

[1]Abramowitz, M. and I.A. Stegun (Ed.), "Handbook of Mathematical Functions," National Bureau of Standards, U.S. Government Printing Office, Washington, D.C. (1954), p. 360.

2. The asymptotic series expansion[2] for X/L large.

3. The recurrence relation.[3,4]

The power series converges always. However, for large values of X/L, a large number of terms is necessary to reach a given precision. More seriously, the terms initially increase in magnitude, starting to decrease only after going through a maximum. When the terms are summed by employing floating point techniques, using a fixed number of significant digits, the accuracy of the final answer decreases by the same order as the order of magnitude of the maximum term. In other words, the final accuracy corresponds to the number of digits available, counted from the most significant digit of the maximum term.

The asymptotic series has an additional problem. Since it eventually starts to diverge, the minimum term may not be small enough to reach the required precision. In that case this method cannot be applied without further refinements. Even if the required precision can be reached, the asymptotic series may still be inappropriate because of the number of terms involved or because of the magnitude of its maximum term, which affects the accuracy the same way as described for the power series in the previous paragraph.

---

[2] Abramowitz, M. and I.A. Stegun (Ed.), "Handbook of Mathematical Functions," National Bureau of Standards, U.S. Government Printing Office, Washington, D.C. (1954), p. 364.

[3] Abramowitz, M. and I.A. Stegun, "Generation of Bessel Functions on High Speed Computers,"Mathematical Tables and Aids to Computation," Vol. II (1957), pp. 255-257.

[4] British Association for the Advancement of Science, "Mathematical Tables," Vol. X, Bessel Functions, Part II, Functions of Positive Integer Order, Cambridge University Press, Cambridge (1952), p. XIV.

The algorithm based on the recurrence relation appears to be the least sensitive method. The only disadvantage is that the normalizing factor of the algorithm consists of the sum of all even order Bessel functions, down from approximately twice the value of the argument. For large arguments, this number may become prohibitive. The effect of such long sums on the accuracy of the method has not been determined explicitly in this paper, but some checks[5] showed that the absolute accuracy was always within three decimals of the machine precision. Usually it was much better.

The program given in Appendix A applies bounds that are based on a number of considerations described in detail in the chapters that follow. These considerations concern the number of terms required and the magnitudes of the maximum and minimum terms. The precisions for which the bounds are derived correspond to the single (10^{-14}) and double (10^{-29}) precision modes of the Control Data Corporation (CDC) 6000 series computers.

Repeated use will be made of a form of the Stirling[6] asymptotic approximation to the factorial:

$$K! \sim \sqrt{2\pi K}\left(\frac{K}{e}\right)^K = \sqrt{2\pi}\,\frac{K^{K+\frac{1}{2}}}{e^K} \tag{1}$$

This approximation is already accurate to 2 percent when K = 4.

[5] Hayashi, K, "Tafeln der Besselschen, Theta, Kugel and anderer Funktionen," Springer Verlag, Berlin, Germany (1940).

[6] Feller, W., "An Introduction to Probability Theory and Its Applications," John Wiley and Sons, Inc., New York, Vol. I (1968), p. 52.

POWER SERIES EXPANSION

The power series expansion for $J_L(X)$ is[1]

$$J_L(X) \approx \frac{\left(\frac{X}{2}\right)^{L}}{L!} - \frac{\left(\frac{X}{2}\right)^{2+L}}{1!\,(1+L)!} + \frac{\left(\frac{X}{2}\right)^{4+L}}{2!\,(2+L)!} \cdots (-1)^{K} \frac{\left(\frac{X}{2}\right)^{2K+L}}{K!\,(K+L)!} + \cdots \tag{2a}$$

$$= \frac{\left(\frac{X}{2}\right)^{L}}{L!}\left[1 - \frac{\left(\frac{X}{2}\right)^{2}}{1+(1+L)} + \cdots\right] + (-1)^{K} \frac{\left(\frac{X}{2}\right)^{2(K-1)+L}}{(K-1)!\,(K-1+L)!}\; \frac{\left(\frac{X}{2}\right)^{2}}{K(K+L)} + \cdots \tag{2b}$$

The desired precision q is reached when the last term T_K becomes less than that precision. The condition is thus

$$T_K \le q \tag{3}$$

where

$$T_K = \frac{\left(\frac{X}{2}\right)^{2K+L}}{K!\;(K+L)!} \sim \frac{\left(\frac{X}{2}\right)^{2K+L} e^{K+K+L}}{2\pi\; K^{K+\frac{1}{2}}\;(K+L)^{K+L+\frac{1}{2}}} \tag{4}$$

Stirling's approximation to the factorial has been used to obtain the final expression in Equation (4).

Instead of solving for the K that satisfies Equation (3), it takes in general less computer time to test if the last term T_K has indeed reached the desired precision q.

The precision of the last term does not guarantee the accuracy of the final answer. The magnitude of the maximum term may be several orders of magnitude larger than one, so that an equal amount of accuracy is lost in the final answer of this alternating sign series if floating point techniques are employed. Therefore, the term with the largest magnitude must be determined. This term occurs when the incremental multiplier in the last term in Equation (2b) first becomes less than one:

$$\frac{\left(\frac{X}{2}\right)^2}{K(K+L)} < 1 \tag{5}$$

This implies that

$$X^2/4 < K^2 + K\,L$$

or

$$K > K_1 \tag{6}$$

where

$$K_1 = -\tfrac{1}{2}L + \tfrac{1}{2}\sqrt{L^2 + X^2} \tag{7}$$

The magnitude T_{K_1} of this maximum term follows from Equations (4) and (7).

$$TK_1 = \frac{\left(\frac{Xe}{2}\right)^{2K_1+L}}{2\pi[K_1(K_1+L)]^{K_1+\frac{1}{2}}(K_1+L)^L} = \frac{\left(\frac{Xe}{2}\right)^{2K_1+L}}{2\pi[(-\frac{1}{2}L+\frac{1}{2}\sqrt{L^2+X^2})(\frac{1}{2}L+\frac{1}{2}\sqrt{L^2+X^2})]^{K_1+\frac{1}{2}}(K_1+L)^L}$$

$$= \frac{\left(\frac{X}{2}\right)^{2K_1+L}\exp(-L+\sqrt{L^2+X^2}+L)}{2\pi\left(\frac{X}{2}\right)^{2K_1+1}(\frac{1}{2}L+\frac{1}{2}\sqrt{L^2+X^2})^L} = \frac{\exp[X\sqrt{(L/X)^2+1}]}{\pi X[L/X+\sqrt{(L/X)^2+1}]^L} \tag{8}$$

The behavior of the magnitude of the maximum term T_{K_1} of Equation (8) is shown in Table 1.

TABLE 1 - MAGNITUDE T_{K_1} OF MAXIMUM TERM IN POWER SERIES OF $J_L(X)$

L \ X	1	10	100	1000
1	0.5 (K=1)	600 (K=5)	10^{41} (K=50)	10^{431} (K=500)
10	10^{-9} (K=1)	6 (K=2)	10^{40} (K=45)	10^{430} (K=495)
100	10^{-187} (K=1)	10^{-88} (K=1)	10^{20} (K=21)	10^{429} (K=452)
1000	0 (K=0)	0 (K=1)	10^{-868} (K=2)	10^{228} (K=207)

A more detailed table is given in Appendix B. If a loss in accuracy of no more than two or three decimals is required, the maximum term must be less than $10^{2.5}$ or approximately 300.

ASYMPTOTIC SERIES EXPANSION

The asymptotic series expansion for $J_L(X)$ is[2]

$$J_L(X) = \sqrt{\frac{2}{\pi X}}\,[P(L,X)\cos Y - Q(L,X)\sin Y] \tag{9}$$

with

$$Y = X - (\tfrac{1}{2}L + \tfrac{1}{4})\pi$$

$$P(L,X) \sim 1 - \frac{(4L^2-1^2)(4L^2-3^2)}{2!\,(8X)^2} + \frac{(4L^2-1^2)(4L^2-3^2)(4L^2-5^2)(4L^2-7^2)}{4!\,(8X)^4} - \cdots \tag{10}$$

$$Q(L,X) \sim \frac{(4L^2-1^2)}{1!\,8X} - \frac{(4L^2-1^2)(4L^2-3^2)(4L^2-5^2)}{3!\,(8X)^3} + \cdots \tag{11}$$

Consider the term

$$R_K(L,X) = \frac{\left|[(2L)^2-1^2][(2L)^2-3^2]\cdots[(2K-1)^2-(2L)^2]\right|}{K!\ (8X)^K} \tag{12a}$$

$$= R_{K-1}(L,X)\ \frac{\left|(2K-1)^2-(2L)^2\right|}{K\ 8X} \tag{12b}$$

These terms decrease as long as

$$\frac{\left|(2K-1)^2-(2L)^2\right|}{K\ 8X} < 1 \tag{13}$$

This yields

$$4L^2 - 4K^2 + 4K - 1 - 8XK < 0$$

$$\text{and}\quad 4K^2 - 4K + 1 - 4L^2 - 8XK < 0 \tag{14}$$

Solving Equation (14) gives for the interval of K for which the terms R_K decrease from maximum to minimum magnitude

$$K_2 < K < K_3 \tag{15}$$

where

$$K_2 = -\tfrac{1}{2}(2X-1) + \tfrac{1}{2}\sqrt{(2X-1)^2 + 4L^2 - 1} \tag{16}$$

and

$$K_3 = \tfrac{1}{2}(2X+1) + \tfrac{1}{2}\sqrt{(2X+1)^2 + 4L^2 - 1} \tag{17}$$

It is obvious that $K_2 < L < K_3$, so that $K_3 - L$ and $L - K_2$ are both positive, a fact that will be used in the four following equations. From Equation (12a) it follows that the K^{th} term, if $K > L$, can be written as

$$R_K(L,X) = \frac{[(1+2L)(3+2L)\cdot\cdot(2K-1+2L)][(2L-1)(2L-3)\cdot\cdot 3.1.1.3\cdot\cdot(2K-1-2L)]}{K\ (8X)^K}$$

$$= \frac{(2K+2L)!\ L!}{(2L)!\ 2^K\ (K+L)!}\ \frac{(2L)!\ (2K-2L)!}{2^L\ L!\ 2^{K-L}\ (K-L)!}\ \frac{1}{K!\ (8X)^K}$$

$$= \frac{(2K+2L)!\ (2K-2L)!}{(K+L)!\ (K-L)!}\ \frac{1}{2^{2K}}\ \frac{1}{K}\ \frac{1}{2^{3K}}\ \frac{1}{X^K}\ (K > L) \qquad (18)$$

Similarly, for $K < L$

$$R_K(L,X) = \frac{[(2L+1)(2L+3)\cdot\cdot(2L+2K-1)][(2L-1)(2L-3)\cdot\cdot(2L-2K+1)]}{K!\ (8X)^K}$$

$$= \frac{(2L+2K)!\ L!}{(2L)!\ 2^K\ (L+K)!}\ \frac{(2L)!\ (L-K)!}{(2L-2K)!\ 2^K\ L!}\ \frac{1}{K!}\ \frac{1}{(8X)^K}$$

$$= \frac{(2L+2K)!\ (L-K)!}{(L+K)!\ (2L-2K)!}\ \frac{1}{2^{5K}}\ \frac{1}{K!}\ \frac{1}{X^K}\ (K < L) \qquad (19)$$

Using Stirling's formula, we find asymptotically

$$R_K(L,X) \sim \sqrt{\frac{2}{\pi K}}\ \frac{(K+L)^{K+L}(K-L)^{K-L}}{K^K\ 2^K\ e^K\ X^K} = \sqrt{\frac{2}{\pi K}}\ \left(\frac{K^2-L^2}{K2\ Xe}\right)^K\left(\frac{K+L}{K-L}\right)^L (K > L) \quad (20)$$

and

$$R_K(L,X) \sim \frac{1}{\sqrt{2\pi K}}\ \left(\frac{L^2-K^2}{K2Xe}\right)^K\left(\frac{L+K}{L-K}\right)^L \quad (K < L) \qquad (21)$$

By substituting K_2 and K_3 from Equations (16) and (17) in Equations (21) and (20), respectively, we obtain the maximum and minimum magnitude of the asymptotic terms. The minimum term should be within the precision required of the answer, while the accuracy is determined by the number of decimals lost in floating point arithmetic, i.e., by the order of magnitude of the maximum term R_{K_2}.

If either of these requirements cannot be met, a different algorithm must be used.

Tables of terms with maximum and minimum magnitudes are given in Appendix C.

RECURRENCE RELATION

The recurrence relation for Bessel functions is[3,4]

$$J_{L-1}(X) = \frac{2L}{X} J_L(X) - J_{L+1}(X) \tag{22}$$

with the normalization constraint

$$J_0(X) + 2 \sum_{N=1}^{\infty} J_{2N}(X) = 1 \tag{23}$$

The algorithm based on this recurrence relation starts with setting $J_{L+1}(X)$ equal to zero and $J_L(X)$ equal to a (small) constant. A running count is kept of the normalizing sum, Equation (23). The final answer is obtained by dividing the value resulting from the recursive iteration by the normalizing factor. The method is remarkably insensitive to the starting point and the starting value. It is advisable to start the recursion at an order L for which $J_L(X)$ equals approximately the desired precision q, which is usually the machine precision available.

This starting order is readily estimated from the power series expansion given by Equation (2b):

$$\frac{\left(\frac{X}{2}\right)^L}{L!} \sim \frac{1}{\sqrt{2\pi L}} \left(\frac{Xe}{2L}\right)^L = q \tag{24}$$

and, again from Equation (2b),

$$\frac{\left(\frac{X}{2}\right)^2}{1+L} < 1 \tag{25}$$

The latter inequality ensures that the series in parentheses in Equation (2b) converges from the start. Since the normalizing factor of Equation (23) has to be carried all the way down to the zero order function $J_0(X)$, a too high starting order may be computationally unacceptable; however, in that case an asymptotic expansion may already perform well. An additional advantage of the recurrence relation method is that function values for all integer descending orders may be obtained by the same effort.

Estimates based on Equation (24) for the starting order L of the recurrence relation of Equation (22) are given in Appendix D. From the results of Appendix D it follows that $\underline{L=1.4X + 25}$ (for single precision $q=10^{-14}$) or $\underline{L=1.6X + 40}$ (for double precision $q=10^{-29}$) are good estimates for starting orders of the recurrence relation algorithm for values of X up to 200. In general, one would not like to extend the recursion over more than 200 terms, in the first place because of possible loss of accuracy involving the normalizing sum of Equation (23), and in the second place because of computational efficiency. From the requirement that L be less than 200, we can derive the requirement that X be less than 125 (for single precision $q=10^{-14}$), or that X be less than 100 (for double precision $q=10^{-29}$).

How accurate the obtained results are for orders close to their starting order could not be established exactly, but it is probably close to or better than the machine precision. Where the values obtained by the recurrence relation algorithm could be checked against Reference 5, it was generally found that they were more accurate than the corresponding values obtained by the power series or asymptotic series algorithms, which always suffer from the loss of accuracy due to the magnitude of the maximum term occurring early in this alternating series, as was explained in the previous chapters.

The high accuracy obtained by the recurrence method over the full range of orders L, together with the simplicity of the algorithm itself (three multiplications and two additions per step, following by two memory exchanges), makes the recurrence relation algorithm very attractive for the computation of $J_L(X)$. For this reason, some of the boundaries indicating when the power series expansion and when the asymptotic series expansion can be used may be relaxed with respect to the results from Appendices B and C. The boundaries actually used in the program are given in the next chapter.

ORDER-ARGUMENT DOMAIN COVERED BY ALGORITHMS

The three algorithms into which the calculation of the Bessel function $J_L(X)$ is divided occupy the following regions in the order-argument (L-X) domain:

1. Power series expansion when

   <u>X less than 1 or X less than $\frac{1}{2}$L</u> .

2. Asymptotic series expansion when:

<u>X greater than 50 and X greater than $\frac{1}{4}L^2$</u> (for mach.prec.10^{-29})

or <u>X greater than 30 and X greater than $\frac{1}{4}L^2$</u> (for mach.prec.10^{-14})

3. Recurrence relation method when:

<u>X less than 100 and L less than 200</u> (for mach.prec.10^{-29})

or <u>X less than 125 and L less than 200</u> (for mach.prec.10^{-14}).

The coverage of the different algorithms is shown graphically in Figure 1. A wedged area uncovered by any of the algorithms remains for large orders and arguments of approximately equal magnitude.

Outside the rectangular area in the L-X domain for which the recurrence relation algorithm seems indicated, the boundaries for the power series and asymptotic series expansions could be tightened again, but this has not been attempted in the program presented in Appendix A. Instead, the program notifies the user of any entries into this forbidden zone, but returns with a Stirling approximation to the first term in the power series - which may be completely wrong - as an exit value.

CHECKING THE RESULTS

To check the accuracy of the algorithms, Bessel functions $J_L(X)$ were calculated by the recurrence relation method for arguments from 1 to 10 in steps of 1, and from 10 to 100 in steps of 10. The region covered in the L-X domain in this way is pictured as the hatched area in Figure 2.

Also indicated in Figure 2 are the points in the L-X domain for which values of the Bessel function were calculated by the power series expansion or by the asymptotic series expansion. The values of L and X were chosen close to the boundaries assumed by the FORTRAN program of Appendix A.

The required accuracy — three decimals less than the machine precision — was indeed obtained; this is in accordance with the predictions of the previous chapters. The accuracy obtained by the recurrence relation method was generally better.

Some of the results obtained are given below.

$J_o(1)$	(Ref.5)	0.76519	76865	57966	55144	97175	26103
(recurrence rel.)							2612
(power series)							2604
$J_o(10)$	(Ref.5)	-0.24593	57644	51348	33519	77608	62485
(recurrence rel.)							6253
(power series)							4585
$J_1(1)$	(Ref.5)	0.44005	05857	44933	51595	96822	03719
(recurrence rel.)							0372
(power series)							0367
$J_1(40)$	(Ref.5)	0.12603	83180	37584	99920	56027	21839
(recurrence rel.)							2185
(asympt.series)							2179
$J_1(50)$	(Ref.5)	-0.09751	18281	25175	13766	14589	53873
(recurrence rel.)							53721
(asympt.series)							53782
$J_{16}(50)$	(Ref.5)	0.00489	81607	77813	78173	17342	69265
(recurrence rel.)							68979
(asympt.series)							69234
$J_{30}(1)$	(Ref.5)	$.0_{41}$	3482	8697			
(recurrence rel.)			3482	86979	42514		
(power series			3482	75946	71184		
$J_{128}(64)^*$		$.0_{26}$	3241	50085	84477	63106	
(recurrence rel.)			3241	50085	84477	63102	
(power series)			3241	82782			

* From a 60-term power series expansion; also from a recurrence relation calculation starting at $J_{150}(64)$.

The results given here show for example that $J_0(10)$, when computed by using the power series expansion algorithm, is accurate only to 25 decimal places. This is consistent with the fact that the maximum terms ($T_4=T_5=678=10^{2.83}$, see Table 4 in Appendix B) decrease the accuracy by almost 3 decimal places. Note that the program of Appendix A would have chosen the recurrence relation method to compute $J_0(10)$, giving a result that is accurate to 28 decimal places.

Taking $J_{16}(50)$ as another example, we can derive from the chapter on coverage of the L-X domain that the program would again have chosen the recurrence relation algorithm. The asymptotic series algorithm would in this case have been slightly more accurate, but both results are good to more than 26 decimal places as was desired.

$J_{30}(1)$ begins to show a difference at the fifth significant digit of the power series result. This is because of the fact that the algorithm takes one more term after detecting that the machine precision has been reached, which in this case is the very first term. The third term is $3\text{x}10^{-5}$ times the first term and this accounts for the difference.

The last result, $J_{128}(64)$, again would have been calculated by using the recurrence relation method; according to the rule of thumb given in Appendix D, the recursion starts with $L = 1.6\text{x}64 + 40 = 142$, i.e., with $J_{142}(64) = 0$ and $J_{141}(64) = 10^{-29}$. The maximum term in the power series expansion is $T_7 = 10^{-20.4}$ according to Table 4 in Appendix B, so no loss of accuracy is expected from this effect. However, the summation of the power series algorithm ends when the terms become smaller than

10^{-29} ($=T_{30}$ approximately); we find indeed that the power-series result is accurate to 30 decimal places. By continuing the power-series expansion further (to 60 terms), the more accurate result of the footnote on page 13 was obtained.

ACKNOWLEDGMENT

I gratefully acknowledge the stimulating interest of Dr. J.W. Wrench, Jr. in the course of development of this study, and his help in providing—not without effort sometimes—independent answers to check the results.

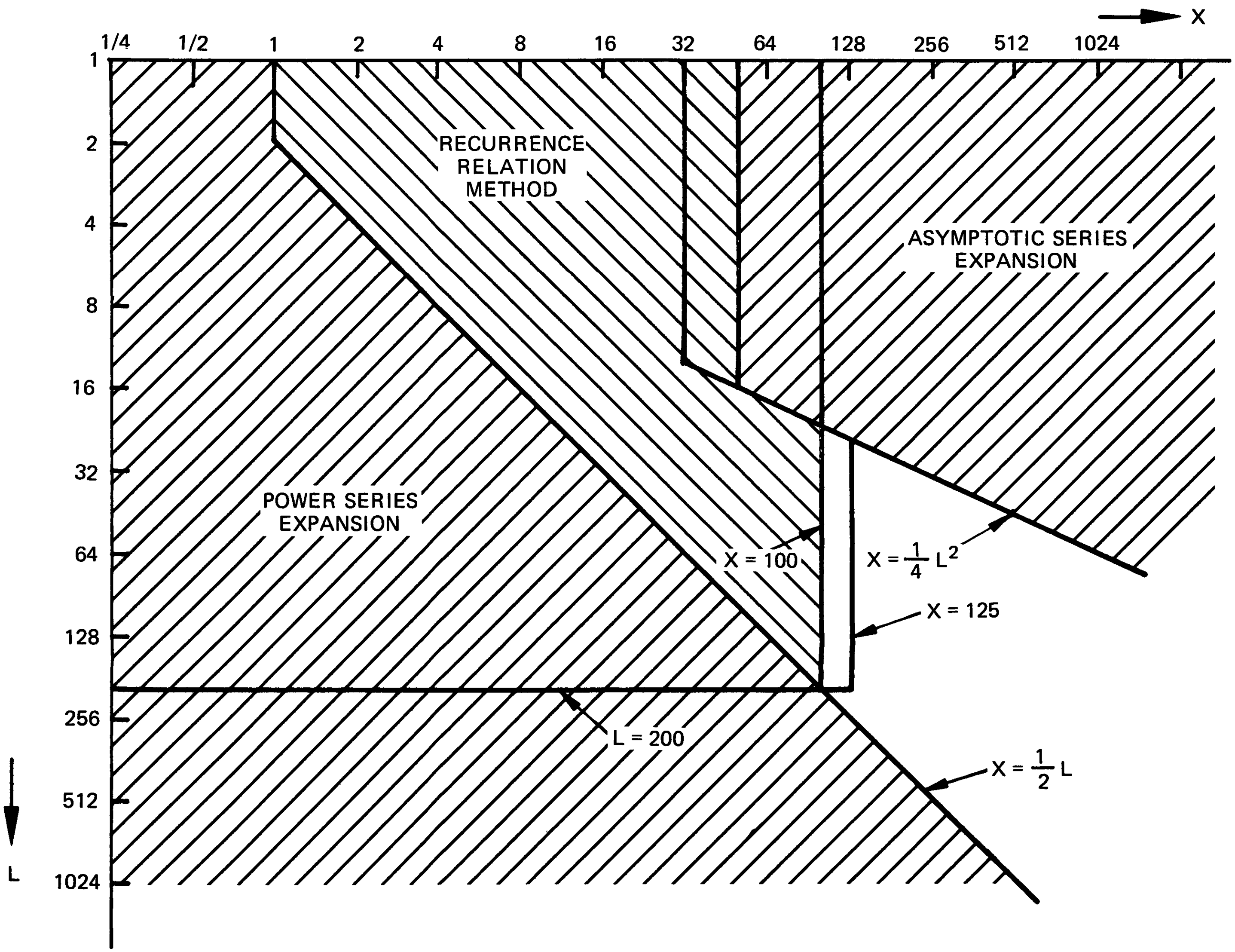

Figure 1 – Order-Argument Domain Covered by Algorithms for $J_L(X)$

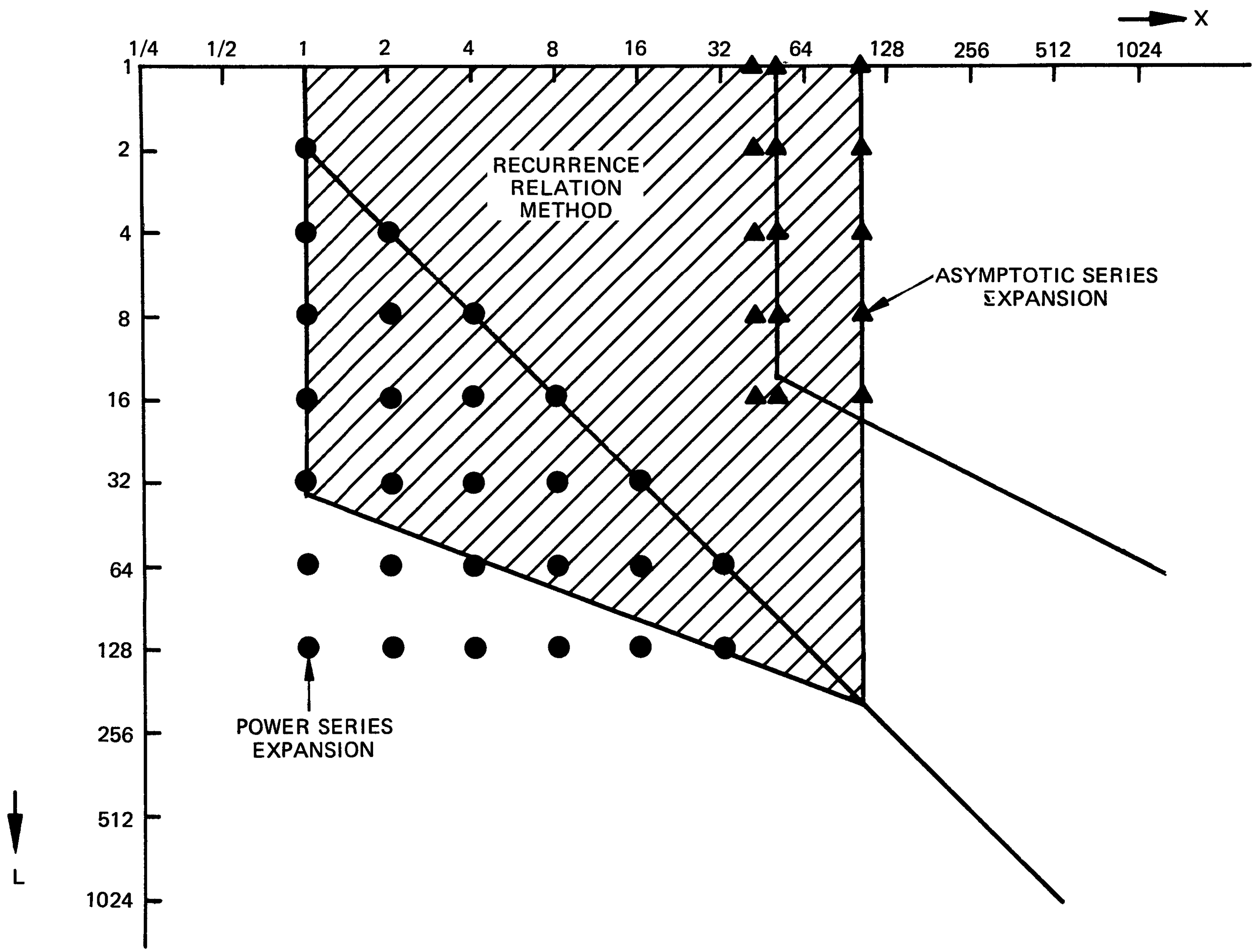

Figure 2 – Results Checked or Compared

APPENDIX A

FORTRAN IV PROGRAM

The FORTRAN IV program is used to compute Bessel functions of the first kind of positive integer order and positive argument.

The program is written as a double precision function, with indications of the changes involved when a single precision program is desired. The program has been checked on the NSRDC CDC 6700 system.

The calling sequence, variable parameters, and options are explained in comments at the beginning of the program.

```
   C       BESSEL FUNCTION                                         1931,KOOIJ
   C ------------------------------------------------------------------
   C
     DOUBLE PRECISION FUNCTION DBSL(LP,XP,IOPT)
 5 C
   C BESSEL FUNCTION J(L,X) OF NON-NEGATIVE INTEGER ORDER L
   C   AND NON-NEGATIVE ARGUMENT X.
   C
   C ------------------------------------------------------------------
10 C
     INTEGER LP,IOPT,L,ISW,IX,IM,I,K,KK,KM,LL,LP1,M
     REAL ALM1,TX
     DOUBLE PRECISION XP,X,MACHEP,PI,S,S2,Y,Z,ZZ
     DOUBLE PRECISION AJ,AJLM1,AJL,AJLP1,AL,ALF,ALK,AM,AKK
15 C INTEGER IABS. REAL SQRT. DOUBLE PRECISION DABS,DCOS,DSIN,DSQRT.
   C
   C ------------------------------------------------------------------
   C
   C  ABSOLUTE VALUES OF L AND X ARE TAKEN BY THIS PROGRAM.
20   X=DABS(XP)
     L=IABS(LP)
   C
   C ------------------------------------------------------------------
   C
25 C MACHEP IS A MACHINE PRECISION PARAMETER. CHANGES AFFECT LINES
   C   27,68,72, AND 86 IN THIS PROGRAM.
     MACHEP=2.**(-97)
   C
   C ------------------------------------------------------------------
30 C
   C THE FOLLOWING OPTIONS ARE AVAILABLE
   C IOPT = 1
   C   ALGORITHM IS BASED ON RECURRENCE RELATION
   C   J(L-1,X)=(2L/X)*J(L,X) - J(L+1,X), STARTING WITH SUFFICIENTLY
35 C   HIGH L FOR WHICH APPROXIMATELY J(L,X)=MACHEP, AND J(L+1,X)=0,
   C   AND NORMALIZED BY THE SCALING FACTOR S=J(0,X)+2*SUML(J(2L,X)),
   C   WHICH SHOULD HAVE EQUALED ONE FOR THE RIGHT VALUES OF J.
   C IOPT = 2
   C   ALGORITHM IS BASED ON POWER SERIES EXPANSION
40 C   SUMK((-1)**K *(X/2)**(2K+L) /(KFACT*(K+L)FACT) )
   C IOPT = 3
   C   ALGORITHM IS BASED ON ASYMPTOTIC SERIES EXPANSION
   C   SQRT(2/(PI*X)) *(P(L,X)COS(ALF) - Q(L,X)SIN(ALF) ), WHERE
   C   ALF=X - (.5L+.25)PI, AND, ASYMPTOTICALLY, WITH MU=(2L)**2
45 C   P(L,X)=1 - (MU-1)(MU-9)/(1*2 *(8X)**2) + ....
   C   Q(L,X)=(MU-1)/(1 *(8X)) - (MU-1)(MU-9)(MU-25)/(1*2*3 *(8X)**3))
   C    (-1)**INT(K/2)*(MU-1)..(MU-(2K-1)**2)/(1*2*..*K *(8X)**K) +..
   C IOPT = NOT EQUAL TO 1,2 OR 3
   C   ALGORITHM DETERMINES MOST SUITABLE OF ABOVE ALGORITHMS, BUT
50 C   MAY LEAD TO UNSATISFACTORY RESULTS, ESPECIALLY IF L AND X
   C   ARE OF SAME ORDER AND LARGER THAN 100.
   C
   C ------------------------------------------------------------------
   C
55 C
```

```
      PI=.31415926535897932384626433833D+01
      AL=L
      AX=X
      IX=AX
      ISW=IOPT
      IF (ISW.LE.0) ISW=4
      IF (ISW.GE.4) ISW=4
      GOTO (10,20,30,40), ISW
C
C     ----------------------------------------------------------------
C
C  40 FINDS MOST SUITABLE ALGORITHM IF POSSIBLE.
   40 IF (IX.GE.50 .AND. 4*IX.GE.L*L) GOTO 30
C     INEQUALITY MAY BE REDUCED TO
C       IX.GE.30 .AND. 4*IX.GE.L*L  IF MACHEP=2**(-47).
      IF (IX.LT.1 .OR. 2*IX.LT.L) GOTO 20
      IF (IX.LT.100 .AND. L.LT.200) GOTO 10
C     INEQUALITY MAY BE INCREASED TO
C       IX.LT.125 .AND. L.LT.200  IF MACHEP=2**(-47).
      WRITE(6,15)
C     SIGNALS THAT NO SUITABLE METHOD IS AVAILABLE, BUT PROVIDES
C       STIRLING APPROXIMATION TO FIRST TERM OF POWER SERIES.
   15 FORMAT(13H WRONG BESSEL)
      S=(AX*1.359140914?/AL)**AL/SQRT(6.283185307*AL)
      GOTO 47
C
C     ----------------------------------------------------------------
C
C  10 RECURRENCE RELATION.
   10 LP1=L+1
      KK=(8*IX)/5 + 40
C     FINDS ORDER L FOR WHICH APPROXIMATELY J(L,X)=MACHEP. BECOMES
C       KK=(7*IX)/5 + 25  IF MACHEP=2**(-47).
      K=2*(KK/2)-1
C     STARTS NORMALIZING SUM S (OF TERMS OF EVEN ORDER) WITH
C       HIGHEST NON-ZERO TERM.
      AJLP1=0.
      AJL=MACHEP
      S=0.
      DO 23 I=1,K,2
      LL=K-I+2
      AL=LL
      AJLM1=AJL*2.*AL/X - AJLP1
      AJLP1=AJL
      AJL=AJLM1
      IF (LL.EQ.LP1) AJ=AJL
      LL=K-I+1
      AL=LL
      AJLM1=AJL*2.*AL/X - AJLP1
      S=S+AJLP1
C     ADDS THE NEXT LOWER EVEN ORDER TERM TO THE NORMALIZING SUM S.
      AJLP1=AJL
      AJL=AJLM1
      IF (LL.EQ.LP1) AJ=AJL
   23 CONTINUE
```

```
      S=2.*S+AJL
C     COMPLETES THE NORMALIZING SUM S=J(0,X)+2*SUML(J(2L,X)).
      S=AJ/S
      GOTO 47
C
C     ----------------------------------------------------------------
C
C 20  POWER SERIES EXPANSION.
   20 Y=X/2.
      ALF=1.
      IF (L.EQ.0) GOTO 37
      DO 33 IM=1,L
      AM=IM
   33 ALF=ALF*AM
C     ALF IS (K+L)-FACTORIAL. NOW SET UP FOR AND EXECUTE SUMMATION.
   37 Z=1./ALF
      Z=Z*Y**L
      S=Z
      KK=1
   57 AKK=KK
      ALK=L+KK
      Z=-Z*Y*Y/(AKK*ALK)
      S=S+Z
      ZZ=DABS(Z)
      IF (ZZ.LT.MACHEP) GOTO 47
      IF (KK.GE.60) GOTO 47
      KK=KK+1
      GOTO 57
C
C     ----------------------------------------------------------------
C
C 30  ASYMPTOTIC SERIES EXPANSION.
   30 Y=8.*X
      ALM1=4*L*L-1
      TX=2.*X+1.
      KM=.5*(TX+SQRT(TX*TX+ALM1))
C     FINDS MINIMUM TERM IN CASE IT OCCURS BEFORE MACHEP IS REACHED.
      AM=4*L*L
      Z=1.
      S=1.
      S2=0.
      KK=1
   77 AKK=KK
      M=4*KK*KK - 4*KK + 1
      ALK=M
      Z=Z*(AM-ALK)/(AKK*Y)
      S2=S2+Z
      IF (KK+1.GE.KM) GOTO 167
      AKK=KK+1
      ALK=M + 8*KK
```

APPENDIX B

POWER SERIES TABLES

TABLE 2 - POWER SERIES FINAL TERM AND COMMON LOGARITHM OF ITS MAGNITUDE FOR MACHINE PRECISION 10^{-14}

(Algorithm continues for 60 terms or until required precision is reached)

L \ X	1	2	4	8	16	32	64	128	256	512	1024
	8	10	14	21	33	55	99	186	360	707	1403
1	-15.3	-14.2	-14.3	-14.9	-14.9	-14.3	-14.4	-14.3	-14.5	-13.8	-14.2
	8	10	14	21	33	55	98	185	359	707	1403
2	-16.6	-15.2	-15.2	-15.6	-15.5	-14.8	-13.9	-13.9	-14.0	-14.3	-14.6
	6	9	13	20	32	54	97	184	358	706	1402
4	-14.2	-15.3	-15.3	-15.7	-15.6	-14.9	-13.9	-13.9	-14.0	-14.3	-14.6
	4	7	11	17	30	51	95	182	356	704	1400
8	-14.9	-15.8	-15.7	-14.5	-15.7	-13.9	-14.0	-13.9	-14.0	-14.3	-14.6
	1	1	5	13	25	48	91	178	352	700	1396
16	-19.9	-14.5	-14.0	-15.5	-15.1	-15.3	-14.2	-14.0	-14.1	-14.3	-14.6
	1	1	1	1	15	38	82	170	344	692	1387
32	-47.1	-36.9	-26.7	-16.4	-15.5	-14.8	-14.1	-14.5	-14.3	-14.4	-13.8
	1	1	1	1	1	14	62	152	327	675	1371
64	-110.7	-90.9	-71.0	-51.1	-31.3	-15.2	-13.9	-14.4	-14.3	-14.0	-14.1
	1	1	1	1	1	1	1	111	291	641	1338
128	-256.8	-217.7	-178.5	-139.4	-100.3	-61.1	-22.0	-14.3	-14.6	-14.1	-14.1
	1	1	1	1	1	1	1	1	207	568	1270
256	-587.0	-509.3	-431.6	-354.0	-276.3	-198.6	-121.0	-43.3	-14.1	-14.0	-14.5
	1	1	1	1	1	1	1	1	1	399	1123
512	-1323.9	-1169.2	-1014.5	-859.8	-705.0	-550.3	-395.6	-240.8	-86.1	-13.8	-13.7
	1	1	1	1	1	1	1	1	1	1	784
1024	-2951.6	-2642.7	-2333.9	-2025.0	-1716.1	-1407.3	-1098.4	-789.6	-480.7	-171.9	-13.9

TABLE 3-POWER SERIES FINAL TERM AND COMMON LOGARITHM OF ITS MAGNITUDE FOR MACHINE PRECISION 10^{-29}

(Algorithm continues for 60 terms or until required precision is reached)

L \ X	1	2	4	8	16	32	64	128	256	512	1024
1	14 -31.8	17 -30.4	22 -29.9	30 -29.6	44 -30.1	68 -29.7	113 -29.1	201 -28.8	376 -29.2	724 -29.0	1420 -29.2
2	13 -30.3	16 -29.1	22 -31.0	30 -30.5	43 -29.4	67 -29.0	113 -29.6	201 -29.3	376 -29.6	724 -29.5	1419 -28.7
4	12 -30.4	15 -29.2	21 -31.0	29 -30.6	42 -29.4	66 -29.1	112 -29.7	200 -29.3	375 -29.6	723 -29.5	1418 -28.7
8	10 -30.8	13 -29.5	19 -31.3	27 -30.7	40 -29.5	64 -29.1	110 -29.7	198 -29.3	373 -29.7	721 -29.5	1416 -28.7
16	5 -29.6	9 -30.7	14 -30.1	22 -29.6	36 -30.0	60 -20.4	106 -29.9	194 -29.5	369 -29.7	717 -29.5	1412 -28.7
32	1 -47.1	1 -36.9	3 -29.3	12 -29.4	27 -30.5	51 -29.4	97 -29.5	186 -29.9	360 -29.0	709 -29.6	1404 -28.8
64	1 -110.7	1 -90.9	1 -71.0	1 -51.1	1 -31.3	31 -30.2	78 -29.4	168 -29.5	343 -29.0	692 -29.2	1388 -29.0
128	1 -256.8	1 -217.7	1 -178.5	1 -139.4	1 -100.3	1 -61.1	31 -30.4	128 -28.9	307 -28.9	658 -29.2	1355 -29.1
256	1 -587.0	1 -509.3	1 -431.6	1 -354.0	1 -276.3	1 -198.6	1 -121.0	1 -43.3	226 -29.2	585 -28.7	1287 -29.3
512	1 -1323.9	1 -1169.2	1 -1014.5	1 -859.8	1 -705.0	1 -550.3	1 -395.6	1 -240.8	1 -86.1	419 -29.0	1141 -29.0
1024	1 -2951.6	1 -2642.7	1 -2333.9	1 -2025.0	1 -1716.1	1 -1407.3	1 -1098.4	1 -789.6	1 -480.7	1 -171.9	804 -28.7

TABLE 4-POWER SERIES MAXIMUM TERM AND COMMON LOGARITHM OF ITS MAGNITUDE

(Dashed line indicates boundary at which loss of accuracy may be 2 decimals; algorithm chooses power series if X is less than $\frac{1}{2}$ L or if X is less than 1)

L \ X	1	2	4	8	16	32	64	128	256	512	1024
1	1 -1.2	1 -.2	1 .7	3 2.1	7 5.2	15 11.9	31 25.5	63 53.0	127 108.3	255 219.2	511 441.2
2	1 -1.9	1 -.7	1 .5	3 2.0	7 5.2	15 11.9	31 25.5	63 53.0	127 108.3	255 219.2	511 441.2
4	1 -3.8	1 -2.0	1 -.2	2 1.7	6 5.0	14 11.8	30 25.4	62 53.0	126 108.3	254 219.1	510 441.2
8	1 -8.5	1 -5.5	1 -2.5	1 .5	4 4.4	12 11.5	28 25.3	60 52.9	124 108.2	252 219.1	508 441.2
16	1 -19.9	1 -14.5	1 -9.1	1 -3.7	3 2.0	9 10.2	24 24.6	56 52.6	120 108.1	248 219.0	504 441.2
32	1 -47.1	1 -36.9	1 -26.7	1 -16.4	1 -6.2	6 5.4	19 22.1	49 51.3	112 107.4	240 218.7	496 441.0
64	1 -110.7	1 -90.9	1 -71.0	1 -51.1	1 -31.3	3 -11.0	13 12.5	39 46.2	99 104.8	225 217.4	480 440.3
128	1 -256.8	1 -217.7	1 -178.5	1 -139.4	1 -100.3	1 -61.1	7 -20.4	26 27.0	79 94.6	199 212.2	451 437.7
256	1 -587.0	1 -509.3	1 -431.6	1 -354.0	1 -276.3	1 -198.6	3 -120.6	15 -38.8	53 56.3	158 191.9	399 427.4
512	1 -1323.9	1 -1169.2	1 -1014.5	1 -859.8	1 -705.0	1 -550.3	1 -395.6	7 -239.2	30 -75.3	106 115.3	316 386.7
1024	1 -2951.6	1 -2642.7	1 -2333.9	1 -2025.0	1 -1716.1	1 -1407.3	1 -1098.4	3 -789.2	15 -476.1	60 -148.0	212 233.5

APPENDIX C

ASYMPTOTIC SERIES TABLES

TABLE 5-ASYMPTOTIC SERIES MAXIMUM TERM AND COMMON LOGARITHM OF ITS MAGNITUDE

(Dashed line indicates boundary at which loss of accuracy may be 2 decimals; algorithm chooses asymptotic series if X is greater than $L^2/4$)

L \ X	1	2	4	8	16	32	64	128	256	512	1024
1	1 -.5	1 -.8	1 -1.1	1 -1.4	1 -1.7	1 -2.0	1 -2.3	1 -2.6	1 -2.9	1 -3.2	1 -3.5
2	1 .3	1 -.0	1 -.3	1 -.6	1 -.9	1 -1.2	1 -1.5	1 -1.8	1 -2.1	1 -2.4	1 -2.7
4	3 1.6	2 .8	1 .3	1 .0	1 -.3	1 -.6	1 -.9	1 -1.2	1 -1.5	1 -1.8	1 -2.1
8	7 5.8	6 3.8	5 2.1	3 1.0	1 .3	1 .0	1 -.3	1 -.6	1 -.9	1 -1.2	1 -1.5
16	15 16.6	14 12.2	12 8.2	10 4.8	6 2.4	3 1.0	1 .3	1 .0	1 -.3	1 -.6	1 -.9
32	31 43.2	30 34.0	28 25.2	25 17.2	20 10.4	13 5.5	7 2.6	3 1.0	1 .3	1 .0	1 -.3
64	63 106.2	62 87.4	60 69.0	56 51.4	50 35.3	39 21.7	26 11.9	15 5.8	7 2.6	3 1.0	1 .3
128	127 251.6	126 213.5	124 175.9	120 139.0	113 103.8	100 71.6	79 44.5	53 24.7	30 12.5	15 5.9	7 2.6
256	255 581.2	254 504.6	252 428.4	248 353.1	240 279.4	226 209.0	200 144.5	158 90.3	106 50.5	60 26.0	31 12.7
512	511 1317.6	510 1163.9	508 1010.6	504 858.2	496 707.6	481 560.2	452 419.4	400 290.5	316 181.9	212 102.3	120 53.1
1024	1023 2944.6	1022 2636.8	1020 2329.4	1016 2022.9	1008 1718.1	992 1416.7	962 1122.0	904 840.3	799 582.5	633 365.4	424 206.0

TABLE 6-ASYMPTOTIC SERIES MINIMUM TERM AND COMMON LOGARITHM OF ITS MAGNITUDE

(Dashed lines indicate boundaries at which magnitude attains single machine precision (10^{-14}) and double machine precision (10^{-29}); algorithm chooses asymptotic series if X is greater than 30—single precision—or if X is greater than 50—double precision)

L \ X	1	2	4	8	16	32	64	128	256	512	1024
1	3 -1.0	5 -2.0	9 -4.0	17 -7.6	33 -14.7	65 -28.8	129 -56.7	257 -112.5	513 -223.8	1025 -446.3	2049 -891.2
2	3 -.5	5 -1.8	9 -3.8	17 -7.5	33 -14.7	65 -28.8	129 -56.7	257 -112.5	513 -223.8	1025 -445.3	2049 -891.2
4	5 1.0	7 -.8	10 -3.3	17 -7.2	33 -14.5	65 -28.7	129 -56.7	257 -112.5	513 -223.8	1025 -445.3	2049 -891.2
8	9 5.1	10 2.2	13 -1.3	20 -6.1	34 -13.9	65 -28.4	129 -56.5	257 -112.4	513 -223.8	1025 -446.3	2049 -891.2
16	17 16.0	18 10.7	21 4.9	26 -2.0	39 -11.5	68 -27.1	130 -55.9	257 -112.0	513 -223.6	1025 -446.2	2049 -891.2
32	33 42.6	34 32.5	36 22.0	41 10.4	52 -3.4	78 -22.3	136 -53.3	260 -110.8	514 -222.9	1025 -445.9	2049 -891.0
64	65 105.6	66 85.9	68 65.8	73 44.7	82 21.6	104 -6.0	155 -43.8	272 -105.7	520 -220.4	1028 -444.6	2050 -890.3
128	129 251.1	130 212.1	132 172.7	136 132.4	145 90.2	164 44.0	207 -11.0	309 -86.6	543 -210.2	1040 -439.4	2056 -887.7
256	257 580.7	258 503.2	260 425.2	264 346.4	273 265.8	290 181.4	328 89.1	414 -20.8	618 -171.9	1085 -419.1	2080 -877.4
512	513 1317.0	514 1162.5	516 1007.5	520 851.6	528 693.9	545 532.7	580 364.0	656 179.5	829 -40.3	1236 -342.5	2169 -836.7
1024	1025 2944.0	1026 2635.4	1028 2326.2	1032 2016.2	1040 1704.5	1057 1389.2	1090 1066.7	1160 729.3	1312 360.3	1657 -79.3	2473 -683.5

TABLE 7-ASYMPTOTIC SERIES FINAL TERM AND COMMON LOGARITHM OF ITS MAGNITUDE FOR MACHINE PRECISION 10^{-14}

(Algorithm continues for 60 terms or until required precision is reached)

L \ X	1	2	4	8	16	32	64	128	256	512	1024
1					31 -14.7	12 -14.5	9 -14.8	7 -14.4	6 -14.6	5 -14.1	5 -15.6
2					31 -14.7	12 -14.4	9 -14.7	7 -14.3	6 -14.4	6 -16.2	5 -15.3
4					31 -14.5	13 -14.8	9 -14.1	8 -15.2	7 -15.6	6 -15.2	6 -17.0
8					32 -13.9	15 -14.7	11 -14.3	10 -15.8	8 -14.0	8 -16.4	7 -15.3
16						22 -14.9	18 -16.4	15 -15.0	13 -15.3	11 -15.1	10 -16.2
32						39 -13.0	32 -15.0	27 -15.5	21 -14.9	17 -15.3	14 -15.6
64							67 -14.8	55 -14.9	40 -14.4	29 -14.4	22 -14.8
128							227 -10.1	127 -16.0	96 -15.2	65 -15.3	43 -14.7
256								338 -12.4	244 -14.6	174 -15.3	109 -15.1
512									639 -12.5	478 -15.4	327 -15.1
1024										1249 -12.8	943 -14.6

TABLE 8-ASYMPTOTIC SERIES FINAL TERM AND COMMON LOGARITHM OF ITS MAGNITUDE FOR MACHINE PRECISION 10^{-29}

(Algorithm continues for 60 terms or until required precision is reached)

L \ X	1	2	4	8	16	32	64	128	256	512	1024
1						45 -27.3	25 -29.4	18 -29.3	15 -30.2	13 -30.9	11 -30.3
2						45 -27.3	26 -30.0	18 -29.2	15 -30.1	13 -30.8	11 -30.2
4						46 -27.4	26 -29.8	19 -30.1	15 -29.7	13 -30.4	11 -29.7
8						48 -27.2	27 -29.7	20 -30.1	16 -29.9	14 -30.7	12 -30.1
16						66 -27.1	31 -29.6	23 -29.5	19 -29.3	18 -32.7	16 -32.0
32							43 -28.4	35 -30.6	31 -31.5	26 -29.8	22 -29.5
64							82 -27.8	64 -29.1	54 -30.1	43 -30.2	34 -29.8
128								134 -28.8	111 -29.9	82 -29.6	60 -29.8
256									254 -30.6	193 -30.0	131 -30.2
512									693 -27.4	490 -30.5	349 -29.9
1024										1290 -27.7	957 -30.4

APPENDIX D

RECURRENCE RELATION TABLE

TABLE 9-BESSEL FUNCTIONS WHOSE MAGNITUDE APPROXIMATELY EQUALS MACHINE PRECISION

(Fast rule: $L \approx 1.4X + 25$, mach. prec. 10^{-14}; $L \approx 1.6X + 40$, mach. prec. 10^{-29})

X	L	Log $J_L(X)$	X	L	Log $J_L(X)$
1	13	-12.7	1	23	-28.3
10	34	-13.5	10	50	-28.3
20	51	-13.9	20	70	-28.8
30	66	-13.8	30	87	-28.6
40	81	-14.0	40	103	-28.6
50	95	-13.8	50	119	-29.0
60	109	-13.7	60	134	-28.9
70	123	-13.7	70	149	-29.0
80	137	-13.8	80	163	-28.7
90	151	-13.8	90	178	-29.0
100	165	-13.9	100	192	-28.8
200	302	-13.8	200	332	-28.8
400	575	-14.0	400	607	-29.1

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13. ABSTRACT

Three different algorithms—the power series, the asymptotic series, and the recurrence relation method—are investigated with special attention to the single (10^{-14}) and double precision (10^{-29}) of Computer Data Corporation (CDC) 6000 Series computers. The final accuracy of each method depends partly on the magnitudes of the largest and smallest terms when floating point additions are involved. Another consideration is the number of terms required for each algorithm. Combination of all considerations leads to a partitioning of the order-argument domain into partially overlapping areas in which each algorithm is most appropriate. A wedged area not covered by any of the algorithms remains for large order and argument of approximately equal size.

Orders and arguments up to 1024 were investigated and checked where possible. A FORTRAN IV program in the form of an external function is included.

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