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WASHINGTON, D.C.

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A SUMMARY OF SUBMARINE STRUCTURAL RESEARCH
PART I
CONVENTIONAL HULL CONFIGURATIONS
CHAPTER 5
STRESSES IN PRESSURIZED SHELLS OF REVOLUTION

by

Thomas E. Reynolds

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STRUCTURAL MECHANICS LABORATORY
RESEARCH AND DEVELOPMENT REPORT

December 1966

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FOREWORD

The results of submarine structural research are now so scattered among numerous reports and papers that it is difficult for investigators to locate and evaluate information. Steps have recently been taken to consolidate this material in the form of a research summary which should be of some value particularly to those concerned with pressure hull designs. The summary is intended to be a source of information rather than a textbook and is to include an explanation of the development and use of important theory as well as a critical examination of its merit. As presently planned, the first part of the summary, which deals with conventional hull configurations, will consist of nine chapters, as follows:

Chapter 1 – Axisymmetric Behavior of Ring-Stiffened Cylindrical Shells
2 – Asymmetric Behavior of Ring-Stiffened Cylindrical Shells
3 – Fatigue of Submarine Hull Structures
4 – Interpretation and Significance of Submarine Model and Full-Scale Experimental Studies
5 – Stresses in Pressurized Shells of Revolution
6 – Pressure Hull Materials
7 – Frame Strength of Ring-Stiffened Cylindrical Shells
8 – Hemispherical Closures
9 – Hull Penetrations

Preliminary versions of Chapters 1 through 4 have been issued as David Taylor Model Basin Reports C-1569-1–C-1569-4; the remaining chapters are to be issued in unclassified form.

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NOTATION

A_r	Cross-sectional area of ring
$A_1 \dots A_4$	Arbitrary coefficients
$C_1 \dots C_4$	Arbitrary coefficients
c_A	$1 + \left(\frac{1 - 2\nu}{2\rho}\right) \cot \phi_A$ (spherical shell parameter)
c_B	$1 - \left(\frac{1 - 2\nu}{2\rho}\right) \cot \phi_B$ (spherical shell parameter)
d_A	$1 + \frac{\sqrt{2}\,(1 - 4\nu)}{\xi_A}$ (conical shell parameter)
d_B	$1 - \frac{\sqrt{2}\,(1 - 4\nu)}{\xi_B}$ (conical shell parameter)
E	Young's modulus
F_A, F_B, G_A, G_B	Conical shell parameters
$f_1 \dots f_4$	Special functions for cylindrical shells
H	Edge force (perpendicular to axis of revolution) per unit length
h	Shell thickness
I_z	Moment of inertia of ring
J_1, J_2, K_1, K_2	Special functions for spherical shells
K	$2\,R^2\theta/EhL$
K_A, K_B	$2\,R^2\beta/Eh \cos\alpha$
k	$2\,R^2\theta/EhL\,(\sinh^2\theta - \sin^2\theta)$
L	Shell length (also differential operator)
l	Horizontal dimension in ring
M	Bending moment per unit length
$m_c, m_d, n_c, n_d, t_c, t_d$	Special functions for spherical shells

N	Normal stress resultant
p	Pressure (positive external)
Q	Shear force (normal to shell) per unit length
$\overline{Q}$	$Q \sqrt{\sin \phi}$
q	$\sqrt{z}\ \overline{Q}$
R	Radius
U	$\sqrt[4]{12\,(1 - \nu^2)}$
W_c, X_c, X_d, X_e	Functions for spherical shells
w	Deflection normal to shell
$\overline{w}$	Deflection normal to axis of shell
x	Coordinate along generator (cylinders and cones)
y	$x_B - x$
z	$2\sqrt{x}$
α	Angle between axis of cone and generator
β	$\sqrt[4]{3\,(1 - \nu^2)} \Big/ \sqrt{\cos\alpha\,/Rh}$
ζ	$e^{-\beta x} \sin \beta x$
η	$e^{-\beta x} \cos \beta x$
λ	$\sqrt[4]{3\,(1 - \nu^2)}/\sqrt{h \tan\alpha}$
$\lambda_1 \ldots \lambda_6$	Special functions for cylindrical shells
ν	Poisson's ratio
ξ	$2\sqrt{x/h \tan\alpha}\ \sqrt[4]{12\,(1 - \nu^2)}$
ρ	$\sqrt[4]{(3R^2/h^2)\,(1 - \nu^2)}$
σ	Normal stress
ϕ	Angular coordinate in meridional direction

NOTATION (Continued)

χ	Angular rotation in meridional direction
$\psi_1 \ldots \psi_4$	Schleicher functions
$\Omega_1 \ldots \Omega_6$, $\omega_1 \ldots \omega_6$	Functions for conical shells
A, B	Subscripts denoting left- and right-hand edges of shell
θ	Subscript denoting circumferential direction

ABSTRACT

Analytical methods utilizing the linear bending theory of thin shells are presented which are judged the most useful for determining stresses in submarine pressure hulls. Only axisymmetric deformations are considered, and emphasis is placed on methods that are sufficiently simple to be applied at the practical level. Attention is restricted to analytical solutions for cylindrical, conical, and spherical shells. Efforts have been made to employ a uniform system of notation and sign conventions wherever possible.

ADMINISTRATIVE INFORMATION

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INTRODUCTION

The industrial uses of shell structures are so extensive today that it is not surprising to find such industries as petroleum, aerospace, and atomic power represented in the technical literature on shells. Nor is it unexpected that with such widespread interest, the volume of this literature should have grown to staggering proportions. One consequence is that an effective literature search is becoming increasingly important and correspondingly more difficult. The special needs of individual research or development groups often narrow their fields of search considerably, but some problems are so frequently encountered and of such importance that they have universal interest. For instance, during flights a rocket motor case must withstand certain dynamic loads that would never be experienced by a submarine hull. Yet both structures are also subjected to static, rotationally symmetric loads, and to the extent that the configurations of both have basic similarities, the presence of these loads creates structural problems common to rockets and submarines.

The present chapter is concerned with this common area of interest, i.e., the axially symmetric deformations and stresses in those shells of revolution frequently used in submarine pressure hulls. Because so much has been written in this general field, the contributions of submarine structural research can be counted as much by what has been accomplished in the selection and adaptation of available theory as by what has been done in the way of original work. It would seem that one of the main purposes of this research should be to develop methods that can find practical application in improving the structural efficiency of submarine hulls. A summary of the significant results of such research would therefore emphasize the practical over the precise where a choice must be made. The material has thus been chosen (it is hoped) to reflect the needs of the practicing engineer who must frequently "make do" with formulas and methods that are somewhat inexact but sufficiently simple to permit application at the practical level. It is not expected, therefore, that this chapter will have much to interest the mathematician or the research worker engaged in developing new shell theory; in any case, they will have been over most of the ground before.

The main portion of a submarine hull is the easiest to deal with analytically since it is usually nearly cylindrical in form. Noncylindrical shells such as cones, spheres, and toroids present problems of considerably greater difficulty. Figure 5-1 shows a composite of some typical shell elements commonly used in submarine hulls. Spherical or spheroidal domes are often used as end closures; conical or toroidal sections, or combinations thereof, frequently serve as the connection between two cylinders of different diameters. Ring stiffeners are employed both to increase stability and to reduce stresses near junctures of dissimilar shells. The methods described in this chapter are those considered most useful for analyzing these structures. All are based on linear bending theory, and their application is restricted to thin shells having no imperfections. As general reference texts, the author recommends Timoshenko's "Plates and Shells"[5-1*] and Flügge's "Stresses in Shells"[5-2] where most of the analytical work and much more can be found.

It should be emphasized that only analytical solutions to shell problems are considered here. This puts definite restrictions on the types of shells to be covered since analytical solutions to the differential equations of equilibrium are known for only a few simple geometric shapes. Accurate solutions for shells of arbitrary shape can be accomplished only through numerical methods. Today this can be done with comparative ease through the use of high-speed computers, and efforts along these lines have become widespread. The trouble is that such computer programs usually are written for the special needs of individual groups and as yet are not of such scope and flexibility that they are readily usable by the general public. Krause[5-3] gives an interesting account of the pitfalls that can be encountered when one group tries to use a program developed by another. His paper also discusses some of the numerical techniques commonly used in this kind of programming and includes an extensive bibliography. Perhaps the day is not far off when shell analysis will be performed exclusively by computers, but until it arrives, some engineers will have to go on using the methods described in this chapter.

THE METHOD OF EDGE COEFFICIENTS

When loaded only by hydrostatic pressure, a shell of revolution responds in what might be called its natural pattern of deformation, a pattern peculiar to the type of shell in question. Two dissimilar shells will in general therefore deform naturally in different amounts under the same pressure load. Should the two shells be joined, they must deform equally at the circle of contact and it is clear that under such conditions, the deformations of each shell will be disturbed from their "natural" form. The result is a deformation pattern that varies in the axial

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*References are listed on page 64.

direction (but not circumferentially so long as the loading is axisymmetric) accompanied by bending as well as membrane stresses. These bending stresses (usually critical at or near junctures) are of considerable interest to designers, and the problem of their determination is often not an easy one.

When two dissimilar shells are joined, one can find the stresses developed in each under hydrostatic pressure by enforcing conditions of compatible deformations and equilibrium of forces and moments at the juncture. This process is often lengthy and tedious and, so far, few short cuts have been devised, although the principles involved are not complicated. The method of edge or influence coefficients has had wide acceptance in the solutions of such problems, and it is the method to be employed in the development that follows.

Figure 5-2 shows two shells of revolution having arbitrary meridional shapes and subjected to normal pressure. The edge forces and moments represent the interactions of the shells per unit of circumference at the juncture. It is usually the practice to resolve the forces into components parallel and perpendicular to the axis of revolution as indicated in the figure. When normal pressure is the only external load, the axial force is simply that due to the axial component of the pressure, or $p \frac{R}{2}$, R being the local radius from the axis of revolution to the midthickness of the shell. For Shell 1, the meridional stress σ_{x1} and the circumferential stress $\sigma_{\theta 1}$ are functions of x, a meridional coordinate and the edge loads:

$$\begin{aligned} \sigma_{x1} &= \sigma_{x1}\,(x,\ H_{1A},\ M_{1A},\ H_{1B},\ M_{1B},\ p) \\ \sigma_{\theta 1} &= \sigma_{\theta 1}\,(x,\ H_{1A},\ M_{1A},\ H_{1B},\ M_{1B},\ p) \end{aligned} \qquad [5\text{-}1]$$

Similar relationships will, of course, hold for Shell 2. To determine the edge forces and moments, one must introduce the deformations of the shell juncture, and this is usually done in terms of the deflection $\bar{w}$* normal to the axis of revolution and the rotation χ of a shell element in the meridional plane. These quantities can be expressed as functions of the edge forces and moments by means of edge coefficients:

$$\begin{aligned} \bar{w}_{1B} &= \frac{\partial \bar{w}_{1B}}{\partial H_{1B}} H_{1B} + \frac{\partial \bar{w}_{1B}}{\partial M_{1B}} M_{1B} + \frac{\partial \bar{w}_{1A}}{\partial H_{1A}} H_{1A} + \frac{\partial \bar{w}_{1B}}{\partial M_{1A}} M_{1A} + \frac{\partial w_{1B}}{\partial p} p \\ \chi_{1B} &= \frac{\partial \chi_{1B}}{\partial H_{1B}} H_{1B} + \frac{\partial \chi_{1B}}{\partial M_{1B}} M_{1B} + \frac{\partial \chi_{1B}}{\partial H_{1A}} H_{1A} + \frac{\partial \chi_{1B}}{\partial M_{1A}} M_{1A} + \frac{\partial \chi_{1B}}{\partial p} p \end{aligned} \qquad [5\text{-}2]$$

*The deflection normal to the shell will be designated by w.

where $\bar{w}_{1B}$ and χ_{1B} denote the deflection and rotation of Shell 1 at Point B.* Similar relationships hold for Point A and for Shell 2. Conditions of compatibility at Point B require that

$$\begin{aligned} \bar{w}_{1B} &= \bar{w}_{2A} \\ \chi_{1B} &= \chi_{2A} \end{aligned} \tag{5-3}$$

and for equilibrium at Point B, we must have

$$\begin{aligned} H_{1B} &= H_{2A} \\ M_{1B} &= M_{2A} \end{aligned} \tag{5-4}$$

The unknown forces and moments can be determined from these equations and then be substituted into Equation [5-1]. It is frequently found in shell problems that the deformations at one edge (e.g., Point B) are insensitive to the forces and moments applied at a remote edge (Point A). In the present example, this would mean that the coefficients $\frac{\partial \bar{w}_{1B}}{\partial H_{1A}}$, $\frac{\partial \bar{w}_{1B}}{\partial M_{1A}}$, $\frac{\partial \chi_{1B}}{\partial H_{1A}}$, $\frac{\partial \chi_{1B}}{\partial M_{1A}}$ could be neglected and the analysis would be greatly simplified.

Whether or not such simplifications can be made, the crux of the problem is to find the edge coefficients appearing in Equations [5-2]. Usually the differential equations for the shell in question must be solved, and this is where many of the difficulties are encountered. The value of any analysis is therefore in providing the necessary edge coefficients together with the corresponding equations ([5-1]) relating stresses and edge loads. Once these relationships have been established, the remainder of the analysis, although often tedious, is usually straightforward.

CYLINDRICAL SHELLS

SHORT SHELLS

The equations for a cylindrical shell have already been presented in Chapter 1. However, in that case they were not arranged in such a way that the edge coefficients could be easily discerned. If the second-order "beam-column" effect is neglected, the differential equation for a cylindrical shell under hydrostatic pressure is

*The inclusion of $\frac{\partial \bar{w}}{\partial p}$ and $\frac{\partial \chi}{\partial p}$ as separate coefficients is somewhat artificial since all quantities are linearly dependent on p, when that is the only external load. What they represent are the membrane deformations (i.e., those that would occur in the absence of structural discontinuities) and those that arise from an axial force due to pressure acting against the ends of the structure.

$$\frac{d^4 w}{dx^4} + \frac{12\,(1-\nu^2)}{R^2 h^2}\, w = \frac{12\,(1-\nu^2)\,(1-\nu/2)}{Eh^3}\, p \qquad [5\text{-}5]$$

where w is positive inward and p is the pressure taken positive when external. The general solution to this equation is

$$\begin{aligned} w = {} & C_1 \cos\frac{\theta x}{L}\cosh\frac{\theta x}{L} + C_2 \sin\frac{\theta x}{L}\sinh\frac{\theta x}{L} \\ & + C_3 \cos\frac{\theta x}{L}\sinh\frac{\theta x}{L} + C_4 \sin\frac{\theta x}{L}\cosh\frac{\theta x}{L} + (1-\nu/2)\,\frac{pR^2}{Eh} \end{aligned} \qquad [5\text{-}6]$$

(Reference 5-1, pp. 399 ff.) where the term multiplied by p is a particular solution of Equation [5-5]. The C's are unknown coefficients whose values are determined by the boundary conditions and θ is $\frac{\sqrt[4]{3\,(1-\nu^2)}\,L}{\sqrt{Rh}}$ as defined in Chapter 1. It should not be confused with the circumferential coordinate having the same designation. If the shell is considered to be acted on by the edge forces and moments with directions as shown in Figure 5-3 (Edges A and B are separated by the distance L), then the coefficients are as follows:

$$\begin{aligned} C_1 = {} & \frac{2R^2\theta}{EhL\,(\sinh^2\theta-\sin^2\theta)} \Big[H_A(\sinh\theta\cosh\theta-\sin\theta\cos\theta) + \frac{\theta M_A}{L}(\cosh^2\theta-\cos^2\theta) \\ & + H_B(\sin\theta\cosh\theta-\cos\theta\sinh\theta) - \frac{2\theta M_B}{L}\sin\theta\sinh\theta \Big] \\[1ex] C_2 = {} & \frac{2R^2\theta^2 M_A}{EhL^2} \\[1ex] C_3 = {} & \frac{2R^2\theta}{EhL\,(\sinh^2\theta-\sin^2\theta)} \Big[-H_A\sinh^2\theta - \frac{\theta M_A}{L}(\cosh\theta\sinh\theta+\cos\theta\sin\theta) \\ & - H_B\sin\theta\sinh\theta + \frac{\theta M_B}{L}(\cos\theta\sinh\theta+\sin\theta\cosh\theta) \Big] \\[1ex] C_4 = {} & \frac{2R^2\theta}{EhL\,(\sinh^2\theta-\sin^2\theta)} \Big[-H_A\sin^2\theta - \frac{\theta M_A}{L}(\cosh\theta\sinh\theta+\cos\theta\sin\theta) \\ & - H_B\sin\theta\sinh\theta + \frac{\theta M_B}{L}(\cos\theta\sinh\theta+\sin\theta\cosh\theta) \Big] \end{aligned} \qquad [5\text{-}7]$$

The stresses in the shell at any location can be written in terms of these coefficients:

$$\sigma_\theta = -\frac{pR}{h} - \frac{E}{R}\left[\left(C_1 \pm \nu\sqrt{\frac{3}{1-\nu^2}}\,C_2\right)\cos\frac{\theta x}{L}\cosh\frac{\theta x}{L} + \left(C_2 \mp \nu\sqrt{\frac{3}{1-\nu^2}}\,C_1\right)\sin\frac{\theta x}{L}\sinh\frac{\theta x}{L} + \left(C_3 \pm \nu\sqrt{\frac{3}{1-\nu^2}}\,C_4\right)\cos\frac{\theta x}{L}\sinh\frac{\theta x}{L} + \left(C_4 \mp \nu\sqrt{\frac{3}{1-\nu^2}}\,C_3\right)\sin\frac{\theta x}{L}\cosh\frac{\theta x}{L}\right] \quad [5\text{-}8]$$

$$\sigma_x = -\frac{pR}{2h} \pm \frac{E}{R}\sqrt{\frac{3}{1-\nu^2}}\left[C_2\cos\frac{\theta x}{L}\cosh\frac{\theta x}{L} - C_1\sin\frac{\theta x}{L}\sinh\frac{\theta x}{L} + C_4\cos\frac{\theta x}{L}\sinh\frac{\theta x}{L} - C_3\sin\frac{\theta x}{L}\cosh\frac{\theta x}{L}\right]$$

In these equations, the upper and lower signs refer to the outer and inner surfaces, respectively. Here and throughout this chapter, tensile stresses are considered positive.

Finally, the displacements w and rotations of the edges $\chi = \frac{\partial w}{\partial x}$ can be obtained using the following edge coefficients:*

$$\frac{\partial w_A}{\partial H_A} = k(\sinh\theta\cosh\theta - \sin\theta\cos\theta) \qquad \frac{\partial \chi_A}{\partial H_A} = -\frac{k\theta}{L}(\cosh^2\theta - \cos^2\theta)$$

$$\frac{\partial w_A}{\partial M_A} = \frac{k\theta}{L}(\cosh^2\theta - \cos^2\theta) \qquad \frac{\partial \chi_A}{\partial M_A} = -\frac{2k\theta^2}{L^2}(\cosh\theta\sinh\theta + \cos\theta\sin\theta) \quad [5\text{-}9]$$

---

*It will be noted that there are a number of identities among these equations. The relations $\frac{\partial w_A}{\partial M_A} = -\frac{\partial \chi_A}{\partial H_A}$, $\frac{\partial w_A}{\partial H_B} = -\frac{\partial w_B}{\partial H_A}$, $\frac{\partial w_A}{\partial M_B} = \frac{\partial \chi_B}{\partial H_A}$, $\frac{\partial \chi_B}{\partial H_B} = -\frac{\partial w_B}{\partial M_B}$, $\frac{\partial \chi_A}{\partial H_B} = \frac{\partial w_B}{\partial M_A}$, $\frac{\partial \chi_A}{\partial M_B} = -\frac{\partial \chi_B}{\partial M_A}$ follow directly from the theorem of reciprocity.[5-4] The others, $\frac{\partial w_A}{\partial H_A} = -\frac{\partial w_B}{\partial H_B}$, $\frac{\partial w_A}{\partial M_A} = \frac{\partial w_B}{\partial M_B}$, $\frac{\partial w_A}{\partial M_B} = \frac{\partial w_B}{\partial M_A}$, $\frac{\partial \chi_A}{\partial H_A} = \frac{\partial \chi_B}{\partial H_B}$, $\frac{\partial \chi_A}{\partial M_A} = -\frac{\partial \chi_B}{\partial M_B}$, $\frac{\partial \chi_A}{\partial H_B} = \frac{\partial \chi_B}{\partial H_A}$, $\frac{\partial w_A}{\partial p} = \frac{\partial w_B}{\partial p}$, result from the geometrical symmetry of the problem.

$$\frac{\partial w_A}{\partial H_B} = k\,(\sin\theta\cosh\theta - \cos\theta\sinh\theta) \qquad \frac{\partial \chi_A}{\partial H_B} = -\frac{2k\theta}{L}\sin\theta\sinh\theta$$

$$\frac{\partial w_A}{\partial M_B} = -\frac{2k\theta}{L}\sin\theta\sinh\theta \qquad \frac{\partial \chi_A}{\partial M_B} = \frac{2k\theta^2}{L^2}(\cos\theta\sinh\theta + \sin\theta\cosh\theta)$$

$$\frac{\partial w_A}{\partial p} = \left(1 - \frac{\nu}{2}\right)\frac{R^2}{Eh} \qquad \frac{\partial \chi_A}{\partial p} = 0$$

$$\frac{\partial w_B}{\partial H_B} = -k\,(\sinh\theta\cosh\theta - \sin\theta\cos\theta) \qquad \frac{\partial \chi_B}{\partial H_B} = -\frac{k\theta}{L}(\cosh^2\theta - \cos^2\theta)$$

[5- 9]

$$\frac{\partial w_B}{\partial M_B} = \frac{k\theta}{L}(\cosh^2\theta - \cos^2\theta) \qquad \frac{\partial \chi_B}{\partial M_B} = \frac{2k\theta^2}{L^2}(\cosh\theta\sinh\theta + \cos\theta\sin\theta)$$

$$\frac{\partial w_B}{\partial H_A} = -k\,(\sin\theta\cosh\theta - \cos\theta\sinh\theta) \qquad \frac{\partial \chi_B}{\partial H_A} = -\frac{2k\theta}{L}\sin\theta\sinh\theta$$

$$\frac{\partial w_B}{\partial M_A} = -2k\theta\sin\theta\sinh\theta \qquad \frac{\partial \chi_B}{\partial M_A} = -\frac{2k\theta^2}{L^2}(\cosh\theta\sinh\theta + \sin\theta\cosh\theta)$$

$$\frac{\partial w_B}{\partial p} = \left(1 - \frac{\nu}{2}\right)\frac{R^2}{Eh} \qquad \frac{\partial \chi_B}{\partial p} = 0$$

where

$$k = \frac{2R^2\theta}{EhL\,(\sinh^2\theta - \sin^2\theta)}$$

LONG SHELLS

Equations [5-9] are exact in the sense that they include the effects of the loads at one edge on the deformations at the other. The magnitudes of these effects are directly dependent on the shell parameter θ. It is necessary to use the equations in their exact form only when considering short shells (i.e., $\theta \leq 3$). The effects are small, for long shells, and one can make the approximations:

$$\frac{\partial w_A}{\partial H_B} = \frac{\partial w_B}{\partial H_A} = \frac{\partial w_A}{\partial M_B} = \frac{\partial w_B}{\partial M_A} = \frac{\partial \chi_A}{\partial H_B} = \frac{\partial \chi_B}{\partial H_A} = \frac{\partial \chi_A}{\partial M_B} = \frac{\partial \chi_B}{\partial M_A} = 0$$

Furthermore, in such cases, $\sinh\theta \approx \cosh\theta >> \sin\theta, \cos\theta$ so that the remaining equations can be greatly simplified:

$$\frac{\partial w_A}{\partial H_A} = K \qquad \frac{\partial \chi_A}{\partial H_A} = -\frac{K\theta^2}{L}$$

$$\frac{\partial w_A}{\partial M_A} = \frac{K\theta}{L} \qquad \frac{\partial \chi_A}{\partial M_A} = -2\frac{K}{L^2}\theta^2$$

$$\frac{\partial w_B}{\partial H_B} = -K \qquad \frac{\partial \chi_B}{\partial H_B} = -\frac{K\theta^2}{L} \qquad [5\text{-}10]$$

$$\frac{\partial w_B}{\partial M_B} = \frac{K\theta}{L} \qquad \frac{\partial \chi_B}{\partial M_B} = 2\frac{K}{L^2}\theta^2$$

where

$$K = \frac{2R^2\theta}{EhL}$$

The coefficients $\frac{\partial w_A}{\partial p}$ and $\frac{\partial w_B}{\partial p}$ are, of course, unaltered. At the edges of a long shell, the stresses given by Equations [5-8] can be closely approximated by

$$\sigma_\theta = -\frac{pR}{h} - \frac{2R\theta}{hL}\left(\frac{M_A\theta}{L} + H_A\right) \pm \frac{6\nu M_A}{h^2}$$

$$\sigma_x = -\frac{pR}{2h} \pm \frac{6M_A}{h^2} \qquad \text{at Edge } A\ (x = 0)$$

$$\sigma_\theta = -\frac{pR}{h} - \frac{2R\theta}{hL}\left(\frac{M_B\theta}{L} - H_B\right) \pm \frac{6\nu M_B}{h^2} \qquad [5\text{-}11]$$

$$\sigma_x = -\frac{pR}{2h} \pm \frac{6M_B}{h^2} \qquad \text{at Edge } B\ (x = L)$$

SERIES REPRESENTATIONS

It is unfortunate that no such simplifications can be made for short shells. The exact equations are complicated not only because of their length but because they require frequent reference to tables for values of the circular and hyperbolic functions—a process that is often tedious and inconvenient. Raetz has sought to overcome this difficulty by making use of Maclaurin expansions for these functions.* Such expansions are best applied in the short shell range where rapid convergence is assured. A few of the expansions that have been found useful are as follows:

$$\lambda_1 = \frac{\sinh^2\theta + \sin^2\theta}{\sinh^2\theta - \sin^2\theta} = \frac{3}{\theta^2\eta}\left[1 + 2\,\frac{(2\theta)^4}{6!} + 2\,\frac{(2\theta)^8}{10!} + 2\,\frac{(2\theta)^{12}}{14!} + \cdots\right]$$

$$\lambda_2 = \frac{\cosh\theta\sinh\theta + \cos\theta\sin\theta}{\sinh^2\theta - \sin^2\theta} = \frac{3}{\theta^3\eta}\left[1 + \frac{(2\theta)^4}{5!} + \frac{(2\theta)^8}{9!} + \frac{(2\theta)^{12}}{13!} + \cdots\right]$$

$$\lambda_3 = \frac{\cosh\theta\sinh\theta - \cos\theta\sin\theta}{\sinh^2\theta - \sin^2\theta} = \frac{2}{\theta\eta}\left[1 + 3!\,\frac{(2\theta)^4}{7!} + 3!\,\frac{(2\theta)^8}{11!} + 3!\,\frac{(2\theta)^{12}}{15!} + \cdots\right]$$

[5-12]

$$\lambda_4 = \frac{2\sinh\theta\sin\theta}{\sinh^2\theta - \sin^2\theta} = \frac{3}{\theta^2\eta}\left[1 - \frac{1}{2}\,\frac{(2\theta)^4}{6!} + \left(\frac{1}{2}\right)^3\frac{(2\theta)^8}{10!} - \left(\frac{1}{2}\right)^5\frac{(2\theta)^{12}}{14!} + \cdots\right]$$

$$\lambda_5 = \frac{\cosh\theta\sin\theta - \sinh\theta\cos\theta}{\sinh^2\theta - \sin^2\theta} = \frac{1}{\theta\eta}\left[1 - \left(\frac{1}{2}\right)^2 3!\,\frac{(2\theta)^4}{7!} + \left(\frac{1}{2}\right)^4 3!\,\frac{(2\theta)^8}{11!} + \cdots\right]$$

$$\lambda_6 = \frac{\cosh\theta\sin\theta + \sinh\theta\cos\theta}{\sinh^2\theta - \sin^2\theta} = \frac{3}{\theta^3\eta}\left[1 - \left(\frac{1}{2}\right)^2\frac{(2\theta)^4}{5!} + \left(\frac{1}{2}\right)^4\frac{(2\theta)^8}{9!} - \cdots\right]$$

$$\eta = 1 + 4!\,\frac{(2\theta)^4}{8!} + 4!\,\frac{(2\theta)^8}{12!} + 4!\,\frac{(2\theta)^{12}}{16!} + \cdots$$

An alternative form is

$$f_1 = \frac{\cosh\theta + \cos\theta}{\sinh\theta + \sin\theta} = \frac{1}{\theta}\left[\frac{1 + \dfrac{\theta^4}{4!} + \dfrac{\theta^8}{8!} + \cdots}{1 + \dfrac{\theta^4}{5!} + \dfrac{\theta^8}{9!} + \cdots}\right]$$

[5-13]

$$f_2 = \frac{\cosh\theta - \cos\theta}{\sinh\theta - \sin\theta} = \frac{3}{\theta}\left[\frac{1 + 2\,\dfrac{\theta^4}{6!} + 2\,\dfrac{\theta^8}{10!} + \cdots}{1 + 6\,\dfrac{\theta^4}{7!} + 6\,\dfrac{\theta^8}{11!} + \cdots}\right]$$

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*These appear in a paper as yet unpublished.

$$f_3 = \frac{\cosh\theta - \cos\theta}{\sinh\theta + \sin\theta} = \frac{\theta}{2}\left[\frac{1 + 2\,\dfrac{\theta^4}{6!} + 2\,\dfrac{\theta^8}{10!} + \cdots}{1 + \dfrac{\theta^4}{5!} + \dfrac{\theta^8}{9!} + \cdots}\right]$$

$$f_4 = \frac{\cosh\theta + \cos\theta}{\sinh\theta - \sin\theta} = \frac{6}{\theta^3}\left[\frac{1 + \dfrac{\theta^4}{4!} + \dfrac{\theta^8}{8!} + \cdots}{1 + 6\,\dfrac{\theta^4}{7!} + 6\,\dfrac{\theta^8}{11!} + \cdots}\right] \qquad [5\text{-}13]$$

where the functions appearing in Equations [5-12] can be found from the relations

$$\lambda_1 = f_1 f_2 \qquad \lambda_4 = f_2 (f_3 - f_1)$$

$$\lambda_2 = \frac{f_3 + f_4}{2} \qquad \lambda_5 = \frac{f_2 - f_1}{2} \qquad [5\text{-}14]$$

$$\lambda_3 = \frac{f_1 + f_2}{2} \qquad \lambda_6 = \frac{f_4 - f_3}{2}$$

The functions f_1 through f_4 can also be expressed in another series form:

$$\frac{1}{f_1} = \frac{8\theta}{\pi^2} \sum_{m=1}^{\infty} \frac{m^2}{m^4 + \dfrac{4\theta^4}{\pi^4}} \qquad m = 1, 3, 5, \cdots$$

$$\frac{1}{f_2} = \frac{2\theta}{\pi^2} \sum_{m=1}^{\infty} \frac{m^2}{m^4 + \dfrac{\theta^4}{4\pi^4}} \qquad m = 1, 2, 3, \cdots$$

$$\frac{1}{f_3} = \frac{2}{\theta}\left[1 + \frac{1}{2\pi^4} \sum_{m=1}^{\infty} \frac{1}{m^4 + \dfrac{\theta^4}{4\pi^4}}\right] \qquad m = 1, 2, 3, \cdots \qquad [5\text{-}15]$$

$$\frac{1}{f_4} = \frac{16\theta^3}{\pi^4} \sum_{m=1}^{\infty} \frac{1}{m^4 + \dfrac{4\theta^4}{\pi^4}} \qquad m = 1, 3, 5, \cdots$$

For all of the series expressions listed, convergence is more rapid as θ is reduced. The choice of which form is most convenient will usually depend on the nature of the computations to be performed. None appears to have any distinct advantages over the others. Equations [5-7] and [5-9] can be rewritten in terms of the new functions as follows:

$$C_1 = K\left[\lambda_3 H_A + \frac{\theta}{L}\lambda_1 M_A + \lambda_5 H_B - \frac{\theta}{L}\lambda_4 M_B\right]$$

$$C_2 = K\frac{\theta M_A}{L}$$

$$C_3 = K\left[-\frac{(\lambda_1+1)}{2}H_A - \frac{\theta}{L}\lambda_2 M_A - \frac{\lambda_4}{2}H_B + \frac{\theta}{L}\lambda_6 M_B\right] \qquad [5\text{-}7a]$$

$$C_4 = K\left[\frac{(1-\lambda_1)}{2}H_A - \frac{\theta}{L}\lambda_2 M_A - \frac{\lambda_4}{2}H_B + \frac{\theta}{L}\lambda_6 M_B\right]$$

$$\frac{\partial w_A}{\partial H_A} = K\lambda_3 \qquad \frac{\partial \chi_A}{\partial H_A} = -K\frac{\theta}{L}\lambda_1$$

$$\frac{\partial w_A}{\partial M_A} = K\frac{\theta}{L}\lambda_1 \qquad \frac{\partial \chi_A}{\partial M_A} = -2K\left(\frac{\theta}{L}\right)^2\lambda_2$$

$$\frac{\partial w_A}{\partial H_B} = K\lambda_5 \qquad \frac{\partial \chi_A}{\partial H_B} = -K\frac{\theta}{L}\lambda_4$$

$$\frac{\partial w_A}{\partial M_B} = -K\frac{\theta}{L}\lambda_4 \qquad \frac{\partial \chi_A}{\partial M_B} = -2K\left(\frac{\theta}{L}\right)^2\lambda_6 \qquad [5\text{-}9a]$$

$$\frac{\partial w_A}{\partial p} = \left(1-\frac{\nu}{2}\right)\frac{R^2}{Eh} \qquad \frac{\partial \chi_A}{\partial p} = 0$$

$$\frac{\partial w_B}{\partial H_B} = -K\lambda_3 \qquad \frac{\partial \chi_B}{\partial H_B} = -K\frac{\theta}{L}\lambda_1$$

$$\frac{\partial w_B}{\partial M_B} = K\frac{\theta}{L}\lambda_1 \qquad \frac{\partial \chi_B}{\partial M_B} = 2K\left(\frac{\theta}{L}\right)^2\lambda_2$$

$$\frac{\partial w_B}{\partial H_A} = -K\lambda_5 \qquad \frac{\partial \chi_B}{\partial H_A} = -K\frac{\theta}{L}\lambda_4$$

$$\frac{\partial w_B}{\partial M_A} = -K\frac{\theta}{L}\lambda_4 \qquad \frac{\partial \chi_B}{\partial M_A} = -2K\left(\frac{\theta}{L}\right)^2\lambda_6 \qquad [5\text{-}9a]$$

$$\frac{\partial w_B}{\partial p} = \left(1 - \frac{\nu}{2}\right)\frac{R^2}{Eh} \qquad \frac{\partial \chi_B}{\partial p} = 0$$

EVALUATION

In discussing these equations, one inevitably must turn to the question of reliability. The best means of evaluation is direct comparison with experimental measurement. The difficulty with making such a comparison is that there are so much experimental data from which to choose. One could refer to Chapter 1 of this summary as a place to start looking for test reports, but there is no need. The equations presented here (if properly transcribed) are correct to the extent that linear thin shell theory is correct, and submarine hull geometries are well within the limits of this theory. Agreement with experiment will be found to be as good or as bad as the specimens tested and the quality of the measurements. Using reliable strain gages, properly installed on accurately machined shells, one is apt to find only very small discrepancies between theory and experiment. In such cases, the magnitude of the discrepancies will depend pretty much on how accurately the gages can be positioned, the length of the gage over which the measured strain is necessarily averaged, and the magnitude of the strain gradient in the region of interest. Positioning errors can be practically eliminated through photoelasticity studies, but with a considerable sacrifice in sensitivity.

What this comes down to, then, is that the size of the shell relative to the size of the measuring elements is all-important. In practice, of course, the real hull is extremely large but full of imperfections unavoidably introduced during construction. Because of them, the deformations under pressure are to some extent rotationally nonsymmetric—an effect not covered in the present chapter. It is possible to account analytically for this kind of behavior in an approximate way (see, for example, Chapters 2 and 7), but to do it with precision would require exhaustive measurements and analytical efforts far beyond the scope of this chapter. Measurements have shown, however, that averages of measured deformations in such cases conform closely to calculated values for the perfect (symmetric) case.

EXACT SOLUTION (TAYLOR-WENK FORM)

The differential equation of equilibrium for a conical shell of constant thickness under hydrostatic pressure can be stated in the form[5-5]

$$x^2 \frac{d^4w}{dx^4} + 2x \frac{d^3w}{dx^3} - 2 \frac{d^2w}{dx^2} + \frac{12\,(1-\nu^2)\,w}{h^2 \tan^2\alpha} = \frac{9\,(1-\nu^2)\,px^2}{Eh^3} \qquad [5\text{-}16]$$

Referring to Figure 5-4, x is the distance from the apex measured along a generator, α is one half the apex angle, and w is the deflection normal to the shell. The "beam-column" effect which has been discussed in connection with cylinders is not included here.

This equation has been investigated by many authors. Dubois[5-6] appears to have been the first to obtain a solution to the homogenous form of the equation (pressure term omitted). He expressed his result in terms of special infinite series and asymptotic formulas. Much of the subsequent effort on the problem has been devoted to reducing this highly intricate solution to a more simple, usable form and to its application for specific types of loading. Flügge (Reference 5-2, pp. 371 ff.), for example, solved the equation directly in terms of Bessel functions of second order which he expressed as functions tabulated by Schleicher.[5-7] A solution equivalent to this was obtained by Watts and Burrows[5-8] which they then applied to the stress analysis of the juncture of a cylinder and a truncated cone. There have been many other papers on the subject, mention of which will have to be omitted because of space limitations. Attention instead will be confined to two or three papers which are particularly useful for analyzing submarine structures.

The Watts-Burrows solution to Equation [5-16]* is given as follows:

$$\chi = \frac{dw}{dx} = \frac{2\sqrt{3(1-\nu^2)}}{Eh}\left(C_1\, kei_2\,\xi - C_2\, ker_2\,\xi + C_3\, bei_2\,\xi - C_4\, ber_2\,\xi\right) + \frac{3px\tan^2\alpha}{2Eh}$$

$$\begin{aligned}\overline{w} = -\frac{\sin\alpha}{Eh}\Big[&C_1\left(\frac{\xi}{2}\, ker_2'\,\xi - \nu\, ker_2\,\xi\right) + C_2\left(\frac{\xi}{2}\, kei_2'\,\xi - \nu\, kei_2\,\xi\right)\\ &+ C_3\left(\frac{\xi}{2}\, ber_2'\,\xi - \nu\, ber_2\,\xi\right) + C_4\left(\frac{\xi}{2}\, bei_2'\,\xi - \nu\, bei_2\,\xi\right)\Big]\end{aligned} \qquad [5\text{-}17]$$

***Actually their solution is derived from two second-order equations that are equivalent to Equation [5-16]:**

$$L(\overline{Q}) + Eh\,\chi\cot\alpha = \frac{3}{2}\,px\tan\alpha$$

$$L(\chi) - \frac{12\,(1-\nu^2)}{Eh^3}\,\overline{Q}\cot\alpha = 0$$

$$L(\;) = x\frac{d^2}{dx^2}(\;) + \frac{d}{dx}(\;) - \frac{(\;)}{x}$$

$$\overline{Q} = Qx\tan\alpha$$

where Q is the shear force acting in a direction normal to the shell.

$$+ \frac{px^2 \sin\alpha \ \tan\alpha}{Eh} \left(1 - \frac{\nu}{2}\right)$$

[5-17]

$$\xi = 2\sqrt{\frac{x}{h \tan\alpha}} \ \sqrt[4]{12(1-\nu^2)} = \frac{2\sqrt{2}\ \sqrt[4]{3(1-\nu^2)}}{\sin\alpha} \sqrt{\frac{R \cos\alpha}{h}}$$

ber_2, bei_2, ker_2 and kei_2 are Kelvin (Thomson) functions of second order. The primes indicate differentiation with respect to ξ, and the C's are constants to be determined from the boundary conditions. In form, this solution is exact, but approximations can be, and frequently are, made in determining the constants since their determination is a formidable problem. In most cases, the determination of the constants involves the evaluation of the Kelvin functions at the edges. The difficulty is that the practical range of interest for submarine structures and pressure vessels covers large values of the argument ξ for which tables of these functions are not readily available. Hence no solution is really useful unless it includes tables or graphs of the functions in this range.

In Reference 5-5, Taylor and Wenk have provided one of the most complete, yet practical solutions. To obtain values of the constants, they made use of one approximation—that deformations near one edge are unaffected by forces and moments acting at the other. In this sense, the shell is assumed to be long or "single-ended." This often used approximation greatly simplifies the computation. It is generally considered accurate so long as $\xi_B - \xi_A \geq \sqrt{2}\,\pi$ where A and B designate the two edges.* The results of the Taylor-Wenk analysis are as follows:

$$C_1 = \frac{Q_A x_A h \tan\alpha \ (\xi_A \ ker_2' \ \xi_A + 2\nu \ ker_2 \ \xi_A) + 2 M_A U^2 x_A \ kei_2 \ \xi_A}{h(F_A + 2\nu G_A)}$$

$$C_2 = \frac{Q_A x_A h \tan\alpha \ (\xi_A \ kei_2' \ \xi_A + 2\nu \ kei_2 \ \xi_A) - 2 M_A U^2 x_A \ ker_2 \ \xi_A}{h(F_A + 2\nu G_A)}$$

[5-18]

$$C_3 = \frac{Q_B x_B h \tan\alpha \ (\xi_B \ ber_2' \ \xi_B + 2\nu \ ber_2 \ \xi_B) + 2 M_B U^2 x_B \ bei_2 \ \xi_B}{h(F_B + 2\nu G_B)}$$

$$C_4 = \frac{Q_B x_B h \tan\alpha \ (\xi_B \ bei_2' \ \xi_B + 2\nu \ bei_2 \ \xi_B) - 2 M_B U^2 x_B \ ber_2 \ \xi_B}{h(F_B + 2\nu G_B)}$$

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*Baltrukonis[5-9] has the equivalent short or "double-ended" shell solution where this approximation is not made. Throughout this chapter (except for the case of the cylinder), equations are written out in detail only for the single-ended case. The corresponding equations for the double-ended case are usually too complicated to be presented in this way, and it is easier to work them out numerically.

$$U^2 = 2\sqrt{3(1 - \nu^2)}$$

$$F_A = \xi_A\,(ker_2\,\xi_A\,ker_2'\,\xi_A + kei_2\,\xi_A\,kei_2'\,\xi_A)$$

$$G_A = (ker_2\,\xi_A)^2 + (kei_2\,\xi_A)^2 \qquad [5\text{-}18]$$

$$F_B = \xi_B\,(ber_2\,\xi_B\,ber_2'\,\xi_B + bei_2\,\xi_B\,bei_2'\,\xi_B)$$

$$G_B = (ber_2\,\xi_B)^2 + (bei_2\,\xi_B)^2$$

It should be remembered that in the notation of this report, A refers to the small end and B to the large end of the truncated cone. In Reference 5-5, some of the symbols and sign conventions are different, and subscripts 1 and 2 are used to denote the large and small ends, respectively. The coefficients, it will be observed, are written in terms of the shear forces Q_A and Q_B rather than the vertical forces H_A and H_B. For both ends of the cone, these are related by:

$$Q = H\cos\alpha + \frac{pR}{2}\sin\alpha \qquad [5\text{-}19]$$

The stresses σ_θ (circumferential) and σ_x (along a generator) at any point can then be found from the following relations as given in Reference 5-10:

$$\sigma_x = \frac{Eh^2}{12(1-\nu^2)}\tan\alpha\left(\frac{d^3w}{dx^3} + \frac{1}{x}\frac{d^2w}{dx^2} - \frac{1}{x^2}\frac{dw}{dx}\right) - \frac{px}{2h}\tan\alpha$$

$$\pm\frac{Eh}{2(1-\nu^2)}\left(\frac{d^2w}{dx^2} + \frac{\nu}{x}\frac{dw}{dx}\right) \qquad [5\text{-}20]$$

$$\sigma_\theta = \frac{1}{h}\frac{d}{dx}(xN_x) \pm \frac{Eh}{2(1-\nu^2)}\left(\nu\frac{d^2w}{dx^2} + \frac{1}{x}\frac{dw}{dx}\right)$$

$$N_x = \frac{Eh^3}{12(1-\nu^2)}\tan\alpha\left(\frac{d^3w}{dx^3} + \frac{1}{x}\frac{d^2w}{dx^2} - \frac{1}{x^2}\frac{dw}{dx}\right) - \frac{px}{2}\tan\alpha$$

where, as before, a positive stress indicates tension, and the upper and lower signs in the expression for σ_x refer to the outer and inner surfaces, respectively. The determination of the stresses using Equations [5-17] to [5-20] is usually laborious and not always necessary since it is often the deformations and stresses at the boundaries of the shell rather than at an interior point that are required. It is here that the analysis of Taylor and Wenk[5-5] is particularly useful since it presents the necessary edge coefficients in a convenient, yet exact form. Furthermore, the accuracy of other more approximate solutions is readily assessable from an examination of these coefficients. Usually such approximations are based on the condition that $\xi(x)$ exceeds a certain value at the small end of the cone. No such approximation is involved in the Taylor-Wenk coefficients which were obtained from the exact solution. As previously noted, the only approximation involved is that conditions at both edges were not considered simultaneously. This was again applied in determining the edge coefficients; i.e., C_3 and C_4 were neglected when considering deformations at Edge A and C_1 and C_2 were neglected for Edge B. In the notation of this report, the coefficients are as follows:

$$\frac{\partial \bar{w}_A}{\partial H_A} = K_A \Omega_6 \cos\alpha \qquad \frac{\partial \bar{w}_B}{\partial H_B} = -K_B \Omega_3 \cos\alpha$$

$$\frac{\partial \bar{w}_A}{\partial M_A} = K_A \beta_A \Omega_5 \qquad \frac{\partial \bar{w}_B}{\partial M_B} = K_B \beta_B \Omega_2$$

$$\frac{\partial \bar{w}_A}{\partial p} = \frac{R_A K_A}{2}\left[\Omega_6 \sin\alpha + \frac{1-\nu/2}{R_A \beta_A} + \frac{R_A \beta_A}{4(1-\nu)}\left(\frac{h}{R_A}\right)^2 \Omega_5 \tan^2\alpha\right] \qquad \frac{\partial \bar{w}_B}{\partial p} = \frac{R_B K_B}{2}\left[-\Omega_3 \sin\alpha + \frac{1-\nu/2}{R_B \beta_B} - \frac{R_B \beta_B}{4(1-\nu)}\left(\frac{h}{R_B}\right)^2 \Omega_2 \tan^2\alpha\right] \tag{5-21}$$

$$\frac{\partial \chi_A}{\partial H_A} = -K_A \beta_A \Omega_5 \qquad \frac{\partial \chi_B}{\partial H_B} = -K_B \beta_B \Omega_2$$

$$\frac{\partial \chi_A}{\partial M_A} = -\frac{2 K_A \beta_A^2}{\cos\alpha} \Omega_4 \qquad \frac{\partial \chi_B}{\partial M_B} = \frac{2 K_B \beta_B^2}{\cos\alpha} \Omega_1$$

$$\frac{\partial \chi_A}{\partial p} = -\frac{K_A}{2} \tan\alpha \left[R_A \beta_A \Omega_5 - \frac{3}{2 R_A \beta_A} - \frac{3(1+\nu)\sin\alpha}{2(R_A \beta_A)^2} \Omega_4\right] \qquad \frac{\partial \chi_B}{\partial p} = -\frac{K_B}{2} \tan\alpha \left[R_B \beta_B \Omega_2 - \frac{3}{2 R_B \beta_B} + \frac{3(1+\nu)\sin\alpha}{2(R_B \beta_B)^2} \Omega_1\right]$$

$$\beta_A = \sqrt[4]{3(1-\nu^2)}\sqrt{\frac{\cos\alpha}{R_A h}} = \frac{\xi_A \sin\alpha}{2\sqrt{2}R_A}$$

$$\beta_B = \sqrt[4]{3(1-\nu^2)}\sqrt{\frac{\cos\alpha}{R_B h}} = \frac{\xi_B \sin\alpha}{2\sqrt{2}R_B}$$

$$K_A = \frac{2R_A^2\beta_A}{Eh\cos\alpha} = \frac{R_A\xi_A}{Eh\sqrt{2}}\tan\alpha$$

$$K_B = \frac{2R_B^2\beta_B}{Eh\cos\alpha} = \frac{R_B\xi_B}{Eh\sqrt{2}}\tan\alpha$$

$$\Omega_4 = \frac{-\xi_A G_A}{\sqrt{2}(F_A + 2\nu G_A)} \qquad \Omega_1 = \frac{\xi_B G_B}{\sqrt{2}(F_B + 2\nu G_B)}$$

$$\Omega_5 = \frac{-C_A}{F_A + 2\nu G_A} \qquad \Omega_2 = \frac{-C_B}{F_B + 2\nu G_B}$$

$$\Omega_6 = \frac{-\xi_A D_A - \dfrac{4\nu^2 G_A}{\xi_A}}{\sqrt{2}(F_A + 2\nu G_A)} \qquad \Omega_3 = \frac{\xi_B D_B - \dfrac{4\nu^2 G_B}{\xi_B}}{\sqrt{2}(F_B + 2\nu G_B)}$$

$$C_A = \xi_A(ker_2'\,\xi_A\, kei_2\,\xi_A - kei_2'\,\xi_A\, ker_2\,\xi_A)$$

$$D_A = (ker_2'\,\xi_A)^2 + (kei_2'\,\xi_A)^2$$

$$C_B = \xi_B(ber_2'\,\xi_B\, bei_2\,\xi_B - bei_2'\,\xi_B\, ber_2\,\xi_B)$$

$$D_B = (ber_2'\,\xi_B)^2 + (bei_2'\,\xi_B)^2$$

Using series expansions for the Bessel and Kelvin functions, Taylor and Wenk obtained values for the Ω's over a wide range of the argument ξ. They presented the results in the form of graphs which are reproduced from Reference 5-5 here in Figure 5-5. These curves clearly show how the Ω-functions tend toward unity for large values of ξ and the extent of the error involved when they are taken to be unity as is done in many of the approximate solutions.

APPROXIMATE SOLUTION (RAETZ-PULOS)

Long Shells

One of the most useful of the simplified solutions is based on an approximation that has been employed by Geckeler[5-11] in dealing with problems of this type. When it is permissible to neglect the variations in radius with the meridional coordinate, the governing differential equations can usually be simplified. In the case of a segment of a conical shell along which the total change in radius is not large, this is a reasonable approximation. Furthermore, Wenk and Taylor[5-10] have shown that for the range of interest of pressure vessel designers, the terms in Equation [5-16] involving $\frac{d^3w}{dx^3}$ and $\frac{d^2w}{dx^2}$ can be neglected. With the origin of x taken at one edge of the segment rather than at the cone apex, the differential equation in its homogeneous form is:

$$\frac{d^4w}{dx^4} = 4\ \beta^4\ w = 0 \qquad [5\text{-}22]$$

where

$$\beta^4 = \frac{3\ (1 - \nu^2)\ \cos^2\alpha}{h^2\ R^2}$$

and R is a constant average radius.

Since this equation is identical in form to Equation [5-5], it is referred to as the equivalent cylinder approximation.* Its solution in terms of exponential rather than hyperbolic functions is

$$w_b = e^{-\beta x}\ (C_1 \cos \beta x + C_2 \sin \beta x) + e^{\beta x}\ (C_3 \cos \beta x + C_4 \sin \beta x) \qquad [5\text{-}23]$$

The subscript "b" indicates that this is the deflection resulting from the bending of the shell by edge forces and moments and does not include "membrane" deflections (independent of x) as would be produced by normal pressure. Raetz and Pulos[5-12] have considered the long shell case treated by Taylor and Wenk, i.e., where the edges of the shell are sufficiently separated that deformations near one edge are unaffected by conditions at the other. In that case when determining the deflection w_b near Edge A, Raetz and Pulos have shown that the coefficients C_3 and C_4 can be neglected (here the use of the exponential form makes this

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*Geekeler's approach was to reduce the Love-Meissner differential equations for arbitrary shells of revolution to a simplified equation of the form [5-22] by making certain approximations. Hence, the equivalent "Geekeler" equation for any shell, regardless of its shape can always be represented by Equation [5-22], and its solution by [5-23]. The definitions of w, x, and β can vary according to the type of shell being considered.

simplification possible) and the shell parameters can be defined in terms of the local geometry. Their solution is

Near Edge A:

$$w_b = \frac{K_A}{\cos\alpha} e^{-\beta_A x} [Q_A \cos \beta_A x + \beta_A M_A (\cos \beta_A x - \sin \beta_A x)] \qquad [5\text{-}24a]$$

where K_A and β_A have the same meanings as in Equation [5-13]. At the other end of the shell, C_1 and C_2 can likewise be neglected, so that:

Near Edge B:

$$w_b = \frac{-K_B}{\cos\alpha} e^{\beta_B (x - x_B)} \{\cos \beta_B x [Q_B \cos \beta_B x_B + \beta_B M_B (\sin \beta_B x_B - \cos \beta_B x_B)]$$

$$+ \sin \beta_B x [Q_B \sin \beta_B x_B - \beta_B M_B (\sin \beta_B x_B + \cos \beta_B x_B)]\} \qquad [5\text{-}24b]$$

or

$$w_b = \frac{-K_B}{\cos\alpha} e^{-\beta_B y} [Q_B \cos \beta_B y + M_B \beta_B (\sin \beta_B y - \cos \beta_B y)]$$

$$y = x_B - x$$

It can be noted that R_A and R_B are related by:

$$R_B = R_A + x_B \sin\alpha \qquad [5\text{-}25]$$

where $\alpha > 0$ for Edge A at small end,

$\alpha < 0$ for Edge A at large end, and

x_B is the slant length of the cone frustrum.

It is also important to remember that the sign conventions used here are those indicated in Figure 5-4 and not those followed in Reference 5-12. With the present conventions:

$$Q_A = H_A \cos\alpha + \frac{pR_A}{2} \sin\alpha$$

$$Q_B = H_B \cos\alpha + \frac{pR_B}{2} \sin\alpha \qquad [5\text{-}26]$$

The reader is cautioned that unless one system of sign conventions is consistently followed, errors can easily result, especially when several connected shell elements are under consideration.

To obtain the total stress at any point in the shell, one must determine the stress associated with the bending deflection w_B and add to it the corresponding membrane stress. To do this, Raetz and Pulos have made use of simplified relations for the stress resultants consistent with the approximations already made:

$$\sigma_x = -\frac{pR}{2h\cos\alpha} - \frac{Eh^2}{12(1-\nu^2)}\tan\alpha \cdot \frac{d^3 w_b}{dx^3} \pm \frac{Eh}{2(1-\nu^2)}\frac{d^2 w}{dx^2}$$

$$\sigma_\theta = -\frac{pR(1-\nu/2)}{h\cos\alpha} - \frac{Ew_b}{R}\cos\alpha + \nu\sigma_x \qquad [5\text{-}27]$$

In each of these equations, the first term is the membrane stress. To get an idea of the kind of approximations involved, one can compare these relations with the exact Equations [5-20]. By combining Equations [5-27] with [5-24a] and [5-24b], Raetz and Pulos obtained expressions for the stresses than can be written as follows:

Near Edge A,

$$\sigma_x = -\frac{pR_A}{2h\cos\alpha} + \frac{\tan\alpha}{h}\left[Q_A(\zeta_A - \eta_A) + 2M_A\beta_A\zeta_A\right]$$

$$\pm \frac{6}{h^2\beta_A}\left[Q_A\zeta_A + M_A\beta_A(\eta_A + \zeta_A)\right]$$

$$[5\text{-}28a]$$

$$\sigma_\theta = -\frac{pR_A}{h\cos\alpha}(1-\nu/2) + \frac{2\beta_A}{h\cos\alpha}\left[-Q_A\eta_A + M_A\beta_A(\zeta_A - \eta_A)\right] + \nu\sigma_x$$

Near Edge B,

$$\sigma_x = -\frac{pR_B}{2h\cos\alpha} + \frac{\tan\alpha}{h}\left[Q_B(\zeta_B - \eta_B) - 2M_B\zeta_B\right]$$

$$\pm \frac{6}{h^2\beta_B}\left[-Q_B\zeta_B + M_B\beta_B(\eta_B + \zeta_B)\right]$$

$$[5\text{-}28b]$$

$$\sigma_\theta = -\frac{pR_B}{h\cos\alpha}(1-\nu/2) + \frac{2\beta_B}{h\cos\alpha}\left[Q_B\zeta_B + M_B\beta_B(\zeta_B - \eta_B)\right]$$

where

$$\eta_A = e^{-\beta_A x} \cos \beta_A x \qquad \zeta_A = e^{-\beta_A x} \sin \beta_A x$$

$$\eta_B = e^{-\beta_B y} \cos \beta_B y \qquad \zeta_B = e^{-\beta_B y} \sin \beta_B y$$

Graphs of the functions η and ζ are reproduced here from Reference 5-10 in Figure 5-5.

To get the corresponding edge coefficients, Raetz and Pulos have employed an approximation for the radial deflection $\bar{w}$ which is discussed in Reference 5-12:

$$\bar{w} = w_b \cos\alpha + \frac{pR^2}{Eh\cos\alpha}(1 - \nu/2) \qquad [5\text{-}29]$$

The edge coefficients can then be written:

$$\frac{\partial \bar{w}_A}{\partial H_A} = K_A \cos\alpha \qquad \frac{\partial \bar{w}_B}{\partial H_B} = -K_B \cos\alpha$$

$$\frac{\partial \bar{w}_A}{\partial M_A} = K_A \beta_A \qquad \frac{\partial \bar{w}_B}{\partial M_B} = K_B \beta_B$$

$$\frac{\partial \bar{w}_A}{\partial p} = \frac{K_A}{2\beta_A}(1 - \nu/2) + \frac{K_A R_A}{2}\sin\alpha \qquad \frac{\partial \bar{w}_B}{\partial p} = \frac{K_B}{2\beta_B}(1 - \nu/2) - \frac{K_B R_B}{2}\sin\alpha \ .$$

$$[5\text{-}30]$$

$$\frac{\partial \chi_A}{\partial H_A} = -K_A \beta_A \qquad \frac{\partial \chi_B}{\partial H_B} = -K_B \beta_B$$

$$\frac{\partial \chi_A}{\partial M_A} = -\frac{2K_A \beta_A^2}{\cos\alpha} \qquad \frac{\partial \chi_B}{\partial M_B} = \frac{2K_B \beta_B^2}{\cos\alpha}$$

$$\frac{\partial \chi_A}{\partial p} = -\frac{K_A \tan\alpha}{2}\left(R_A \beta_A - \frac{3}{2R_A \beta_A}\right) \qquad \frac{\partial \chi_B}{\partial p} = -\frac{K_B \tan\alpha}{2}\left(R_B \beta_B - \frac{3}{2R_B \beta_B}\right)$$

where χ_B has the meaning $\frac{dw}{dx}$, not $\frac{dw}{dy}$. It can be seen that these coefficients bear a close similarity to those for the cylindrical shell (Equations [5-10]) and that they are almost identical to those given by Taylor and Wenk (Equations [5-21]) for large arguments of ξ where the Ω-functions approach unity. The exceptions are the third terms in $\frac{\partial \bar{w}}{\partial p}$ and $\frac{\partial \chi}{\partial p}$ which do not appear in the Raetz-Pulos coefficients. It might also be mentioned that the expressions for $\frac{\partial \chi_A}{\partial p}$ and $\frac{\partial \chi_B}{\partial p}$ appearing in Equations [5-12] include within the parentheses the term $-\frac{3}{2R\beta}$ which results from taking the radius R to be a function of x rather than as a constant. This term is usually negligible compared with the first term.

The foregoing equations can be used with confidence so long as $\beta_A x_B$ and $\beta_B x_B$ are no smaller than 3.* In this range, conditions at one edge do not significantly affect deformations at the other. However, it should be realized that near the middle of the shell, conditions at both edges are apt to have influences that cannot be disregarded unless $\beta_A x_B$ and $\beta_B x_B$ are greater than 6. In other words, the effects of both discontinuities can be expected to overlap (and over a considerable portion of the shell if $\beta x_B \approx 3$). Raetz and Pulos have shown that the total effect of both discontinuities can be accurately accounted for by superposition. To calculate the stress at any point using Equations [5-28], one adds to the membrane stresses the additional stresses resulting from M and Q at both edges. If $\beta_A x_B$ or $\beta_B x_B < 3$, this procedure will result in inaccuracies because in that range, the effects of the discontinuities overlap throughout the entire length of the shell so that M_A and Q_A are influenced by M_B and Q_B, and vice versa. An analysis that takes this interaction into account is described in the next section.

Short Shells

This case has been covered in a number of papers including, for example, the work of Linkous and Horvay[5-13] and that of Raetz.** Equations developed in this latter paper are convenient for presentation here because they are developed for the particular loading conditions of interest.

For the short shell, it is sufficiently accurate to dispense with β_A and β_B and use the single parameter

$$\beta = \sqrt[4]{3\,(1-\nu^2)}\,\sqrt{\frac{\cos\alpha}{Rh}} \qquad [5\text{-}31]$$

where R is an average radius given by $\frac{R_A + R_B}{2}$; x is still measured from Edge A so that x_B denotes the length of the shell. The "bending" deflection, is again obtained as a solution to the differential Equation [5-22]. However, in this case, it is convenient to express w_b in terms of hyperbolic instead of exponential functions since it is not permissible

*This is approximately the same as saying $\xi_B - \xi_A \geq 3\sqrt{2} \approx \pi\sqrt{2}$, the limit placed on the Taylor-Wenk analysis (page 14).

**Unpublished work on short shells.

to make the approximations leading to Equations [5-24]. The form of w_b is then the same as that for the solution [5-6] of the cylindrical shell equations (excluding the membrane term):

$$\begin{aligned} w_b = \; & C_1 \cos\beta x \cosh\beta x + C_2 \sin\beta x \sinh\beta x \\ & + C_3 \cos\beta x \sinh\beta x + C_4 \sin\beta x \cosh\beta x \end{aligned} \qquad [5\text{-}32]$$

If θ is allowed to represent βx_B, the coefficients can be expressed by Equations [5-7a] with H replaced by Q:

$$\begin{aligned} C_1 &= K\,[\lambda_3 Q_A + \beta\lambda_1 M_A + \lambda_5 Q_B - \beta\lambda_4 M_B] \\ C_2 &= K\beta M_A \\ C_3 &= K\left[-\frac{(\lambda_1+1)}{2}Q_A - \beta\lambda_2 M_A - \frac{\lambda_4}{2}Q_B + \beta\lambda_6 M_B\right] \\ C_4 &= K\left[\frac{(1-\lambda_1)}{2}Q_A - \beta\lambda_2 M_A - \frac{\lambda_4 Q_B}{2} + \beta\lambda_6 M_B\right] \end{aligned} \qquad [5\text{-}33]$$

Here K represents $\frac{2R^2\beta}{Eh}$, and the series relations [5-12]–[5-15] all apply in the present case.

The radial deflection $\bar{w}$ is again obtained using the approximation [5-29] and the stresses are given by

$$\begin{aligned} \sigma_x &= -\frac{pR}{2h\cos\alpha} \pm \frac{Eh}{2(1-\nu^2)}\frac{d^2w_b}{dx^2} \\ \sigma_\theta &= -\frac{pR\,(1-\nu/2)}{h\cos\alpha} - \frac{Ew_b}{R}\cos\alpha + \nu\sigma_x \end{aligned} \qquad [5\text{-}34]$$

It may be noticed that the equation for σ_x contains one less term than does Equation [5-27]. Raetz has found that this term $\left(\text{involving } \frac{d^3w_b}{dx^3}\right)$ is much smaller than the others and can be neglected.

In terms of the λ-functions, the edge coefficients for this case are as follows:

$$\frac{\partial \bar{w}_A}{\partial H_A} = K\lambda_3 \qquad \frac{\partial \bar{w}_B}{\partial H_B} = -K\lambda_3$$

$$\frac{\partial \bar{w}_A}{\partial M_A} = \frac{K\beta\lambda_1}{\cos\alpha} \qquad \frac{\partial \bar{w}_B}{\partial M_B} = \frac{K\beta\lambda_1}{\cos\alpha}$$

$$\frac{\partial \bar{w}_A}{\partial H_B} = K\lambda_5 \qquad \frac{\partial \bar{w}_B}{\partial H_A} = -K\lambda_5$$

$$\frac{\partial \bar{w}_A}{\partial M_B} = \frac{-K\beta\lambda_4}{\cos\alpha} \qquad \frac{\partial \bar{w}_B}{\partial M_A} = -\frac{K\beta\lambda_4}{\cos\alpha}$$

$$\frac{\partial \bar{w}_A}{\partial p} = \frac{K}{2\cos\alpha}\left[\frac{1-\nu/2}{\beta} + (\lambda_3+\lambda_5)R\sin\alpha\right] \qquad \frac{\partial \bar{w}_B}{\partial p} = \frac{K}{2\cos\alpha}\left[\frac{1-\nu/2}{\beta} - (\lambda_3+\lambda_5)R\sin\alpha\right] \tag{5-35}$$

$$\frac{\partial \chi_A}{\partial H_A} = -\frac{K\beta\lambda_1}{\cos\alpha} \qquad \frac{\partial \chi_B}{\partial H_B} = -\frac{K\beta\lambda_1}{\cos\alpha}$$

$$\frac{\partial \chi_A}{\partial M_A} = -\frac{2K\beta^2\lambda_2}{\cos^2\alpha} \qquad \frac{\partial \chi_B}{\partial M_B} = \frac{2K\beta^2\lambda_2}{\cos^2\alpha}$$

$$\frac{\partial \chi_A}{\partial H_B} = -\frac{K\beta\lambda_4}{\cos\alpha} \qquad \frac{\partial \chi_B}{\partial H_A} = -\frac{K\beta\lambda_4}{\cos\alpha}$$

$$\frac{\partial \chi_A}{\partial M_B} = \frac{2K\beta^2\lambda_6}{\cos^2\alpha} \qquad \frac{\partial \chi_B}{\partial M_A} = -\frac{2K\beta^2\lambda_6}{\cos^2\alpha}$$

$$\frac{\partial \chi_A}{\partial p} = -\frac{KR\beta\tan\alpha}{\cos\alpha}(\lambda_1+\lambda_4) \qquad \frac{\partial \chi_B}{\partial p} = -\frac{KR\beta\tan\alpha}{\cos\alpha}(\lambda_1+\lambda_4)$$

ANOTHER APPROXIMATE FORM (AFTER HETÉNYI)

A later section of this report presents an approximate solution due to Hetényi[5-14] for spherical shells which is remarkable for its simplicity and accuracy. The same approach permits an equivalent solution for conical shells. The author was unable to find such a solution in the literature, but believes this only to be the sign of an incomplete search since the method is so readily adaptable.

The two second-order homogeneous equations for the conical shell can be written:

$$L(\bar{Q}) + Eh\,\chi \cot\alpha = 0$$

$$L(\chi) - \frac{12(1-\nu^2)}{Eh^3}\,\bar{Q}\cot\alpha = 0 \qquad [5\text{-}36]$$

in which

$$\bar{Q} = x\,Q\tan\alpha$$

$$L(\ ) = x\frac{d^2}{dx^2}(\ ) + \frac{d(\ )}{dx} - \frac{(\ )}{x}$$

The combination of Equations [5-36] produces the fourth-order equation (see Reference 5-1, p. 458),

$$L\,L(\bar{Q}) + 4\lambda^4\bar{Q} = 0 \qquad [5\text{-}37a]$$

which can be expressed as the two second-order equations

$$L(\bar{Q}) \pm 2i\lambda^2\bar{Q} = 0 \qquad [5\text{-}37b]$$

where

$$\lambda = \frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{h\tan\alpha}}$$

with the transformations

$$\bar{Q} = \frac{q(z)}{\sqrt{z}}$$

$$z = 2\sqrt{x} \qquad [5\text{-}38]$$

Equations [5-37b] takes the form

$$\frac{d^2q}{dz^2} - \frac{15q}{4z^2} \pm 2i\lambda^2 q = 0 \qquad [5\text{-}39]$$

which are easily solved if the term $-15q/4z^2$ can be neglected. We can expect in general that $\frac{d^2q}{dz^2} >> q$ so that the approximation ought to be reasonable except for small values of z.

Thus, the apex region is excluded from consideration.* With this approximation, Equations [5-37a] and [5-37b] are reduced to

$$\frac{d^2 q}{dz^2} \pm 2 i \lambda^2 q = 0 \qquad [5\text{-}40a]$$

$$\frac{d^4 q}{dz^4} + 4 \lambda^4 q = 0 \qquad [5\text{-}40b]$$

It will be seen that these are the cylindrical shell equations whose solution is

$$\begin{aligned} q = {} & e^{-\lambda z} (C_1 \cos \lambda z + C_2 \sin \lambda z) \\ & + e^{\lambda z} \ (C_3 \cos \lambda z + C_4 \sin \lambda z) \end{aligned} \qquad [5\text{-}41]$$

The shear force is thus obtained from the relation

$$\begin{aligned} \overline{Q} = \frac{z^2}{4} Q \tan \alpha = \frac{1}{\sqrt{z}} \ & [e^{-\lambda z} (C_1 \cos \lambda z + C_2 \sin \lambda z) \\ & + e^{\lambda z} \ (C_3 \cos \lambda z + C_4 \sin \lambda z)] \end{aligned} \qquad [5\text{-}42]$$

The first of Equations [5-28] can be solved for the rotation χ:

$$\begin{aligned} \chi & = - \frac{\tan \alpha}{E h} \left[\frac{d^2 \overline{Q}}{dz^2} + \frac{1}{z} \frac{d\overline{Q}}{dz} - \frac{4 \overline{Q}}{z^2} \right] \\ & = - \frac{\tan \alpha}{E h \sqrt{z}} \left[\frac{d^2 q}{dz^2} - \frac{15 q}{4 z^2} \right] \end{aligned} \qquad [5\text{-}43]$$

As before, we neglect the quantity $\dfrac{15 q}{4 z^2}$ and have the results

$$\chi = - \frac{\tan \alpha}{E h \sqrt{z}} \frac{d^2 q}{dz^2} \qquad [5\text{-}44]$$

---

*Hetényi's solution for the spherical shell excludes the polar region on the same grounds.

Knowing the basic variables $\bar{Q}$ and χ, we can determine all quantities necessary for obtaining the stresses and deflections. With the aid of Reference 5-1 (pp. 454 ff.) and making allowances for differences in sign conventions, we get the relations:

$$N_x = \frac{\bar{Q}}{x} - \frac{p\,x\tan\alpha}{2} = \frac{4\,q}{z^2\sqrt{z}} - \frac{pz^2}{8}\tan\alpha$$

$$N_\theta = \frac{d\bar{Q}}{dx} - \frac{pR\,(1-\nu/2)}{\cos\alpha} = \frac{2}{z\sqrt{z}}\left(\frac{dq}{dz} - \frac{q}{2\,z}\right) - \frac{pz^2}{4}\,(1-\nu/2)\tan\alpha$$

$$M_x = \frac{Eh^3}{12\,(1-\nu^2)}\left(\frac{d\chi}{dx} + \frac{\nu\chi}{x}\right) = \frac{Eh^3}{6\,(1-\nu^2)\,z}\left(\frac{d\chi}{dz} + \frac{2\,\nu}{z}\,\chi\right)$$

$$M_\theta = \frac{Eh^3}{12\,(1-\nu^2)}\left(\frac{\chi}{x} + \nu\,\frac{d\chi}{dx}\right) = \nu M_x - \frac{h^2\tan\alpha}{3\,z^2\sqrt{z}}\;\frac{d^2q}{dz^2} \qquad [5\text{-}45]$$

$$\bar{w} = -\,x\,\frac{\sin\alpha}{Eh}\left(\frac{d\bar{Q}}{dx} + \nu\,\frac{\bar{Q}}{x}\right) = -\,\frac{\sqrt{z}\,\sin\alpha}{2\,Eh}\left[\frac{dq}{dz} - \frac{q}{2\,z}\,(1+4\,\nu)\right]$$

$$+\,\frac{p\,z^4\sin^2\alpha\;(1-\nu/2)}{16\,Eh\cos\alpha} + \frac{pR^2\,(1-\nu/2)}{Eh\cos\alpha}$$

in which the membrane stress resultants and deflection have been included. For the case of a long or single-ended shell, near Edge A the solution is

$$q = e^{-\lambda z}\,(C_1\cos\lambda z + C_2\sin\lambda z) \qquad [5\text{-}46a]$$

and near Edge B

$$q = e^{\lambda z}\,(C_3\cos\lambda z + C_4\sin\lambda z) \qquad [5\text{-}46b]$$

The C's can then be written in terms of the edge moments and forces.

$$C_1 = \frac{z_A\sqrt{z_A}}{d_A}\,e^{\lambda z_A}\left[\frac{Q_A\,z_A\tan\alpha}{4}\,(\sin\lambda z_A + d_A\cos\lambda z_A) + M_A\,\lambda\tan\alpha\,\sin\lambda z_A\right]$$

[5-47]

$$C_2 = \frac{z_a\sqrt{z_a}}{d_A}\,e^{\lambda z_A}\left[\frac{Q_A\,z_A\tan\alpha}{4}\,(-\cos\lambda z_A + d_A\sin\lambda z_A) - M_A\,\lambda\tan\alpha\,\cos\lambda z_A\right]$$

$$C_3 = \frac{z_B\sqrt{z_B}}{d_B}\, e^{\lambda z_B}\left[\frac{Q_B z_B \tan\alpha}{4}\,(-\sin\lambda z_B + d_B\cos\lambda z_B) + M_B\,\lambda\tan\alpha\,\sin\lambda z_B\right]$$

[5-47]

$$C_4 = \frac{z_B\sqrt{z_B}}{d_B}\, e^{\lambda z_B}\left[\frac{Q_B z_B \tan\alpha}{4}\,(\cos\lambda z_B + d_B\sin\lambda z_B) - M_B\,\lambda\tan\alpha\,\cos\lambda z_B\right]$$

$$d_A = 1 + \frac{1 - 4\nu^*}{2\lambda z_A}$$

$$d_B = 1 - \frac{1 - 4\nu}{2\lambda z_B}$$

The corresponding edge coefficients in which the effects of lateral pressure have been included are as follows:

$$\frac{\partial \bar{w}_A}{\partial H_A} = \frac{K_A\cos\alpha}{d_A}\left[1 + \frac{\sin\alpha}{4\beta_A R_A} + \frac{(1-16\nu^2)}{2}\left(\frac{\sin\alpha}{4\beta_A R_A}\right)^2\right]$$

$$\frac{\partial \bar{w}_B}{\partial H_B} = -\frac{K_B\cos\alpha}{d_B}\left[1 - \frac{\sin\alpha}{4\beta_B R_B} + \frac{(1-16\nu^2)}{2}\left(\frac{\sin\alpha}{4\beta_B R_B}\right)^2\right]$$

$$\frac{\partial \bar{w}_A}{\partial M_A} = \frac{K_A\beta_A}{d_A} \qquad \frac{\partial \bar{w}_B}{\partial M_B} = \frac{K_B\beta_B}{d_B}$$

$$\frac{\partial \bar{w}_A}{\partial p} = \frac{R_A\tan\alpha}{2}\,\frac{\partial \bar{w}_A}{\partial H_A} + \frac{K_A(1-\nu/2)}{2\beta_A} \qquad \frac{\partial \bar{w}_B}{\partial p} = \frac{R_B\tan\alpha}{2}\,\frac{\partial \bar{w}_B}{\partial H_B} + \frac{K_B(1-\nu/2)}{2\beta_B}$$

$$\frac{\partial \chi_A}{\partial H_A} = -\frac{K_A\beta_A}{d_A} \qquad \frac{\partial \chi_B}{\partial H_B} = -\frac{K_B\beta_B}{d_B} \qquad [5\text{-}48]$$

$$\frac{\partial \chi_A}{\partial M_A} = -\frac{2K_A\beta_A^2}{d_A\cos\alpha} \qquad \frac{\partial \chi_B}{\partial M_B} = \frac{2K_B\beta_B^2}{d_B\cos\alpha}$$

$$\frac{\partial \chi_A}{\partial p} = -\frac{R_A K_A\beta_A}{2\,d_A}\tan\alpha \qquad \frac{\partial \chi_B}{\partial p} = -\frac{R_B K_B\beta_B}{2\,d_B}\tan\alpha$$

---

*The reader may observe that it is possible for d_A to be zero, in which case the solution is meaningless. However this can happen only if $\frac{x_A}{h} = \frac{(4\nu-1)^2\tan\alpha}{16\sqrt{3(1-\nu^2)}}$, which is extremely close to the apex and therefore well outside the region in which the solution is intended to be applicable.

The symbols β_A, β_B, K_A, and K_B are as defined in the section covering the Wenk-Taylor analysis. One can see at a glance the close correspondence between these equations and those obtained by Hetényi (Equations [5-75]) for the spherical shell.

It is also instructive to express the edge coefficients in terms of the parameter ξ employed by Taylor and Wenk. Using the relations

$$\frac{\sin\alpha}{2\beta_A R_A} = \frac{\sqrt{2}}{\xi_A} \; ; \quad \frac{\sin\alpha}{2\beta_B R_B} = \frac{\sqrt{2}}{\xi_B}$$

$$d_A = 1 + \frac{\sqrt{2}(1-4\nu)}{2\xi_A} \; ; \quad d_B = 1 - \frac{\sqrt{2}(1-4\nu)}{2\xi_B} \tag{5-49}$$

one can write Equations [5-48] as follows:

$$\frac{\partial \bar{w}_A}{\partial H_A} = K_A \cos\alpha \cdot \omega_6 \qquad \frac{\partial \bar{w}_B}{\partial H_B} = -K_B \cos\alpha \cdot \omega_3$$

$$\frac{\partial \bar{w}_A}{\partial M_A} = K_A \beta_A \omega_5 \qquad \frac{\partial \bar{w}_B}{\partial M_B} = K_B \beta_B \omega_2$$

$$\frac{\partial \bar{w}_A}{\partial p} = \frac{R_A K_A \sin\alpha}{2} \cdot \omega_6 + \frac{K_A}{2\beta_A}(1-\nu/2) \qquad \frac{\partial \bar{w}_B}{\partial p} = -\frac{R_B K_B \sin\alpha}{2} \cdot \omega_3 + K_B \frac{(1-\nu/2)}{2\beta_B}$$

$$\frac{\partial \chi_A}{\partial H_A} = -K_A \beta_A \omega_5 \qquad \frac{\partial \chi_B}{\partial H_B} = -K_B \beta_B \omega_2 \tag{5-50}$$

$$\frac{\partial \chi_A}{\partial M_A} = -\frac{2K_A \beta_A^2 \omega_4}{\cos\alpha} \qquad \frac{\partial \chi_B}{\partial M_B} = \frac{2K_B \beta_B^2 \omega_1}{\cos\alpha}$$

$$\frac{\partial \chi_A}{\partial p} = -\frac{R_A K_A \beta_A}{2} \tan\alpha \; \omega_5 \qquad \frac{\partial \chi_B}{\partial p} = -\frac{R_B K_B \beta_B}{2} \tan\alpha \; \omega_2$$

where

$$\omega_1 = \omega_2 = \frac{1}{d_B} \; ; \quad \omega_4 = \omega_5 = \frac{1}{d_A}$$

$$\omega_3 = \frac{1}{d_B}\left[1 - \frac{\sqrt{2}}{\xi_B} + \frac{(1-16\nu^2)}{8\xi_B^2}\right] ; \quad \omega_6 = \frac{1}{d_A}\left[1 + \frac{\sqrt{2}}{\xi_A} + \frac{(1-16\nu^2)}{8\xi_A^2}\right]$$

Expansion of $\frac{1}{d_A}$ and $\frac{1}{d_B}$ in power series in $\frac{1}{\xi}$ results in:

$$\frac{1}{d_B} = 1 - \frac{\sqrt{2}}{2\,\xi_B}(1-4\,\nu) + \frac{(1-4\,\nu)^2}{2\,\xi_B^2} - \cdots$$

$$\frac{1}{d_A} = 1 + \frac{\sqrt{2}}{2\,\xi_A}(1-4\,\nu) + \frac{(1-4\,\nu)^2}{2\,\xi_A^2} + \cdots$$

$$\omega_3 = 1 - \frac{2\sqrt{2}\,\nu}{\xi_B} + \frac{(1-4\,\nu)^2}{4\,\xi_B^2} - \cdots \qquad [5\text{-}51]$$

$$\omega_6 = 1 + \frac{2\sqrt{2}\,\nu}{\xi_A} + \frac{(1-4\,\nu)^2}{4\,\xi_A^2} + \cdots$$

For the case $\nu = 0.3$, the values are

$$\omega_1 = \omega_2 = 1 - \frac{\sqrt{2}}{10\,\xi_B} + \frac{0.02}{\xi_B^2} - \cdots$$

$$\omega_3 = 1 - \frac{0.84853}{\xi_B} + \frac{0.01}{\xi_B^2} - \cdots$$

$$\omega_4 = \omega_5 = 1 + \frac{\sqrt{2}}{10\,\xi_A} + \frac{0.02}{\xi_A^2} + \cdots \qquad [5\text{-}52]$$

$$\omega_6 = 1 + \frac{0.84853}{\xi_A} + \frac{0.01}{\xi_A^2} - \cdots$$

The first two terms in such series are identical with corresponding terms given by Wenk and Taylor in their asymptotic expansions of the Ω-functions for large ξ (see Figure 5-5).

$$\Omega_1 = 1 - \frac{\sqrt{2}}{10\,\xi_B} - \frac{1.855}{\xi_B^2}$$

$$\Omega_2 = 1 - \frac{\sqrt{2}}{10\,\xi_B} - \frac{3.730}{\xi_B^2} \qquad [5\text{-}53]$$

$$\Omega_3 = 1 - \frac{0.84853}{\xi_B} - \frac{1.865}{\xi_B^2}$$

$$\Omega_4 = 1 + \frac{\sqrt{2}}{10\,\xi_A} - \frac{1.855}{\xi_A^2}$$

$$\Omega_5 = 1 + \frac{\sqrt{2}}{10\,\xi_A} - \frac{3.730}{\xi_A^2} \qquad [5\text{-}53]$$

$$\Omega_6 = 1 + \frac{0.84853}{\xi_A} - \frac{1.865}{\xi_A^2}$$

Since the ω's tend toward unity for large values of ξ just as the Ω's do, it is easily shown that the edge coefficients given by Equations [5-50] reduce essentially to those given by Raetz and Pulos (Equations [5-30]). The only differences are the second terms in the Raetz-Pulos expressions for $\frac{\partial \chi_A}{\partial p}$ and $\frac{\partial \chi_B}{\partial p}$ which do not appear in Equations [5-50].

The stresses near the edges can be found by substituting the solution [5-41] into the relations [5-45]:

Near Edge A,

$$N_x = \frac{4}{z^2\sqrt{z}}\, e^{-\lambda z}\,(C_1 \cos \lambda z + C_2 \sin \lambda z) - \frac{p z^2}{8} \tan \alpha$$

$$N_\theta = \frac{2\lambda e^{-\lambda z}}{z\sqrt{z}} \left\{ - C_1 \left[\sin \lambda z + \left(1 + \frac{1}{2\lambda z}\right) \cos \lambda z\right] + C_2 \left[\cos \lambda z - \left(1 + \frac{1}{2\lambda z}\right) \sin \lambda z\right]\right\} + \frac{p z^2}{4}(1 - \nu/2) \tan \alpha \qquad [5\text{-}54]$$

$$M_x = \frac{e^{-\lambda z}}{\lambda z\sqrt{z}\tan \alpha} \left\{ C_1 \left[- \cos \lambda z + \left(1 + \frac{1 - 4\nu}{2\lambda z}\right) \sin \lambda z\right] - C_2 \left[\sin \lambda z + \left(1 + \frac{1 - 4\nu}{2\lambda z}\right) \cos \lambda z\right]\right\}$$

$$M_\theta = \nu M_x + \frac{2(1 - \nu^2)\, e^{-\lambda z}}{\lambda^2 z^2 \sqrt{z} \tan \alpha} (C_2 \cos \lambda x - C_1 \sin \lambda x)$$

Near Edge B,

$$N_x = \frac{4}{z^2\sqrt{z}} e^{\lambda z} (C_3 \cos \lambda z + C_4 \sin \lambda z) - \frac{pz^2}{8} \tan \alpha$$

$$N_\theta = \frac{2\lambda e^{\lambda z}}{z\sqrt{z}} \left\{ C_3 \left[-\sin \lambda z + \left(1 - \frac{1}{2\lambda z}\right) \cos \lambda z \right] + C_4 \left[\cos \lambda z + \left(1 - \frac{1}{2\lambda z}\right) \sin \lambda z \right] \right\} + \frac{pz^2}{4} (1 - \nu/2) \tan \alpha \qquad [5\text{-}54]$$

$$M_x = \frac{e^{\lambda z}}{\lambda z\sqrt{z} \tan \alpha} \left\{ C_3 \left[\cos \lambda z + \left(1 - \frac{1 - 4\nu}{2\lambda z}\right) \sin \lambda z \right] + C_4 \left[\sin \lambda z - \left(1 - \frac{1 - 4\nu}{2\lambda z}\right) \cos \lambda z \right] \right\}$$

$$M_\theta = \nu M_x - \frac{2(1 - \nu^2)\, e^{\lambda x}}{\lambda^2 z^2 \sqrt{z} \tan \alpha} (C_3 \sin \lambda x - C_4 \cos \lambda x)$$

EVALUATION

A good deal of experimentation has been done with conical shells to evaluate the analyses just discussed. It is not practical here to cite all sources of information, and no effort will be made to do so. Instead mention will be made only of some work carried on at the Model Basin which may be regarded as typical and provides enough data to draw general conclusions as to the applicability of theory.

Investigations by Borg,[5-15] Krenzke,[5-16] and Raetz[5-17] have produced a large amount of data that support the general theory. In particular, they have shown the Raetz-Pulos method to be adequate for values of ξ_A as small as 11, and this includes values of α up to 60 deg. In addition, the data indicate that the rule that superposition of edge effects is unnecessary when $\beta_A x_B$ and $\beta_B x_B$ exceed 6 is a good one. The general results of these studies are encouraging since they show that the relatively simple Raetz-Pulos equations are reliable over a wide range of geometry. Figure 5-7 is a typical plot of strain distribution taken from Reference 5-15 for a shell consisting of two cylinders of different diameters connected by a conical section with $\alpha = 60$ deg. Since the Hetényi-type solution was only recently obtained, there was not time to include it in the numerical comparisons.

SPHERICAL SHELLS

Spherical shells present some of the more interesting mathematical problems to be found in the field of stress analysis. One might suppose that because of the high degree of symmetry possessed by the spherical shell, its analysis would be mathematically simple. Unfortunately this is not the case. All known forms of the exact solution are so cumbersome and complex that they are impractical to use except when programmed into a high-speed computer. Because of this, much effort has been devoted to obtaining approximate solutions and it is these that will be of particular interest here.

Figure 5-8 is a general representation of a spherical shell element in a pressure vessel. As drawn, it could depict a transition component between two cylindrical sections of different diameters; ϕ, the meridional coordinate, is the angle between the axis of symmetry and the radius to a point on the shell. If ϕ_A is small and ϕ_B is 90 deg, the case could be that of a hemispherical closure or end cap with a small crown opening like a hatch. With ϕ_A zero and ϕ_B equal to 90 deg, we have a hemisphere. For ϕ_B slightly less than 180 deg and ϕ_A equal to zero, the case could be a sphere with a small hatch. Since this chapter covers only axisymmetric deformations, noncircular openings and openings whose centers do not lie on the axis of symmetry are excluded. These and other more difficult cases are taken up in the chapter on hull penetrations.

The basic theory for spherical shells can be found in References 5-1, 5-2, and 5-4. In addition, the reader may wish to refer to an excellent summary[5-18] by Tsui and Stern of the important methods of solution and a discussion by Galletly[5-19] of the applicability of some of these methods to open-crown hemispheres.

With Q_ϕ denoting the shear force per unit length on a shell element (Figure 5-8) and χ the rotation in the meridional plane, the governing differential equations for no surface loading can be written as follows:

$$\begin{aligned} &L(Q_\phi) + \nu Q_\phi + Eh\,\chi = 0 \\ &L(\chi) - \nu\chi - \frac{12(1-\nu^2)}{Eh^3}\,Q_\phi = 0 \end{aligned} \qquad [5\text{-}55]$$

where L is the operator,

$$L(\ ) = \frac{d^2(\ )}{d\phi^2} + \frac{d(\ )}{d\phi}\cot\phi - (\ )\cot^2\phi$$

Equations [5-47] can be combined to yield the equation

$$LL(Q_\phi) + 4\rho^4 Q_\phi = 0 \qquad [5\text{-}56a]$$

which is equivalent to the two second-order equations:

$$L(Q_\phi) \pm 2 i \rho^2 Q_\phi = 0 \qquad [5\text{-}56b]$$

where

$$\rho^4 = \frac{3R^2}{h^2}(1-\nu^2) - \frac{\nu^2}{4}$$

$$= \frac{3R^2}{h^2}(1-\nu^2) \quad \text{for thin shells}$$

$$i = \sqrt{-1}$$

The corresponding set of equations for χ are identical in form.

EXACT SOLUTIONS

Evidently it was Reissner[5-20] who first recognized the advantages of taking as the dependent variables Q_ϕ and χ rather than the displacements u and w, as is a common practice. Once a solution to the equations has been obtained, one can get the displacements, rotations, and stresses from the following relations (which include the membrane deformations due to external pressure):

$$\chi = -\frac{1}{Eh}\left[L(Q_\phi) + \nu Q_\phi\right]$$

$$= -\frac{1}{Eh}\left(\frac{d^2 Q_\phi}{d\phi^2} + \frac{dQ_\phi}{d\phi}\cot\phi - Q_\phi \cot^2\phi + \nu Q_\phi\right)$$

$$\overline{w} = -\frac{R\sin\phi}{Eh}\left(\frac{dQ_\phi}{d\phi} - \nu Q_\phi \cot\phi\right) + \frac{pR^2(1-\nu)}{2Eh}\sin\phi \qquad [5\text{-}57]$$

$$\sigma_\phi = \frac{N_\phi}{h} \pm \frac{6M_\phi}{h^2}$$

$$\sigma_\theta = \frac{N_\theta}{h} \pm \frac{6M_\theta}{h^2}$$

$$M_\phi = \frac{Eh^3}{12\,(1-\nu^2)\,R}\left(\frac{d\chi}{d\phi} + \nu\,\chi\cot\phi\right)$$

$$M_\theta = \frac{Eh^3}{12\,(1-\nu^2)\,R}\left(\nu\,\frac{d\chi}{d\phi} + \chi\cot\phi\right) = \nu M_\phi + \frac{Eh^3}{12}\,\chi\cot\phi$$

$$N_\phi = Q_\phi\cot\phi - \frac{pR}{2}$$

$$N_\theta = \frac{dQ_\phi}{d\phi} - \frac{pR}{2}$$

The terms involving p are the membrane components.

Two forms of the exact solution are frequently found in the literature. One, due to Meissner[5-21] and Bolle,[5-22] is expressed in hypergeometrical series; the other, introduced by Love,[5-4] is in terms of Legendre polynomials. Both forms are too unmanageable* to be of much use in practice and will not be discussed in detail (they are completely described in References 5-1, 5-2, 5-18, and 5-19). Furthermore, they exhibit slow convergence for certain ranges of ϕ where the computational task becomes formidable. It has been found that acceptable accuracy can be obtained using far simpler approximate solutions. The importance of the exact solutions is that they are the standard against which the accuracy of all approximate solutions can be assessed. The remainder of this section explains some of these simpler methods.

APPROXIMATE SOLUTIONS

Various approaches have been used to obtain workable solutions to the differential Equations [5-56]. So numerous are the papers on this subject that it would be difficult even to compile a complete bibliography, and to present a detailed discussion of each method would require a treatise of far greater scope than was intended for this chapter. Instead it will be the objective here to present a few methods that give results of good accuracy and can be put to practical use without the aid of high-speed computing machines.

Geckeler's Solution for the Polar Region

When the angle ϕ is sufficiently small, certain approximations can be made that greatly simplify the solution of the differential equations. Cases that fall within this range are illustrated in Figure 5-9. One type of solution is based on shallow shell theory; it will not be

*Galletly,[5-23] in a discussion of Reference 5-19, estimated that the desk computation time for one set of edge coefficients using Love's method is about one month.

discussed here since it is covered in some detail by Lomacky in Chapter 9 (Hull Penetrations) of this summary. Another solution was originally obtained by Geckeler[5-24*] and has been carried out in detail by Esslinger[5-25] and Galletly.[5-19] Of the cases represented in Figure 5-9, Case *b*, that of a spherical shell with a small polar opening, is the one of primary interest here, and the only one for which detailed expressions are presented. Case *a*, the shallow cap, is not apt to be encountered frequently in conventional submarine construction. The same is true of Case *c*. Moreover, since ϕ is restricted to small angles, the interaction in that case between Edges *A* and *B* would have to be considered—a situation that would lead to severe analytical complications.

If the angle ϕ is restricted to small values, its trigonometric functions can be approximated by the first term of the appropriate series representations. Thus cot ϕ can be replaced by $\frac{1}{\phi}$ and Equations [5-56b] take the form

$$\frac{d^2 Q_\phi}{d\phi^2} + \frac{1}{\phi}\,\frac{dQ_\phi}{d\phi} - \frac{Q_\phi}{\phi^2} \pm 2 i \rho^2 Q_\phi = 0 \qquad [5\text{-}58]$$

Each of the equations is reducible to the modified Bessel's equation of first order. The solution can be written in terms of the first order Kelvin functions:

$$Q_\phi = A_1\, ber'(\xi) + A_2\, bei'(\xi) + A_3\, ker'(\xi) + A_4\, kei'(\xi) \qquad [5\text{-}59]$$

where $\xi = \rho\sqrt{2}\,\phi$ and the primes indicate differentiation with respect to ξ.
These functions can be found tabulated in many references in the literature, among them Flügge's "Four-Place Tables of Transcendental Functions."[5-26] Several other forms of the solution are possible. Esslinger has chosen to write it in terms of Schleicher functions:

$$Q_\phi = C_1\, \psi_1'(\xi) + C_2\, \psi_2'(\xi) + C_3\, \psi_3'(\xi) + C_4\, \psi_4'(\xi) \qquad [5\text{-}60]$$

*This should not be confused with the well-known "Geckeler approximation"[5-11] previously discussed in the section on conical shells. The application of this approximation to spherical shells is described in a later article.

To dispel any impression that the Kelvin and Schleicher functions are fundamentally different, the following relations are noted:*

$$\psi_1(x) = \; ber(x) \qquad \psi_3(x) = -\frac{2}{\pi}\, kei(x)$$
$$\psi_2(x) = -\, bei(x) \qquad \psi_4(x) = -\frac{2}{\pi}\, ker(x) \tag{5-61}$$

They can be found in the Jahnke-Emde tables[5-27] as real and imaginary parts of Bessel and Hankel functions:

$$\psi_1(x) = R_e J_0 \; (x\sqrt{i})$$
$$\psi_2(x) = I_m J_0 \; (x\sqrt{i})$$
$$\psi_3(x) = R_e H_0^1 \; (x\sqrt{i}) \tag{5-62}$$
$$\psi_4(x) = I_m H_0^1 \; (x\sqrt{i})$$

Carrying the approximation $\cot\phi = \dfrac{1}{\phi}$ over to the relations [5-57], we get the following expressions:

$$\chi = -\frac{1}{Eh}\left(\frac{d^2Q_\phi}{d\phi^2} + \frac{1}{\phi}\,\frac{dQ_\phi}{d\phi} - \frac{Q_\phi}{\phi^2} - \nu Q_\phi\right)$$

$$\bar{w} = -\frac{R\phi}{Eh}\left(\frac{dQ_\phi}{d\phi} - \frac{\nu Q_\phi}{\phi}\right)^{**} + \frac{pR^2}{2Eh}(1-\nu)\,\phi \tag{5-63}$$

$$M_\phi = \frac{Eh^3}{12\,(1-\nu^2)\,R}\left[\frac{d\chi}{d\phi} + \frac{\nu\chi}{\phi}\right]$$

---

*This multiplicity of names and symbols assigned somewhat indiscriminately to the same basic functions can often lead to needless confusion. One who looks for tables of Kelvin functions and finds Thomson functions instead may face a long search unless he knows that Lord Kelvin was Sir William Thomson.

**Galletly[5-19] gives this as $-\dfrac{R\sin\phi}{Eh}\left(\dfrac{dQ_\phi}{d\phi} - \dfrac{\nu Q_\phi}{\phi}\right)$, but this seems inconsistent with the basic approximation.

$$M_\theta = \nu M_\phi + \frac{Eh^3}{12} \frac{\chi}{\phi}$$

$$N_\phi = \frac{Q_\phi}{\phi} - \frac{pR}{2}$$

$$N_\theta = \frac{dQ_\phi}{\phi} - \frac{pR}{2}$$

where $\frac{pR}{2}$ is the membrane stress resultant. Unfortunately, from here on, the equations have been developed by Esslinger only for the case $\nu = 0.25$. In Chapter 9 of this summary, Lomacky has rewritten them in terms of general ν, and it is these equations that are presented here.

For Case *b* of Figure 5-9 it is assumed that the interaction between Edges *A* and *B* is negligible. This means that C_1 and C_2 can be set equal to zero. The resulting edge coefficients are:

$$\frac{\partial \bar{w}_A}{\partial H_A} = \frac{W_c}{E\phi_A} \qquad \frac{\partial \chi_A}{\partial H_A} = -\frac{X_c}{Eh\phi_A}$$

$$\frac{\partial \bar{w}_A}{\partial M_A} = \frac{X_c}{Eh\phi_A} \qquad \frac{\partial \chi_A}{\partial M_A} = -\frac{X_d}{Eh^2\phi_A} \qquad [5\text{-}64]$$

$$\frac{\partial \bar{w}_A}{\partial p} = \frac{R^2(1-\nu)}{2Eh}\phi_A \qquad \frac{\partial \chi_A}{\partial p} = 0$$

where

$$X_c = -\xi_A^2 \frac{(\psi_3 \psi_3' + \psi_4 \psi_4')}{X_e}$$

$$X_d = \xi_A \frac{(\psi_3'^2 + \psi_4'^2)\, 2\sqrt{3(1-\nu^2)}}{X_e}$$

$$X_e = \psi_3 \psi_4' - \psi_4 \psi_3' + \frac{1-\nu}{\xi_A}[\psi_3'^2 + \psi_4'^2]$$

$$W_c = \xi_A^3 \frac{(\psi_3^2 + \psi_4^2) - 2\xi_A^2(\psi_3'\psi_4 - \psi_4'\psi_3) + (1-\nu^2)\,\xi_A(\psi_3'^2 + \psi_4'^2)}{X_e\, 2\sqrt{3(1-\nu^2)}}$$

$$\xi_A = \rho\sqrt{2}\,\phi_A$$

In these equations, the approximation has been made that $H_A = N_{\phi_A}$ and $H_B = N_{\phi_B}$.

Finally, using the expressions for σ_ϕ and σ_θ given in Equations [5-57], these stresses can be found from the relations

$$N_\phi = n_c(\xi) J + n_d(\xi) K - \frac{pR}{2}$$

$$N_\theta = t_c(\xi) J + t_d(\xi) K - \frac{pR}{2}$$

$$\frac{M_\phi}{h} = m_\theta(\xi) J + m_d(\xi) K \qquad [5\text{-}65]$$

$$\frac{M_\theta}{h} = \frac{\nu M_\phi}{h} + \frac{\sqrt{3(1-\nu^2)}}{3\sqrt{2}\,\xi} (\psi_4' J - \psi_3' K)^*$$

in which

$$J = J_1 H_A \cos\phi_A - \frac{J_2 M_A}{h} \qquad K = K_1 H_A \cos\phi_A - \frac{K_2 M_A}{h}^{**}$$

$$J_1 = \frac{\xi_A \psi_4(\xi_A) - (1-\nu)\psi_3'(\xi_A)}{X_e\sqrt{2}} \qquad J_2 = \frac{\psi_4'(\xi_A) \cdot 2\sqrt{3(1-\nu^2)}}{X_e\sqrt{2}}$$

$$K_1 = -\frac{\xi_A \dot{\psi}_3(\xi_A) + (1-\nu)\psi_4'(\xi_A)}{X_e\sqrt{2}} \qquad K_2 = -\frac{\psi_3'(\xi_A) \cdot 2\sqrt{3(1-\nu^2)}}{X_e\sqrt{2}}$$

$$n_c(\xi) = \frac{\sqrt{2}\,\psi_3'(\xi)}{\xi} \qquad n_d(\xi) = \frac{\sqrt{2}\,\psi_4'(\xi)}{\xi}$$

$$t_c(\xi) = \sqrt{2}\,\psi_4(\xi) - n_c(\xi) \qquad t_d(\xi) = -\sqrt{2}\,\psi_3(\xi) - n_d(\xi)$$

$$m_c(\xi) = -\frac{1}{\sqrt{6(1-\nu^2)}}\left[\psi_3(\xi) + \frac{(1-\nu)}{\xi}\psi_4'(\xi)\right]$$

$$m_d(\xi) = -\frac{1}{\sqrt{6(1-\nu^2)}}\left[\psi_4(\xi) - \frac{(1-\nu)}{\xi}\psi_3'(\xi)\right]$$

---

*Esslinger does not give an expression for $\frac{M_\theta}{h}$; Galletly's expression differs from the one presented here, but it appears to contain a sign error.

**Esslinger does not make the approximation $\cos\phi_A = 1$, although this would seem to be called for if the equations are to be consistent.

Notice that J and K are constants involving functions evaluated at Edge A ($\xi_A = \rho\sqrt{2}\,\phi_A$) whereas n_c, n_d, t_c, t_d, m_c, and m_d are all functions of the variable ξ. The equations are valid, of course, only over a small range of ξ because of the limits placed on ϕ. However, one usually finds that the bending effects decay rapidly with increasing ξ, so that where the solution is no longer valid the membrane solution alone may be sufficiently accurate.

Hetényi's Solution (Polar Region Excluded)

Because of its simplicity and accuracy, this solution (already mentioned in the section on conical shells) is of great practical interest. Its range of validity is not limited to large values of ϕ, but it does exclude the polar region in which Geckeler's solution is applicable. Figure 5-10 therefore represents two general cases for which Hetényi's solution should be good. The only restriction in either case is that the opening angles must not be too small.

Hetényi makes the transformation

$$Q_\phi = q(\phi) \cdot f(\phi)$$
$$f(\phi) = \frac{1}{\sqrt{\sin\phi}} \qquad [5\text{-}66]$$

$q(\phi)$ now replaces Q as the dependent variable, and Equations [5-56b] take the form

$$\frac{1}{\sqrt{\sin\phi}}\left[\frac{d^2q}{d\phi^2} + q\left(\frac{1}{2} - \frac{3}{4}\cot^2\phi \pm 2i\rho^2\right)\right] = 0 \qquad [5\text{-}67]$$

The advantage in using the transformation [5-66] is that Equations [5-67] have no terms involving $\frac{dq}{d\phi}$. Equations [5-67] are therefore a good deal less complicated than Equations [5-56b] and can be solved easily if one neglects the term $q\left(\frac{1}{2} - \frac{3}{4}\cot^2\phi\right)$.* This is acceptable so long as $\cot^2\phi$ is not large (i.e., ϕ is not small). The resulting equations to be solved are:

$$\frac{d^2q}{d\phi} \pm 2i\rho^2 q = 0 \qquad [5\text{-}68a]$$

or, the single fourth-order equation:

$$\frac{d^4q}{d\phi^4} + 4\rho^4 q = 0 \qquad [5\text{-}68b]$$

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*This means that $q\left(\frac{1}{2} - \frac{3}{4}\cot^2\phi\right) << \frac{d^2q}{d\phi^2}$ and it is the one basic approximation made in arriving at the solution.

Its solution is

$$q = e^{-\rho\phi} (C_1 \cos \rho\phi + C_2 \sin \rho\phi) + e^{\rho\phi} (C_3 \cos \rho\phi + C_4 \sin \rho\phi) \qquad [5\text{-}69]$$

It is seen that Equations [5-68b] and [5-69] are identical to those for the cylindrical shell. The complete solution to the spherical shell is

$$Q_\phi = \frac{1}{\sqrt{\sin\phi}} [\, e^{-\rho\phi} (C_1 \cos \rho\phi + C_2 \sin \rho\phi) + e^{\rho\phi} (C_3 \cos \rho\phi + C_4 \sin \rho\phi)] \qquad [5\text{-}70]$$

With the aid of the first of Equations [5-57], one gets for the rotation

$$\chi = -\frac{1}{Eh\sqrt{\sin\phi}} \left[\frac{d^2 q}{d\phi^2} + q\left(\frac{1}{2} - \frac{3}{4}\cot^2\phi + \nu\right)\right] \qquad [5\text{-}71a]$$

Neglecting the coefficient of q in this equation is consistent with the approximation already made in arriving at Equation [5-49a]. Hence one can write

$$\chi = -\frac{1}{Eh\sqrt{\sin\phi}} \frac{d^2 q}{d\phi^2} \qquad [5\text{-}71b]$$

with the assurance that no additional approximations have been introduced. The other quantities given by Equations [5-57] can then be expressed in terms of q as follows:

$$\begin{aligned}
w &= -\frac{R\sqrt{\sin\phi}}{Eh}\left[\frac{dq}{d\phi} - q\left(\frac{1}{2} + \nu\right)\cot\phi\right] + \frac{pR^2(1-\nu)}{2Eh}\sin\phi \\
M_\phi &= -\frac{R}{4\rho^4\sqrt{\sin\phi}}\left[\frac{d^3 q}{d\phi^3} - \frac{d^2 q}{d\phi^2}\left(\frac{1}{2} - \nu\right)\cot\phi\right] \\
M_\theta &= \nu M_\phi - \frac{R(1-\nu^2)\cot\phi}{4\rho^4\sqrt{\sin\phi}}\frac{d^2 q}{d\phi^2} \\
N_\phi &= q\frac{\cot\phi}{\sqrt{\sin\phi}} - \frac{pR}{2} \\
N_\theta &= \frac{1}{\sqrt{\sin\phi}}\left[\frac{dq}{d\phi} - \frac{q}{2}\cot\phi\right] - \frac{pR}{2}
\end{aligned} \qquad [5\text{-}72]$$

For the case of the long or "single-ended" shell, we have near Edge A:

$$q = e^{-\rho\phi}\,(C_1 \cos \rho\phi + C_2 \sin \rho\phi) \qquad [5\text{-}73a]$$

and near Edge B:

$$q = e^{\rho\phi}\,(C_3 \cos \rho\phi + C_4 \sin \rho\phi) \qquad [5\text{-}73b]$$

The C's can then be written in terms of the edge moments and rotations:

$$C_1 = \frac{e^{\rho\phi_A}\sqrt{\sin\phi_A}}{c_A}\left\{ Q_A\,[\sin\rho\phi_A + c_A \cos\rho\phi_A] + 2\rho\,\frac{M_A}{R}\sin\rho\phi_A \right\}$$

$$C_2 = \frac{e^{\rho\phi_A}\sqrt{\sin\phi_A}}{c_A}\left\{ Q_A\,[-\cos\rho\phi_A + c_A \sin\rho\phi_A] - 2\rho\,\frac{M_A}{R}\cos\rho\phi_A \right\}$$

[5-74]

$$C_3 = \frac{e^{-\rho\phi_B}\sqrt{\sin\phi_B}}{c_B}\left\{ Q_B\,[-\sin\rho\phi_B + c_B \cos\rho\phi_B] + 2\rho\,\frac{M_B}{R}\sin\rho\phi_B \right\}$$

$$C_4 = \frac{e^{-\rho\phi_B}\sqrt{\sin\phi_B}}{c_B}\left\{ Q_B\,[\cos\rho\phi_B + c_B \sin\rho\phi_B] - 2\rho\,\frac{M_B}{R}\cos\rho\phi_B \right\}$$

in which $c_A = 1 + \dfrac{(1-2\nu)}{2\rho}\cot\phi_A$ and $c_B = 1 - \dfrac{(1-2\nu)}{2\rho}\cot\phi_B$.*

The corresponding edge coefficients for the load parameters H, M, and p are as follows:

$$\frac{\partial \bar{w}_A}{\partial H_A} = \frac{2\rho R \sin^2\phi_A}{Eh\,c_A}\left[1 + \frac{\cot\phi_A}{2\rho} + \frac{\cot^2\phi_A}{8\rho^2}(1-4\nu^2)\right]$$

[5-75]

$$\frac{\partial \bar{w}_B}{\partial H_B} = -\frac{2\rho R \sin^2\phi_B}{Eh\,c_B}\left[1 - \frac{\cot\phi_B}{2\rho} + \frac{\cot^2\phi_B}{8\rho^2}(1-4\nu^2)\right]$$

---

*Here, as in the case of the Hetenyi-type solution for conical shells, we have a denominator term which can become zero. For the case $\nu = 0.3$ and $h/R = 0.05$, this would happen if $\phi_B \approx 2$ deg; for thinner shells, the angle would be even smaller. Such cases are clearly outside the region in which the solution is applicable.

$$\frac{\partial \bar{w}_A}{\partial M_A} = \frac{2\rho^2 \sin \phi_A}{Eh\, c_A} \qquad \frac{\partial \bar{w}_B}{\partial M_B} = \frac{2\rho^2 \sin \phi_B}{Eh\, c_B}$$

$$\frac{\partial \bar{w}_A}{\partial p} = \frac{R \cot \phi_A}{2} \frac{\partial \bar{w}_A}{\partial H_A} + \frac{R^2}{2Eh}(1-\nu) \qquad \frac{\partial \bar{w}_B}{\partial p} = \frac{R \cot \phi_B}{2} \frac{\partial \bar{w}_B}{\partial H_B} + \frac{R^2}{2Eh}(1-\nu)$$

$$\frac{\partial \chi_A}{\partial H_A} = -\frac{2\rho^2 \sin \phi_A}{Eh\, c_A} \qquad \frac{\partial \chi_B}{\partial H_B} = -\frac{2\rho^2 \sin \phi_B}{Eh\, c_B} \tag{5-75}$$

$$\frac{\partial \chi_A}{\partial M_A} = -\frac{4\rho^3}{EhR\, c_A} \qquad \frac{\partial \chi_B}{\partial M_B} = \frac{4\rho^3}{Eh\, c_B}$$

$$\frac{\partial \chi_A}{\partial p} = -\frac{\rho^2 R \cos \phi_A}{Eh\, c_A} \qquad \frac{\partial \chi_B}{\partial p} = -\frac{\rho^2 R \cos \phi_B}{Eh\, c_B}$$

In forming these coefficients, use has been made of the relations

$$Q_A = H_A \sin \phi_A + \frac{pR}{2} \cos \phi_A \ ; \ Q_B = H_B \sin \phi_B + \frac{pR}{2} \cos \phi_B \tag{5-76}$$

For the moments and stress resultants, we have

Near Edge A:

$$N_\phi = \frac{e^{-\rho\phi} \cot \phi}{\sqrt{\sin \phi}} [C_1 \cos \rho\phi + C_2 \sin \rho\phi] - \frac{pR}{2}$$

$$\begin{aligned} N_\theta = -\frac{\rho\, e^{-\rho\phi}}{\sqrt{\sin \phi}} \Bigg[C_1 \left(\cos \rho\phi + \sin \rho\phi + \frac{\cot \phi}{2\rho} \cos \rho\phi \right) \\ + C_2 \left(-\cos \rho\phi + \sin \rho\phi + \frac{\cot \phi}{2\rho} \sin \rho\phi \right) \Bigg] - \frac{pR}{2} \end{aligned} \tag{5-77a}$$

$$M_\phi = -\frac{R\,e^{-\rho\phi}}{2\rho\sqrt{\sin\phi}}\left\{C_1\left[\cos\rho\phi - \sin\rho\phi - \frac{(1-2\nu)}{2\rho}\cot\phi\sin\rho\phi\right]\right.$$

$$\left. + C_2\left[\cos\rho\phi + \sin\rho\phi + \frac{(1-2\nu)}{2\rho}\cot\phi\cos\rho\phi\right]\right\} \qquad [5\text{-}77a]$$

$$M_\theta = \nu M_\phi - \frac{R(1-\nu^2)\cot\phi}{2\rho^2\sqrt{\sin\phi}}\,e^{-\rho\phi}\,[C_1\sin\rho\phi - C_2\cos\rho\phi]$$

Near Edge B:

$$N_\phi = \frac{e^{\rho\phi}\cot\phi}{\sqrt{\sin\phi}}\,[C_3\cos\rho\phi + C_4\sin\rho\phi] - \frac{pR}{2}$$

$$N_\theta = \frac{\rho\,e^{\rho\phi}}{\sqrt{\sin\phi}}\left[C_3\left(\cos\rho\phi - \sin\rho\phi - \frac{\cot\phi}{2\rho}\cos\rho\phi\right)\right.$$

$$\left. + C_4\left(\cos\rho\phi + \sin\rho\phi - \frac{\cot\phi}{2\rho}\sin\rho\phi\right)\right] - \frac{pR}{2}$$

[5-77b]

$$M_\phi = \frac{R\,e^{\rho\phi}}{2\rho\sqrt{\sin\phi}}\left\{C_3\left[\cos\rho\phi + \sin\rho\phi - \frac{(1-2\nu)}{2\rho}\cot\phi\sin\rho\phi\right]\right.$$

$$\left. - C_4\left[\cos\rho\phi - \sin\rho\phi - \frac{(1-2\nu)}{2\rho}\cot\phi\cos\rho\phi\right]\right\}$$

$$M_\theta = \nu M_\phi + \frac{R(1-\nu^2)\cot\phi}{2\rho^2\sqrt{\sin\phi}}\,e^{\rho\phi}\,[C_3\sin\rho\phi + C_4\cos\rho\phi]$$

It is interesting to note that all relations obtained from Hetényi's solution can be reduced exactly to those resulting from the application of Geckeler's approximate method[5-11] to the case of a spherical shell.* If ϕ is in the neighborhood of 90 deg, $\sin\phi$ can be approximated by 1 and $\cot\phi$ by zero. In that case, Equation [5-66] is simply

$$Q_\phi = q \qquad [5\text{-}78]$$

and Equation [5-68b] reduces to

$$\frac{d^4 Q_\phi}{d\phi^4} + 4\rho^4 Q_\phi = 0 \qquad [5\text{-}79]$$

*This approximation is discussed in References 5-1, 5-18, and 5-19.

In all subsequent equations, Q_ϕ replaces q, 1 replaces $\sin\phi$, and all terms multiplied by $\cot\phi$ do not appear. The result thus can be regarded as an approximate form of Hetényi's solution. Since this latter solution is itself relatively simple, one may wonder whether the slight additional simplification gained by going to the Geckeler form is worth the accuracy lost.

Leckie's Solution (All Regions)

Since it is intended to be applicable for all ranges of the angle ϕ, this solution is considered to be the most accurate of the approximate forms described here. Leckie[5-28] obtained his solution as a special result of some general theoretical work on the antisymmetric response of spherical shells to localized loads. The solution has received further discussion in a paper by Leckie and Penny.[5-29] Baker and Cline[5-30] independently arrived at the identical solution in developing theory for the axisymmetric bending of general shells of revolution. Their paper is perhaps the more informative because they have carried out the solution in greater detail. A valuable discussion is also given in Tsui and Stern.[5-18]

With the transformation

$$Q_\phi = q(\phi)\, f(\phi)$$
$$f(\phi) = \sqrt{\frac{\phi}{\sin\phi}} \tag{5-80}$$

Equations [5-56b] become

$$\sqrt{\frac{\phi}{\sin\phi}}\left\{\frac{d^2 z}{d\phi^2} + \frac{1}{\phi}\frac{dz}{d\phi} + z\left[-\frac{1}{\phi^2} \pm i\cdot 2\rho^2 + \left(\frac{3}{4\phi^2} + \frac{1}{2} - \frac{3\cot^2\phi}{4}\right)\right]\right\} = 0 \tag{5-81}$$

Since the term $\frac{3}{4\phi^2} + \frac{1}{2} - \frac{3\cot^2\phi}{4}$ is small compared with $2\rho^2$ for all values of ϕ,* it can be omitted. The result is

$$\frac{d^2 q}{d\phi^2} + \frac{1}{\phi}\frac{dq}{d\phi} - \frac{q}{\phi^2} \pm 2\, i\, \rho^2\, q = 0 \tag{5-82}$$

---

*When ϕ is small, $\frac{3}{4\phi^2} - \frac{3\cot^2\phi}{4} \approx 0$.

which is identical in form to Equation [5-58] obtained by Geckeler, hence solvable in terms of the Kelvin functions. The complete solution is

$$
\begin{aligned}
Q_\phi &= \sqrt{\frac{\phi}{\sin\phi}}\,[A_1\, ber'(\xi) + A_2\, bei'(\xi) + A_3\, ker'(\xi) + A_4\, kei'(\xi)] \\
&= \sqrt{\frac{\phi}{\sin\phi}}\,[C_1\, \psi_1'(\xi) + C_2\, \psi_2'(\xi) + C_3\, \psi_3'(\xi) + C_4\, \psi_4'(\xi)]
\end{aligned}
\qquad [5\text{-}83]
$$

where, as before, ξ has the meaning $\rho\sqrt{2}\,\phi$. It can be seen that this solution bears similarities to the solutions of both Geckeler and Hetényi. In the polar region $\frac{\phi}{\sin\phi} \approx 1$, and the solution becomes exactly that of Geckeler. On the other hand, Leckie has found through asymptotic expansions of the Kelvin functions that the solution is reducible to the simple Hetényi form for $\xi > 6$. This means that when $\frac{h}{R} = 0.01$ and $\nu = 0.3$, for example, ϕ should exceed 19 deg. For this case we find:

$$\cot 19 \text{ deg} = 2.904$$

$$\frac{1}{\phi} = 3.016$$

so that the approximation $\cot\phi = \frac{1}{\phi}$ results in an error of only 4 percent. This indicates that there may be a range in ϕ the extent of which depends on $\left(\frac{h}{R}\right)$ over which both the Geckeler and the Hetényi solutions are reasonably accurate.

For these reasons, the author has not considered it necessary to carry out the more cumbersome Leckie equations in detail. The Geckeler equations (which are complicated enough) should suffice up to a value of ξ where the Hetényi equations can be used. This is not to downgrade the value of Leckie's solution, for without it there would be difficulty in finding a connection between the Geckeler and Hetényi solutions and a consequent range of possible overlap. Moreover, Tsui and Stern[5-18] have found excellent agreement between the Leckie edge coefficients as given by Baker and Cline and those obtained from the exact solution of Meissner[5-21] over a wide range of ξ. It is thus indicated that the Leckie solution can serve as a useful substitute for the unwieldy exact solutions of Meissner and Love wherever a standard of accuracy is needed. Tsui and Stern have also presented an extensive series of curves for the edge coefficients obtained from the Meissner solution and as given by Baker and Cline.

EVALUATION

One way to get an idea of "how approximate" some of these solutions are is to compare them with an exact solution in terms of edge coefficients. Table 5-1 makes such a comparison for an open-crown hemisphere with the following properties:

$$R = 181 \text{ in.}$$
$$h = 2 \text{ in.}$$
$$\phi_A = 10.5 \text{ deg}$$
$$\nu = 0.3$$

TABLE 5-1

Edge Coefficients for Spherical Shells Compared

	$Eh \dfrac{\partial \bar{w}_A}{\partial H_A}$	$Eh \dfrac{\partial \bar{w}_A}{\partial M_A}$	$-Eh \dfrac{\partial \chi_A}{\partial H_A}$	$-Eh \dfrac{\partial \chi_A}{\partial M_A}$
Geckeler Approximation[5-11]	147.01 (–11.0)*	54.50 (14.0)	54.50 (14.0)	40.41 (11.5)
Hetényi[5-14]	167.14 (1.5)	50.18 (5.0)	50.18 (5.0)	37.24 (3.0)
Geckeler-Esslinger[5-24] (Polar Region)	163.45 (–1.0)	46.62 (–2.0)	46.62 (–2.0)	35.35 (–2.5)
Exact (Love)[5-4]	164.66	47.64**	47.71**	36.24
$R = 181$ in. $h = 2$ in. $\phi_A = 10.5$ deg $\nu = 0.3$				

*Numbers in parentheses are percentage deviations from the exact solution.

**From reciprocity considerations, these two coefficients should be identical. Presumably the differences arise from small round-off error.

Values of the coefficients from the Geckeler approximation,[5-11] from Geckeler's solution for the polar region,[5-24] and from Love's exact solution[5-4] were found in Galletly's paper.[5-19] The Love coefficients were calculated using a hypergeometric series representation. It can be seen that the Geckeler (polar region)[5-24] coefficients are in excellent agreement with the exact solution. More striking perhaps is the close agreement shown for the Hetényi coefficients since the polar angle ϕ_A is relatively small, and it is for small angles that this solution is least accurate. Even the Geckeler approximation[5-11] gives reasonably good results, considering the size of the angle.

Such a comparison, of course, does not give a complete picture. A more thorough examination might be to present plots of the stress distributions for each solution. In Reference 5-1 Timoshenko has presented a plot of M_ϕ and N_θ versus ϕ for the case of a clamped spherical cap under external normal pressure where

$$R = 90 \text{ in.}$$
$$h = 3 \text{ in.}$$
$$\phi_B = 35 \text{ deg}$$
$$\nu = \frac{1}{6}$$

In this case, the difference between Hetényi's solution and the exact solution is scarcely discernible, and the curve representing the Geckeler approximation[5-11] is sufficiently close for engineering purposes.

The author was able to find very little in the way of experimental data (again possibly because of an inadequate literature search) with which to evaluate theory. One source is a recent Model Basin study by Nishida[5-31] which is devoted primarily to the strength of layered spherical shells but which does have some data on monolithic shells. Figure 5-11 shows experimental strains measured on two 90-deg machined aluminum spherical caps (ϕ_B = 45 deg) having clamped edges and subjected to normal pressure on the convex surface. The strains are plotted for a unit pressure; the theoretical curves are obtained from Hetényi's solution. In both cases, the points tend generally to follow the shape of the curves, widest disagreement being for the thicker of the two shells. However, it would be unwise to make any sweeping conclusions from such a limited amount of data.

OTHER TYPES OF SHELLS

The preceding sections have dealt with the three types of shells for which solutions are sufficiently simple for practical engineering applications. Two other shells, the torus and the spheroid (or ellipsoid), also have relatively simple geometric shapes and are frequently used in pressure vessels and in submarine hulls. However, the analytical solutions for these shells are far too complex for practical use and will not be discussed here. In practice, the differential equations are usually solved numerically using high-speed computers. The best-known solution for the spheroid is due to Naghdi and de Silva,[5-32] that for the torus is due to Clark.[5-33]

Probably the easiest way to get a workable analytical solution for these shells is to make use of Geckeler's approximation,[5-11] and this can be subject to serious errors. It can yield useful results, however, if used with care, even in cases where the shell geometries cannot be simply expressed. What this amounts to is that the section of shell to be investigated is replaced by an equivalent cylinder–usually one whose radius is equal to the average radius of the actual shell. The analysis then proceeds as described in the section on cylindrical shells. It takes a fair amount of engineering experience to know when this approximation can be safely applied, and no attempt will be made here to set down rules of practice.

Other approximations are, of course, possible. For instance, a shell might be replaced by an equivalent conical segment. In such a case, one would do well to use the simplest possible solutions for the segment since the exact conical shell solutions are so complicated as to defeat the purpose of the approximation.

Where a long continuous shell is to be analyzed, it can be broken up into a series of segments. Since such segments often will be relatively short, they must usually be considered as "double-ended," thus requiring the simultaneous determination of all four coefficients for each segment. Because this greatly complicates the analysis, it is well to treat the element in the simplest possible way. A good discussion of such finite element techniques together with a useful bibliography can be found in Reference 5-34.

STIFFENING RINGS

A stiffening ring such as the one shown in Figure 5-12 can be dealt with in a simple manner provided certain assumptions can be made, i.e.:

1. The ring is sufficiently short that bending in the meridional plane can be neglected.
2. The cross section is not so deep that thick ring theory must be employed.

With these conditions, the ring can deflect radially and rotate within the plane of its cross section but does not undergo any change in cross-sectional shape.

The drawing in Figure 5-12 is supposed to represent a typical stiffening ring to which two shells are joined. R_A is the radius to the mean surface of the shell on the left; R_B is the corresponding radius of the shell to the right. The appropriate forces and moments are drawn with the same conventions used for shells. The deformations of the ring may be described as follows:

$$\begin{aligned} \overline{w}_A &= \overline{w}_0 - \ell_A \chi_0 \\ \overline{w}_B &= \overline{w}_0 + \ell_B \chi_0 \\ \chi_A &= \chi_0 = \chi_B \end{aligned} \qquad [5\text{-}84]$$

where $\overline{w}$ and χ have the usual meanings and the subscript "0" refers to the center of gravity of the section.

From simple geometrical relations, we have

$$\overline{w}_0 = \frac{R_0}{EA_r} \left[H_A R_A - H_B R_B + p \, \frac{(\ell_A + \ell_B)(R_A + R_B)}{2} \right] \qquad [5\text{-}85]$$

$$\chi_0 = \frac{R_0}{EI_z}\left[-H_A R_A \ell_A - H_B R_B \ell_B - M_A R_A - M_B R_B + \frac{p}{2}(\ell_B^2 R_B - \ell_A^2 R_A) + \frac{pR_A^2}{2}(R_A - R_0) - \frac{pR_B^2}{2}(R_B - R_0)\right] \qquad [5\text{-}85]$$

A_r indicates the cross-sectional area of the ring and I_z its moment of inertia about the z-axis, through the center of gravity. The two expressions contain some slight approximations. The pressure term in $\bar{w}_0$ comes from taking the lateral pressure to be applied at an average radius equal to $\frac{R_A + R_B}{2}$. In χ_0 the term $\frac{p}{2}(\ell_B^2 R_B - \ell_A^2 R_A)$ was gotten from the approximation that the pressure to the left of the center of gravity acts at the radius R_A and that to the right acts at R_B. In most cases, this term will be so small that it can be neglected altogether.

From the relations [5-84] and [5-85], the following edge coefficients are obtained:

$$\frac{\partial \bar{w}_A}{\partial H_A} = \frac{R_0 R_A}{EA_r} + \frac{\ell_A^2 R_0 R_A}{EI_z} \qquad \frac{\partial \bar{w}_B}{\partial H_B} = -\frac{R_0 R_B}{EA_r} - \frac{\ell_B^2 R_0 R_B}{EI_z}$$

$$\frac{\partial \bar{w}_A}{\partial M_A} = \frac{\ell_A R_0 R_A}{EI_z} \qquad \frac{\partial \bar{w}_B}{\partial M_B} = \frac{\ell_B R_0 R_B}{EI_z}$$

$$\frac{\partial \bar{w}_A}{\partial H_B} = -\frac{R_0 R_B}{EA_r} + \frac{\ell_A \ell_B R_0 R_B}{EI_z} \qquad \frac{\partial \bar{w}_B}{\partial H_A} = \frac{R_0 R_A}{EA_r} - \frac{\ell_A \ell_B R_0 R_A}{EI_z}$$

$$\frac{\partial \bar{w}_A}{\partial M_B} = -\frac{\ell_A R_0 R_B}{EI_z} \qquad \frac{\partial \bar{w}_B}{\partial M_A} = -\frac{\ell_B R_0 R_A}{EI_z} \qquad [5\text{-}86]$$

$$\frac{\partial \bar{w}_A}{\partial p} = \frac{R_0}{2EA_r}(R_A + R_B)(\ell_A + \ell_B) - \frac{\ell_A R_0}{2EI_z}[R_A^2(R_A - R_0) - R_B^2(R_B - R_0) + \ell_B^2 R_B - \ell_A^2 R_A]$$

$$\frac{\partial \bar{w}_B}{\partial p} = \frac{R_0}{2EA_r}(R_A + R_B)(\ell_A + \ell_B) + \frac{\ell_B R_0}{2EI_z}[R_A^2(R_A - R_0) - R_B^2(R_B - R_0) + \ell_B^2 R_B - \ell_A^2 R_A]$$

$$\frac{\partial \chi_A}{\partial H_A} = -\frac{\ell_A R_A R_0}{EI_z} = \frac{\partial \chi_B}{\partial H_A} \qquad \frac{\partial \chi_A}{\partial M_A} = -\frac{R_A R_0}{EI_z} = \frac{\partial \chi_B}{\partial M_A}$$

$$\frac{\partial \chi_A}{\partial H_B} = -\frac{\ell_B R_B R_0}{EI_z} = \frac{\partial \chi_B}{\partial H_B} \qquad \frac{\partial \chi_A}{\partial M_B} = \frac{R_B R_0}{EI_z} = \frac{\partial \chi_B}{\partial M_B}$$

$$\frac{\partial \chi_A}{\partial p} = \frac{R_0}{2EI_z} \left[R_A^2 (R_A - R_0) - R_B^2 (R_B - R_0) + \ell_B^2 R_B - \ell_A^2 R_A\right] = \frac{\partial \chi_B}{\partial p} \qquad [5\text{-}86]$$

Sometimes use is made of flexible tapered rings like that shown in Figure 5-13 to reduce stresses at cone-cylinder junctures. Because of their flexibility, these rings must be treated as tapered shells and their behavior cannot be adequately described by the simple Equations [5-86]. For a detailed treatment of this case, the reader is referred to an analysis by Raetz.[5-35]

COMPOSITE SHELLS

The complete analysis of a composite shell structure such as a submarine hull is a formidable problem because of the number of junctures involved. Since each discontinuity must be treated as a juncture, an attempt to analyze the structure all at once could require the solution of dozens of simultaneous equations—a situation which taxes a computer's storage capacity and can lead to such large numerical errors that results are sometimes meaningless. By following such a course, one may find that conditions at Frame 1 have, numerically, a significant effect on the deformations, say, at Frame 75. There are many ways around this. One is to use some engineering judgment to limit the number of junctures to be considered at one time. On either side of a deep frame, for example, it might not be necessary to carry the analysis beyond three small frame spacings. That is, the third frame is so little affected by the presence of the deep frame that it deforms essentially like a typical frame in a uniformly stiffened shell of infinite length. Errors can also be reduced by a judicious choice of the order in which the equations are solved. The number of equations to be solved in any structure depends on the number of consecutive "double-ended" elements. Fortunately only a few unknowns appear in any one of the equations. Consider, for example, a simple hull configuration containing N junctures in which only two elements meet at each interior juncture. If we want to eliminate $\bar{w}$ and χ as unknowns so that H and M are to be found, the system of equations might look something like this: For the nth juncture (represented by Figure 5-14),

$$
\begin{aligned}
0 = \; & H_{n-1}\left[\frac{\partial \bar{w}_{nB}}{\partial H_{nA}}\right] + M_{n-1}\left[\frac{\partial \bar{w}_{nB}}{\partial M_{nA}}\right] \\
& + H_n \left[\frac{\partial \bar{w}_{nB}}{\partial H_{nB}} - \frac{\partial \bar{w}_{(n+1)A}}{\partial H_{(n+1)A}}\right] + M_n \left[\frac{\partial \bar{w}_{nB}}{\partial M_{nB}} - \frac{\partial \bar{w}_{(n+1)A}}{\partial M_{(n+1)A}}\right] \\
& - H_{n+1}\left[\frac{\partial \bar{w}_{(n+1)A}}{\partial H_{(n+1)B}}\right] - M_{n+1}\left[\frac{\partial \bar{w}_{(n+1)A}}{\partial M_{(n+1)B}}\right] + \alpha_n p
\end{aligned}
\qquad [5\text{-}87]
$$

$$
\begin{aligned}
0 = \; & H_{n-1}\left[\frac{\partial \chi_{nB}}{\partial H_{nA}}\right] + M_{n-1}\left[\frac{\partial \chi_{nB}}{\partial M_{nA}}\right] \\
& + H_n \left[\frac{\partial \chi_{nB}}{\partial H_{nB}} - \frac{\partial \chi_{(n+1)A}}{\partial H_{(n+1)A}}\right] + M_n \left[\frac{\partial \chi_{nB}}{\partial M_{nB}} - \frac{\partial \chi_{(n+1)A}}{\partial M_{(n+1)A}}\right] \\
& - H_{n+1}\left[\frac{\partial \chi_{(n+1)A}}{\partial H_{(n+1)B}}\right] - M_{n+1}\left[\frac{\partial \chi_{(n+1)A}}{\partial M_{(n+1)B}}\right] + \beta_n p
\end{aligned}
$$

Here, it has been assumed that the only axial forces are due to pressure; the terms $\alpha_n p$ and $\beta_n p$ therefore represent known quantities. The coefficients of the H's and M's can be represented by a matrix that has a good many zeroes and is banded about the main diagonal:

$$
\begin{bmatrix}
(\;)(\;) & & & & \\
(\;)(\;)(\;) & & & & \\
 & \ddots & & & \\
 & & (\;)(\;)(\;) & & \\
 & & & \ddots & \\
 & & & & (\;)(\;)(\;) \\
 & & & & (\;)(\;)
\end{bmatrix}
\begin{bmatrix}
H_1 \\ M_1 \\ H_2 \\ M_2 \\ \vdots \\ H_n \\ M_n \\ \vdots \\ H_{N-1} \\ M_{N-1} \\ H_N \\ M_N
\end{bmatrix}
\qquad [5\text{-}88]
$$

Each ( ) indicates a small submatrix of the coefficients appearing in the pair of equations for each juncture. The band is three elements wide because the equations contain three unknown H's and three unknown M's at each interior juncture. This is not presented as the best scheme for solving the equations, but only as an example. It is worth noting, however, that certain computer routines are especially well adapted to the handling of banded matrices of this type.

In addition, some computational effort can be avoided if one remembers that for the linear systems considered here, the theorem of reciprocity ensures that

$$\frac{\partial \bar{w}_{nA}}{\partial H_{nB}} = -\frac{\partial w_{nB}}{\partial H_{nA}}\left(\frac{R_B}{R_A}\right) \qquad \frac{\partial \chi_{nA}}{\partial H_{nB}} = \frac{\partial \bar{w}_{nB}}{\partial M_{nA}}\left(\frac{R_B}{R_A}\right)$$

$$\frac{\partial \bar{w}_{nA}}{\partial M_{nA}} = -\frac{\partial \chi_{nA}}{\partial H_{nA}} \qquad \frac{\partial \bar{w}_{nB}}{\partial M_{nB}} = -\frac{\partial \chi_{nB}}{\partial H_{nB}} \tag{5-89}$$

$$\frac{\partial \bar{w}_{nA}}{\partial M_{nB}} = \frac{\partial \chi_{nB}}{\partial H_{nA}}\left(\frac{R_B}{R_A}\right) \qquad \frac{\partial \chi_{nA}}{\partial M_{nB}} = -\frac{\partial \chi_{nB}}{\partial M_{nA}}\left(\frac{R_B}{R_A}\right)$$

One notable achievement in the analysis of a complete submarine pressure hull is a computer program recently developed at the Naval Construction Research Establishment in Scotland. This program, described in References 5-36, 5-37, and 5-38, is capable of determining stresses and deformations at every juncture of interest in a typical pressure hull as well as stress distributions in areas selected by the programmer. The program can handle virtually any type of shell; the only real limitations are that the number of junctures and the number of elements that can be entered as input are restricted by the capacity of the computer. Recent comparisons with independent calculations done at the Model Basin showed excellent correlation.

ACKNOWLEDGMENTS

The author is greatly indebted to Mr. R.V. Raetz and to Mr. O. Lomacky for their valuable suggestions and advice.

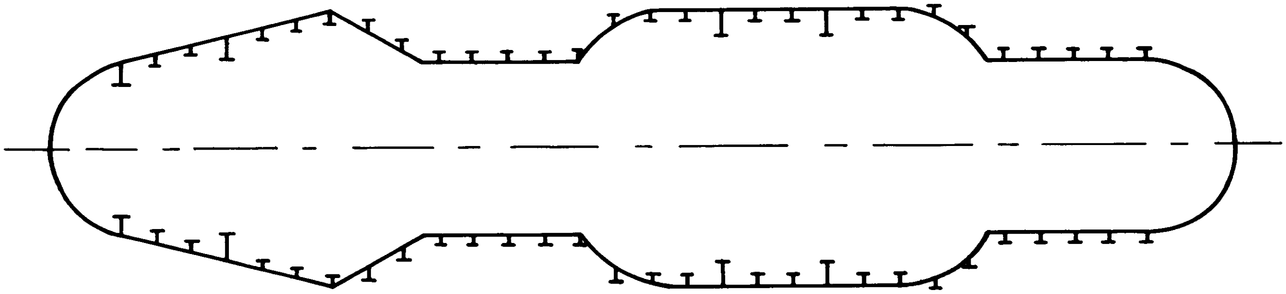


Figure 5-1 – Composite of Typical Shells Used in Submarine Hulls

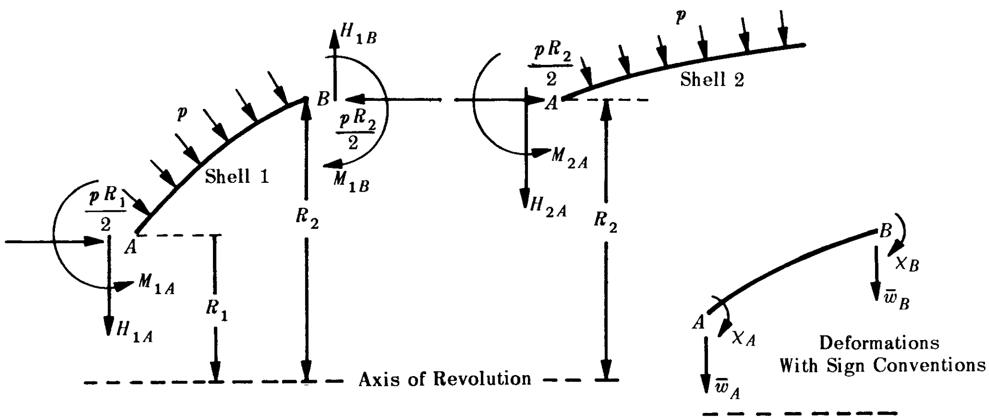


Figure 5-2 – Loads and Deformations in Intersecting Shells

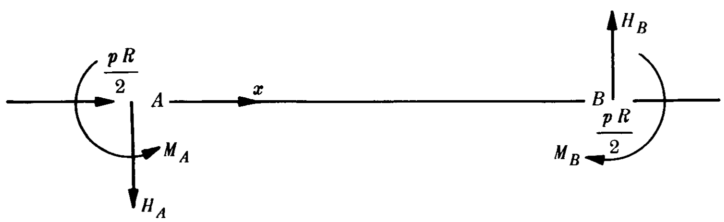


Figure 5-3 – Cylindrical Shell with Edge Loads

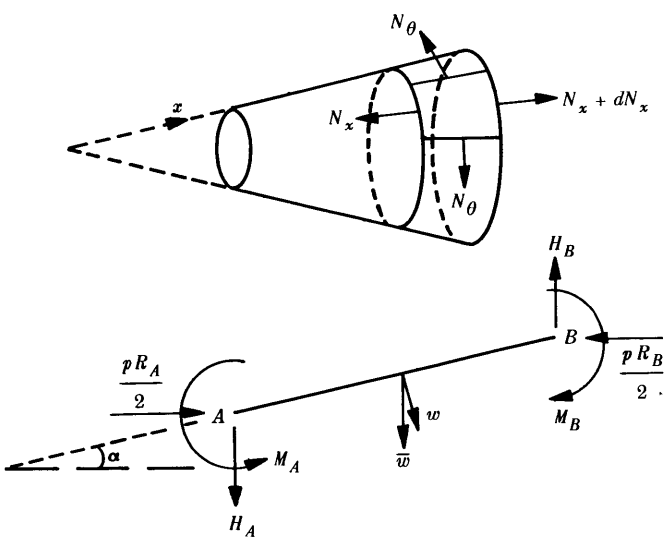


Figure 5-4 – Conical Shell with Edge Loads

Figure 5-5 – Taylor-Wenk Functions for Conical Shells

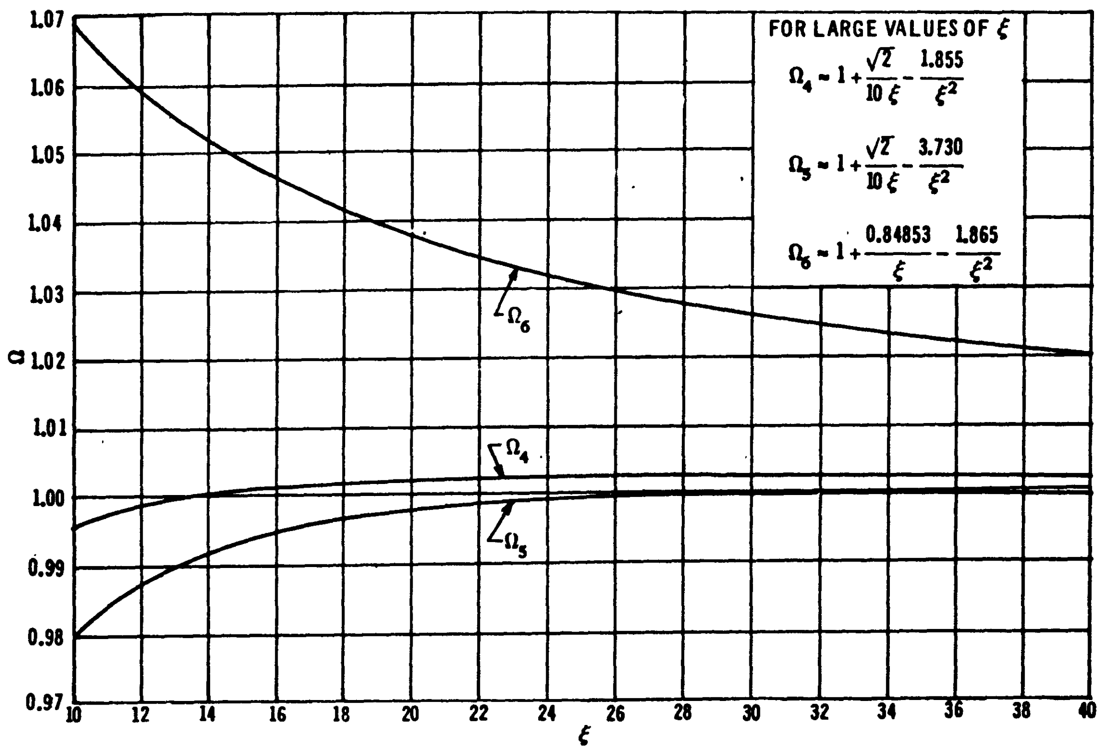


Figure 5-5a – Edge *A* ($10 \leq \xi \leq 40$)

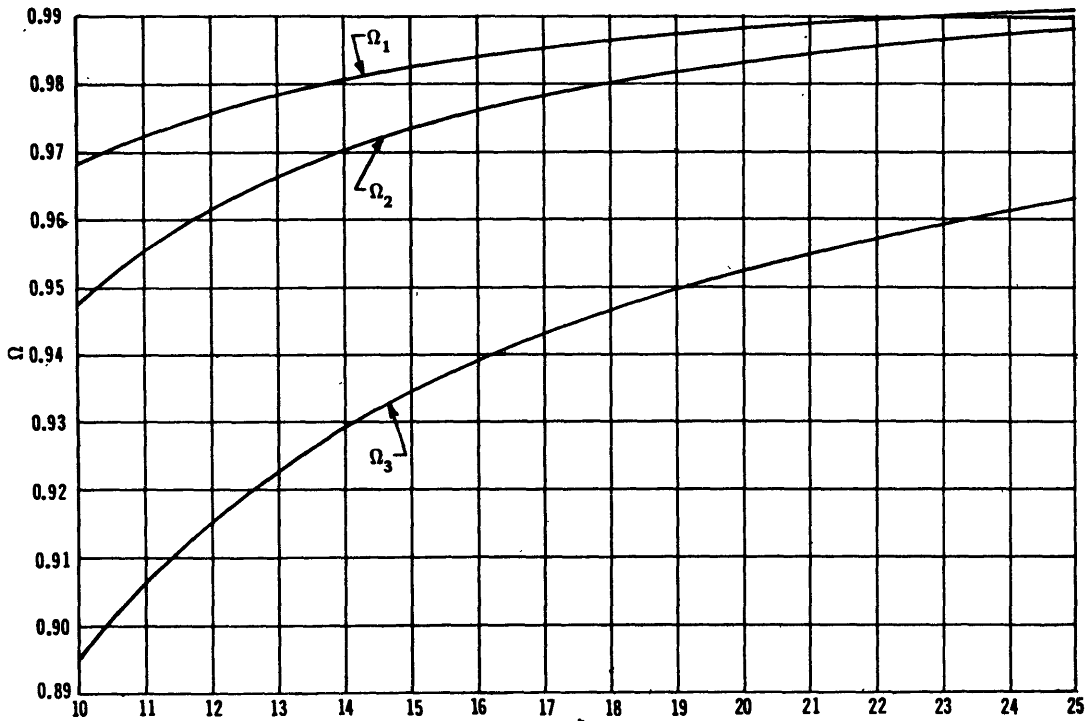


Figure 5-5b – Edge B ($10 \leq \xi \leq 25$)

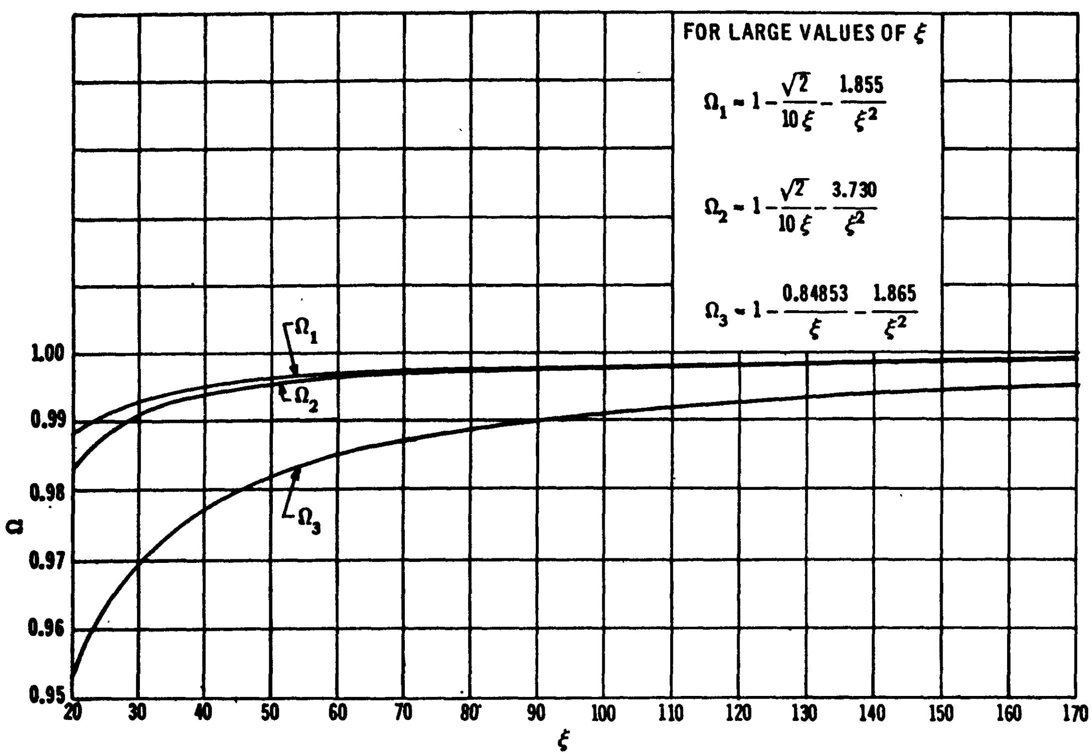


Figure 5-5c – Edge B ($20 \leq \xi \leq 170$)

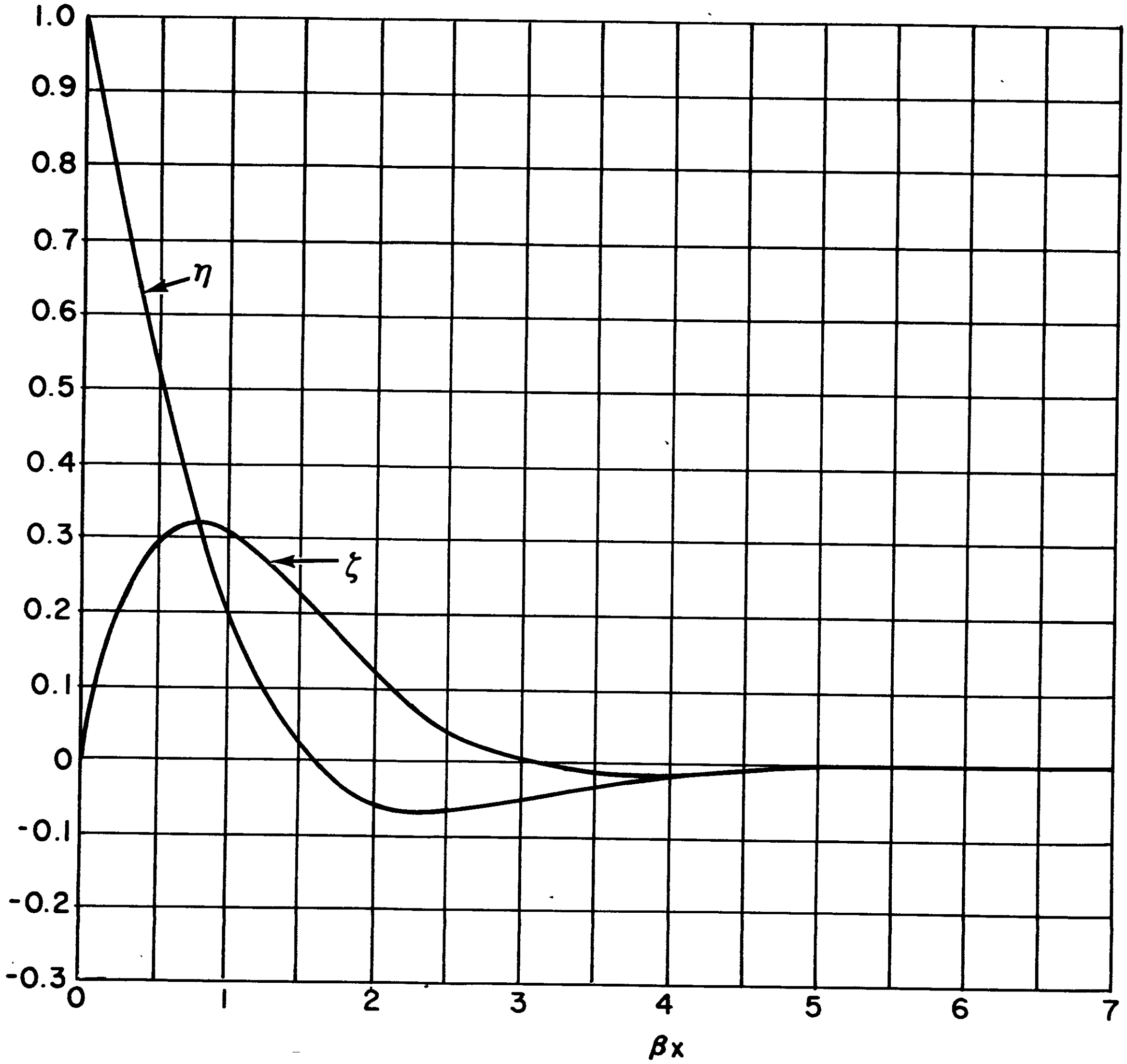


Figure 5-6 – Raetz-Pulos Functions

Figure 5-7 – Comparison of Raetz-Pulos Solution with Strain Data for Cone-Cylinder Model

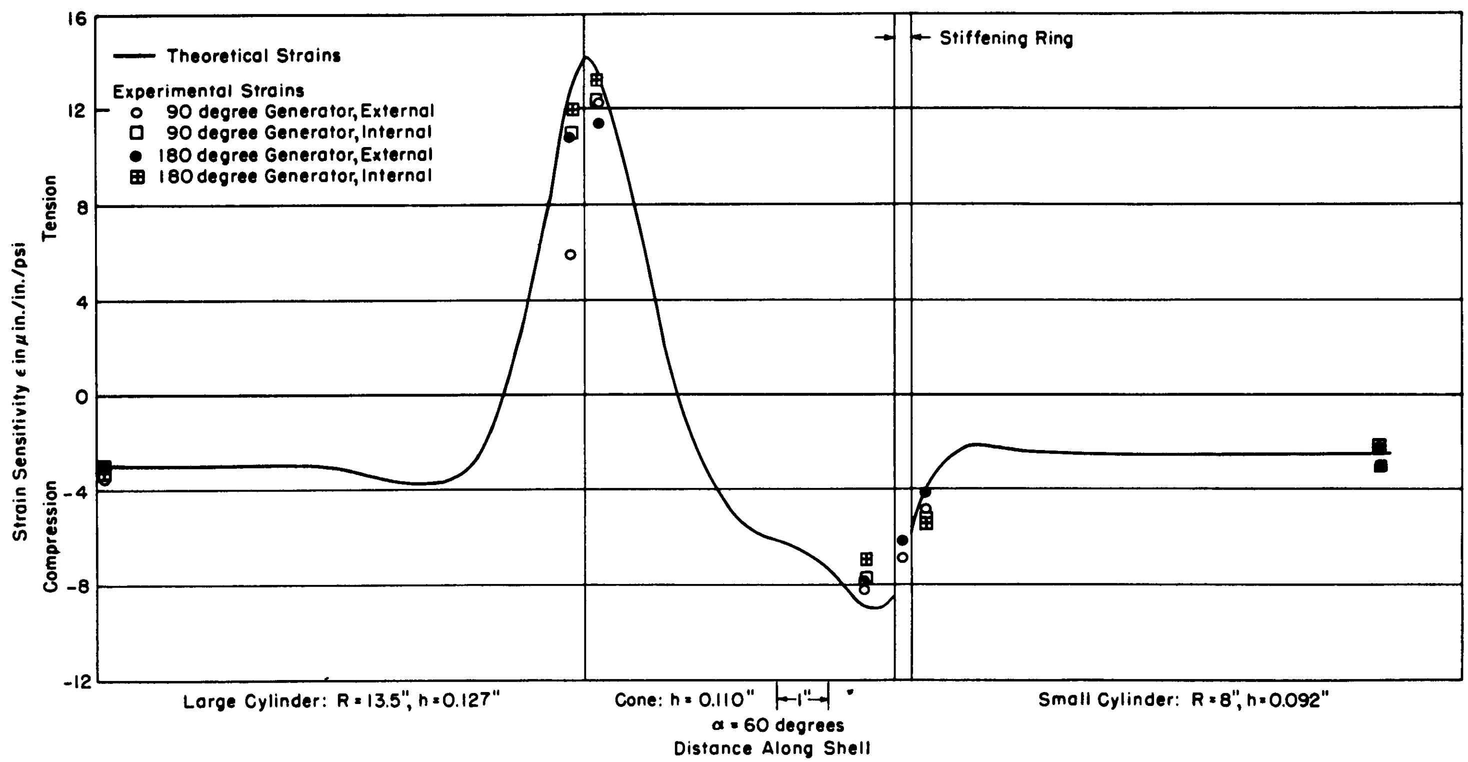


Figure 5-7a – Circumferential Strains

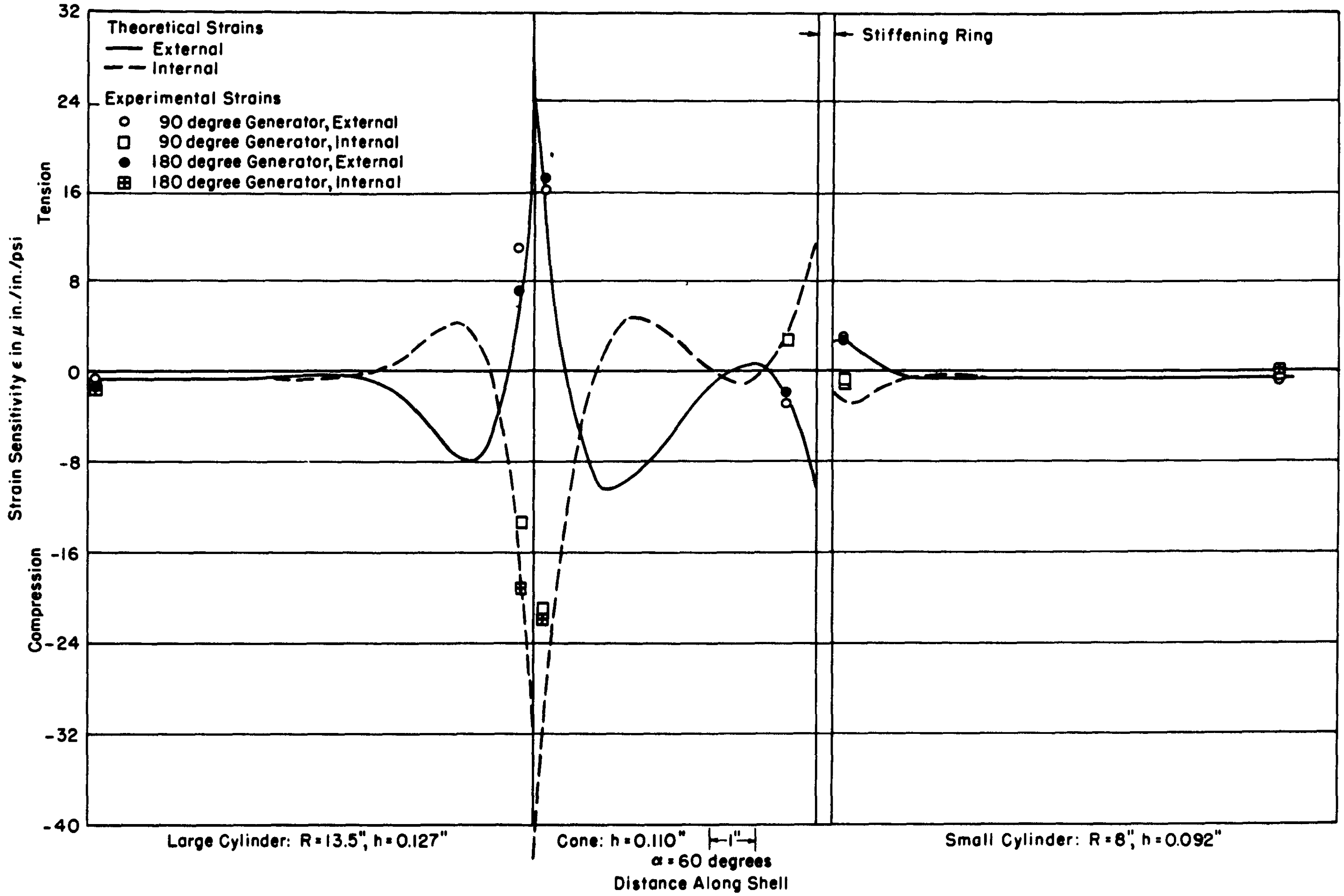


Figure 5-7b — Longitudinal Strains

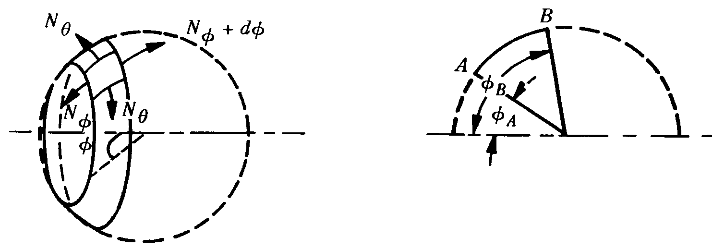


Figure 5-8 – Spherical Shell Element

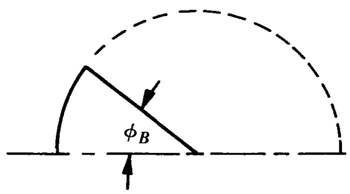


Figure 5-9a – Shallow Cap

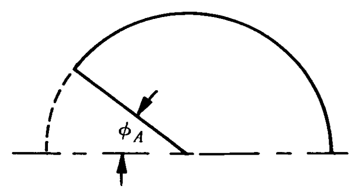


Figure 5-9b – Complement

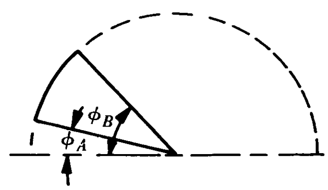


Figure 5-9c – Shallow Segment

Figure 5-9 – Spherical Shells Bounded by Small Polar Angles

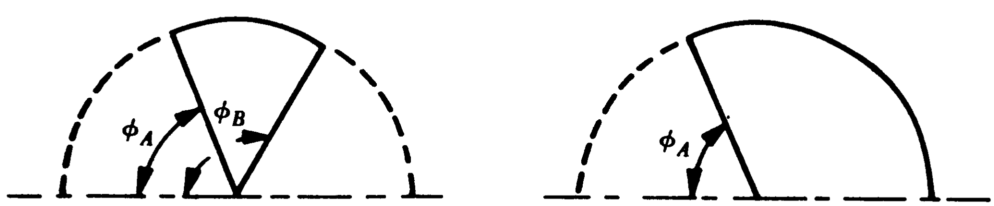


Figure 5-10 – Spherical Shells Bounded by Large Polar Angles

Figure 5-11 – Experimental Strains for Pressurized Clamped Spherical Cap Compared with Hetényi's Solution

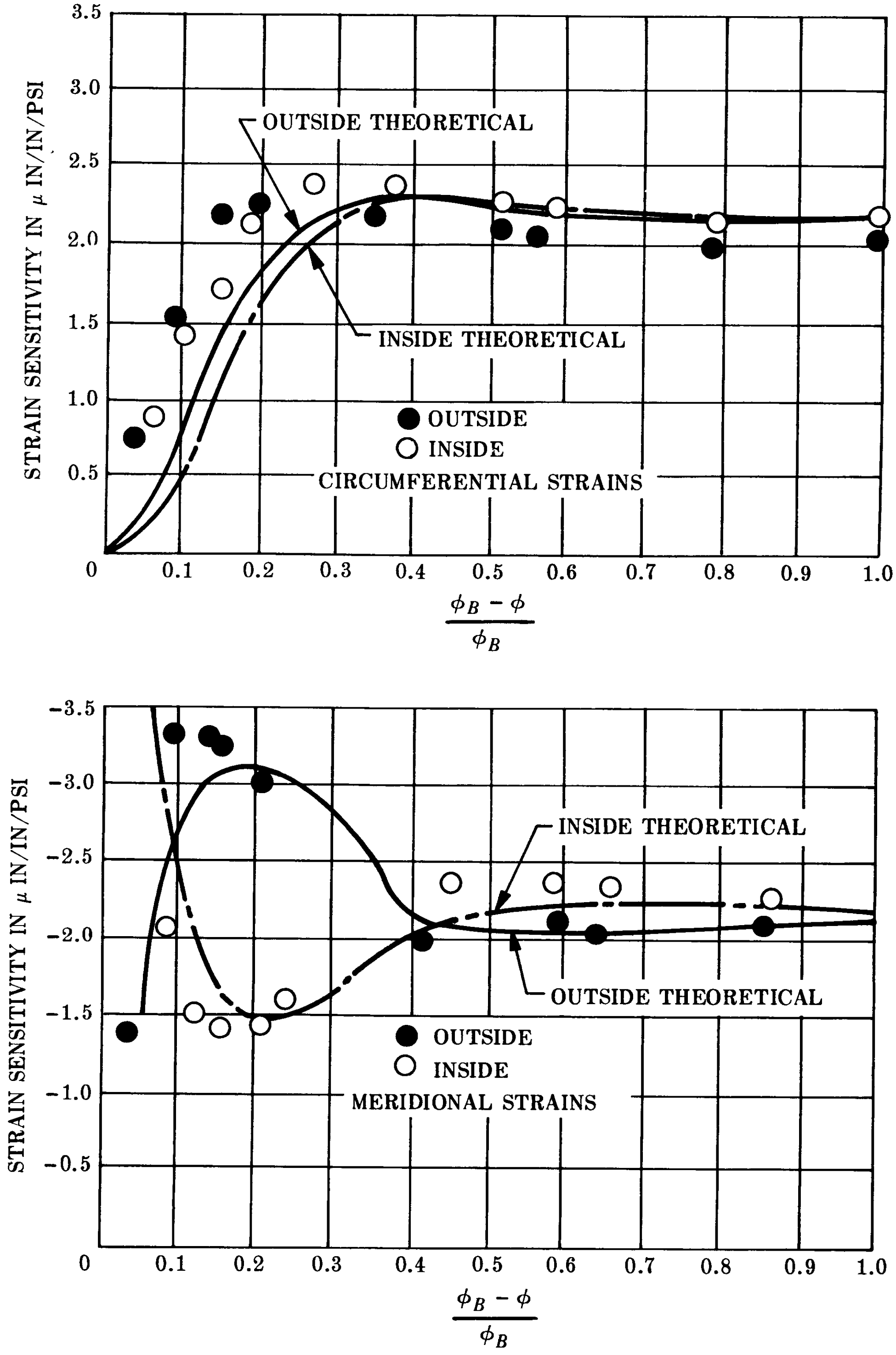
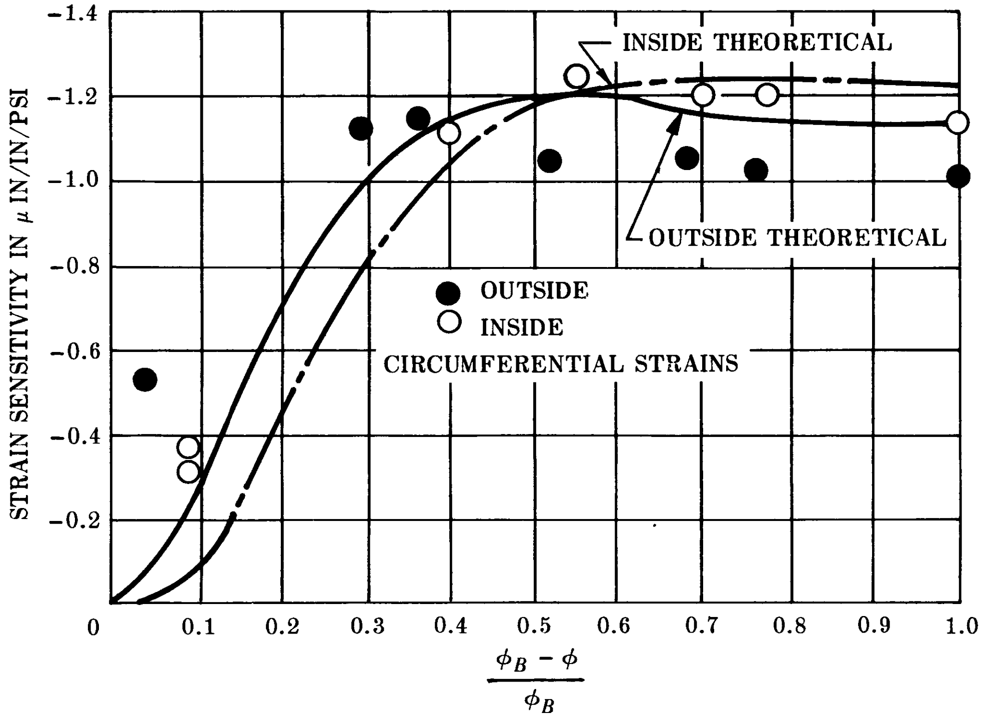


Figure 5-11a – $h/R = 0.0149$, $\phi_B = 45$ Degrees, $\nu = 0.3$



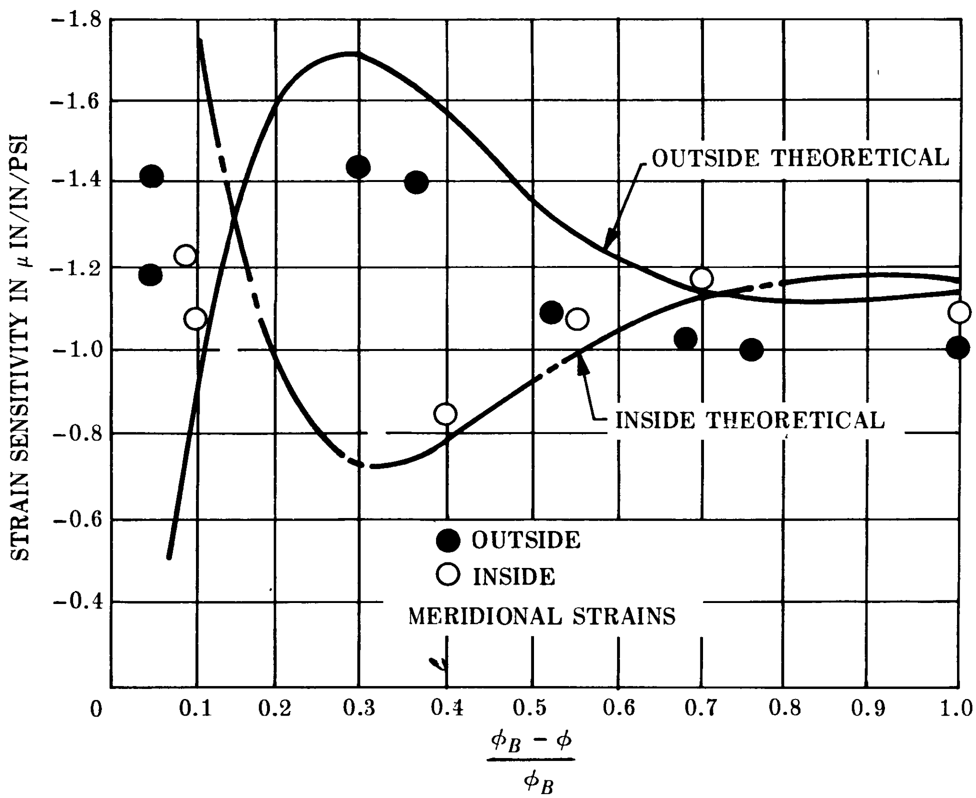


Figure 5-11b — $h/R = 0.0296$, $\phi_B = 45$ Degrees, $\nu = 0.3$

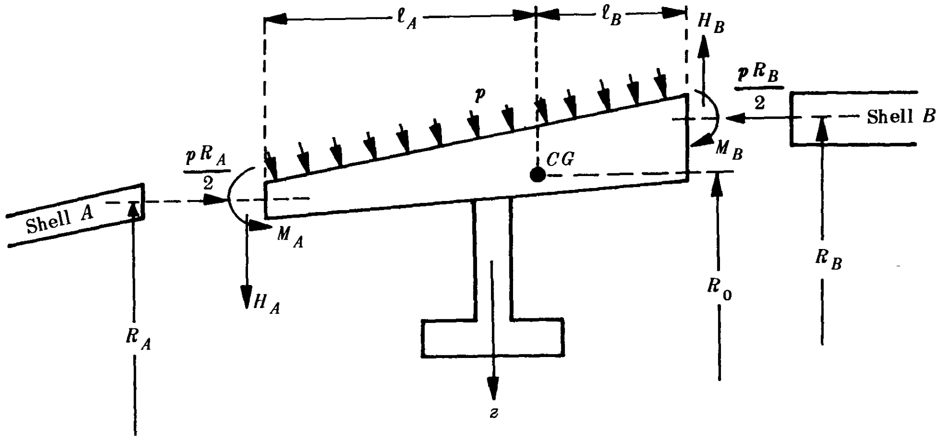


Figure 5-12 – Stiffening Ring with Edge Loads

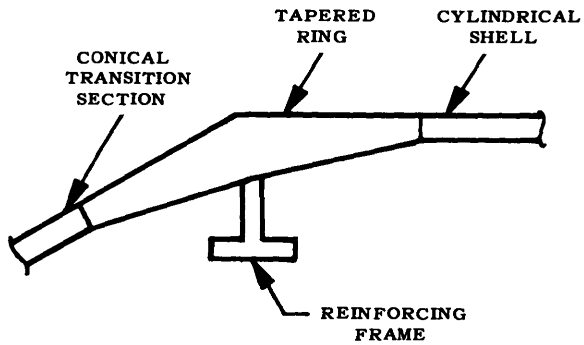


Figure 5-13 – Cross Section of Cone-Cylinder Juncture with Tapered Ring

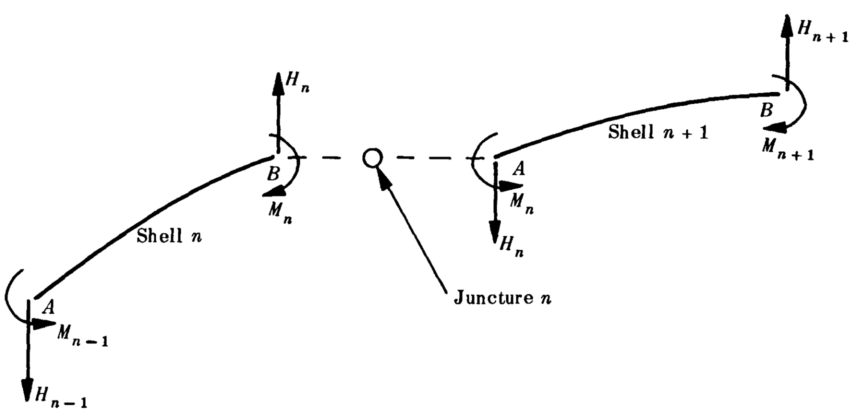


Figure 5-14 – nth Juncture in a Chain of Connected Shells

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2. Submarines--Stresses--Research
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4. Cylindrical shells (Stiffened)--Stresses--Research
5. Conical shells (Stiffened)--Stresses--Research
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