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A CONCISE PROOF OF THE ALGORITHM TO GENERATE MULTIDIMENSIONAL HERMITE POLYNOMIALS AND GRAM-CHARLIER COEFFICIENTS

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Dr. S. Berkowitz

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COMPUTATION AND MATHEMATICS DEPARTMENT
RESEARCH AND DEVELOPMENT REPORT

July 1971

Report 3697

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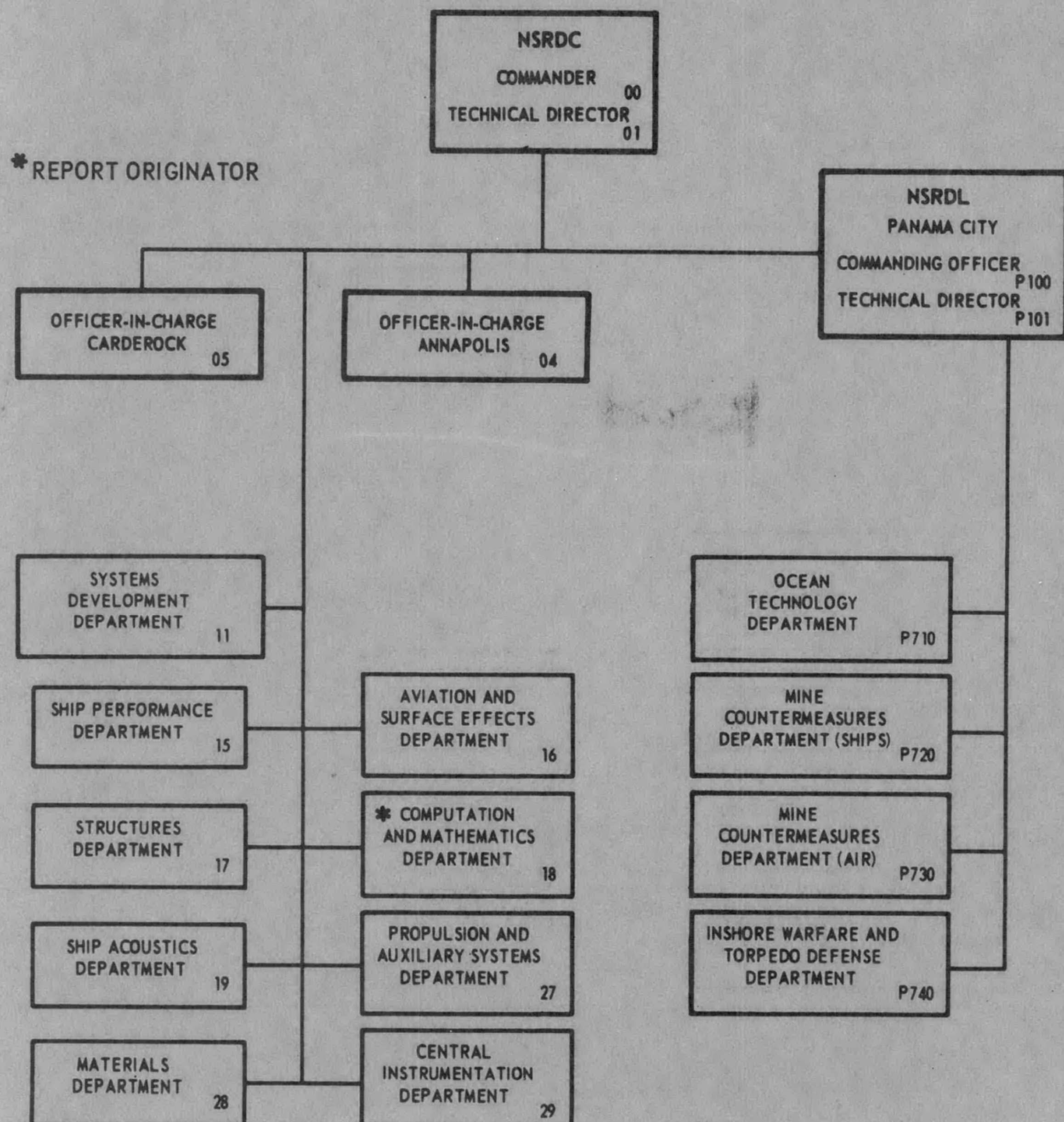

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Dr. S. Berkowitz

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ABSTRACT

This report supplements earlier documentation on the generation of multidimensional Hermite polynomials and Gram-Charlier coefficients. The report contains a new concise proof of the inequality employed in an efficient algorithm to choose suitable suborders in the calculation of a polynomial or coefficient of a given order.

ADMINISTRATIVE INFORMATION

This work was carried out in the Computation and Mathematics Department under Task Area SF 14532107, Task 15329, Speech Recognition Project.

INTRODUCTION

Hermite polynomials and Gram-Charlier coefficients can serve to expand a near-Gaussian probability density distribution used in problems of communications, pattern classification, and turbulence. Recursive methods for calculating the polynomials and coefficients have been derived and documented by the author.[1] An extension of this effort which provided an efficient algorithm that minimized the number of polynomials and/or coefficients required by the recurrence relations of Reference 1 for the calculation of a polynomial or coefficient of a given order was subsequently published.[2] The report at hand provides a more concise proof of part of the algorithm given in Reference 2.

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1 References are listed on page 6.

RECURRENCE RELATIONS FOR MULTIDIMENSIONAL HERMITE POLYNOMIALS AND GRAM-CHARLIER COEFFICIENTS

Reference 1 established the following recurrence relations for Hermite polynomials.

$$H_{\underline{m}}(\underline{x}) = \left(\sum_{j=1}^{n} q_{kj}x_j\right) H_{\underline{m}-\underline{e}_k}(\underline{x}) - \sum_{j\neq k} q_{kj}m_j\, H_{\underline{m}-\underline{e}_k-\underline{e}_j}(\underline{x}) - q_{kk}(m_k-1)H_{\underline{m}-2\underline{e}_k}(\underline{x}) \quad (1)$$

$$G_{\underline{m}}(\underline{x}) = x_k\, G_{\underline{m}-\underline{e}_k}(\underline{x}) - \sum_{j\neq k} r_{kj}\, m_j\, G_{\underline{m}-\underline{e}_k-\underline{e}_j}(\underline{x}) - (m_k-1)r_{kk}\, G_{\underline{m}-2\underline{e}_k}(\underline{x}) \quad (2)$$

where $k = 1,\ldots, n$

$\underline{x}$, $\underline{m}$ are n-dimensional vectors, with components x_j, m_j, respectively

$\underline{e}_j$ is an n-dimensional vector whose $i^{\underline{th}}$ component is δ_{ij}

$[r_{ij}] = [q_{ij}]^{-1}$ is a positive definite matrix, namely, the covariance matrix of $\underline{x}$.

If $p_o(\underline{x})$ is the Gaussian density distribution of mean $\underline{\mu} = (\mu_1,\ldots, \mu_n)$ and covariance matrix $[r_{ij}]$, then one can expand any probability density $p(x)$ in a Gram-Charlier series

$$p(\underline{x}) = p_o(\underline{x}) \sum_{m=0}^{\infty} A_{\underline{m}}\, H_{\underline{m}}(\underline{y}) \quad (3)$$

or

$$p(\underline{x}) = p_o(\underline{x}) \sum_{m=0}^{\infty} B_{\underline{m}}\, G_{\underline{m}}(\underline{y}) \quad (4)$$

where $m = \sum_{j=1}^{n} m_j$

$y_i = (x_i - \mu_i)/\sigma_i$ and σ_i is the standard deviation of x_i.

The coefficients A_m, B_m, as shown in Reference 1, are generated by the following recurrence relations.

$$A_{\underline{m}} = \frac{1}{M} [b_{\underline{m}} \sum_{i=1}^{M} y_k^{(i)} G_{\underline{m}-\underline{e}_k}(\underline{y}^{(i)})] - \sum_{j=1}^{n} r_{kj} m_k^{-1} A_{\underline{m}-\underline{e}_k-\underline{e}_j} \quad (5)$$

$$B_{\underline{m}} = \frac{1}{M} [b_{\underline{m}} \sum_{i=1}^{M} \sum_{j=1}^{n} q_{kj} y_j^{(i)} H_{\underline{m}-\underline{e}_k}(\underline{y}^{(i)})] - \sum_{j=1}^{n} q_{kj} m_k^{-1} B_{\underline{m}-\underline{e}_k-\underline{e}_j} \quad (6)$$

where $k = 1, \ldots, n$

$b_{\underline{m}} = \prod_{i=1}^{n} (m_i!)^{-1}$

$\underline{y}^{(i)} = (y_1^{(i)}, \ldots, y_n^{(i)})$ is the i^{th} normalized sample vector of a set of M such vectors

A CONCISE PROOF OF THE GENERATION ALGORITHM

In Reference 1, we established by a somewhat lengthy proof that a polynomial or a coefficient with order vector $\underline{m}$ could be computed by a recurrence relation using only those coefficients or polynomials of sub-order $\underline{v} = v_1, \ldots, v_n$ if, and only if,

1. A permutation $\sigma(j)$ is defined in such a way that $i \leq j$ iff $m_{\sigma(i)} \geq m_{\sigma(j)}$;
2. The suborder vectors are generated as vectors $(v_{\sigma(1)}, \ldots, v_{\sigma(n)}$ in the ascending sequence of integers $v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}$ in a base equal to the greatest m_j;
3. A suborder vector $\underline{v}$ of the form

$$\underline{v} = (v_{(1)}, \ldots, v_{(p)}, 0, \ldots, 0) \quad (7)$$

implies that the corresponding coefficient or polynomial may be computed (if needed) only by the p^{th} recurrence relation (p is called the <u>pivot</u>);

4. For a pivot p,

$$\sum_{j=1}^{p-1} (m_{\sigma(j)} - v_{\sigma(j)}) \le (m_{\sigma(p)} - v_{\sigma(p)}) + \sum_{j=p+1}^{n} m_{\sigma(j)}; \quad (8)$$

5. For computing coefficients only, $\sum_{j=1}^{n} (m_j - v_j)$ must be even.

A concise proof of Equation (8) is now given.

<u>PROOF:</u>

There are three kinds of index decrements involved in Equations (3), (4), (5), (6), namely:

(i) $\underline{m}-\underline{e}_j-\underline{e}_p$

(ii) $\underline{m}-\underline{e}_p$

(iii) $\underline{m}-2\underline{e}_p$

Suppose that α decrements of type (i) from $\underline{m}$ are considered so that

$$0 \le m_{\sigma(p)} - v_{\sigma(p)} = \alpha \quad (9)$$

Then α decrements are also distributed among the p-1 components to the left of $m_{\sigma(p)}$ so that

$$\sum_{j=1}^{p-1} (m_{\sigma(j)} - v_{\sigma(j)}) = \alpha + \kappa \quad (10)$$

where κ is the number of decrements distributed among the p-1 components to the left of $m_{\sigma(p)}$ by decrementing from the n-p+1 pivot components to the right of $m_{\sigma(p)}$. Clearly,

$$\sum_{j=p+1}^{n} m_{\sigma(j)} = \max \kappa \quad (11)$$

where the maximum is taken over the possible decrements from $\underline{m}$ to $(v'_{\sigma(1)}, \ldots, v'_{\sigma(p-1)}, m_{\sigma(p)}, 0, \ldots, 0)$, and

$$\sum_{j=1}^{p-1} (v'_{\sigma(j)} - v_{\sigma(j)}) = \alpha$$

From Equations (9), (10), (11), one sees that (8) is satisfied.

Similarly, for decrement type (ii), Equations (9) and (11) still hold, but Equation (10) becomes

$$\sum_{j=1}^{p-1} (m_{\sigma(j)} - v_{\sigma(j)}) = \kappa \tag{12}$$

so that

$$\sum_{j=1}^{p-1} (m_{\sigma(j)} - v_{\sigma(j)}) \leq \sum_{j=p+1}^{n} m_{\sigma(j)} \tag{13}$$

But this inequality is included in the inequality of Equation (8).

Again, for decrement (iii), Equation (13) holds and is implied by Equation (8).

Thus, for any pivot, and regardless of decrement type, the suborder vectors which are reachable from $\underline{m}$ will be specified by Equation (8).

END OF PROOF

REFERENCES

1. Berkowitz, S., "The Formulation of Multidimensional Hermite Polynomials and Gram-Charlier Coefficients," Naval Ship Research and Development Center Report 3614 (Jan 1971).

2. Berkowitz, S., and Garner, F.J., "The Calculation of Multidimensional Hermite Polynomials and Gram-Charlier Coefficients," Mathematics of Computation, Vol.24, No.111, pp.537-45 (Jul 1970).

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