Report 1706

DAVID TAYLOR MODEL BASIN

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DEPARTMENT OF THE NAVY

HYDROMECHANICS

O

AERODYNAMICS

O

STRUCTURAL MECHANICS

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APPLIED MATHEMATICS

O

ACOUSTICS AND VIBRATION

A METHOD FOR PREDICTING THE PLATE-HULL GIRDER RESPONSE OF A SHIP INCIDENT TO SLAM

by

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and

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ACOUSTICS AND VIBRATION LABORATORY

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NOTATION

Symbol	Definition	Dimension
A_p	Area of plate	ft^2
A_γ, A_δ	γ^{th} and δ^{th} elemental area of plate	ft^2
$A_m(t)$	Amplitude factor	ft/sec^2
a	Width of plate (shorter side)	ft
b	Length of plate (longer side)	ft
C_{1m}, C_{2m}	Constants of integration	ft
C_n	Distance from neutral axis to location at which stress is desired	ft
C_0	Sound velocity in water	ft/sec
C_1, C_2, C_3, C_4	Constants determined from end conditions	
D_x, D_y, D_1, H	Elastic constants for bending of stiffened or sandwich plate	ton-ft
E	Modulus of elasticity in tension and compression for plate material	ton/ft^2
$f(t)$	Time load distribution	ton/ft^2
G	Modulus of elasticity in shear for plate material	ton/ft^2
$G_1(x)$, $G_2(y)$	Space load distribution	dimensionless
I	Area moment of inertia of cross section of a beam	ft^4
I_n	Areal moment of inertia of ship cross section about a principal axis perpendicular to the plane of bending at station n	ft^4
ij	Mode of vibration	
i_{w_x}, i_{w_y}	Moments of inertia of web about neutral planes of equivalent orthotropic plate	ft^3
i_x', i_y', i_x'', i_y'', i_{xy}, i_{yx}	Defined by Equations [B.15] - [B.20]	ft^3

Symbol	Definition	Dimension
K	Structural viscous damping force per unit area per unit velocity	ton/ft^2/ft/sec
k_{ij}	Defined by Equation [A.30]	
l	Length of beam	ft
M_n	Bending moment in hull girder cross section at Station n	ft-ton
M_x, M_y	Bending moments per unit length of sections of a plate perpendicular to x and y axes, respectively	$\frac{\text{ft-ton}}{\text{ft}}$
M_{xy}, M_{yx}	Twisting moments per unit length of section of a plate perpendicular to x and y axes, respectively	$\frac{\text{ft-ton}}{\text{ft}}$
m_{w_x}, m_{w_y}	Unit bending moments in webs in two coordinate directions x and y, respectively	$\frac{\text{ton-ft}}{\text{ft}}$
n	Station numbers n = 0, 1....N	dimensionless
n_{ij}, n_m	Damping constant for ij^{th} (or m^{th}) mode	1/sec
$\overline{P}_m(x,y)$	Water pressure due to vibrating plate in the m^{th} mode of vibration	ton/ft^2
$\overline{P}_m(x,y)_{\gamma\delta}$	Average pressure on γ^{th} elemental area A_γ due to modal vibration of the δ^{th} elemental area A_δ	ton/ft^2
$P(x,y,t)$	Pressure (load/unit area) applied normal to the plate	ton/ft^2
$\overline{P}(x,y,t)$	Forcing pressure	ton/ft^2
p_{ij}, p_m	Circular frequency for i^{th} (or m^{th}) mode of vibration	rad/sec
p_{lm}	Defined by Equation [A.19]	rad/sec
q_{ij}, q_m	A function of time such that $w_{ij}\ q_{ij}$ or $w_m\ q_m$ satisfies the orthotropic plate equation	ft

Symbol	Definition	Dimensions
$r_{n'x}$, $r_{n'y}$	Scalar distance of centroid of near flange from neutral axis of bending in x or y directions, respectively	ft
$r_{f'x}$, $r_{f'y}$	Scalar distance of centroid of far flange from neutral axis of bending in x or y directions, respectively	ft
S_x, S_y	Distance between repeating sections y and x directions, respectively	ft
t	Time	sec
t_{nx}, t_{ny}	Thickness of near flanges (see Figure 5)	ft
t_{fx}, t_{fy}	Thickness of far flanges (see Figure 5)	ft
V	Total energy per unit area in a repeating section of a plate	ton/ft
V_0	Bending energy per unit area of an orthotropic plate	ton/ft
V_p	Bending and twisting energy per unit area in a repeating section of a plate	ton/ft
V_w	Energy in the webs per unit area for a repeating section of a plate	ton/ft
w(x,y,t)	Lateral deflection of plate	ft
w_{ij}, w_m	Deflection in the ijth (or mth) mode of a rectangular plate	dimensionless
$(w_\delta)_{ij}$, $(w_\delta)_m$	Deflection in the ijth (or mth) mode of the δ^{th} elemental area normalized to the maximum deflection	dimensionless
X_i, Y_j	Normal mode shapes of a uniform beam	dimensionless
X_i'', Y_j''	$\frac{d^2X}{dx^2}$ and $\frac{d^2Y}{dy^2}$, respectively	$1/ft^2$
x, y, z	Rectangular coordinate axes	

Symbol	Definition	Dimensions
$(Z_{\gamma\delta})_{ij}, (Z_{\gamma\delta})_m$	Mutual radiation impedance between the γ^{th} and δ^{th} elemental areas on the plate for the ij^{th} (or m^{th}) mode of vibration	$\frac{\text{ton-sec}}{\text{ft}}$
α	Rotational compliance or stiffness	$\frac{\text{ft-ton}}{\text{radian}}$
β_i, β_j	Frequency numbers for a uniform beam	dimensionless
$(\Delta x)_i$	Section length	ft
$\varepsilon_{n'x}, \varepsilon_{n'y}$	Unit elongations for the near flange in the x and y directions, respectively	ft/ft
$\varepsilon_{f'x}, \varepsilon_{f'y}$	Unit elongations for the far flange in the x and y directions, respectively	ft/ft
θ	Frequency number for a uniform beam	dimensionless
$(\theta_{\gamma\delta})_{ij}, (\theta_{\gamma\delta})_m$	Resistive (radiation damping) component of impedance for the ij^{th} (or m^{th}) mode	dimensionless
μ_p	Mass per unit area of stiffened plate	$\frac{\text{ton-sec}^2}{\text{ft}^3}$
ν	Poisson's ratio for plate material	
ρ_0	Density of water	ton-sec^2/ft^4
$(\sigma_{P.S.})_n$	Primary bending stress in ship at Station n	
σ_x, σ_y	Bending stresses in x and y directions, respectively	ton/ft^2
$(\sigma_x)_p, (\sigma_x)_s, (\sigma_x)_t$	Primary, secondary and tertiary stresses, respectively	ton/ft^2
σ_{nx}, σ_{ny}	Bending stresses in the near flanges of the stiffeners in the x and y directions, respectively	ton/ft^2
σ_{fx}, σ_{fy}	Bending stresses in the far flanges of the stiffeners in the x and y directions, respectively	ton/ft^2

Symbol	Definition	Dimensions
τ_{xy}, τ_{yx}	Shear stress in plate	ton/ft^2
$(\tau_{xy})_{nx}$, $(\tau_{xy})_{fx}$	Shear stress in near and far flanges of plate respectively	ton/ft^2
$(\chi_{\gamma\delta})_{ij}$, $(\chi_{\gamma\delta})_m$	Reactive (virtual mass) component of impedance for the ijth(or mth) mode	dimensionless

ABSTRACT

A method is presented for computing the component and overall dynamic response and stress patterns for a ship subject to slam in heavy seas. These patterns include primary, secondary, and tertiary deflections and stresses produced in the hull girder stiffened plate sections of the hull and plating between stiffeners, respectively; the analyses for a vibrating free-free, continuous non-uniform beam and for a vibrating cross-stiffened plate with elastic rotational constraints at the boundary, are combined. Experimental programs to check the theory are recommended. Plastic deformation considerations for the plates are also treated.

INTRODUCTION

For several years the David Taylor Model Basin has been investigating the problem of slamming loads on ships and their response in heavy seas in order to devise improved ship structural design criteria. One aspect of this problem involves the determination of and relationship between cause (sea excitation) and effect (vibratory deformations and their associated stresses) for the system (ship) under consideration.* In particular, one

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*This relationship describes ship and fluid medium interaction, i.e., cause and effect are interdependent. The system is then a closed one in which part of the system response to an excitation is fed back to the input of the system (see pages 2 and 85 of Reference 1).

of the authors has been carrying out work for computing the primary beam stresses in a ship hull girder during slamming.[1,2] * Here, the model for the ship system is a nonuniform beam, and the motions of this model consist of combined rigid-body and flexural oscillations. Using this model, a comparison between the theoretical and measured time histories of the mid-ship stress in the hull girder keel of a Dutch Destroyer showed good agreement.[1]

This model, however, is inadequate for determining the stresses in the bottom stiffened plates or in the local plating of the ship. These stresses combined with the hull girder stresses represent a more realistic picture of the actual stress pattern for any section of the ship. Hence, an *additional* model for the ship system is the plating. Theories have been devised by one of the authors [3,4,5] for determining the dynamic stresses induced in unstiffened and stiffened plates under impact loading with emphasis toward the slamming problem.

The objective of this report is to extend these plate theories to include rotational constraint and then to integrate this analysis with the beam type analysis to compute the total stress picture consisting of combined primary, secondary, and tertiary components of stress at any point of a given ship undergoing a given slamming condition. The equations developed are used to outline a computer program for computing the total stress pattern in a ship during slamming in heavy seas. Experimental programs to check the theory are recommended. A solution of the plate

* References are listed on page 87.

equations in a form suitable for coding, equations to be programmed, a method for evaluating plate parameters, and the required input data for the digital computer are presented in the appendixes.

METHOD OF ATTACK

The total (i.e., combined) stress pattern, at any time, in a ship moving in a seaway is computed by summing (algebraically) the primary stresses due to beam action (denoted by P.S.), see Figure 1; the secondary stresses on bottom stiffened plates* (denoted by S.S.), see Figure 2; and the tertiary stresses in local plating (denoted by T.S.), see Figure 3.**

Total Stress*** = P.S. + S.S. + T.S.

The stress components are computed as follows:

1. Primary Stresses (Figure 1).

The primary stresses are computed from the beam bending moment determined by the methods of References 1 and 2. This stress at Station n in the ship is

$$(\sigma_{P.S.}) = \frac{M_n C_n}{I_n}$$

* Actually, the secondary and tertiary stresses in any of the stiffened plates can be computed. However, the bottom plating is of major interest in impact loading.

** This stress division is described in a report by St. Denis; see Reference 6.

*** The stress due to slam is superposed on the static stresses and the slowly changing dynamic stresses which occur as the ship moves through the waves.

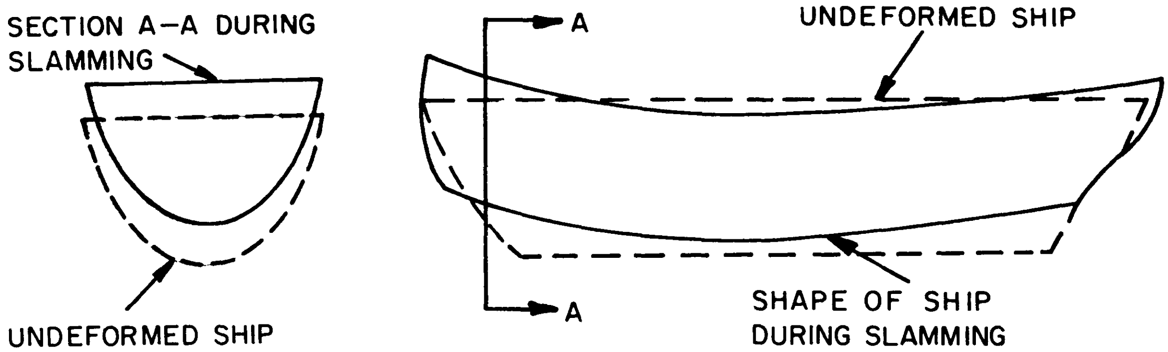

Figure 1 – Primary Stresses Due to Beam Action (Denoted by P.S.)

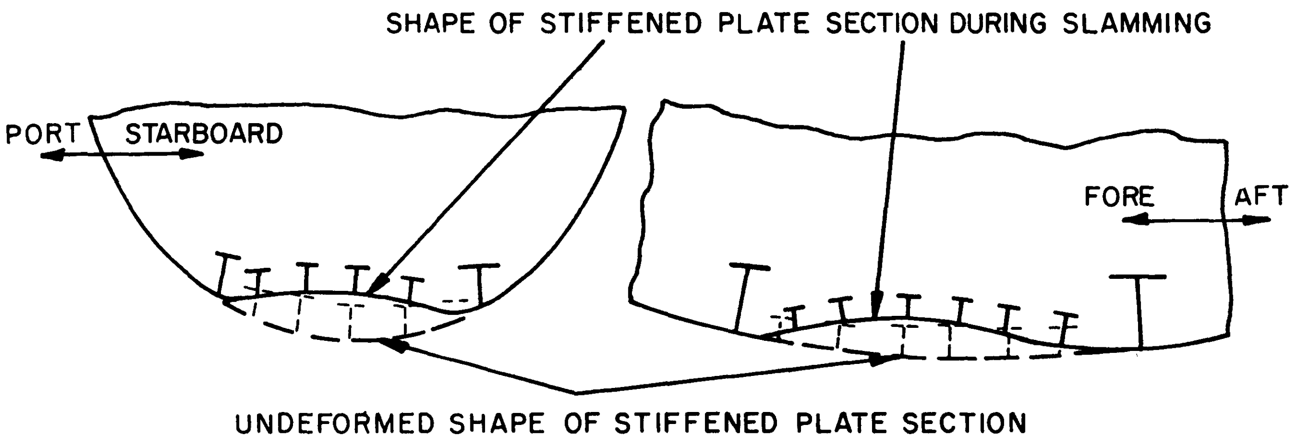

Figure 2 – Secondary Stresses on Bottom Stiffened Plates (Denoted by S.S.)

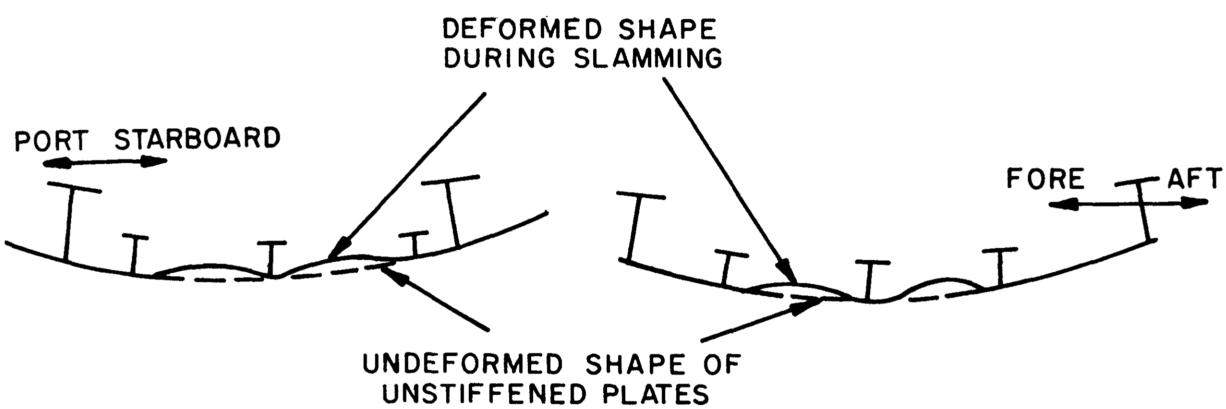

Figure 3 – Tertiary Stresses in the Local Plating (Denoted by T.S.)

where M_n is the bending moment at the n^{th} station,

C_n is the distance from the neutral axis to the location at which the stress is desired, and

I_n is the moment of inertia at the n^{th} station.

2. Secondary Stresses (Figure 2)

The stresses in stiffened plate sections are known as the secondary stresses. The response of clamped and simply supported stiffened plates is contained in Reference 5. The analysis used in that reference is generalized here to include elastic rotational constraint at the boundary so that any case between simple and clamped supports can also be solved. A modal type method, in which the response of the plate is represented by the sum of the response in the individual modes, is used to compute the secondary stresses; the modal response is computed by approximate methods. Comparison of theoretical results with *static* test results on stiffened and sandwich plates indicates good agreement of this theory with experiment.[5]

3. Tertiary Stresses (Figure 3).

The stresses in the local plating are known as the tertiary stresses. The method of analysis used here is that given in Reference 3. Dynamic tertiary stresses in ship plates incident to slamming were computed in Reference 4 and were in good agreement with stresses determined from observations for the limited experiments performed.

THEORETICAL ANALYSIS OF PROBLEM

PLATE ANALYSIS

The theory for thin plates with small deflection will be used in treating the forced vibration of orthotropic plates;* see Reference 7. Details of analysis are presented in Appendix A. Solution of the differential equation of motion for these plates is obtained by approximating the deflection of the plate by the sum of the product of the normal modes of lateral vibration of a *beam* with rotational constraint; simply supported and clamped end conditions are then included as limiting cases of the rotational constraint. The effects of virtual mass and damping of the plates are included.

For a stiffened plate (Figures 4 and 5), let D_x, D_y, D_1, D_{xy} be the elastic constants (calculated according to the methods of Reference 8 and presented for completeness of this report in Appendix B), K the structural viscous damping force per unit area per unit velocity, μ_p the mass per unit area, P(x,y,t) the external lateral pressure (load/unit area) applied to the plate, and w(x,y,t) the deflection of the plate. Then, as shown in detail in Appendix A, the differential equation of motion for an orthotropic plate is

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D \frac{\partial^4 w}{\partial y^4} + \mu_p \frac{\partial^2 w}{\partial t^2} + K \frac{\partial w}{\partial t} = P(x,y,t) \qquad [1]$$

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* Small deflection theory is applicable for deflections up to about 0.7 of the thickness of a flat plate.

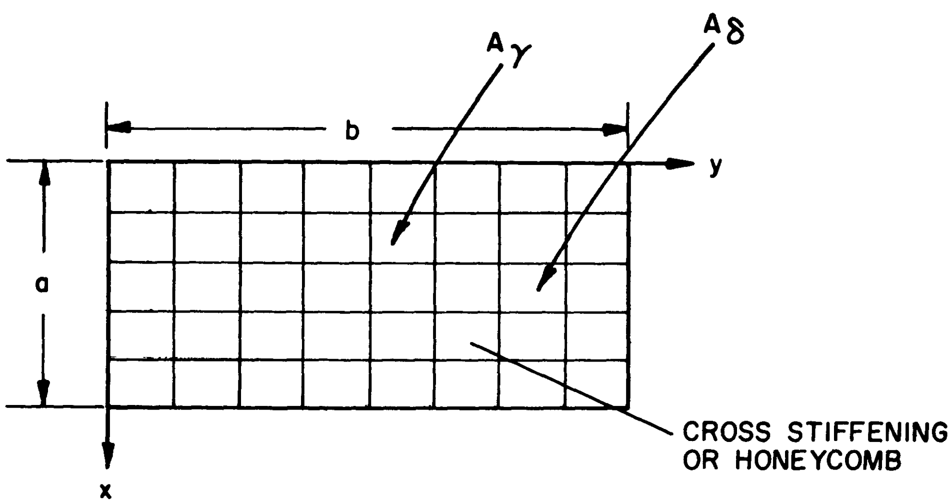

Figure 4 – Cross-Stiffened or Honeycomb Rectangular Plate

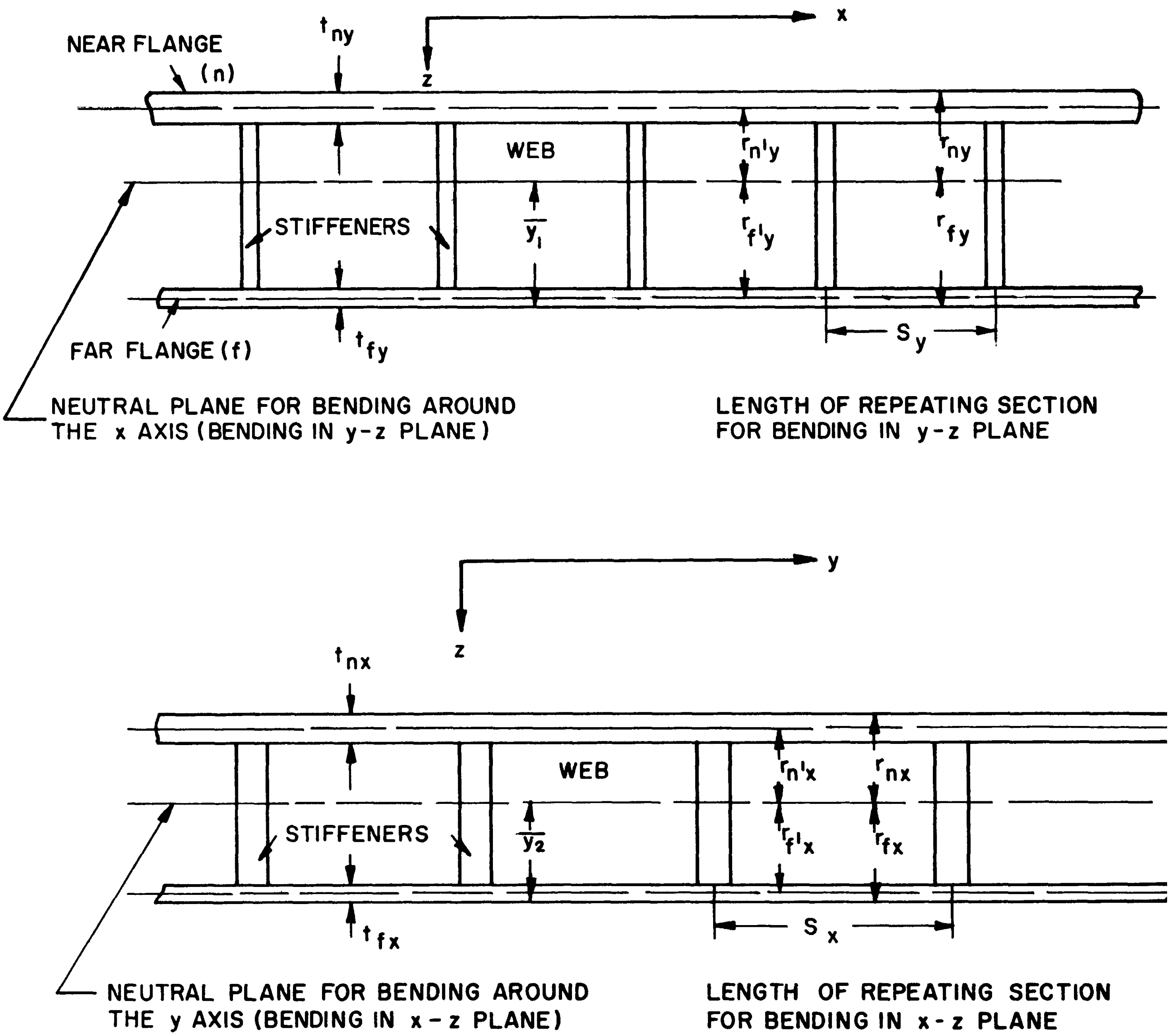

Figure 5 - Cross-Sectional Views of Stiffened Plate

t_{nx}, t_{ny} is required to allow for different types of stiffeners with two different flange thicknesses running in orthogonal directions.

where

$$H = D_1 + 2D_{xy}$$

Let the solution be represented by

$$w(x,y,t) = \sum_i \sum_j w_{ij}(x,y)\, q_{ij}(t) \qquad [2]$$

Corresponding expressions obtained for the bending stresses σ_x and σ_y in the x and y directions, respectively, are found in Appendixes B and C.

In an actual problem, w, σ_x, σ_y, and other quantities treated in Appendixes A, B, and C are obtained by means of a digital computer using the equations to be programmed. Expressions for the orthotropic constants and the total input of the program are given in Appendixes B and D. Plastic deformation considerations for the plates are treated very briefly in Appendixes E and G. In particular, the procedure for computing the approximate plastic deformations and the failure loads for the plates is given in Appendix E. A basic approach for computing plastic deformations of stiffened and unstiffened plates is given in Appendix G.

COMBINED ANALYSIS

The actual mathematical analysis instructing the programmer and the engineer how to compute the total stress at any point in the ship at any time is given in Appendix F. This total stress consists of the sum of the primary, secondary, and tertiary stresses.

If the stresses are such that the Von Mises yield condition is satisfied at a given point, i.e., if

$$\sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2} = \sigma_o$$

then the ship plating can be expected to go into the plastic region at that point. In the above equation σ_x, σ_y, τ_{xy} are the total bending and shearing stresses and σ_o is the yield stress in pure tension.[9] Although the yield condition is not used explicitly in this report, it will be used in the digital program to determine the locations where plasticity commences. It will be used in future work on the plasticity of plates due to slamming loads.

ANALYTICAL CONSIDERATIONS

In the theory used here rotatory inertia and shear deformation of the plate have been neglected. A quantitative estimate of the accuracy of plate theory neglecting these items is found in Mindlin's paper.[10] When the wave length of the wave is greater than ten times the thickness of the plate, the classical plate theory as used here should be accurate to within several percent in predicting natural frequencies, deflections, and stresses of a simply supported plate. For a clamped plate, additional errors due to boundary condition approximations are discussed elsewhere in the present report. Although it is difficult to extrapolate expected errors in stiffened plates by omitting rotatory inertia and shear, we can take the maximum depth of stiffener as the thickness of the plate for estimating this error. Naturally, the error will be greater in the higher modes. These higher modes are of significance when plates are subjected to excitation by pulses whose duration are only a fraction of the fundamental period of the plate. The *actual* effect of omitting rotatory inertia and shear will be ascertained both quantitatively and practically when experiments are conducted and results of the theory are compared with the test results.

In the plate analysis performed here a modal approach was used because it has proved to be successful in predicting frequencies, deflections, and stresses in isotropic plates;[3] in predicting slamming strains;[4] and in predicting static deflections and stresses in cross-stiffened plates.[5] The accuracy of the modal approach depends upon how accurately the modes can be approximated. This approach is best for problems where the pulse duration is longer than the fundamental period because a rapidly converging series is obtained for this case. The value of the modal approach lies in the fact that relatively few equations have to be solved, and these are completely separable if normal modes are used. The consideration of only a few modes has proved to be very successful in previous problems.[3,4,5] Theoretically, no approximation is involved in using the modal approach if the true modes can be exactly represented. The solutions are then exact. Practically, approximations do often exist in the representation of the modes. For simply supported plates, the modes can be predicted exactly. For clamped plates, the modal representation and, therefore, the results are not as good; see Reference 3. For predicting the primary stresses in the hull girder a finite difference approach has been used.[1] This approach is quite accurate and involves only representation of the ship treated as a beam by 20 mass-elastic elements. If a finite difference approach were used for the plates, it would involve 20 x 20 or 400 elements - too many for ease of calculation.

Therefore, this approach is not the most practical for plates, especially since representation of plate vibration by a few modes seems to give reasonably good results, thereby lending weight to the modal approach in this case.

A ship is essentially composed of a series of flat and curved plates welded to each other. Stiffeners in turn are welded to these plates. We are considering only the flat plate portions of the hull in the present report. For the slamming problem, most of the critical areas exposed to large slamming loads will be composed of flat plates.

To consider curved plate segments, thereby achieving more accurate results, a shell type approach must be followed. This type of approach is outlined in Reference 8. However, the theory presented here, which treats only flat plate segments, is expected to give results sufficient for design purposes.

Only structural damping and radiation damping (resistive component of impedance) for the plates and only beam structural damping for the primary stresses (see Reference 1) are treated here.* The resistive component of impedance is neglected in beam theory since lower frequencies are involved. It is taken into account approximately in plate theory by the introduction of $\theta_{\gamma\delta}$. A further simplification is made in Reference 12 by finding an equivalent piston impedance for each mode of the plate.

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* For completeness, the equations in References 1 and 2 for the rigid-body motions of the hull should also contain a term for hydrodynamic damping. This refinement was considered in a report by R. H. MacNeal.[11]

The same approximation may be sufficient here for the slamming problem, but imposition of this limitation does not seem desirable at this time.

DISCUSSION

Digital computation of the component and total motional response and stress pattern for a ship subject to intense arbitrary hydrodynamic loading associated with rough seas, i.e., slam, can be made using the methods presented in this report together with those in References 1 and 2. The primary component corresponds to the main hull girder motion whereas the secondary and tertiary components* are associated with motions of the bottom stiffened plating and local plating between stiffeners, respectively (see Figures 1, 2, and 3); summation of the components yields the total pattern which is a time history of the total motion and stress in the ship's hull.

Water pressures associated with the higher modes of plate vibration are included in the computation by treating the interaction of mutual impedance effect of all the plate area elements, considering each element to act like a piston in an infinite baffle.

Application of this type of computation to ship design is imperative in view of the following empirical facts:

The forward bottom plating of several ships in service has been dished in or has failed completely because of high stresses developed in the ship plating due to large pressures incident to slam. Moreover,

* This report describes only the computation of the secondary and tertiary components. The results are then combined with corresponding results for the main hull girder computed by the methods of References 1,2.

the pressure distribution induced by slamming usually subjects only a relatively limited area of bottom plating to large loading at a particular time;[4] the area of maximum loading is invariably closer to the keel than to the turn of the bilge keel. Local loads and stresses may or may not be accompanied by large hull girder stresses. Finally, as stated in Reference 13, "there is a strong possibility that the peak of a wave-induced stress may occur simultaneously with a "slamming" stress at a time when cargo dead-load stresses are appreciable and under conditions of temperature favoring brittle fractures." In such a situation slamming may well lead to a major failure.

It is apparent, therefore, that prediction of the slamming loads that ship plating can tolerate without large permanent set (i.e., plastic deformation) or failure is a significant determinant, as a limiting factor, of the ships operational capabilities in severe seas. For ships already in service, the combined plate-hull girder analysis can ascertain the degree to which particular areas of a ship subject to prescribed severe seas or slamming loads require additional strengthening to withstand the excitation. The loads can be experimentally determined from records and can be either explicitly or statistically (Reference 11, Part I) formulated in mathematical terms. On the other hand, for ships in the design stage, the bottom panels and/or the main hull girder in the area exposed to slamming can be designed either to experience stresses within the elastic limit for most slam-induced pressures or for a given set.

Stated in design terms, specification that the plate stresses lie within the elastic limit or that the plates sustain a given set (and, similarly, for the hull girder) will permit the designer to realistically and quite accurately select a minimum thickness of plate and optimize the dimensions for the hull girder in areas susceptible of intense sea wave damage as dictated by experience.* The actual plate edge condition can be determined by measurement or, if the plate is one part of a larger plate containing stiffeners, a fixed edge approximation can be made for the plate boundaries.** The maximum stresses experienced under severe sea wave excitation can be compared with static stresses ordinarily computed for the trochoidal wave.

In evaluating the accuracy of the method presented here for computing the response of plates to arbitrary excitation forces, the work of Reference 15 is cited. This reference shows that the displacements and strains in rectangular plates due to transient forces can be predicted with reasonable accuracy using a small number of modes for which the nodal patterns (mode shapes) are known only approximately. Hence, the inclusion of a large number of modes and a more accurate expression for the natural frequencies may be expected to produce very accurate results.

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* The specification presumes a knowledge of plate-hull girder hydrodynamic loadings, types of edge support for the plates, and hull mass-elastic parameters and geometry.

** In general, the theory will be better for simply supported plates than for clamped plates since sinusoidal beam functions are exact solutions for simply supported orthotropic plating (see Appendix A).

CONCLUSIONS

This paper presents a method for computing the component and overall response and stress patterns for a ship subject to slam in heavy seas. These patterns include primary, secondary, and tertiary deflections and stresses produced in the hull girder, stiffened plate sections of the hull and plating between stiffeners, respectively. The results of such computations can be of great utility in designing the ship plating and hull girder for good performance of these structures under actual operating conditions in heavy seas.

RECOMMENDATIONS

It is recommended that experiments be performed on a ship undergoing slam which will permit comparison between each type (P.S., S.S., T.S.) of measured stress and the combined sum with corresponding theoretical calculations based upon the methods presented in this report. Conceivable instrumentation suitable for making measurements yielding stress discrimination and combination includes:

1. Placement of strain gage bridges on each side of the hull plating to separate the tertiary bending in the plates from the beam and secondary stress.*

2. Placement of strain gages on stiffeners to obtain secondary bending stresses in a fashion similar to that in 1.

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* Tertiary plate slamming stresses obtained for a very extensive slamming test conducted on a U. S. Coast Guard Cutter are reported in References 4, 13, and 14. A test similar to that reported in Reference 4 could be conducted again, this time with emphasis on measuring all types of stresses.

3. Placement of strain gages on main hull girders and cancelling of local bending stresses obtained by 1 and 2 to obtain beam stresses in the hull.

4. Placement of a single strain gage at various locations on the hull to measure total strain.

This instrumentation must be placed on a ship which is expected to experience rough weather and, therefore, a good deal of slamming. The U. S. Coast Guard Cutter used for previous tests[4,13,14] operated in the North Atlantic during the winter months. Moreover, for a ship to be chosen, the ship personnel must be able to participate in the test and the skipper must be scientifically oriented.

From a computation of the total elastic stress picture in the ship as outlined in this report, we can find out whether or not certain points on the ship will go plastic. We can also obtain an approximation to the failure load of stiffened plates subjected to the slamming load. However, to the authors' knowledge, the only existing method for predicting small dynamic plastic deformation of the plates is that of Nagai.[16,17,18] The utility of such a computation arises from the fact that proposals have been made for designing the plating of ships using small permanent sets as a design criteria. It is recommended that this problem be considered in the near future and integrated into the design procedure for ships.

APPENDIX A

SOLUTION OF EQUATIONS OF MOTION OF CROSS-STIFFENED PLATES SUBMERGED IN A FLUID AND SUBJECT TO ARBITRARY LOADING

The differential equation of motion for the deflection of an orthotropic plate, that is, one whose material has three planes of symmetry with respect to the elastic properties, can be written as (see Figures 4 and 5 and Reference 7, page 365):

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + \mu_p \frac{\partial^2 w}{\partial t^2} + K \frac{\partial w}{\partial t} = P(x,y,t) \qquad [A.1]$$

where

$$H = D_1 + 2D_{xy}$$

and D_x, D_y, D_1, D_{xy} are the elastic constants for the stiffened plate, K is the structural viscous damping force per unit area per unit velocity,* μ_p is the mass per unit area of the stiffened plate,** and P(x,y,t) is the external lateral pressure (load/unit area) applied to the plate. Derivation of the method for calculating the elastic constants is given in Appendix B.

Let the deflection w of the plate be expressed by the following summation of normal modes:[5,12]

* K is calculated by assuming a given ratio of critical damping for the plate; see Equations [A.7c] - [A.7f].

** μ_p is the total mass divided by the projected surface area (a x b).

$$w = \sum_{r=1}^{\infty} w_r(x,y)\, q_r\,(t) \qquad [A.2]$$

where w_r is the dimensionless space-dependent normal mode function,

q_r is a time dependent function having the dimensions of length, and

r is the mode number.

Substitution of Equation [A.2] into Equation [A.1] yields:

$$D_x \frac{\partial^4}{\partial x^4} \sum_{r=1}^{\infty} w_r q_r + 2H \frac{\partial^4}{\partial x^2 \partial y^2} \sum_{r=1}^{\infty} w_r q_r + D_y \frac{\partial^4}{\partial y^4} \sum_{r=1}^{\infty} w_r q_r$$

$$+ \mu_P \frac{\partial^2}{\partial t^2} \sum_{r=1}^{\infty} w_r q_r + K \frac{\partial}{\partial t} \sum_{r=1}^{\infty} w_r q_r = P(x,y,t) \qquad [A.3]$$

Integration of the product of Equation [A.3] *and one of the normal mode functions* w_m over the plate area A_p gives:

$$\int_{A_p} w_m \left[D_x \frac{\partial^4}{\partial x^4} \sum_{r=1}^{\infty} w_r q_r + 2H \frac{\partial^4}{\partial x^2 \partial y^2} \sum_{r=1}^{\infty} w_r q_r + D_y \frac{\partial^4}{\partial y^4} \sum_{r=1}^{\infty} w_r q_r \right] dA_p$$

$$+ \mu_p \int_{A_p} w_m \frac{\partial^2}{\partial t^2} \sum_{r=1}^{\infty} w_r q_r dA_p + K \int_{A_p} w_m \frac{\partial}{\partial t} \sum_{r=1}^{\infty} w_r q_r dA_p = \int_{A_p} w_m P(x,y,t) dA_p \quad [A.4]$$

If the plate boundary is wholly or partly clamped, simply supported or free, then for $r \neq m$ the last two terms in the left member are:

$$\mu_p \int_{A_p} w_m \frac{\partial^2}{\partial t^2} \sum_{r=1}^{\infty} w_r q_r dA_p + K \int_{A_p} w_m \frac{\partial}{\partial t} \sum_{r=1}^{\infty} w_r q_r dA_p$$

$$= \mu_p \int_{A_p} \frac{\partial^2 q_r}{\partial t^2} \sum_{r=1}^{\infty} w_r w_m dA_p + K \int_{A_p} \frac{\partial q_r}{\partial t} \sum_{r=1}^{\infty} w_r w_m dA_p \qquad [A.5]$$

which are equal to zero because for the stated boundary conditions, when $r \neq m$ (interchanging the integral and time derivatives in the right-hand members of Equation [A.5]), the following orthogonality relations hold:[19]

$$\int_{A_p} w_m w_r \, dA_p = 0 \qquad \text{[A.6]}$$

If the plate is vibrating freely in one of its modes so that $P(x,y,t) = 0$, then from Equation [A.5] the first term (consisting of the integral of three terms) in the left-hand member of Equation [A.4] is zero for $m \neq r$. Hence, with the forcing function $P(x,y,t)$ included, Equation [A.4] can be written ($r = m$):*

$$q_m \int_{A_p} w_m \left[D_x \frac{\partial^4 w_m}{\partial x^4} + 2H \frac{\partial^4 w_m}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w_m}{\partial y^4} \right] dA_p + \dot{q}_m \left[K \int_{A_p} w_m^2 \, dA_p \right]$$

$$+ \ddot{q}_m \left[\mu_p \int_{A_p} w_m^2 \, dA_p \right] = \int_{A_p} w_m P(x,y,t) \, dA_p \qquad \text{[A.7a]}$$

or

$$\ddot{q}_m + \dot{q}_m \left\{ \frac{K \int_{A_p} w_m^2 \, dA_p}{\mu_p \int_{A_p} w_m^2 \, dA_p} \right\} + q_m \left\{ \frac{\int_{A_p} w_m \left[D_x \frac{\partial^4 w_m}{\partial x^4} + 2H \frac{\partial^4 w_m}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w_m}{\partial y^4} \right]}{\mu_p \int_{A_p} w_m^2 \, dA_p} \right\}$$

$$= \frac{\int_{A_p} w_m P(x,y,t) \, dA_p}{\mu_p \int_{A_p} w_m^2 \, dA_p} \qquad \text{[A.7b]}$$

* Equation [A.4] holds for all $P(x,y,t)$, hence, for $P(x,y,t) = 0$. But when $P(x,y,t) = 0$, Equation [A.4] has a nonzero value only for $r = m$, which relationship must hold for all $P(x,y,t)$.

For free vibration, the equation takes the form:

$$\ddot{q}_m + 2n\dot{q}_m + p_m^2 \; q_m = 0 \qquad [A.7c]$$

where

$$2n = \frac{K}{\mu_p} \qquad [A.7d]$$

The ratio of damping to critical damping can be written:

$$\frac{C}{C_c} = \frac{n}{p_m} \qquad [A.7e]$$

Thus

$$K = 2\mu_p p_m \frac{C}{C_o} = \mu_p \frac{\delta}{\pi} p_m \qquad [A.7f]$$

To determine K, a value of $\frac{C}{C_c}$ must be assumed, calculated, or measured and p_m calculated, where

$$= \sqrt{\frac{\int_{A_p} w_m \left[D_x \frac{\partial^4 w_m}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w_m}{\partial y^4} \right] dA_p}{\mu_p \int_{A_p} w_m^{\;2} dA_p}}$$

or the logarithmic decrement of ship plating can be used. (Reference 20 gives $\delta \approx 0.002$ for steel).

The effect of the water on the plate is introduced into the equations of motion through the function P(x,y,t). The pressure P(x,y,t) is divided into two parts:

1. the fluid pressure due to plate motion,* and
2. the forcing pressure $\overline{P}(x,y,t)$.

* The fluid pressure is opposite in direction but not equal to the forcing pressure.

The fluid pressure due to plate motion is

$$- \overline{P}_m (x,y) \, q_m (t)$$

where $\overline{P}_m(x,y)$ is the pressure distribution on the plates vibrating in *any mode* and $q_m(t)$ is defined as before. $\overline{P}_m(x,y)$ is now written in terms of impedance functions.* First divide the plate (Figure 4) into equal elemental areas and let $(\overline{P}_m)_{\gamma\delta}$ be the average pressure on the γ^{th} elemental area A_γ due to modal vibration of the δ^{th} elemental area A_δ; $[w_\delta(x,y)]_m$ is the deflection amplitude (i.e., amplitude maximum in time) of the m^{th} mode of vibration. This average pressure can be determined in terms of the mutual radiation impedance as follows:[21]

* It is assumed that the water impedance calculated for sinusoidal motion can be used for the slamming problem. This seems logical since we know that the effect of water is to add, as a complex quantity (i.e., impedance), virtual mass and damping to the system so that if motions were sinusoidal only, the results would be exact. Furthermore, the water pressure for sinusoidal motion can be expressed as the sum of two components, one depending on $\ddot{q}_m$ (the virtual mass component) and one depending on q_m (the radiation damping component). The fact that the pressure can be expressed in such general form for sinusoidal motions leads one to believe that the result should be approximately true for other motions. The impedance components θ and χ being, strictly speaking, only defined for sinusoidal motions, are functions of frequency. However, θ and χ will be approximated here by average values for each mode independent of the forcing frequency; since θ and χ depend on plate size they vary for each plate. If, in light of the predicted results, a better approximation seems warranted, further investigation of an improved representation for θ and χ should be made.

Let

$$[\overline{w_\delta(x,y,t)}]_m = [w_\delta(x,y)]_m e^{ip_m t}$$

$$[\overline{\dot{w}_\delta(x,y,t)}]_m = ip_m\ [w_\delta(x,y)]_m e^{ip_m t} = ip_m\ [\overline{w_\delta(x,y,t)}]_m$$

$$(Z_{\gamma\delta})_m = \frac{\text{Force}}{\text{Velocity}}\quad \frac{[P_m(x,y,t)]_{\gamma\delta}\ A_\gamma}{[\overline{\dot{w}}(x,y,t)]_m}$$

$$= \frac{[\overline{P}_m(x,y)]_{x\delta} e^{ip_m t}\ A_\gamma}{ip_m[w_\delta(x,y)]_m e^{ip_m t}}$$

$$= \frac{[\overline{P}_m(x,y)]_{\gamma\delta}}{[\dot{w}_\delta(x,y,t)]_m}\ A_\gamma \qquad [A.8]$$

where

$$[\dot{w}(x,y,t)]_m = ip_m\ [w_\delta(x,y)]_m$$

is the velocity amplitude (i.e., maximum in time) of the elemental area in the m^{th} mode of vibration. Thus

$$[\overline{P}_m(x,y)]_{\gamma\delta} = \frac{(Z_{\gamma\delta})_m\ [\dot{w}_\delta(x,y,t)]_m}{A_\gamma} \qquad [A.9]$$

We can also write:[21]

$$(Z_{\gamma\delta})_m = \rho_o C_o A_\gamma\ [(\theta_{\gamma\delta})_m + i\ (\chi_{\gamma\delta})_m] \qquad [A.10]$$

where ρ_o is the density of water,

C_o is the velocity of sound in water, and

$(\theta_{\gamma\delta})_m$ and $(\chi_{\gamma\delta})_m$ are the resistive (radiation damping) and reactive (virtual mass) components of the impedance for the m^{th} mode, respectively.* Then the total pressure on the γ^{th} area due to all other elements vibrating in the m^{th} mode from Equations [A.9] and [A.10] is:

$$[\overline{P}_m(x,y)]_{\gamma\delta} = \sum_{\delta} \rho_o C_o \, [(\theta_{\gamma\delta})_m + i(\chi_{\gamma\delta})_m] \, ip_m \, [w_\delta(x,y)]_m \qquad [A.11]$$

Let

$$q_m(t) = e^{ip_m t}, \; \dot{q}_m(t) = ip_m e^{ip_m t} = ip_m q_m(t), \; \ddot{q}_m(t) = -p_m^2 e^{ip_m t} = -p_m{}^2 q_m(t)$$

so that

$$\frac{\ddot{q}_m(t)}{p_m} = - p_m q_m(t)$$

Therefore, the fluid pressure due to plate motion is:

$$- [\overline{P}_m(x,y)] \, q_m(t) = -\sum_{\delta} \left\{ \rho_o C_o \left[\dot{q}_m(t) \, (\theta_{\gamma\delta})_m + \frac{\ddot{q}_m(t)(\chi_{\gamma\delta})_m}{p_m} \right] [w_\delta(x,y)]_m \right\} [A.12]$$

Substituting $P(x,y,t) = \overline{P}(x,y,t)$ + the right-hand member of Equation [A.12] into Equation [A.7a], we have:

$$q_m \int_{A_p} w_m \nabla_o^4 w_m + \dot{q}_m K \int_{A_p} w_m^2 dA_p + \dot{q}_m \rho_o C_o \int_{A_p} w_m \left[\sum_{\delta} (\theta_{\gamma\delta})_m (w_\delta)_m \right] dA_p$$

[A.13]

$$+ \ddot{q}_m \mu_p \int_{A_p} w_m^2 dA_p + \ddot{q}_m \rho_o C_o \int_{A_p} w_m \left[\sum_{\delta} \frac{(\chi_{\gamma\delta})_m}{p_m} (w_\delta)_m \right] dA_p = \int_{A_p} \overline{P}(x,y,t) w_m dA_p$$

* $(\theta_{\gamma\delta})_m$ and $(\chi_{\gamma\delta})_m$ of a given plate are included in slamming calculations *only when that plate is submerged* (see Appendix D).

where

$$\nabla_o^4 w_m = D_x \frac{\partial^4 w_m}{\partial x^4} + 2H \frac{\partial^4 w_m}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w_m}{\partial y^4}$$

Equation [A.13] can be put into the following form:

$$\ddot{q}_m + \dot{q}_m \left\{ \frac{\rho_o C_o \int_{A_p} w_m \left[\sum_\delta (\theta_{\gamma\delta})_m (w_\delta)_m \right] dA_p + K \int_{A_p} w_m^2 dA_p}{\mu_p \int_{A_p} w_m^2 dA_p + \rho_o C_o \int_{A_p} w_m \left[\sum_\delta \frac{(\chi_{\gamma\delta})_m}{p_m} (w_\delta)_m \right] dA_p} \right\}$$

$$+ q_m \left\{ \frac{\int_{A_p} w_m \nabla_o^4 w_m dA_p}{\mu_p \int_{A_p} w_m^2 dA_p + \rho_o C_o \int_{A_p} w_m \left[\Sigma \frac{(\chi_{\gamma\delta})_m}{p_m} (w_\delta)_m \right] dA_p} \right\}$$

$$= \frac{\int_{A_p} w_m \overline{P}(x,y,t) dA_p}{\mu \int_{A_p} w_m^2 dA_p + \rho_o C_o \int_{A_p} w_m \left[\sum_\delta \frac{(\chi_{\gamma\delta})_m}{p_m} (w_\delta)_m \right] dA_p} \qquad [A.14]$$

Equation [A.14], which can be written in abbreviated form as:

$$\ddot{q}_m + 2n_m \dot{q}_m + p_m^2 q_m = A_m(t) \qquad [A.15]$$

has the general solution:[22]

$$q_m(t) = e^{-n_m t}(C_{1m}\cos p_{1m}t + C_{2m}\sin p_{1m}t)$$

$$+p_{1m}\int_o^t \frac{A_m(\xi)}{p_{1m}^2} e^{-n_m(t-\varepsilon)} \sin p_{1m}(t-\xi)\, d\xi \qquad [A.16]$$

where $2n_m$, the damping constant which depends on the radiation damping and structural damping, is given by

$$2n_m = \left\{ \frac{\rho_o C_o \int_{A_p} w_m \left[\sum_\delta (\theta_{\gamma\delta})_m (w_\delta)_m \right] dA_p + K \int_{A_p} w_m^2 dA_p}{\mu_p \int_{A_p} w_m^2 dA_p + \rho_o C_o \int_{A_p} w_m \left[\sum_\delta \frac{(\chi_{\gamma\delta})_m}{p_m} (w_\delta)_m \right] dA_p} \right\} \qquad [A.17]$$

and p_m, the natural frequency of the vibrating plate in water, is given by the following transcendental equation (note that $(\chi_{\gamma\delta})_m$ and $(\theta_{\gamma\delta})_m$ are functions of frequency p_m):

$$p_m = \sqrt{\frac{\int_{A_p} w_m \nabla_o^4 w_m dA_p}{\mu_p \int_{A_p} w_m^2 dA_p + \rho_o C_o \int_{A_p} w_m \left[\sum_\delta \frac{(\chi_{\gamma\delta})_m}{p_m} (w_\delta)_m \right] dA_p}} \qquad [A.18a]$$

or

$$p_m^2 \mu \int w_m^2 dA_p + p_m \rho_o C_o \int_{A_p} w_m \left[\Sigma (\chi_{\gamma\delta})_m (w_\delta)_m \right] dA_p - \int_{A_p} w_m \nabla_o^4 w_m dA_p = 0 \quad [A.18b]$$

and

$$p_{1m} = \sqrt{p_m^2 - n_m^2} \qquad [A.19]$$

and

$$A_m(t) = \frac{\int_{A_p} w_m \overline{P}(x,y,t)\, dA_p}{\mu \int_{A_p} w_m^2 dA_p + \rho_o C_o \int_{A_p} w_m \left[\sum_\delta \frac{(\chi_{\gamma\delta})_m}{p_m} (w_\delta)_m \right] dA_p} \qquad [A.20]$$

and C_{1m} and C_{2m} are constants of integration.

The mode shape of the vibrating plate (in air or water) is assumed to be represented by a product of beam functions:[3,12*] (See paragraph following Equation [A.32].)

$$w_m = w_{ij}(x,y) = X_i(x) Y_j(y) \qquad [A.21]$$

where w_{ij} is the deflection in the ij^{th} mode of a rectangular plate,

$X_i(x)$, $Y_j(y)$ are the normal mode shapes of a uniform beam in air (assumed to be the same as the mode shape in water), and

$i = 1$, $j = 1$ for the first mode,

$i = 1$, $j = 2$ for the second mode, etc.

In assuming Equation [A.21] to hold, we consider only those modes whose nodal lines are parallel to the sides of the plate. This means that as the plate geometry approaches a square the errors in this theory will be

* See Reference 3 for a discussion of the limitations of this assumption.

greater than for a rectangle because certain modes for the square are being neglected, i.e., the modes with the circular nodal lines. Furthermore, the actual modal patterns for a rectangular plate are somewhat more complicated than those represented by Equation [A.21]. Reference 3 shows comparisons between more exact static theories and the results using Equation [A.21]. The results indicate that the maximum error will be of the order of 20 per cent for the stress and within 10 per cent for the deflection of a stiffened or unstiffened plate.

The total deflection will then be:

$$w(x,y,t) = \sum_i \sum_j w_{ij}(x,y)\, q_{ij}(t) \qquad [A.22]$$

Now let the external load (not including the water load) be written as the following function of x,y,t where G_1, G_2, and f(t) are separated to provide convenience in integrating the equations in which $\overline{P}(x,y,t)$ is used:

$$\overline{P}(x,y,t) = G_1(x)G_2(y)f(t) \qquad [A.23]$$

where G_1, G_2, and f are polynomials of the form:

$$G_1(x) = a_o + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots + a_nx^n \qquad [A.24a]$$

$$G_2(y) = b_o + b_1y + b_2y^2 + \ldots\ldots\ldots\ldots + b_ny^n \qquad [A.24b]$$

$$f(t) = c_o + c_1t + c_2t^2 \ldots\ldots\ldots\ldots\ldots + c_nt^n \qquad [A.24c]$$

G_1 and G_2 are dimensionless but f(t) includes the pressure amplitude and, therefore, has the dimensions of pressure. Using this form to represent the load will greatly facilitate the calculations. It is believed that this general type of representation will approximate slamming loads quite accurately. By choosing the a_i, b_i, c_i appropriately, these functions can be used to fit an arbitrarily selected load (when the actual load is unknown). If, however, the actual slamming load is known, the functions G_1, G_2, and $f(t)$ need not be found because the actual numerical values of the slamming load at each point, at each time, can be used directly.

Substitution of Equation [A.23] into Equation [A.7a] or [A.7b] gives the equation of motion which has the general solution given by Equation [A.16]. If $\overline{P}(x,y,t)$ represents an arbitrary transient load having a finite rise time to which the plate is subjected, the initial conditions for this type of loading are (since the plate is initially at rest and undeflected):

$$\text{at } t = 0 \; w = 0, \; \frac{dw}{dt} = 0 \qquad [A.25]$$

For these conditions, Equation [A.16] gives:

$$C_{1m} = C_{2m} = 0 \qquad [A.26]$$

Substitution of Equation [A.21] into Equations [A.16] [A.17] [A.18] [A.19], and [A.20], with m replaced by ij and $dA_p = dxdy$, yields:

$$q_{ij}(t) = \frac{k_{ij}{}^2\left[\int_0^a G_1(x)X_i\,dx\right]\left[\int_0^b G_2(y)Y_j\,dy\right]}{\mu_p(p_{ij})_1^2\left[\int_0^a X_i{}^2dx\right]\left[\int_0^b Y_j^2dy\right]}R_{ij}(t) \qquad [A.27]$$

where

$$(p_{ij})_1^2 = p_{ij}^2 - n_{ij}^2 \qquad [A.28]$$

and*

$$p_{ij} = \frac{k_{ij}}{a^2}\sqrt{\frac{D_x}{\mu_p}}\sqrt{\beta_i^4 + \frac{D_y}{D_x}\frac{\beta_j^4}{b^4/a^4} + \frac{H}{D_x}\frac{2a^2b^2}{b^2/a^2}\frac{\left[\int_0^a X_i X_i'' dx\right]\left[\int_0^b Y_j Y_j''\right] dy}{\left[\int_0^a X_i^2 dx\right]\left[\int_0^b Y_j^2 dy\right]}} \qquad [A.29]$$

The beam mode functions are given by:

$$X_i = \sin\frac{\beta_i x}{a} + \frac{C_2}{C_1}\cos\frac{\beta_i x}{a} + \frac{C_3}{C_1}\sinh\frac{\beta_i x}{a} + \frac{C_4}{C_1}\cosh\frac{\beta_i x}{a}$$

$$Y_j = \sin\frac{\beta_j y}{b} + \frac{C_2'}{C_1'}\cos\frac{\beta_j y}{b} + \frac{C_3'}{C_1'}\sinh\frac{\beta_j y}{b} + \frac{C_4'}{C_1'}\cosh\frac{\beta_j y}{b}$$

* Note that $\nabla_o^4 w_{ij} = D_x \frac{\partial^4 w_{ij}}{\partial x^4} + 2H \frac{\partial^4 w_{ij}}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w_{ij}}{\partial y^4} = D_x X_i^{IV} Y_j + 2HX_i'' Y_j'' + D_y X_i Y_j^{IV}$. But according to the Euler-Bernouilli beam equation, $X_i^{IV} = \frac{\beta_i^4}{a^4} x_i$ and $Y_j^{IV} = \frac{\beta_j^4}{b^4} Y_j$, so that $\nabla_o^4 w_{ij} = D_x \frac{\beta_i^4}{a^4} X_i Y_j + 2HX_i'' Y_j'' + D_y \frac{\beta_j^4}{b^4} X_i Y_j$ and $w_{ij}\nabla_o^4 w_{ij} = \frac{\beta_i^4}{a^4} D_x X_i^2 Y_j^2 + 2HX_i X_i'' Y_j Y_j''$ $\frac{\beta_j^4}{b^4} D_y X_i^2 Y_j^2$.

β_i and β_j are frequency numbers which depend upon the boundary conditions (see Reference 3). A frequency equation for determining $\beta_i \ell$ or $\beta_j \ell$ is given below for the rotationally constrained beam:

$$k_{ij} = \sqrt{\frac{1}{1 + \dfrac{\rho_o C_o \int_o^a \int_o^b X_i Y_j \left[\sum_\delta \dfrac{(\chi_{\gamma\delta})_{ij}}{p_{ij}} (w_\delta)_{ij} \right] d_x d_y}{\mu \int_o^a \int_o^b w_{ij}^2 d_x d_y}}} \qquad [A.30]$$

Equations [A.29] and [A.30] actually form a transcendental equation for p_{ij}. However, in some cases $(\chi_{\gamma\delta})_{ij}$ can be written as a linear function of p_{ij} (see Reference 12). In this event Equation [A.29] is an explicit formula for p_{ij}.

$$n_{ij} = \frac{1}{2} \left\{ \frac{\rho_o C_o \int_{A_p} w_{ij} \left[\sum_\delta (\theta_{\gamma\delta})_{ij} (w_\delta)_{ij} \right] dA_p + K \int_{A_p} w_{ij}^2 dA_p}{\mu_p \int_{A_p} w_{ij}^2 dA_p + \rho_o C_o \int_{A_p} w_{ij} \left[\sum_\delta \dfrac{(\chi_{\gamma\delta})_{ij}}{p_{ij}} (w_\delta)_{ij} \right] dA_p} \right\} \qquad [A.31]$$

$$R_{ij}(t) = \left(p_{ij}\right)_1 \int_o^t e^{-n_{ij}(t-\xi)} \sin (p_{ij})_1 (t-\xi) f(\xi) d\xi \qquad [A.32]$$

For

$$\rho_o C_o \int_{A_p} w_m \left[\sum \frac{(\chi_{\gamma\delta})_m}{p_m} (w_\delta)_m \right] dA_p$$

and

$$\rho_o C_o \int_{A_p} w_m \left[\Sigma \ (\theta_{\gamma\delta})_m \ (w_\delta)_m \right] dA_p$$

refer to Reference 12 Equations [43], [44], [49], [56], [57], [61], and [62] and addendum to Reference 5 which uses the low frequency approximation as follows:

(In particular, $\chi_p = 0.48k[ab]^{\frac{1}{2}}$ and $\theta_p = 0.13k^2ab = 0.13\frac{p_m^{\ 2}}{C_o^{\ 2}}$ (ab) when

$$\frac{p_m\sqrt{\frac{ab}{\pi}}}{C_o} < 1$$

ρ_o = mass density of water $\left(64.4 \ \frac{\text{lb.}}{\text{ft}^3} \times \frac{1}{1728} \frac{\text{ft}^3}{\text{in.}^3} \times \frac{1}{386} \frac{\text{sec}^2}{\text{in.}} = 0.0000966 \ \frac{\text{lb. sec}^2}{\text{in.}^4}\right)$

C_0 = sound velocity in water (5000 ft/sec)

$$\rho_o \epsilon_o \int_{A_p} w_m \left[\frac{\Sigma (\chi_{\gamma\delta})_m}{p_m} \left(w_\delta\right)_m \right] dA_p = \rho_o C_o \frac{\chi \text{ piston}}{p_m} \left[w_m\right]_{av}^2 A_p$$

$$=(\rho_o C_o)(\frac{0.48}{p_m} \frac{p_m}{C_o}\sqrt{ab} \ A_{ij}^{\ 2}ab)$$

$$= \rho_o \left[0.48 \ (ab)^{3/2} \ A_{ij}^{\ 2}\right]$$

where

$$A_{ij} = \frac{\int_{A_p} w_m dA_p}{ab} = \frac{\int_o^a X_i dx \int_o^b Y_j dy}{ab} = \left[w_m\right]_{av} ; \ A_{ij}^{\ 2}ab = \left[w_m\right]_{av}^{\ 2} A_p$$

X_i and Y_j are the mode shapes for a plate mode represented by $w_{ij} = X_i Y_j$. For simply supported and clamped plates, A_{ij} is given in the table on page 31 for several modes.[12]

$$\rho_o C_o \int_{A_p} w_m \left[\Sigma(\theta_{\gamma\delta}) \ (w_\delta)_m \right] dA_p = \rho_o C_o \theta_{\text{Piston}} \left[w_m \right]^2_{av} A_p$$

$$= (\rho_o C_o)(0.13 \frac{p_m^2}{C_o^2} ab \ A_{ij}^2 ab)$$

$$= (\frac{\rho_o}{C_o})(0.13 \ p_m^2 \ A_{ij}^2 a^2 b^2)$$

Also contained in the analysis is a factor

$$\int_{A_p} w_m^2 dA_p$$

This is calculated as follows:

$$\int_{A_p} w_m^2 dA_p = \int_o^a X_i^2 dx \int_o^b Y_j^2 dy = \beta_{ij} ab$$

or

$$\beta_{ij} = \frac{\int_o^a X_i^2 dx \int_o^b Y_j^2 dy}{ab}$$

For simply supported and clamped plates, β_{ij} is given in the following tabulation:

Mode	A_{ij}	Clamped Plate β_{ij}	Simply Supported Plate	
			A_{ij}	β_{ij}
i = 1 j = 1	0.6904	1	0.4053	0.25
i = 1 j = 2	0	1	0	0.25
i = 1 j = 3	0.3023	1	0.1351	0.25
i = 1 j = 5	0.1924	1	0.0810	0.25
i = 3 j = 1	0.3023	1	0.1351	0.25
i = 3 j = 2	0	1	0	0.25
i = 3 j = 3	0.1324	1	0.0450	0.25

Values for the rotationally constrained plate will be calculated in the future; however, for the time being values of A_{ij} and β_{ij} for the clamped plate will be used for any immediate calculations.

It should be noted that the integral involving θ is frequency dependent; the one involving X is not frequency dependent. This means that the natural frequency in water for the m^{th} mode is first computed, and this frequency is used in the radiation damping term for calculation in that mode.

The q_{ij} and w_{ij} depend upon the *beam* functions that represent the mode shapes.* The beam functions depend, in turn, upon the boundary

* The multiple of beam functions does not generally satisfy the plate equations and is therefore only an approximate expression for the plate modes. The approximation, however, gives good results.[3] For the special case of the plate simply supported at all edges, the solution is exact.

conditions of the *plate*. For this analysis, we shall prescribe elastic rotational constraints on the boundaries of the *plate*;[23] hence, the X_i, Y_j are written for a *beam* with rotational constraint.* The simply suppported and clamped *plates* are special cases of the rotationally constrained *plate* so that their modes are also represented by these equations for X_i and Y_j.

The beam functions X_i and Y_j are the standard solution to the Euler-Bernouilli Equation $EI \frac{\partial^2 w}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} = 0$. These solutions are given by Equation [A.29a]. The boundary conditions satisfied by a beam with rotational constraint are:

$$X_i = 0 \; ; \quad EI \frac{d^2 X_i}{dx^2} = \alpha \frac{dX_i}{dx} \text{ at } X = 0 \qquad \text{[A.33a]}$$

$$X_i = 0 \; ; \quad EI \frac{d^2 X_i}{dx^2} = -\alpha \frac{dX_i}{dx} \text{ at } X = l \qquad \text{[A.33b]}$$

Substituting X_i from Equation [A.29a] into the above boundary conditions, we obtain four homogeneous equations in $C_1 \ldots C_4$. For a non-trivial solution, the determinant of the constants must vanish. Forming this determinant and letting the frequency numbers $\beta_i = \theta$, the following frequency equation is obtained:

$$\frac{EI}{\alpha l} = \frac{1}{2\beta_i} \left[-\frac{1}{\tanh \beta_i} + \frac{1}{\tan \beta_i} + \left(\frac{1}{\sinh \beta_i} - \frac{1}{\sin \beta_i} \right) \right] \qquad \text{[A.33c]}$$

* See Equations [1] through [5] of Reference 23.

where α is associated with the rotational constraint.[23] Substituting β_i back into the four homogeneous equations, we find that the ratios of the constants are:

$$\frac{C_4}{C_1} = \frac{\sinh \beta_i - \sin \beta_i}{2 \frac{EI}{\alpha l} \beta_i \sinh \beta_i + \cosh \beta_i - \cos \beta_i} \qquad [A.34a]$$

$$\frac{C_3}{C_1} = 2 \frac{EI}{\alpha l} \beta_i \frac{C_4}{C_1} - 1 \qquad [A.34b]$$

$$\frac{C_2}{C_1} = - \frac{C_4}{C_1} \qquad [A.34c]$$

In the application of these equations to plates, the $\frac{EI}{\alpha l}$ will not be used as such. Instead (see Appendix D) lumped rotational constraint parameters $\bar{\alpha}_i = \frac{EI}{\alpha_i l}$, $\bar{\alpha}_j = \frac{EI}{\alpha_j l}$ will be employed, and these will be estimated by using results of experiments. Specifically, they can be determined by finding the frequency of the first mode of the plate experimentally and then fitting the proper α parameter as shown in Reference 23.

APPENDIX B

DERIVATION OF METHOD FOR CALCULATING THE ORTHOTROPIC CONSTANTS

For convenience of reference a definition of the symbols used in this Appendix is given here.

E	Modulus of elasticity in tension and compression for plate material
G	Modulus of elasticity in shear for plate material
I	Area moment of inertia of cross section of a beam
i_{w_x}, i_{w_y}	Moments of inertia of web about neutral planes of equivalent orthotropic plate
i_x', i_y', i_x'', i_y'', i_{xy}, i_{yx}	Defined by Equations [B.15] through [B.20]
M_x, M_y	Bending moments per unit length of section of a plate perpendicular to x and y axes, respectively
M_{xy}, M_{yx}	Twisting moments per unit length of section of a plate perpendicular to x and y axes, respectively
m_{w_x}, m_{w_y}	Unit bending moments in webs in two coordinate directions x and y, respectively
$r_{n'x}$, $r_{n'y}$	Scalar distance of centroid of near flange from neutral axis of bending in x or y directions, respectively
$r_{f'x}$, $r_{f'y}$	Scalar distance of centroid of far flange from neutral axis of bending in x or y directions, respectively

S_x, S_y	Distance between repeating sections in y and x directions, respectively
t_{nx}, t_{ny}	Thickness of near flanges (see Figures 4 and 5)
t_{fx}, t_{fy}	Thickness of far flanges (see Figures 4 and 5)
V	Total energy per unit area in a repeating section of a plate
V_o	Bending energy per unit area of an orthotropic plate
V_p	Bending and twisting energy per unit area in a repeating section of a plate
V_w	Energy in the webs per unit area for a repeating section of a plate
$w(x,y,t)$	Lateral deflection of plate
x,y,z	Rectangular coordinate axes
$\varepsilon_{n'x}$, $\varepsilon_{n'y}$	Unit elongations for the near flange in the x and y directions, respectively
$\varepsilon_{f'x}$, $\varepsilon_{f'y}$	Unit elongations for the far flange in the x and y directions, respectively
ν	Poisson's ratio for plate material
σ_x, σ_y	Bending stresses in x and y directions, respectively
τ_{xy}, τ_{yx}	Shear stress in plate

Expressions for the orthotropic bending constants of Equation [1] will be derived by equating the bending and twisting energy in a repeating section of a stiffened plate to the energy in an equivalent orthotropic plate[8] (see Figure 5). Basic expressions used here are found in Reference 7.

1. Energy in the Webs

Assuming that the webs of stiffeners act as simple beams in bending, the unit bending moments in the webs in the two coordinate directions x and y are:*

$$m_{w_x} = Ei_{w_x} \frac{\partial^2 w}{\partial x^2} \quad ; \quad m_{w_y} = Ei_{w_y} \frac{\partial^2 w}{\partial y^2} \qquad [B.1]$$

The energy in the webs of a single repeating section is then:

$$V_w = \frac{1}{2} m_{w_x} \left(\frac{\partial^2 w}{\partial x^2} \right) S_y + \frac{1}{2} m_{w_y} \left(\frac{\partial^2 w}{\partial y^2} \right) S_x$$

$$= \frac{1}{2} Ei_{w_x} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 S_y + \frac{1}{2} Ei_{w_y} \left(\frac{\partial^2 w}{\partial y^2} \right)^2 S_x \qquad [B.2]$$

2. Energy in the Plating

Referring to Figure 5, the basic expressions for the bending and twisting moments are (primes denote distance from neutral plane to center plane of flange):

$$M_x = \int_{r_{f'x} - \frac{t_{fx}}{2}}^{r_{f'x} + \frac{t_{fx}}{2}} \sigma_x \, z dz + \int_{r_{n'x} - \frac{t_{nx}}{2}}^{r_{n'x} + \frac{t_{nx}}{2}} \sigma_x \, z dz \qquad [B.3]$$

* Reference 5 uses the opposite sign convention for curvature.

$$M_y = \int_{r_{f'y} - \frac{t_{fy}}{2}}^{r_{f'y} + \frac{t_{fy}}{2}} \sigma_y \, zdz + \int_{r_{n'y} - \frac{t_{ny}}{2}}^{r_{n'y} + \frac{t_{ny}}{2}} \sigma_y \, zdz \qquad [B.4]$$

$$M_{xy} = \int_{r_{f'x} - \frac{t_{fx}}{2}}^{r_{f'x} + \frac{t_{fx}}{2}} \tau_{xy} \, zdz + \int_{r_{n'x} - \frac{t_{nx}}{2}}^{r_{n'x} + \frac{t_{nx}}{2}} \tau_{xy} \, zdz \qquad [B.5]$$

$$M_{yx} = \int_{r_{f'y} - \frac{t_{fy}}{2}}^{r_{f'y} + \frac{t_{fy}}{2}} \tau_{yx} \, zdz + \int_{r_{n'y} - \frac{t_{ny}}{2}}^{r_{n'y} + \frac{t_{ny}}{2}} \tau_{yx} \, zdz \qquad [B.6]$$

Substitute the following into Equation [B.3] (and analogously into Equation [B.4]:

For the far flange (first term in the right-hand member)

$$\sigma_x = \frac{z}{r_{f'x}} \sigma_{f'x} = \frac{z}{r_{f'x}} \left[\frac{E}{1-\nu^2} \left(\varepsilon_{f'x} + \nu \varepsilon_{f'y} \right) \right] \qquad [B.7]$$

$$= \frac{z}{r_{f'x}} \left[\frac{E}{1-\nu^2} \left(r_{f'x} \frac{\partial^2 w}{\partial x^2} + \nu r_{f'y} \frac{\partial^2 w}{\partial y^2} \right) \right]$$

For the near flange (second term in the right-hand member):

$$\sigma_x = \frac{z}{r_{n'x}} \sigma_{n'x} = \frac{z}{r_{n'x}} \left[\frac{E}{1-\nu^2} \left(\varepsilon_{n'x} + \nu \varepsilon_{n'y} \right) \right]$$

$$= \frac{z}{r_{n'x}} \left[\frac{E}{1-\nu^2} \left(r_{n'x} \frac{\partial^2 w}{\partial x^2} + \nu r_{n'y} \frac{\partial^2 w}{\partial y^2} \right) \right] \qquad [B.8]$$

Substitute the following into Equation [B.5] (and analogously into Equation [B.6]):

For the far flange[24] * (first term in the right-hand member):

$$\tau_{xy} = \frac{z}{r_{f'x}} \left(\tau_{xy} \right)_{f'x} = \frac{z}{r_{f'x}} \left[G \left(r_{f'x} + r_{f'y} \right) \frac{\partial^2 w}{\partial x \partial y} \right] \qquad [B.9]$$

For the near flange[24] (second term in the right-hand member)

$$\tau_{xy} = \frac{z}{r_{n'x}} \left(\tau_{xy} \right)_{n'x} = \frac{z}{r_{n'x}} \left[G \left(r_{n'x} + r_{n'y} \right) \frac{\partial^2 w}{\partial x \partial y} \right] \qquad [B.10]$$

* A linear distribution is assumed in the present report. Reference 18 only considers shear in flanges, $z = r_{n'x}$ or $z = r_{f'x}$.

This yields after integration of Equations [B.3], [B.4], [B.5], and [B.6] and collection of terms (note $G = \frac{E}{2(1+\nu)}$):

$$M_x = \frac{E}{1-\nu^2}\left[i_x' \frac{\partial^2 w}{\partial x^2} + \nu i_x'' \frac{\partial^2 w}{\partial y^2} \right] \qquad [B.11]$$

$$M_y = \frac{E}{1-\nu^2}\left[i_y' \frac{\partial^2 w}{\partial y^2} + \nu i_y'' \frac{\partial^2 w}{\partial x^2} \right] \qquad [B.12]$$

$$M_{xy} = \frac{E}{2(1+\nu)}\left(i_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \qquad [B.13]$$

$$M_{yx} = \frac{E}{2(1+\nu)}\left(i_{yx} \frac{\partial^2 w}{\partial x \partial y} \right) \qquad [B.14]$$

where

$$i_x' = t_{fx} r_{f'x}^2 + t_{nx} r_{n'x}^2 + \frac{t_{fx}^3}{12} + \frac{t_{nx}^3}{12} \qquad [B.15]$$

$$i_y' = t_{fy} r_{f'y}^2 + t_{ny} r_{n'y}^2 + \frac{t_{fy}^3}{12} + \frac{t_{ny}^3}{12} \qquad [B.16]$$

$$i_x'' = t_{fx} r_{f'x} r_{f'y} + t_{nx} r_{n'x} r_{n'y} + \frac{r_{f'y}}{r_{f'x}} \frac{t_{f'x}^3}{12} + \frac{r_{n'y}}{r_{n'x}} \frac{t_{ny}^3}{12} \qquad [B.17]$$

$$i_y'' = t_{fy} r_{f'x} r_{f'y} + t_{ny} r_{n'x} r_{n'y} + \frac{r_{f'x}}{r_{f'y}} \frac{t_{fy}^3}{12} + \frac{r_{n'x}}{r_{n'y}} \frac{t_{ny}^3}{12} \qquad [B.18]$$

$$i_{xy} = t_{fx} r^2_{f'x} + t_{nx} r^2_{n'x} + \frac{t^3_{fx}}{12} + \frac{t^3_{nx}}{12} + t_{fx} r_{f'y} r_{f'x}$$

[B.19]

$$+ t_{nx} r_{n'x} r_{n'y} + \frac{r_{f'y}}{r_{f'x}} \frac{t^3_{fx}}{12} + \frac{r_{n'y}}{r_{n'x}} \frac{t^3_{nx}}{12}$$

$$i_{yx} = t_{fy} r^2_{f'y} + t_{ny} r^2_{n'y} + \frac{t^3_{fy}}{12} + \frac{t^3_{ny}}{12} + t_{fy} r_{f'x} r_{f'y}$$

[B.20]

$$+ t_{ny} r_{n'y} r_{n'x} + \frac{r_{f'x}}{r_{f'y}} \frac{t^3_{f'y}}{12} + \frac{r_{n'x}}{r_{n'y}} \frac{t^3_{ny}}{12}$$

For a plate of constant thickness t without stiffeners $t_{fx} = t_{fy} = \frac{t}{2}$; $t_{nx} = t_{ny} = \frac{t}{2}$; $r_{n'x} = r_{n'y} = \frac{t}{4}$; $r_{f'x} = r_{f'y} = \frac{t}{4}$; $i_{w_x} = i_{w_y} = 0$. By the use of these expressions, the effect of the stiffeners on the plate can be determined. The bending and twisting energy per unit area in a repeating section of plating is then:

$$V_p = \frac{1}{2}\left[M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + M_{xy} \frac{\partial^2 w}{\partial x \partial y} + M_{yx} \frac{\partial^2 w}{\partial x \partial y}\right]$$

$$= \frac{1}{2}\left[\frac{E}{1-\nu^2}\left(i_x' \frac{\partial^2 w}{\partial x^2} + \nu i_x'' \frac{\partial^2 w}{\partial y^2}\right) \frac{\partial^2 w}{\partial x^2}\right.$$

$$+ \frac{E}{1-\nu^2}\left(i_y' \frac{\partial^2 w}{\partial y^2} + \nu i_y'' \frac{\partial^2 w}{\partial x^2} \right) \frac{\partial^2 w}{\partial y^2}$$

$$\left. + \frac{1}{2}\frac{E}{1+\nu}\left(i_{xy} + i_{yx}\right)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] \qquad [B.21]$$

The total energy per unit area in a repeating section of the plate obtained by summating Equations [B.2] and [B.21] is:*

$$V = V_p + V_w = \frac{1}{2}\left[\left(\frac{Ei_x'}{1-\nu^2} + \frac{Ei_{wx}}{S_x} \right)\left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right.$$

$$+ \left(\frac{Ei_y'}{1-\nu^2} + \frac{Ei_{wy}}{S_y} \right)\left(\frac{\partial^2 w}{\partial y^2} \right)^2$$

$$+ \left(\frac{\nu E}{1-\nu^2} i_x'' + \frac{\nu E}{1-\nu^2} i_y'' \right)\left(\frac{\partial^2 w}{\partial x^2} \right)\left(\frac{\partial^2 w}{\partial y^2} \right)$$

$$\left. + \frac{1}{2}\frac{E}{1+\nu}\left(i_{xy} + i_{yx}\right)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] \qquad [B.22]$$

---

* Note that Equation [B.2] is for a repeating section, whereas Equation [B.22] is per unit area.

This is equated to the bending energy per unit area of an orthotropic plate, which may be written as:

$$V_o = \frac{1}{2}\left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_1 \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_y\left(\frac{\partial^4 w}{\partial y^2}\right)^2 + 4D_{xy}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] \qquad [B.23]$$

to give the following expressions for the orthotropic constants. (Note: Equations [B.24] through [B.27] are more accurate than the corresponding equations for D_x, D_y, D_1, D_{xy} given in Reference 5.)

$$D_x = \frac{E}{1-\nu^2}\left[i_x' + \frac{1-\nu^2}{S_x} i_{w_x}\right] \qquad [B.24]$$

$$D_y = \frac{E}{1-\nu^2}\left[i_y' + \frac{1-\nu^2}{S_y} i_{w_y}\right] \qquad [B.25]$$

$$D_1 = \frac{\nu E}{1-\nu^2}\left[i_x'' + i_y''\right] \qquad [B.26]$$

$$D_{xy} = \frac{1}{2}\frac{E}{4(1+\nu)}\left[i_{xy} + i_{yx}\right] \qquad [B.27]$$

APPENDIX C

DERIVATION OF EQUATIONS OF STRESS FOR CROSS-STIFFENED PLATES SUBMERGED IN A FLUID AND SUBJECT TO ARBITRARY LOADING

The maximum bending and shearing stresses in the submerged cross-stiffened plate can be written thus:

$$\sigma_{fx} = \frac{E}{1-\nu^2}\left(r_{fx}\frac{\partial^2 w}{\partial x^2} + \nu r_{fy}\frac{\partial^2 w}{\partial y^2}\right) \qquad [C.1]$$

$$\sigma_{fy} = \frac{E}{1-\nu^2}\left(r_{fy}\frac{\partial^2 w}{\partial y^2} + \nu r_{fx}\frac{\partial^2 w}{\partial x^2}\right) \qquad [C.2]$$

$$\sigma_{nx} = \frac{E}{1-\nu^2}\left(r_{nx}\frac{\partial^2 w}{\partial x^2} + \nu r_{ny}\frac{\partial^2 w}{\partial y^2}\right) \qquad [C.3]$$

$$\sigma_{ny} = \frac{E}{1-\nu^2}\left(r_{ny}\frac{\partial^2 w}{\partial y^2} + \nu r_{nx}\frac{\partial^2 w}{\partial x^2}\right) \qquad [C.4]$$

$$(\tau_{xy})_f = G\ (r_{fx} + r_{fy})\frac{\partial^2 w}{\partial x \partial y} \quad ; \quad G = \frac{E}{2(1+\nu)} \qquad [C.5]$$

$$(\tau_{xy})_n = G\ (r_{nx} + r_{ny})\frac{\partial^2 w}{\partial x \partial y} \quad ; \quad G = \frac{E}{2(1+\nu)} \qquad [C.6]$$

The derivation of these equations proceeds from the general expressions for σ and ε (i.e., $\sigma_{fx} = \frac{E}{1-\nu^2}(\varepsilon_{f'x} + \nu\varepsilon_{f'y})$; $\sigma_{fy} = \frac{E}{1-\nu^2}(\varepsilon_{f'y} + \nu\varepsilon_{f'x})$; $\varepsilon_{f'x} = r_{f'x}\frac{\partial^2 w}{\partial x^2}$; $\varepsilon_{f'y} = r_{f'y}\frac{\partial^2 w}{\partial y^2}$; $\varepsilon_{n'x} = r_{n'x}\frac{\partial^2 w}{\partial x^2}$; $\varepsilon_{n'y} = r_{n'y}\left(\frac{\partial^2 w}{\partial y^2}\right)$; see also Chapter 11 of Reference 7.

APPENDIX D

EQUATIONS TO BE PROGRAMMED AND TOTAL INPUT TO PROGRAM

1. SUMMARY OF BASIC RELATIONSHIPS

We have previously shown that

$$w(x,y,t) = \sum_{i}\sum_{j} w_{ij}(x,y)\, q_{ij}(t)$$

$$\sigma_{fx} = \frac{E}{1-\nu^2}\left(r_{fx}\frac{\partial^2 w}{\partial x^2} + \nu r_{fy}\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_{fy} = \frac{E}{1-\nu^2}\left(r_{fy}\frac{\partial^2 w}{\partial y^2} + \nu r_{fx}\frac{\partial^2 w}{\partial x^2}\right)$$

$$\sigma_{nx} = \frac{E}{1-\nu^2}\left(r_{nx}\frac{\partial^2 w}{\partial x^2} + \nu r_{ny}\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_{ny} = \frac{E}{1-\nu^2}\left(r_{ny}\frac{\partial^2 w}{\partial y^2} + \nu r_{nx}\frac{\partial^2 w}{\partial x^2}\right) \qquad [D.1]$$

$$(\tau_{xy})_f = G\left(r_{fx} + r_{fy}\right)\frac{\partial^2 w}{\partial x \partial y}$$

$$(\tau_{xy})_n = G\left(r_{nx} + r_{ny}\right)\frac{\partial^2 w}{\partial x \partial y}$$

$$G = \frac{E}{2(1+\nu)}$$

$$\sigma_{fx} = \sum_i\sum_j \frac{E}{1-\nu^2} q_{ij}(t) \left(r_{fx}X_i''Y_j + \nu r_{fy}X_iY_j'' \right)$$

$$\sigma_{fy} = \sum_i\sum_j \frac{E}{1-\nu^2} q_{ij}(t) \left(r_{fy}X_iY_j'' + \nu r_{fx}X_i''Y_j \right)$$

$$\sigma_{nx} = \sum_i\sum_j \frac{E}{1-\nu^2} q_{ij}(t) \left(r_{nx}X_i''Y_j + \nu r_{ny}X_iY_j'' \right)$$

[D.2]

$$\sigma_{ny} = \sum_i\sum_j \frac{E}{1-\nu^2} q_{ij}(t) \left(r_{ny}X_iY_j'' + \nu r_{nx}X_i''Y_j \right)$$

$$(\tau_{xy})_f = \sum_i\sum_j G\,(r_{fx} + r_{fy})\, X_i'Y_j'\, q_{ij}(t)$$

$$(\tau_{xy})_n = \sum_i\sum_j G\,(r_{fx} + r_{fy})\, X_i'Y_j'\, q_{ij}(t)$$

where (from Equation [A.27]):

$$q_{ij}(t) = \frac{k_{ij}^{\;2}\left[\int_o^a G_1(x)X_i\,dx\right]\left[\int_o^b G_2(y)Y_j\,dy\right]}{\mu_p(p_{ij})_1^2\left[\int_o^a X_i^{\,2}dx\right]\left[\int_o^b Y_j^{\,2}dy\right]} R_{ij}(t)$$

$$X_i' = \frac{dX_i}{dx}$$

$$Y_j' = \frac{dY_j}{dy}$$

$$X_i'' = \frac{d^2X_i}{dx^2}$$

$$Y_j'' = \frac{d^2Y_j}{dy^2}$$

In the above equations, $f(\xi)$, $G_1(x)$, and $G_2(x)$ are given in terms of power series. The computer will fit a power series to those functions which are given numerically at points across the plate. If analytical functions can be given directly for $f(\xi)$, $G_1(x)$, and $G_2(x)$, then it will not be necessary to compute the series. Instead, the computer can evaluate the integrals with these functions directly.

The remaining parameters are given by Equations [A.28] through [A.34] with:

$$X_i' = \frac{\beta_i}{a}\left[\cos\frac{\beta_i x}{a} - \frac{C_2}{C_1}\sin\frac{\beta_i x}{a} + \frac{C_3}{C_1}\cosh\frac{\beta_i x}{a} + \frac{C_4}{C_1}\sinh\frac{\beta_i x}{a}\right] \qquad [D.3]$$

$$X_i'' = \frac{\beta_i^2}{a^2}\left[-\sin\frac{\beta_i x}{a} - \frac{C_2}{C_1}\cos\frac{\beta_i x}{a} + \frac{C_3}{C_1}\sinh\frac{\beta_i x}{a} + \frac{C_4}{C_1}\cosh\frac{\beta_i x}{a}\right]$$

Y_j', Y_j'' follow similarly.

2. SUMMARY OF EQUATIONS TO BE PROGRAMMED

To determine the response of and stresses in the plating to dynamic loads, a program can be established for calculating w, σ_{fx}, σ_{fy}, σ_{nx}, σ_{ny}, $(\tau_{xy})_f$, and $(\tau_{xy})_n$, where

$$w = \sum_i\sum_j X_i Y_j q_{ij}(t) \qquad [D.4a]$$

$$\sigma_{fx} = \frac{E}{1-\nu^2}\sum_i\sum_j \left(r_{fx}X_i'' Y_j + \nu r_{fy} X_i Y_j''\right) q_{ij}(t) \qquad [D.4b]$$

$$\sigma_{fy} = \frac{E}{1-\nu^2} \sum_{ij}\sum \left(r_{fy} X_i Y_j'' + \nu r_{fx} X_i'' Y_j \right) q_{ij}(t) \tag{D.4c}$$

$$\sigma_{nx} = \frac{E}{1-\nu^2} \sum_{ij}\sum \left(r_{nx} X_i'' Y_j + \nu r_{ny} X_i Y_j'' \right) q_{ij}(t) \tag{D.4d}$$

$$\sigma_{ny} = \frac{E}{1-\nu^2} \sum_{ij}\sum \left(r_{ny} X_i Y_j'' + \nu r_{nx} X_i'' Y_j \right) q_{ij}(t) \tag{D.4e}$$

$$(\tau_{xy})_f = G\,(r_{fx} + r_{fy}) \sum_{ij}\sum X_i' Y_j' q_{ij}(t) \tag{D.4f}$$

$$(\tau_{xy})_n = G\,(r_{nx} + r_{ny}) \sum_{ij}\sum X_i' Y_j' q_{ij}(t) \tag{D.4g}$$

where

$$q_{ij}(t) = k_{ij}^2 \frac{\left[\int_o^a G_1(x) X_i \, dx \right] \left[\int_o^b G_2(y) Y_j \, dy \right]}{\mu_p (p_{ij})_1^2 \left[\int_o^a X_i^2 \, dx \right] \left[\int_o^b Y_j^2 \, dy \right]} R_{ij}(t) \tag{D.4h}$$

$$R_{ij}(t) = (p_{ij})_1 \int_o^t e^{-n_{ij}(t-\xi)} \sin\,(p_{ij})_1\,(t-\xi)\, f(\xi)\, d\xi \tag{D.4i}$$

$$f(\xi) = f(t)$$

$$(p_{ij})_1 = \sqrt{p_{ij}^2 - n_{ij}^2} \tag{D.4j}$$

$$p_{ij} = \frac{k_{ij}}{a^2}\sqrt{\frac{D_x}{\mu_p}}\sqrt{\beta_i^4 + \frac{D_y}{D_x}\frac{\beta_j^4}{b^4/a^4} + \frac{H}{D_x}\frac{2a^2b^2}{b^2/a^2}\frac{\int_0^a\int_0^b X_i X_i'' Y_j Y_j''\, dxdy}{\int_0^a\int_0^b X_i^2 Y_j^2\, dxdy}} \qquad [D.4k]$$

$$H = D_1 + 2D_{xy} \qquad [D.4l]$$

$$n_{ij} = \frac{1}{2}\left\{\frac{\rho_o C_o \int_0^a\int_0^b X_i Y_j \left[\sum_\delta (\theta_{\gamma\delta})_{ij} (w_\delta)_{ij}\right] dxdy + K\int_0^a\int_0^b X_i^2 Y_j^2\, dxdy}{\mu_p \int_0^a\int_0^b X_i^2 Y_j^2\, dxdy + \rho_o C_o \int_0^a\int_0^b X_i Y_j \left[\sum_\delta \frac{(\chi_{\gamma\delta})_{ij}}{p_{ij}} (w_\delta)_{ij}\right] dxdy}\right\} \qquad [D.4m]$$

$$k_{ij} = \sqrt{\frac{1}{1 + \dfrac{\rho_o C_o \int_0^a\int_0^b X_i Y_j \left[\sum_\delta \frac{(\chi_{\gamma\delta})_{ij}}{p_{ij}} (w_\delta)_{ij}\right] dxdy}{\mu_p \int_0^a\int_0^b X_i^2 Y_j^2\, dxdy}}} \qquad [D.4n]$$

and where

$$X_i = \sin\frac{\beta_i x}{a} + \left(\frac{C_2}{C_1}\right)_i \cos\frac{\beta_i x}{a} + \left(\frac{C_3}{C_1}\right)_i \sinh\frac{\beta_i x}{a} + \left(\frac{C_4}{C_1}\right)_i \cosh\frac{\beta_i x}{a} \qquad [D.4o]$$

$$X_i' = \frac{\beta_i}{a}\left[\cos\frac{\beta_i x}{a} - \left(\frac{C_2}{C_1}\right)_i \sin\frac{\beta_i x}{a} + \left(\frac{C_3}{C_1}\right)_i \cosh\frac{\beta_i x}{a} + \left(\frac{C_4}{C_1}\right)_i \sinh\frac{\beta_i x}{a}\right] \quad [D.4p]$$

$$X_i'' = \frac{\beta_i^2}{a^2}\left[- \sin\frac{\beta_i x}{a} - \left(\frac{C_2}{C_1}\right)_i \cos\frac{\beta_i x}{a} + \left(\frac{C_3}{C_1}\right)_i \sinh\frac{\beta_i x}{a} + \left(\frac{C_4}{C_1}\right)_i \cosh\frac{\beta_i x}{a}\right] \quad [D.4q]$$

$$Y_j = \sin\frac{\beta_j y}{b} + \left(\frac{C_2}{C_1}\right)_j \cos\frac{\beta_j y}{b} + \left(\frac{C_3}{C_1}\right)_j \sinh\frac{\beta_j y}{b} + \left(\frac{C_4}{C_1}\right)_j \cosh\frac{\beta_j y}{b} \quad [D.4r]$$

$$Y_j' = \frac{\beta_j}{b}\left[\cos\frac{\beta_j y}{b} - \left(\frac{C_2}{C_1}\right)_j \sin\frac{\beta_j y}{b} + \left(\frac{C_3}{C_1}\right)_j \cosh\frac{\beta_j y}{b} + \left(\frac{C_4}{C_1}\right)_j \sinh\frac{\beta_j y}{b}\right] \quad [D.4s]$$

$$Y_j'' = \frac{\beta_j^2}{b^2}\left[- \sin\frac{\beta_j y}{b} - \left(\frac{C_2}{C_1}\right)_j \cos\frac{\beta_j y}{b} + \left(\frac{C_3}{C_1}\right)_j \sinh\frac{\beta_j y}{b} + \left(\frac{C_4}{C_1}\right)_j \cosh\frac{3_j y}{b}\right] \quad [D.4t]$$

where β_i and β_j satisfy the following frequency equations (let $\bar{\alpha}_i = \frac{EI}{\alpha l}$ for $\theta = \beta_i$; $\bar{\alpha}_j = \frac{EI}{\alpha l}$ for $\theta = \beta_j$):

$$\bar{\alpha}_i = \frac{1}{2\beta_i}\left[- \frac{1}{\tanh \beta_i} + \frac{1}{\tan \beta_i} + \left(\frac{1}{\sinh \beta_i} - \frac{1}{\sin \beta_i}\right)\right] \quad [D.4u]$$

$$\bar{\alpha}_j = \frac{1}{2\beta_j}\left[- \frac{1}{\tanh \beta_j} + \frac{1}{\tan \beta_j} + \left(\frac{1}{\sinh \beta_j} - \frac{1}{\sin \beta_j}\right)\right] \quad [D.4v]$$

$\bar{\alpha}_i$ and $\bar{\alpha}_j$ are assumed values of rotational constraint which will be determined from *experiment*. Also

$$\left(\frac{C_4}{C_1}\right)_i = \frac{\sinh\beta_i - \sin\beta_i}{2\bar{\alpha}_i\beta_i\sinh\beta_i + \cosh\beta_i - \cos\beta_i} \qquad [D.4x]$$

$$\left(\frac{C_3}{C_1}\right)_i = 2\bar{\alpha}_i\beta_i \left(\frac{C_4}{C_1}\right)_i - 1 \qquad [D.4y]$$

$$\left(\frac{C_2}{C_1}\right)_i = -\left(\frac{C_4}{C_1}\right)_i \qquad [D.4z]$$

$$\left(\frac{C_4}{C_1}\right)_j = \frac{\sinh\beta_j - \sin\beta_j}{2\bar{\alpha}_j\beta_j \sinh\beta_j + \cosh\beta_j - \cos\beta_j} \qquad [D.4aa]$$

$$\left(\frac{C_3}{C_1}\right)_j = 2\bar{\alpha}_j\beta_j \left(\frac{C_4}{C_1}\right)_j - 1 \qquad [D.4bb]$$

$$\left(\frac{C_2}{C_1}\right)_j = -\left(\frac{C_4}{C_1}\right)_j \qquad [D.4cc]$$

The roots β_i and β_j, in order, are the modes of the rotationally constrained beam. These will take the place of the values for clamped and simply supported plates described in References 5 and 25.

Equations for D_x, D_y, D_1, and D_{xy} (see Appendix B) are also to be programmed.

3. INPUT TO PROGRAM

To compute the D_x, D_y, D_1, and D_{xy} (see Appendix B), the required input data are E, ν, t_{fx}, t_{fy}, t_{nx}, t_{ny}, $r_{f'x}$, $r_{f'y}$, $r_{n'x}$, $r_{n'y}$, i_{w_x}, i_{w_y}, S_x, and S_y.

For computation of w, σ_{fx}, σ_{fy}, σ_{nx}, σ_{ny}, $\left(\tau_{xy}\right)_f$, and $\left(\tau_{xy}\right)_n$, we supply the following data in addition to the data just given:

$$a,\ b,\ \beta_i,\ \beta_j,\ k_{ij},\ \frac{\int_0^a \int_0^b X_i X_i'' Y_j Y_j'' \, dxdy}{\int_0^a \int_0^b X_i^2 Y_j^2 \, dxdy} \times \frac{2a^2b^2}{b^2/a^2},$$

$$n_{ij},\ \mu_p,\ \frac{\left[\int_0^a G_1(x) X_i \, dx\right]\left[\int_0^b G_2(y) Y_j \, dy\right]}{\left[\int_0^a X_i^2 dx \quad \int_0^b Y_j^2 dy\right]},\ \bar{\alpha}_i$$

$\bar{\alpha}_j$, $f(\xi)$, x, y (point at which stress is to be calculated).

The hull girder program[1,2] indicates the position of the forward plating with respect to the water. When the plating is submerged, the virtual mass χ and radiation damping θ of that plate are fully effective and are to be included in the calculations. When the plating is out of the water, these factors as well as the slamming load on the plate are zero. For plate slamming, only the forward 25 to 30 per cent of the ship need be considered because it appears that this is the area exposed to severe slamming pressures; see DISCUSSION. Additional experimental information on the subject is required.

As is evident from the equations given in Section 2 of this Appendix, some of the coefficients described above can be programmed in order to reduce the amount of calculations to be performed. In this case only basic data need be supplied.

All of the above input will be supplied to the programmer so that he can code the complete equations for the deflections and stresses over the surface of the plate. These deflections and stresses are given by Equations [D.4a] to [D.4g]. The output from the computer will be deflection and stress data versus time, which can be plotted for points on the plate specified in the input. Thus, the design engineer need specify only those points for which the deflections and stresses are desired.

APPENDIX E

PLASTIC DEFORMATION CONSIDERATIONS
(See Reference 26)

1. PRELIMINARY

For the slamming problem we are interested in determining (1) the stress in a member lying within the elastic region and (2) the permanent set (rather than stress) in a member caused by the onset of plasticity in the member.

A method for computing the stresses has been discussed previously. A method for computing the set in a member subject to plastic flow is the objective of this section. The treatment here is introductory; also it is elementary. A more basic approach for computing secondary and tertiary plastic deformations of stiffened and unstiffened plates is given in Appendix G. Analysis of a different phase of the plasticity problem is given by Nagai.[16,17,18]

a. General Discussion.

In treating stresses in ship plating, which is subject to large loads due to slamming or pounding in a seaway, the role of permanent plastic deformations or sets must be considered. The static and dynamic action of the panel in the plastic region will be studied in this Appendix with the objective of obtaining correct order-of-magnitude solutions for the plastic deformation of plates subject to slam. The solutions presented here are based on extensive simplifications of a very complicated problem.

A number of special solutions are available for the small elasto-plastic deflections of homogeneous isotropic plates. Most of these solutions are for the circular plate, and for the most part, they are based upon the more exact theories of plasticity. A large number of such solutions were outlined by Hodge.[27] There are some difficulties with this fundamental approach since only very special problems can be handled by the more exact theories. The piecewise linear theory as outlined by Hodge[28] offers an alternate simplified approach for some special problems when the principal stress directions are known. Less rigorous but more practical methods for handling plate plasticity problems have been outlined by Vasta.[29]

b. Very Large Deflections and Failure of Plates under Slamming Loads.

(1) Homogeneous isotropic plates

A very simple but accurate method for describing the static and dynamic behavior of thin panels under very large plastic deformations is offered by *membrane theory*.[30,31*] In deriving the formulas for plastic deformation based upon this theory the following assumptions are made:[30]

(a) The tension in the panel is constant.

(b) All bending stresses in the panel may be neglected.

(c) Elastic deflections of the panel may be neglected; i.e., no elastic recovery of the deflection takes place after loading.

---

* The only stresses that are of importance in the elastic small deflection region of plates are those corresponding to bending of the panel. As the load is increased and the plate enters the large deflection region, the load is resisted by both bending and tension in the middle plane of the panel. As the load is further increased, the tensile stresses become predominant and ultimately the plate may be considered to act as a membrane. Finally, we observe that small and moderate plastic deformation is a subject in itself, requiring a separate presentation.

(d) The strain in the panel is large compared with the strain at the yield point.

(e) The permanent set after unloading is equal to the deflection just prior to unloading.

(f) The pressure on the panel is uniformly distributed.

The panel is therefore assumed to act as a stretched membrane. The equation governing the static deflection is then as follows:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P_o}{T} \qquad [E.1]$$

where w is the lateral deflection,

P_o is the external uniform static lateral pressure (load per unit area), and

T is the tension per unit length.

$$T = \sigma_u h$$

where h is the panel thickness and

σ_u is the ultimate stress. It is assumed that the stress-strain law is as shown in Figure E.1.

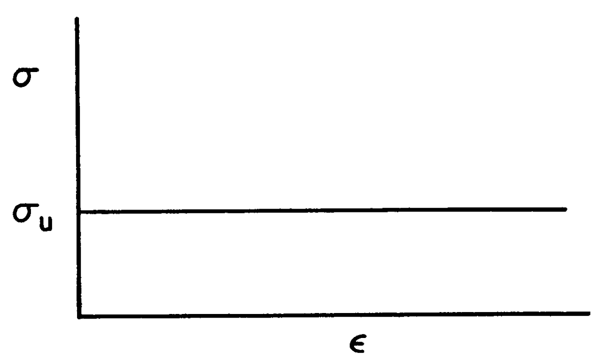

Figure E.1 – Assumed Stress-Strain Law for Panel

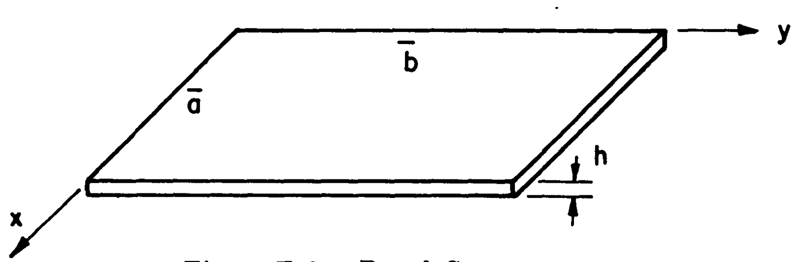

Figure E.2 – Panel Geometry

Assuming that the boundaries of the panel (Figure E.2) remain stationary, the boundary conditions may be written as follows:

$$w = 0;\ x = 0;\ x = \bar{a};\ y = 0;\ y = \bar{b}$$

A first-term approximation for the deflection yields (see Equation [24] of Reference 32):

$$w(x,y) \approx \frac{16P_o\bar{b}^2}{T\pi^4\left(\frac{\bar{b}^2}{\bar{a}^2} + 1\right)} \sin\frac{\pi x}{\bar{a}} \sin\frac{\pi y}{\bar{b}} \qquad [E.2]$$

It has been found[32] that the static load deflection curve is predicted accurately near failure by the above formula at the center of the panel, i.e., at $x = \frac{\bar{a}}{2}$, $y = \frac{\bar{b}}{2}$,

$$w(x,y) = w_{max} \approx \frac{16P_o b^2}{T\pi^4\left(\frac{\bar{b}^2}{\bar{a}^2} + 1\right)} = \frac{0.164P_o a^2}{\sigma_u h\left[1 + \frac{1}{\left(\bar{b}^2/\bar{a}^2\right)}\right]}$$

If we now go back and insert the inertia forces into the membrane equation and allow the load to be time dependent, the basic equation will be:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\rho h}{T}\frac{\partial^2 w}{\partial t^2} - \frac{P_o(t)}{T} \qquad [E.3]$$

If $P_o(t) = P_o f(t)$ is a *uniformly distributed impulse load*, it has been shown that the one-term approximation leads to the following formula (see Equations [20] - [22] of Reference 32):

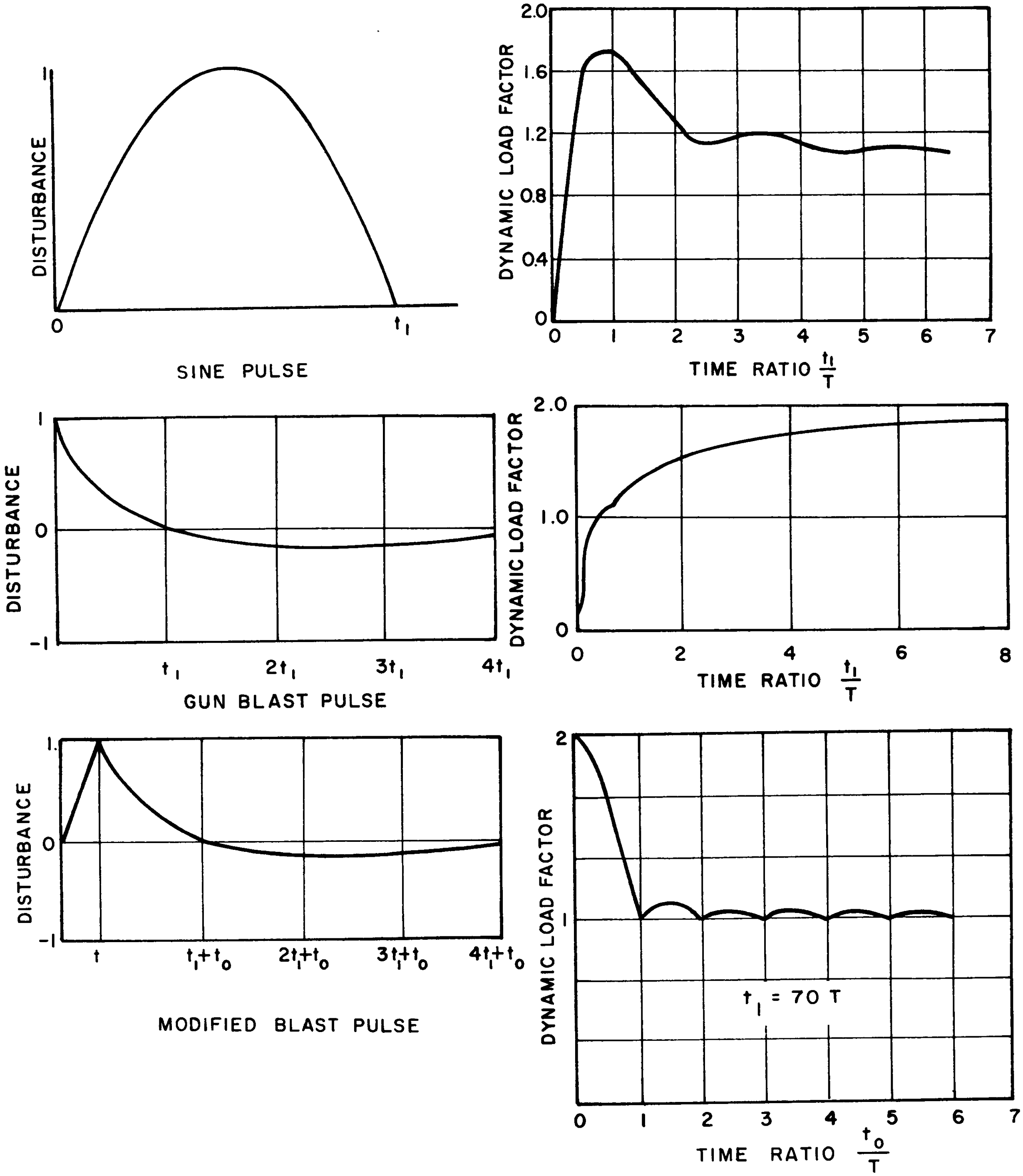

Figure E.3 - Pulse Shapes and Corresponding Load Factor Curves

The procedure for obtaining the load factor for any mode of vibration is as follows:

1. Compute the natural frequency for that mode in cps.
2. Invert this to obtain the period T for that mode.
3. For a given shape and duration of pulse, calculate t_1/T or t_0/T (where t_0 or or t_1 is the time duration as shown in the pulse curves) and use the appropriate curve to determine the load factor for the mode of period T.

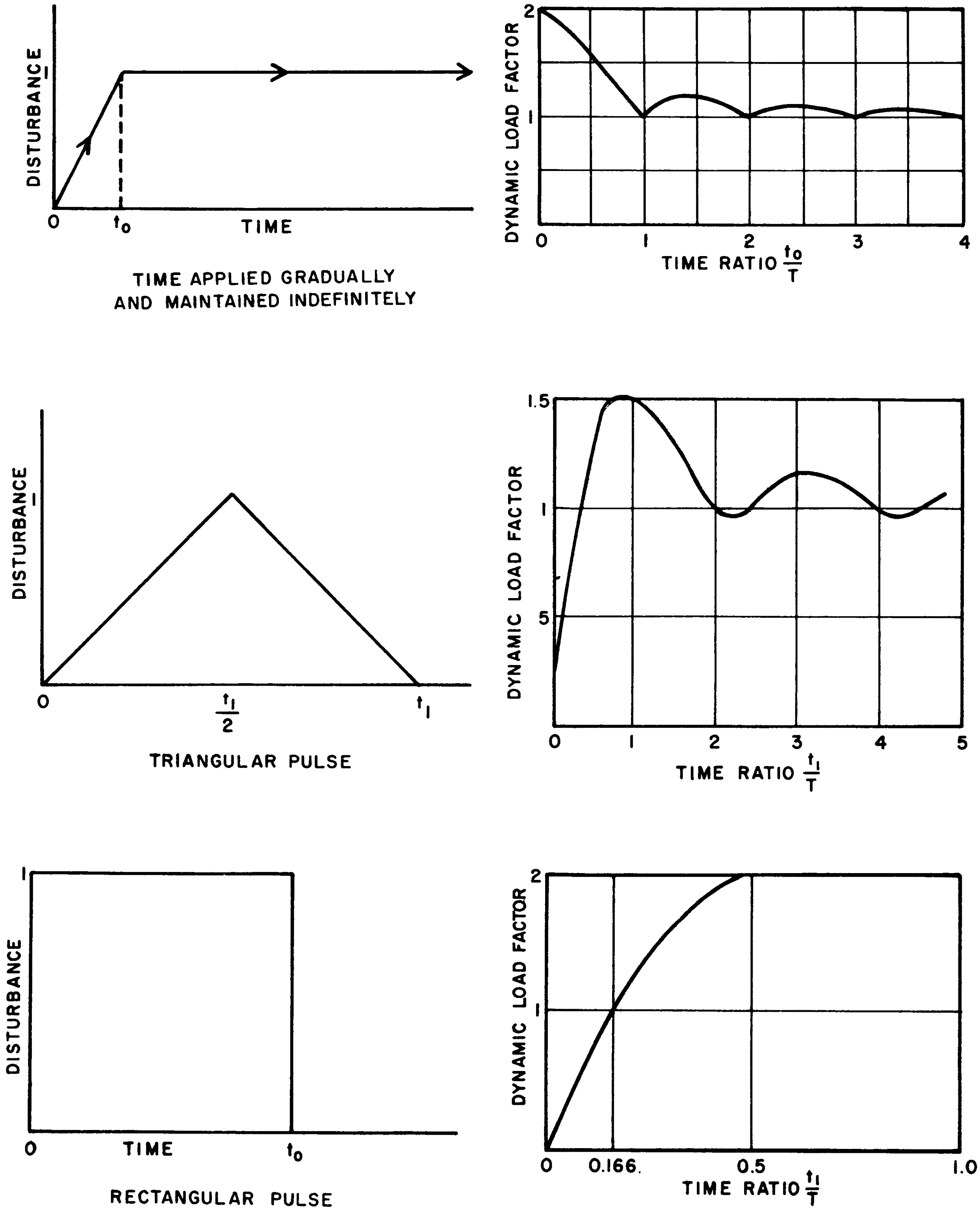

Figure E.3 (continued)

$$w(x,y,t) = \frac{16P_o\bar{b}^{-2}}{T\pi^4\left(\frac{\bar{b}^2}{\bar{a}^{-2}} + 1\right)} \left[p_{11} \int_o^t f(\tau) \sin p_{11} (t-\tau)\, d\tau\right] \sin \frac{\pi x}{\bar{a}} \sin \frac{\pi y}{\bar{b}} \quad [E.4]$$

where p is the fundamental frequency of the membrane under a tension of $\sigma_u h$.

The factor $p_{11} \int_o^t f(\tau)\sin p_{11}(t - \tau)d\tau$ is known as the response factor and has been calculated for various time functions.[33,34] This factor is unity for static loads.[33] The maximum numerical value of the response factor, which is known as the dynamic load factor, has been used in connection with small deflections of plates; see, for example, Reference 25. The dynamic load factor is plotted in Figure E.3 for pulses of various shapes. In other words, the deflection takes the following form:

$$w(x,y,t) = w_{static} \times [\text{Time Dependent Response Factor}] \qquad [E.5]$$

and

$$w_{max} = w_{static} \times [\text{Dynamic Load Factor}] \qquad [E.6]$$

(2) Stiffened Panels

Using the above concepts, we can obtain an approximate solution for a panel containing stiffeners if it does not fail in buckling and if it is assumed that only tension in the panel and stiffeners need be considered when failure is impending. We assume that the tension in the panel is different in the two coordinate directions and depends upon the equivalent thickness for stretching in the two directions. The thicknesses h_x and h_y are defined as follows:

h_x = equivalent thickness for stretching in x direction,

= total cross-sectional area of panel and stiffeners in y-z plane divided by width of panel in that plane,

h_y = equivalent thickness for stretching in y direction,

= total cross-sectional area of panel and stiffeners in x-z plane divided by width of panel in that plane.

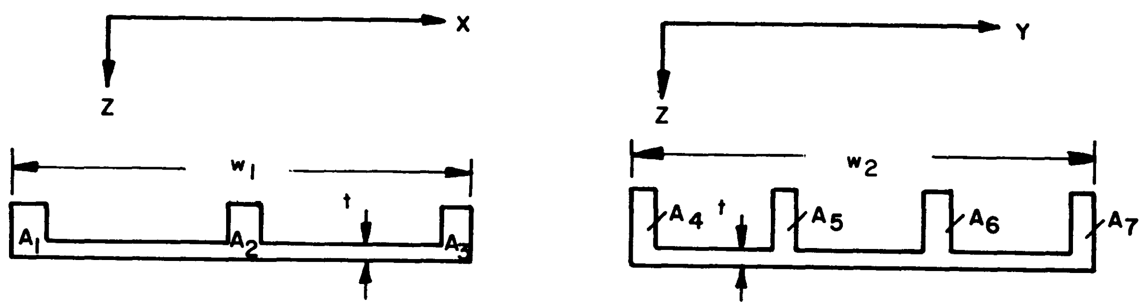

Figure E.4 – Cross Section of Panel

Thus, in Figure E.4:

$$h_x = \frac{A_4+A_5+A_6+A_7+tw_2}{w_2}; \quad h_y = \frac{A_1+A_2+A_3+tw_1}{w_1} \qquad [E.7]$$

The tension in the two directions at very large deformations will be

$$T_x = \sigma_u h_x; \; T_y = \sigma_u h_y \qquad [E.8]$$

where σ_u is the ultimate stress of the material. The equation of motion is

$$T_x \frac{\partial^2 w}{\partial x^2} + T_y \frac{\partial^2 w}{\partial y^2} = \mu \frac{\partial^2 w}{\partial t^2} - P_o(t) \qquad [E.9]$$

where μ is the average mass per unit area of the panel. (It equals the total mass divided by the total surface area.)

Assuming a one-term approximation, we obtain*

$$w = q(t)\ \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \qquad [E.10]$$

where $q(t)$ is a function of time. Substituting this solution into the equation of motion [E.9], the following equation results:

$$-\left[T_x\frac{\pi^2}{a^2}+T_y\frac{\pi^2}{b^2}\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\,q(t) = \mu\ddot{q}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} - P_o(t) \qquad [E.11]$$

Now multiplying both sides of the equation by $\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$ and integrating over the area of the panel, we obtain

$$\int_0^a\int_0^b \sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}\left[\mu\ddot{q}+\left(T_x\frac{\pi^2}{a^2}+T_y\frac{\pi^2}{b^2}\right)q\right]dxdy = P_o(t)\int_0^a\int_0^b \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\,dxdy \qquad [E.12]$$

or

$$\frac{ab}{4}\left[\mu\ddot{q} + \left(T_x\frac{\pi^2}{a^2}+T_y\frac{\pi^2}{b^2}\right)q\right] = \frac{4ab}{\pi^2}P_o(t)$$

$$\therefore\ \ddot{q} + \left(\frac{T_x}{\mu}\frac{\pi^2}{a^2}+\frac{T_y}{\mu}\frac{\pi^2}{b^2}\right)q = \frac{16}{\mu\pi^2}P_o(t)$$

or

$$\ddot{q} + p_{11}{}^2 q = \frac{16}{\mu\pi^2}P_o(t) \qquad [E.13]$$

where

$$p_{11}{}^2 = \frac{T_x}{\mu}\frac{\pi^2}{a^2}+\frac{T_y}{\mu}\frac{\pi^2}{b^2} \qquad [E.14]$$

* A summation of terms would be more exact. Only the first term of the sum is considered here.

p_{11} is the fundamental frequency of the stiffened membrane. The solution for w is therefore (following through in the same manner as for the unstiffened panels):

$$w(x,y,t) = \frac{16P_o}{\mu\pi^2 p_{11}^2}\left[p_{11}\int_o^t f(\tau)\,\sin p_{11}\,(t-\tau)\,d\tau\right]\sin\frac{\pi x}{a}\,\sin\frac{\pi y}{b}$$

$$= \frac{16P_o}{\pi^4\left(\frac{T_x}{\bar a^2}+\frac{T_y}{\bar b^2}\right)}\left[p_{11}\int_o^t f(\tau)\,\sin p_{11}\,(t-\tau)\,d\tau\right]\sin\frac{\pi x}{\bar a}\,\sin\frac{\pi y}{\bar b} \qquad [E.15]$$

$$= \frac{16P_o\bar b^2}{T_x\pi^4\left(\frac{\bar b^2}{\bar a^2}+\frac{T_y}{T_x}\right)}\left[p_{11}\int_o^t f(\tau)\,\sin p_{11}(t-\tau)\,d\tau\right]\sin\frac{\pi x}{\bar a}\,\sin\frac{\pi y}{\bar b}$$

So

$$w_{max} = w_{static} \times \text{(Dynamic Load Factor)} \qquad [E.16]$$

where

$$w_{static} = \frac{16P_o\bar b^2}{T_x\pi^4\left(\frac{\bar b^2}{\bar a^2}+\frac{T_y}{T_x}\right)}\sin\frac{\pi x}{\bar a}\,\sin\frac{\pi y}{\bar b}$$

Admittedly, this is an oversimplification of the dynamic behavior of a stiffened plate near failure; however, there is a reasonable chance that it will give answers of the correct order of magnitude if failure occurs in the same way as in an unstiffened panel, i.e., by tension. On the other hand, a failure analysis based on flow theory or even deformation theory considering all the nonlinear complications would have undoubtedly led to a problem that was mathematically insoluble. Only experiments will be able to bear out whether or not the above simplified formulas give answers which are of the right order of magnitude. If the pulse length is related to the natural period of the membrane in such a manner as to make higher mode contributions important, as in Reference 31, then more terms in the series should be considered.

(3) Failure Criteria

The previous analysis only estimates the load deflection curve for very large plastic deformation. It does not predict when a crack will appear in the panel, starting a complete failure. The failure problem is discussed in Reference 32. For unstiffened panels, it has been found that the ratio of center deflection to shorter side was always greater than 0.1 when failure occurred. Therefore, it is believed that a conservative estimate of the ultimate load in the dynamic problem can be obtained by finding the load (including the effect of the load factor) at which $w_{max}/\bar{a} = 0.1$.

APPENDIX F

INTEGRATION OF PRIMARY, SECONDARY, AND TERTIARY STRESSES

1. GENERAL

The design engineer needs to follow a relatively simple procedure to obtain the sum of the primary, secondary, and tertiary stresses. This procedure will be shown in this section.

a. Primary Stresses

The primary stresses $(\sigma_x)_p$ are obtained as solutions to the finite difference equations for a ship represented as a beam, subject to arbitrary wave excitation.[1,2] In the difference equations, the mechanical properties of the beam are "lumped" at a finite number of points. The beam is divided into a number of sections of equal or unequal length $(\Delta x)_i$ (N = 20 has generally been used but 40 is better); the locations of the ends of the sections, called stations, are denoted by n = 0, 1, 2 ... N, and the midpoints, called midstations, by n = 1/2, 1-1/2 ... N-1/2. The primary stress solutions, computed at n cross sections or stations in accordance with the methods of References 1 and 2, can be plotted to give a continuous stress versus time curve for all ship cross sections. Alternatively, and of greater interest here, the solution data can be plotted to give a continuous stress versus longitudinal position curve for all time steps of interest; see Figure F.1. The equations to be programmed and the required input data for obtaining the primary stresses are presented in References 1 and 2.

b. Secondary Stresses

Between each station for which the primary stress was calculated there are stiffened plate sections (Figures F.1 and F.2). The stresses in

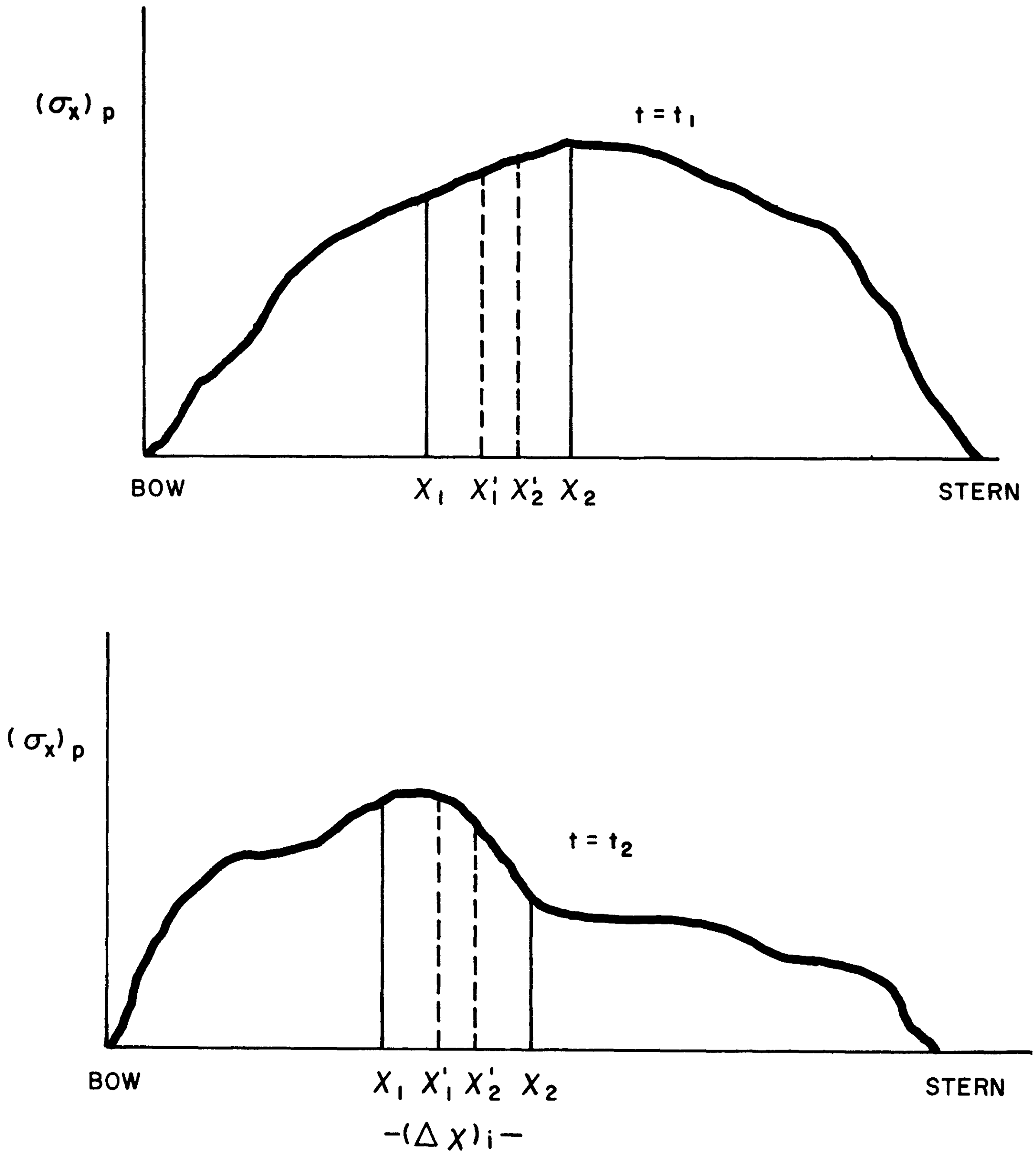

Figure F.1 - Primary Stresses in a Ship's Hull Girder Represented as a Beam at Times t_1 and t_2

each of the stiffened plate sections are computed as outlined in Appendix D. Although for each of these stiffened plates solution data for all secondary stresses, bending and shearing, are obtained, here we will, for purposes of illustration, refer to $(\sigma_x)_s$ only; this is the stress that combines with the primary longitudinal stress. Thus, the secondary stress $(\sigma_x)_s$ in a stiffened plate section $X_1' - X_2'$ lying within a hull girder or beam section $x_1 - x_2$ can be plotted as shown in Figure F.2.

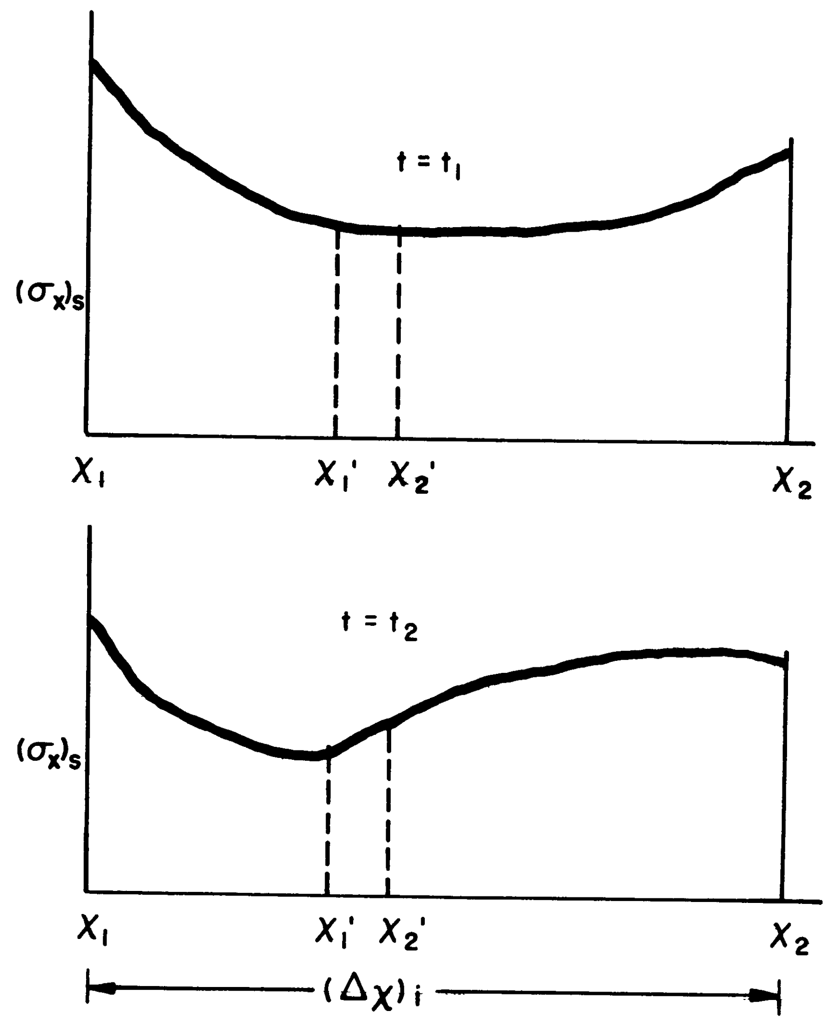

Figure F.2 - Secondary Stresses in Stiffened Plate Sections $x_1' - x_2'$ Lying within a Hull Girder Section $x_1 - x_2$ at Times t_1 and t_2

Secondary Stresses lying outside $x_1' - x_2'$, but within $x_1 - x_2$, are shown.

c. Tertiary Stresses

To complete the stress picture (or pattern), an additional subdivision of the structure, which includes the smallest elastic element stressed by the dynamic slamming load, is required. This element is the plating between the stiffeners of the stiffened plate. The tertiary stresses in such an unstiffened plate lying between X_1' and X_2' (Figure F.3) are calculated by the same formulas used for the stiffened plate except that (see Appendix B):

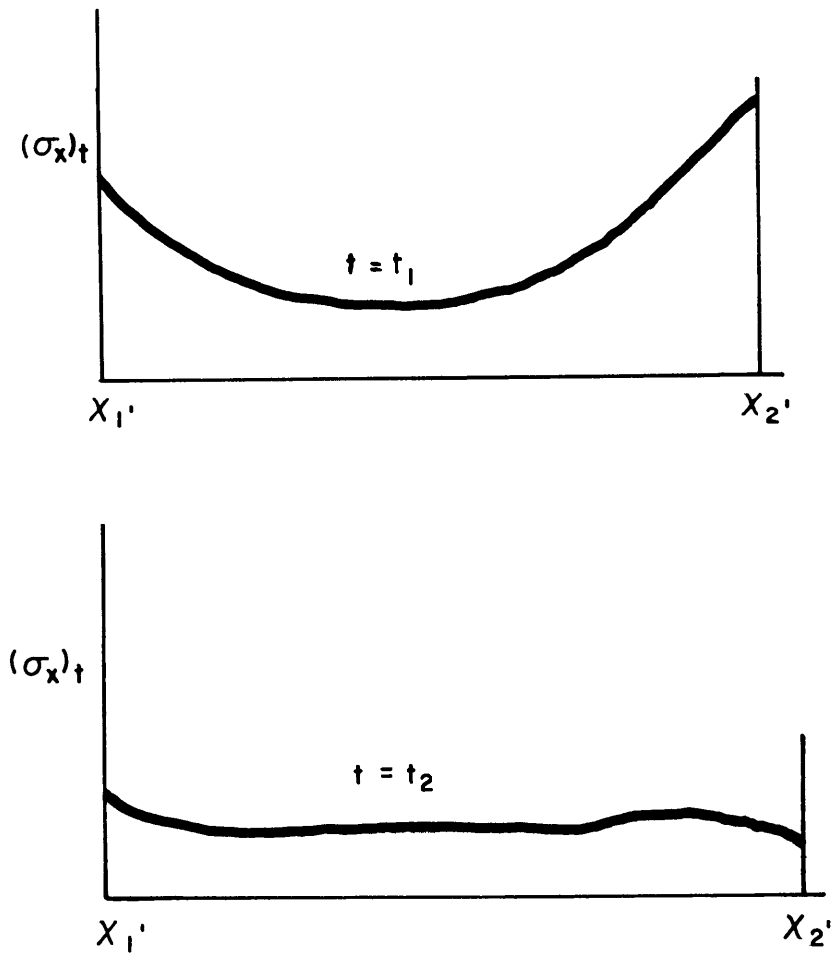

Figure F.3 - Tertiary Stresses in an Unstiffened Plate Lying between the Stiffened Plate Section $x_1' - x_2'$

$t_{fx} = t_{fy} = t_{nx} = t_{ny} = t/2$

$r_{n'x} = r_{n'y} = r_{f'x} = r_{f'y} = t/4$

$i_{w_x} = i_{w_y} = 0$

At each time step, the computer yields solutions for the primary, secondary, and tertiary stresses for all sections or subsections of the hull and adds these stresses algebraically to give the total stress σ_x at any time for all points of the hull.

2. INSTRUCTIONS TO THE PROGRAMMER FOR COMBINING THE STRESSES

Two separate programs for calculating the stresses are available to the programmer. They are the program for calculating the primary stress described in References 1 and 2 and the program for calculating the secondary and tertiary stresses described in Appendix D. To match the values of the computed primary, secondary, and tertiary stresses at a given location, three coordinate systems are used. For primary stresses, the x coordinate designated x_p is measured from the bow (Figure F.4a). For secondary stresses, the x coordinate designated x_s is measured from the edge of the stiffened plate located at $(x_p)_n$ (Figure F.4b). For

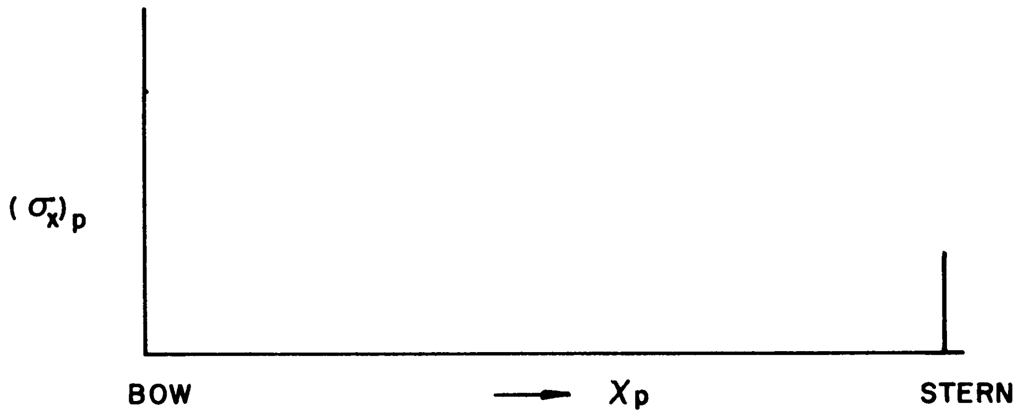

Figure 4a - PRIMARY STRESS COORDINATE SYSTEM

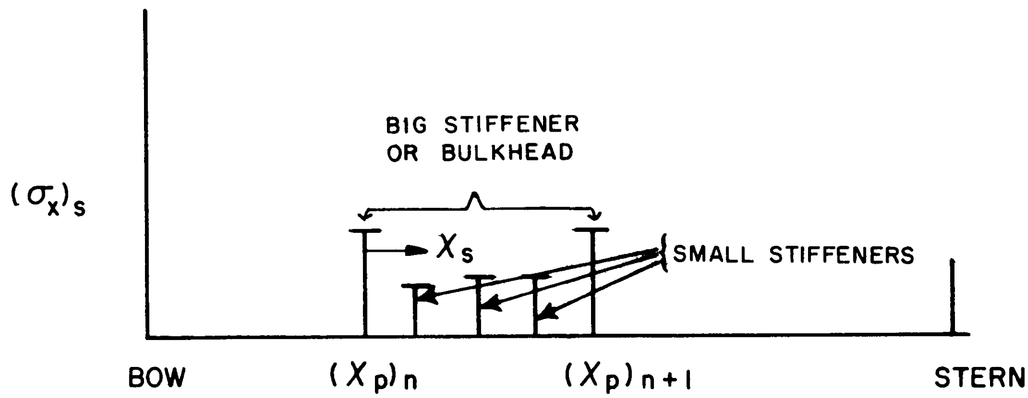

Figure 4b - SECONDARY STRESS COORDINATE SYSTEM

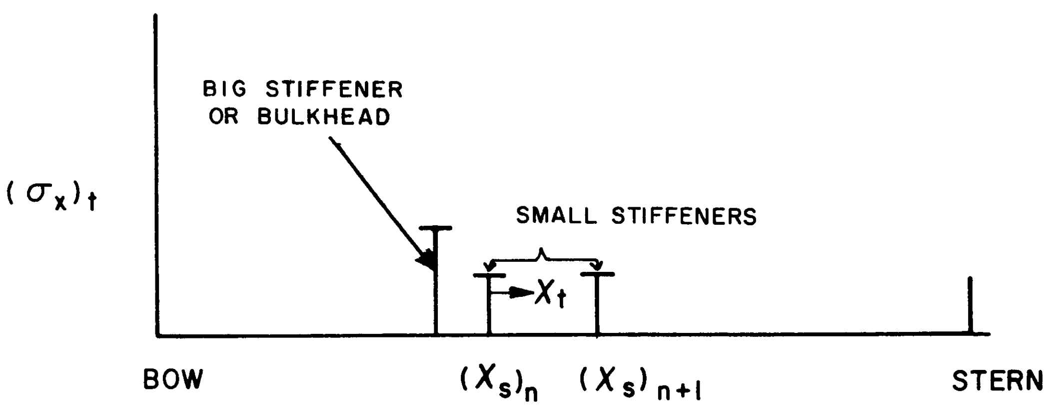

Figure 4c - TERTIARY STRESS COORDINATE SYSTEM

Figure F.4 - Coordinate Systems for Primary, Secondary, and Tertiary Stresses tertiary stresses, the x coordinate designated x_t is measured from the edge of the unstiffened plate located at $(x_s)_n$ (Figure F.4c). Using these coordinate systems, the programmer is then able to provide the designer with solution data consisting of the primary stress at x_p, the secondary stress at x_s lying within section $(x_p)_n - (x_p)_{n+1}$, and the tertiary stress at x_t lying within section $(x_s)_n - (x_s)_{n+1}$. Thus, continuous values of x_p, x_s, x_t along the longitudinal direction can be made available. The programmer will also provide for the algebraic addition of these three stresses at each time step. To facilitate calculation, the programmer should provide for computing each of the stresses for a series of values of x_p, x_s, x_t, and their sum for a corresponding series of values of time. Thus for a given set of input data for the hull girder and plating (stiffened and unstiffened), stresses corresponding to the following locations and times should be obtained:

$(x_p)_{min}$	;	$(\Delta x)_p$	;	$(x_p)_{max}$
$(x_s)_{min}$	;	$(\Delta x)_s$	;	$(x_s)_{max}$
$(x_t)_{min}$	;	$(\Delta x)_t$	;	$(x_t)_{max}$
$(t)_{min}$	;	(Δt)	;	$(t)_{max}$

APPENDIX G

PLASTIC DEFORMATION OF UNSTIFFENED AND STIFFENED RECTANGULAR PLATES

For design purposes an attempt should be made to stipulate the amount of secondary and tertiary deformation that can be allowed. The tertiary or unstiffened plate deformation can result in washboarding of the local forward bottom of the ship and possibly in local failures. The secondary plastic deformation can result in washboarding of large sections of the forward bottom and is probably a more serious problem than the tertiary plastic deformations. The objective of this Appendix is to present a basic approach which can be used to compute the secondary plastic deformations and, as a special case, the tertiary deformations.[35*]

For convenience of reference, definitions of the symbols used in this Appendix are included; the symbols have local value only.

NOTATION

a,b	Length and width of plate
dA	Elemental area of plate
dm	Element of mass
E	Modulus of elasticity of plate material
e_i	Strain factor proportional to octahedral shear strain
e_s	Yield strain in pure tension
f(x,y)	Spatial distribution of the impulse

* Primary plastic deformation is deformation of the entire ship as a beam. It would occur at the center of the ship, where a hinge would form when the critical bending moment is reached at that point, thereby causing a tendency for the ship to break in two.

h	Thickness of one facing of double bottom
$\bar{h}$	Thickness of other facing of double bottom
I	Impulse per unit mass applied to plate
I_{A_r}	Moment of inertia of the r^{th} stiffener about its own neutral axis
I_o	Amplitude of the impulse
I_t	Total impulse on plate
p_o	Amplitude of the applied pressure
T	Initial kinetic energy imparted to plate
T_o	Length of time that pulse is applied
V	Work done by the internal forces during the plate deformation
w	Lateral deflection of plate
$\dot{w}$	Lateral velocity of plate
w_o	Amplitude of the deformation
x,y,z	Rectangular coordinates
x_s	x coordinate of the s^{th} stiffener
y_r	y coordinate of the r^{th} stiffener
$\bar{\bar{z}}$	Distance from neutral plane to any point in face for second part of double bottom
$\bar{z}$	Distance from neutral surface of stiffened plate to midsurface of face
z_p	Distance between midsurface of face and any point in face
β	Shape factor for exponential pulse

γ_{xy} Shear strain

ϵ_x, ϵ_y Tensile or compressive strain in x and y direction, respectively

λ A measure of the slope of the plastic portion of the stress-strain curve of a linear-hardening material

$\bar{\mu}$ Mass per unit area of plate

ρ Mass density of the plate material

σ_i Stress factor proportional to octahedral shear stress

σ_s Yield stress in pure tension

σ_x, σ_y Tensile or compressive stress in x and y direction, respectively

τ_{xy} Shear stress

$\omega(e_i)$ Plastic function in stress-strain law

1. THEORY

Essentially, the approach to this problem follows the work given in two reports.(36),(37) Let the work done by the internal forces of an elastic-plastic stiffened plate under impact be V; let I be the impulse per unit mass applied to the plate. The impulse momentum relation for an elemental mass can be written

$$\dot{w}\ dm = I\ dm \qquad [G.1]$$

where $\dot{w}$ is the lateral velocity imparted to the mass by the impulse (consider only lateral velocity $\dot{w}$; neglect $\dot{u}$ and $\dot{v}$). Thus

$$\dot{w} = I \qquad [G.2]$$

The kinetic energy imparted to the plate is

$$T = \int_A \frac{1}{2} \bar{\mu}\dot{w}^2 dA = \frac{1}{2} \int_A \bar{\mu}\, I^2 dA \qquad [G.3]$$

where $\bar{\mu}$ is the mass per unit area of the plate and dA is an elemental area. The impulse can vary over the surface of the plate; therefore we write:

$$I(x,y) = I_o f(x,y) \qquad [G.4]$$

The kinetic energy becomes:

$$T = \frac{1}{2}\,\bar{\mu} I_o^{\,2} \int_A f^2(x,y)\,dA \qquad [G.5]$$

Equating the initial kinetic energy to the energy of deformation yields the following expression for the impulse per unit mass:

$$\dot{I}_o = \sqrt{V \frac{2}{\bar{\mu} \int_A f^2(x,y)\,dA}} \qquad [G.6]$$

The total impulse on the plate will then be:

$$I_t = \int_A \sqrt{V \frac{2\bar{\mu}}{\int_A f^2(x,y)\,dA}}\; f(x,y)\,dA \qquad [G.7]$$

The work of deformation V per unit volume of an elastic plastic body can be written as:[38]

$$V = \int_o^{e_i} \sigma_i\, de_i + \frac{K\theta^2}{2} \qquad [G.8]$$

where
$$\sigma_i = \frac{\sqrt{2}}{2}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\,(\tau_{xy}^{\,2} + \tau_{yz}^{\,2} + \tau_{zx}^{\,2})}$$

$$e_i = \frac{\sqrt{2}}{3}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + \frac{3}{2}\,(\gamma_{xy}^{\,2} + \gamma_{yz}^{\,2} + \gamma_{zx}^{\,2})}$$

$$\theta = \varepsilon_x + \varepsilon_y + \varepsilon_z \qquad [G.9]$$

The curve of σ_i versus e_i describes the stress-strain law of the material as shown in Figure G.1.

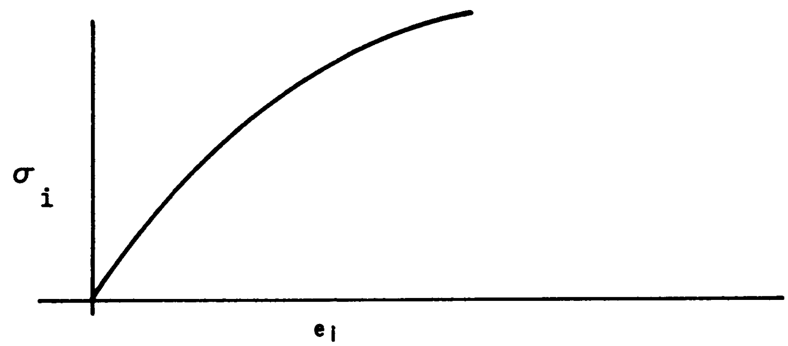

Figure G.1 - General Stress-Strain Law

Assuming an incompressible material $\theta=0$ and considering only plate stresses:

$$V = \int_{V_o} \left[\int_0^{e_i} \sigma_i \; de_i \right] dV_o \qquad [G.10]$$

$$\sigma_i = \sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\,\tau_{xy}^2}$$

$$e_i = \sqrt{\varepsilon_x^2 + \varepsilon_x\varepsilon_y + \varepsilon_y^2 + \frac{1}{4}\gamma_{xy}^2}$$

dV_o = element of volume

Consider only the lateral deflection w to be of significance, and neglect u and v and their derivations. The strains ε_x, ε_y and γ_{xy} then become:[39]

$$\varepsilon_x = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\,\frac{\partial^2 w}{\partial x^2} \quad ; \quad \varepsilon_y = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\,\frac{\partial^2 w}{\partial x^2} \qquad [G.11]$$

$$\gamma_{xy} = \frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y} - 2\,z\,\frac{\partial^2 w}{\partial x \partial y}$$

where z is the distance of any element of the plate from the neutral plane; see Figure G.2.

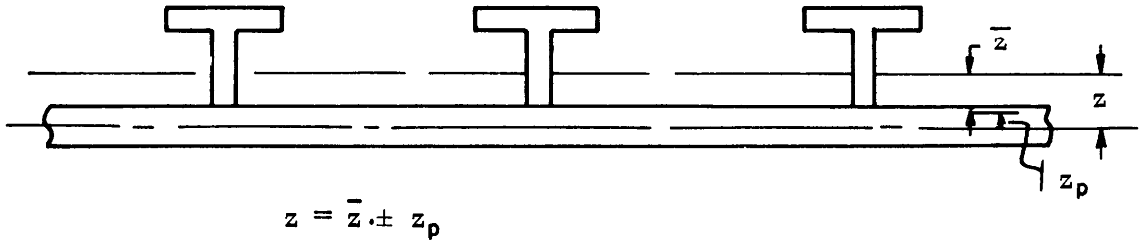

Figure G.2 - Location of Neutral Plane and Elements

Further restrict the material to be an elastic-linear hardening material (although this is not much of a restriction) with the stress-strain law shown in Figure G.3.

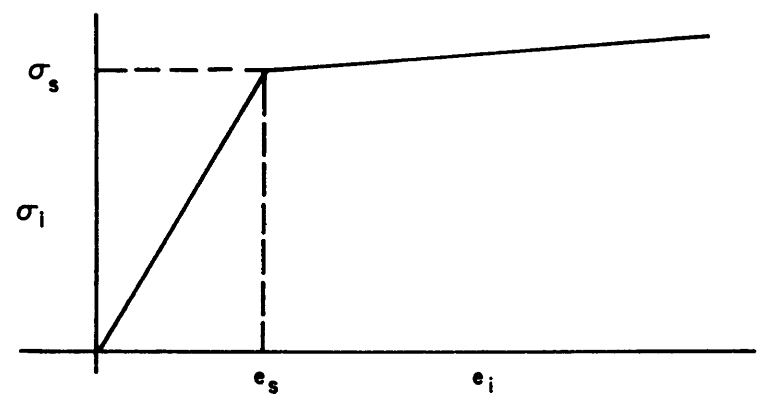

Figure G.3 - Elastic-Linear-Hardening Law

The stresses can be written in terms of the strains as follows (assuming a Poisson's ratio of 1/2):

$$\sigma_x = \frac{4}{3}\frac{\sigma_i}{e_i}\left(\varepsilon_x + \frac{1}{2}\varepsilon_y\right)$$

$$\sigma_y = \frac{4}{3}\frac{\sigma_i}{e_i}\left(\varepsilon_y + \frac{1}{2}\varepsilon_x\right) \qquad [G.12]$$

$$\tau_{xy} = \frac{1}{3}\frac{\sigma_i}{e_i}\gamma_{xy}$$

where the stress-strain law is:

$$\frac{\sigma_i}{e_i} = E\ [1-\omega(e_i)] \qquad [G.13]$$

where

$$\omega(e_i) = 0 \quad \text{for } e_i < e_s \quad \text{(elastic)}$$

$$\omega(e_i) = \lambda\ (1-e_s/e_i) \quad \text{(plastic)} \qquad [G.14]$$

$$\lambda = 1 - \frac{1}{E}\frac{d\sigma_i}{de_i}$$

By substituting the linear-hardening law into the expressions for the stresses and then into the relation for V_1 we obtain the work done by the internal forces on the plating of the stiffened plate:

$$V_1 = \int_0^a \int_0^b \int_{\bar{z}-\frac{h}{2}}^{\bar{z}+\frac{h}{2}} \left[\frac{Ee_i^2}{2}(1-\lambda) + E\lambda e_s e_i\right] dx\,dy\,dz_p$$

$$- \int_0^a \int_0^b \int_{\bar{z}-\frac{h}{2}}^{\bar{z}+\frac{h}{2}} \frac{E\lambda e_s^2}{2}\ dx\,dy\,dz_p \qquad [G.15]$$

where $\bar{z}$ is the distance from neutral plane to midsurface of face,

h is the thickness of plate, and

a,b are the length and width of plate.

If there is a top plate (Figure G.4), then a similar expression V_2 must be added.

If $\bar{\bar{z}}$ is the distance from neutral plane to center of the other face:

$$V_2 = \int_0^a \int_0^b \int_{\bar{\bar{z}} - \frac{\bar{h}}{2}}^{\bar{\bar{z}} + \frac{h}{2}} \left[\text{same expression as in } V_1\right] \qquad [G.16]$$

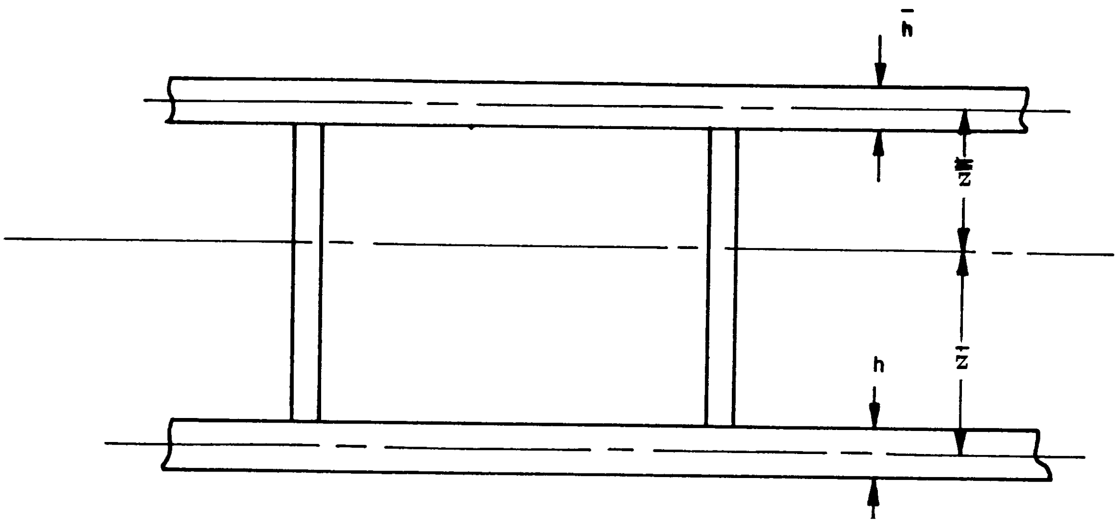

Figure G.4 - Cross Section of Stiffened Plate

Now, substituting the expression for e_i yields:

$$V_1 = \int_0^a \int_0^b \int_{\bar{z} - \frac{h}{2}}^{\bar{z} + \frac{h}{2}} \left[\frac{E}{2}(1-\lambda)\,\frac{4}{3}\,(\varepsilon_x^{\,2} + \varepsilon_x\varepsilon_y + \varepsilon_y^{\,2} + \frac{1}{4}\gamma_{xy}^{\,2}) + \frac{E\lambda e_s}{\sqrt{3}}\, 2\sqrt{\varepsilon_x^{\,2} + \varepsilon_x\varepsilon_y + \varepsilon_y^{\,2} + \frac{1}{4}\gamma_{xy}^{\,2}}\right] dxdydz_p$$

$$-\int_0^a \int_0^b \int_{\bar{z} - \frac{h}{2}}^{\bar{z} + \frac{h}{2}} \frac{E\lambda e_s^{\,2}}{2}\, dxdydz_p \qquad [G.17]$$

Substituting the expressions for the strains in terms of the deflections gives:

$$V_1 = \int_0^a \int_0^b \left\{ \frac{E(1-\lambda)}{2(1-\nu^2)} \left(\alpha z + \gamma\frac{z^2}{2} + \beta\,\frac{z^3}{3}\right)\Bigg|_{z=\bar{z}-\frac{h}{2}}^{z=\bar{z}+\frac{h}{2}} \right.$$

$$+ \frac{2E\lambda e_s}{\sqrt{3}} \left[\frac{(2\beta z+\gamma)\sqrt{\alpha+z\gamma+z^2\beta}}{4\beta} + \frac{4\alpha\beta-\gamma^2}{8\beta}\,\frac{1}{\sqrt{\beta}}\sinh^{-1}\left(\frac{2\beta z+\gamma}{\sqrt{4\alpha\beta-\gamma^2}}\right)\right]_{z=\bar{z}-\frac{h}{2}}^{z=\bar{z}+\frac{h}{2}}$$

$$\left. - \int_0^a \int_0^b \frac{E\lambda e_s^{\,2}}{2}\, h \right\} dxdy \qquad [G.18]$$

and a similar expression for V_2, where

$$\alpha = \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^4 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{4}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\beta = \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$$

$$\gamma = -\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial^2 w}{\partial x^2}\right) - \left(\frac{\partial w}{\partial y}\right)^2\left(\frac{\partial^2 w}{\partial y^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial^2 w}{\partial y^2}\right) - \frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial w}{\partial y}\right)^2$$

$$-\left(\frac{\partial^2 w}{\partial x \partial y}\right)\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \qquad [G.19]$$

For the stiffeners, we use an analysis similar to that employed in Reference 36; see Figure [G.5].

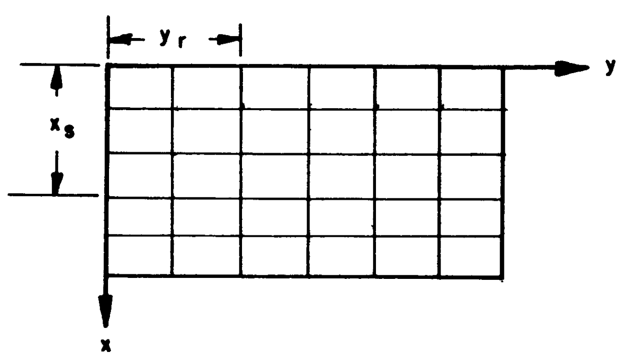

Figure G.5 - Location of Stiffeners in Plate

For a stiffener located at $y=y_r$,

$$V_r = \int_0^a \int_{A_r} \left(\frac{E}{2} (1-\lambda)\ \epsilon_x^{\ 2} + E\lambda e_s e_x \right) dA_r dx - \int_0^a \int_{A_r} \frac{E\lambda e_s^{\ 2}}{2}\ dA_r dx \qquad [G.20]$$

where A_r indicates integration over the cross-sectional area of the r^{th} stiffener. For a stiffener located at $x=x_s$:

$$V_s = \int_0^b \int_{A_s} \left(\frac{E}{2} (1-\lambda)\ \epsilon_y^{\ 2} + E\lambda e_s \epsilon_y \right) dA_s dy - \int_0^b \int_{A_s} \frac{E\lambda e_s^{\ 2}}{2}\ dA_s dy \qquad [G.21]$$

where A_s indicates integration over the cross-sectional area of the s^{th} stiffener. Substituting the expressions for ϵ_x, ϵ_y:

$$V_r = \int_0^a \int_{A_r} \left\{ \frac{E}{2}(1-\lambda) \left[\frac{1}{4} \left(\frac{\partial w}{\partial x} \right)^4 - \left(\frac{\partial w}{\partial x} \right)^2 z\ \frac{\partial^2 w}{\partial x^2} + z^2 \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] \right.$$

$$\left. + E\lambda e_s \left[\frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 - z\ \frac{\partial^2 w}{\partial x^2} \right] \right\} dA_r dx$$

$$- \int_0^a \int_{A_r} \frac{E\lambda e_s^{\ 2}}{2}\ dA_r dx \qquad [G.22]$$

Now

$$\int_{A_r} z^2 dA_r = I_{A_r}$$

the moment of inertia of the stiffener about its own neutral axis, and

$$\int_{A_r} z\ dA_r = z_c A_r$$

where z_c is the distance from the neutral plane of the stiffener to the centroid of the stiffener (equals zero).

So

$$V_r = \int_o^a \left\{ \frac{E}{2} (1-\lambda) \; A_r \; \frac{1}{4} \left(\frac{\partial w}{\partial x} \right)^4 + \frac{E}{2} (1-\lambda) \; I_{A_r} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right.$$

$$\left. + \; E\lambda e_s A_r \; \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right\} dx - \frac{E\lambda e_s{}^2}{2} \; A_r a \qquad [G.23]$$

The terms containing A_r are the contributions from stretching and the term containing I_{A_r} is the contribution of bending; similarly,

$$V_s = \int_o^b \left\{ \frac{E}{2} (1-\lambda) \; A_s \; \frac{1}{4} \left(\frac{\partial w}{\partial y} \right)^4 + \frac{E}{2} (1-\lambda) \; I_{A_s} \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right.$$

$$\left. + \; E\lambda e_s A_s \; \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right\} dy - \frac{E\lambda e_s{}^2}{2} \; A_s b \qquad [G.24]$$

2. SPECIAL CASE OF UNIFORM PLATE

If we consider only the uniform plate and also consider that the plastic deformation is large enough for the work done by membrane action to be of primary significance, then (neglecting the $\frac{E\lambda e_s{}^2}{2}$ hab' term for the time being):

$$V = \int_o^a \int_o^b \left\{ \frac{2}{3} \, E(1-\lambda) \; h\alpha + \frac{2\lambda E e_s}{\sqrt{3}} \; h \; \sqrt{\alpha} \right\} dxdy \qquad [G.25]$$

For a perfectly plastic material, $\lambda=1$ and

$$V = \int_o^a \int_o^b \frac{2\lambda E e_s}{3} \; h \left[\frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] dxdy \qquad [G.26]$$

If a deformation pattern is assumed:

$$w = w_o \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \qquad [G.27]$$

then

$$V = \frac{2Ee_s h}{\sqrt{3}} w_o^2 \int_o^a \int_o^b \left[\frac{1}{2} \frac{\pi^2}{a^2} \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi 2}{b} + \frac{1}{2} \frac{\pi^2}{b^2} \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{b} \right] dxdy \quad [G.28]$$

If the impulse applied to the plate is uniformly distributed, then the impulse per unit area can be written as:

$$\bar{I} = I_o \rho h = \sqrt{V \frac{2\rho h}{ab}} = w_o \sqrt{\frac{\bar{k}\, 2\rho h \psi(a,b)}{ab}} \qquad [G.29]$$

where

$$\psi(a,b) = \int_o^a \int_o^b \left[\frac{1}{2} \frac{\pi^2}{a^2} \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} + \frac{1}{2} \frac{\pi^2}{b^2} \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{b} \right] dxdy \qquad [G.30]$$

$$\bar{k} = \frac{2Ee_s h}{\sqrt{3}}$$

It is found that

$$w_o = \frac{\bar{I}}{\sqrt{\frac{\sigma_s h}{3} \rho h \pi^2 \left[\frac{1}{a^2} + \frac{1}{b^2} \right]}} \qquad [G.31]$$

where σ_s is the yield stress in pure tension. We assume that the plastic work goes on deforming the plate with permanent deformation w_o and that the plastic deformations are very large compared to the elastic deformations.

Consider the two differently shaped pulses applied to the plate; see Figure G.6.

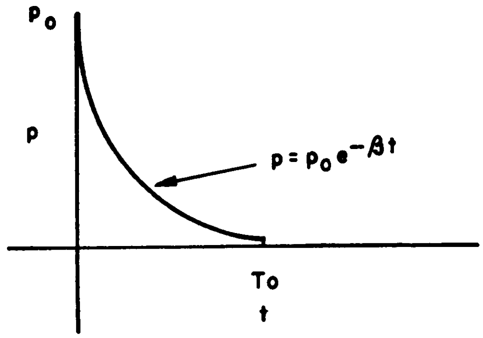

Figure G.6a - Exponential Pulse (Type 1)

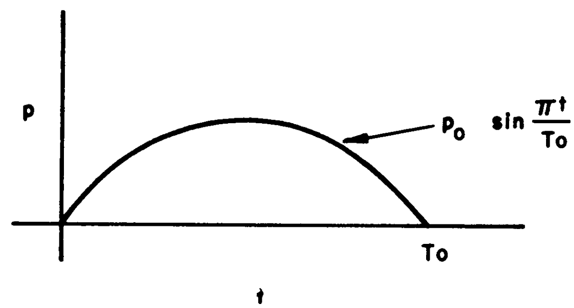

Figure G.6b - Sine Pulse (Type 2)

$$\bar{I} = \int_0^{T_o} p\,dt = \frac{p_o}{\beta}\,[1-e^{-\beta T_o}]$$

$$\bar{I} = \int_0^{T_o} p_o \sin \frac{\pi t}{T_o}\,dt = \frac{2p_oT_o}{\pi}$$

Figure G.6 - Pulses

Referring to Figure G.6a, since $e^{-\beta T_o}$ is usually small compared to 1, $\bar{I} \approx \frac{p_o}{\beta}$.

The pulse type shown in Figure G.6a is typical of an explosion pulse or a severe slamming pulse, whereas that in Figure G.6b is a pulse which has a definite time of rise. According to this theory, the damage will be dependent on $\bar{I}$. The degree of compressibility in the fluid for slamming will be felt in the values of p_o and β.

To answer the question of the importance of compressibility in slamming, approximate values of p_o and β should be evaluated from two theories, one which considers compressibility and one which neglects it. From this it can be seen that damage from Type 1 pulses could equal damage from Type 2 pulses. The relative magnitudes of p_o, β, and T_o are the important factors.

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