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THE FRICTIONAL RESISTANCE AND TURBULENT BOUNDARY LAYER OF ROUGH SURFACES

by

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NOTATION

A	Slope of logarithmic velocity law
a_1, a_2, a_3, a_4	Linearization constants
B_1, B_2, B_3	Intercepts of logarithmic velocity law; see Equations [7], [8], and [10]
$B_{1,s}$	B_1 for smooth surfaces
$B_{2,R}$	B_2 for fully rough regime
B_2'	Derivative of B_2 with respect to ln k^*, Equation [76]
B_2''	Derivative of B_2' with respect to ln k^*, Equation [77]
C_f	Coefficient of total resistance; see Equation [67]
C_τ	Coefficient of local resistance; see Equation [13]
C_1, C_2, C_3, C_4	Linearization constants; see Equations [91] and [95]
D_1, D_2	Velocity profile constants; see Table 2
e	Base of natural logarithms
F	Outer law function, Equation [4]
f_1, f_2	Inner law functions, Equations [2] and [3]
G	Subscript for quantities at junction of inner and outer turbulent sublayers; see Table 2
H	Shape parameter, Equation [63]
I_1, I_2	Integrals of outer law velocity profiles; see Table 2
J_1, J_2	Transitional sublayer factors, Equations [42] and [44]
$k, k_1, k_2, \ldots$	Linear parameters defining roughness
k^*	Roughness Reynolds number; see Equation [2]
k_R^*	Value of k^* at start of fully rough regime, Equation [53]
k_s^*	Value of k^* at end of quasi-smooth regime
L	Subscript for quantities at junction of laminar and transitional sublayers; see Table 2
l	Mixing length
M_1, M_2	Constants in logarithmic resistance formulas for flat plates, Equations [93] and [98]
m	Constant in logarithmic resistance formula for flat plates with engineering roughness, Equation [117]

N_1, N_1', N_2	Constants in logarithmic resistance formulas for flat plates, Equations [94], [99] and [103]
O_1, O_2	Constants in local resistance formulas, Equations [145] and [150]
P_1, P_1', P_2	Constants in local resistance formulas, Equations [146], [156] and [151]
R_f	Frictional resistance
R_x	Reynolds number based on length
R_θ	Reynolds number based on momentum thickness, Equation [61]
r	Factor given in Equation [172]
s	Subscript representing smooth conditions
T	Subscript for quantities at junction of transitional and inner turbulent sublayers; see Table 2
U	Velocity at outer edge of boundary layer
u	Tangential velocity in boundary layer
u'	Fluctuation in u
u_τ	Shear velocity
v'	Fluctuation in normal velocity
x	Distance along boundary layer
x^*	Nondimensional x, Equation [124]
y	Normal distance from wall
y^*	Nondimensional y
α_1, α_2	Velocity profile constants; see Table 2
β_1, β_2	Velocity profile constants; see Table 2
γ	Constant in logarithmic resistance formula for flat plates with engineering roughness, Equation [116]
δ	Boundary layer thickness
δ^*	Displacement thickness
ϵ	Boussinesq eddy viscosity, Equation [22]
θ	Momentum thickness
κ	Slope of ϵ with respect to y
λ	Constant for engineering roughness, Equation [109]

μ	Viscosity of fluid
ν	Kinematic viscosity, μ/ρ
ρ	Density of fluid
σ	Local resistance parameter, U/u_τ
τ	Shearing stress
τ_w	Shearing stress at wall

ABSTRACT

By means of the similarity laws of the turbulent flow in boundary layers, a general relation is developed for the frictional resistance of flat plates with arbitrary roughness from which logarithmic formulas are obtained for the special cases of the fully rough regine, the quasi-smooth regime, and engineering roughness. Among other uses, this general relation permits for the first time a simple rational extrapolation to full-scale conditions of the frictional resistance of model plates covered uniformly with any irregular full-scale roughness encountered in practice. Furthermore, for calculations of turbulent boundary layers in pressure gradients on rough surfaces, relations are derived for the local skin friction and velocity profile shape parameter in terms of momentum thickness Reynolds number. Simplified relations are also derived for the velocity profile of the transitional sublayer in the vicinity of a rough surface.

INTRODUCTION

The analysis and prediction of the frictional resistance or drag of rough surfaces presents in general a more complex problem than that of smooth surfaces owing to the geometric diversity of roughness and the accompanying involved effect on turbulent flow.

Since research into the fundamentals of turbulence has yet to achieve a method for predicting frictional resistance, recourse as in the case of smooth surfaces is made to the similarity characteristics of the mean-velocity profiles of shear flows for a general method of analyzing the resistance of rough surfaces.

For turbulent shear flows, such as fully developed viscous flow in pipes or boundary-layer flow on flat plates, the two laws which provide similarity in the mean-velocity profile by linking it to the wall shearing stress are:

1. The inner law or law of the wall which applies to the flow immediately adjacent to the solid boundary.

2. The outer law or velocity-defect law which applies to the remaining outer region of the shear flow.

The overlapping of the range of application of the two laws requires a logarithmic functional form for both similarity laws within the common region of overlap.

The similarity laws of the velocity profile were originally developed for pipe flows by Prandtl[1] and separately by Von Kármán,[2, 3] though with some significant differences. Prandtl

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[1]References are listed on page 44.

used the logarithmic form for the velocity profile of the whole flow which in effect assumed the region of overlap of the two similarity laws to extend over the whole flow. Von Kármán, on the other hand in a more general presentation, used the logarithmic form for only a limited region of overlap and wrote the outer law as an unspecified functional relation for the flow outside the region of overlap. The difference between the Prandtl and Von Kármán procedures is not significant for pipe flows since a logarithmic form fits very closely most of the flow, but it is significant for boundary-layer flows on flat plates since a logarithmic form fits adequately only about 15 percent of the flow.

Inasmuch as the differences between pipe and flat-plate flows were not apparent at the time, Prandtl and Schlichting[4] used the simpler Prandtl procedure of a logarithmic form for the whole boundary-layer flow as well as numerical values from pipe tests in determining the resistance of rough plates from Nikuradse's[5] tests on pipes coated with sand grains. Consequently, one of the purposes of this report is to determine the resistance of sand-coated plates by applying Von Kármán's procedure of a limited logarithmic region together with the somewhat different numerical values from flat-plate tests.

More importantly, the whole subject of the application of the similarity laws to the resistance of rough plates is developed in detail. The logarithmic form of the similarity laws in the region of overlap is derived in a simple fashion which emphasizes the nature of the resulting constants. Detailed consideration is given to delineating the different sublayers within the boundary layer. Expressions are derived for the velocity profile of transitional sublayers of rough surfaces by an extension of a method of Squire[6] for smooth surfaces. The boundary-layer factors of displacement thickness, momentum thickness, and shape parameter are determined for rough surfaces by combining integrated quantities of the velocity profiles of the separate sublayers.

The relation for momentum thickness is then used to determine the coefficient of total resistance of flat plates as a function of Reynolds number and relative roughness in terms of a local-resistance parameter for the general case of roughness. Then by eliminating the local-resistance parameter and dropping terms which become negligible at larger boundary-layer thicknesses, a simpler logarithmic formula is obtained for the total resistance coefficient of flat plates with arbitrary roughness. From this, special logarithmic resistance formulas are developed for the special cases of the fully rough regime, the quasi-smooth regime, and engineering roughness.

The general logarithmic resistance formula is used to prepare resistance charts from empirical roughness characterizations, to obtain roughness characterizations from total resistance data, and to predict full-scale resistance from plate tests with arbitrary roughness. The last procedure should be especially useful in studying the frictional resistance of ship hulls in newly painted and in fouled conditions. Towing a representative sample of the hull roughness would be required instead of difficult full-scale trials.

Finally relations are derived for the local skin friction and the velocity profile shape parameter for use in calculations of turbulent boundary layers in pressure gradients. A method of obtaining such information from towing tests of plates with arbitrary roughness is also included.

BOUNDARY-LAYER CHARACTERISTICS FROM SIMILARITY LAWS

GENERAL

First, an important distinction should be emphasized between the similarity laws and their historical association with turbulence hypotheses, like the mixing-length theory, which have become somewhat untenable by reason of more detailed experimental investigations[7, 8] into the nature of energy transfer in shear flow. As statements of general functional relationships, the similarity laws can have their basis in purely dimensional arguments supported by experimental data and thus be made independent of the mixing-length and like theories.

In the past it was mainly for the purpose of deriving specific functional relations that hypotheses concerning the turbulence mechanism had been introduced into the analysis. These were the mixing-length theory and assumptions about the laminar sublayer by Prandtl[1] in obtaining the logarithmic form of the inner law, and a similarity hypothesis of the turbulent flow pattern by Von Kármán[3] in formulating his particular expression for the outer law. All this can be avoided since the logarithmic form can be derived from the overlapping of the inner and outer laws and the outer law elsewhere can be analytically treated as an unspecified function in arriving at the desired resistance formulas.

INNER LAW OR LAW OF THE WALL

Close to the wall or solid boundary the distribution of the mean velocity u of the turbulent flow parallel to the wall is considered to depend on the normal distance y away from the wall, the shearing stress τ_w at the wall, the density ρ and viscosity μ of the fluid, and various linear parameters defining the roughness of the wall k, k_1, k_2, . . . , or

$$u = f(y, \tau_w, \rho, \mu, k, k_1, k_2, \ldots) \qquad [1]$$

There is in addition the boundary condition at the wall, $u = 0$ at $y = 0$, to be satisfied. By dimensional analysis the variables can be grouped significantly into the following nondimensional ratios:

$$\frac{u}{u_\tau} = f_1\left(y^*, k^*, \frac{k}{k_1}, \frac{k_1}{k_2}, \ldots\right) \qquad [2]$$

or

$$\frac{u}{u_\tau} = f_2\left(\frac{y}{k}, k^*, \frac{k}{k_1}, \frac{k_1}{k_2}, \ldots\right) \qquad [3]$$

where $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$ is a factor having the dimensions of velocity and hence called the friction or shear velocity, $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity of the fluid, $y^* = \frac{u_\tau y}{\nu}$, and $k^* = \frac{u_\tau k}{\nu}$.

Empirically, Equation [2] or [3] which is the statement of the inner law is uniquely defined for fully developed turbulent flow in pipes regardless of the pipe diameter Reynolds number. A similar but somewhat different numerical relationship holds uniquely for flat plates independent of plate length Reynolds number. In addition the inner law has been found to be independent of pressure gradients.[9]

OUTER LAW OR VELOCITY-DEFECT LAW

For the turbulent stream at some distance away from the wall the velocity-defect ratio $\frac{U-u}{u_\tau}$, where U is the maximum velocity, is found experimentally to be directly independent of viscosity and a function only of its relative position in the flow or

$$\frac{U-u}{u_\tau} = F\left(\frac{y}{\delta}\right) \qquad [4]$$

where δ is the thickness of the shear flow. The boundary condition $u = U$ at $y = \delta$ is to be also satisfied. The function F has been found empirically to be independent of Reynolds number and, most significantly, of the roughness of the wall.[3]

The function F is numerically different for boundary-layers on flat plates than for pipe flow owing mainly to the presence of the free outer boundary of boundary-layer flow which has been found to have undulating characteristics.[9] It is also markedly affected by longitudinal pressure gradients.[10]

LOGARITHMIC VELOCITY LAW

Within the boundary layer there is a region where both the inner and outer laws are considered valid. This overlapping leads to a logarithmic relation which will now be derived by a simplified version of Millikan's[11] original demonstration.

Equating the derivative of velocity u with respect to distance y of the inner and outer laws, Equations [2] and [4], results in

$$\frac{\partial u}{\partial y} = \frac{u_\tau^2}{\nu}\frac{\partial f_1}{\partial y^*} = -\frac{u_\tau}{\delta}\frac{dF}{d(y/\delta)} \qquad [5]$$

or

$$y^* \frac{\partial f_1}{\partial y^*} = -\left(\frac{y}{\delta}\right) \frac{dF}{d(y/\delta)} = A \qquad [6]$$

Since the left-hand side of Equation [6] is only a function of y^* and roughness parameters $k^*, \frac{k}{k_1}, \frac{k_1}{k_2}, \ldots$ and the right-hand side is only a function of $\frac{y}{\delta}$, the only quantity satisfying these conditions is a constant A independent of all these variables.

From the left-hand side there results after integration

$$f_1 = \frac{u}{u_\tau} = A \ln y^* + B_1 \left(k^*, \frac{k}{k_1}, \frac{k_1}{k_2}, \ldots\right) \qquad [7]$$

where the constant of integration B_1 is necessarily a function of $k^*, \frac{k}{k_1}, \frac{k_1}{k_2}, \ldots$ from the integration of a partial derivative. Equation [7] is the statement of the inner law in the region of overlap. An alternate expression is

$$\frac{u}{u_\tau} = A \ln \frac{y}{k} + B_2 \left(k^*, \frac{k}{k_1}, \frac{k_1}{k_2}, \ldots\right) \qquad [8]$$

where

$$B_2 = B_1 + A \ln k^* \qquad [9]$$

From the right-hand side of Equation [6] the outer law in the region of overlap is

$$F = \frac{U-u}{u_\tau} = -A \ln \frac{y}{\delta} + B_3 \qquad [10]$$

This establishes the logarithmic form of both the inner and outer laws in the region of overlap. A very important result is that the factor B_1 or B_2 is seen to be solely a function of roughness Reynolds number k^* for a particular roughness configuration, $\frac{k}{k_1}, \frac{k_1}{k_2}, \ldots,$ and that A is independent of roughness. Hence the frictional effects of any particular roughness may be considered defined when B_1 or B_2 is experimentally determined as a function of k^*. This will be termed the *resistance characterization* of a roughness configuration.

Now equating the velocities from the inner and outer laws in the region of overlap gives important relations for the local skin friction.

From Equations [7] and [10]

$$\sigma \equiv \frac{U}{u_\tau} = A \ln \frac{u_\tau \delta}{\nu} + B_3 + B_1 \qquad [11]$$

or from Equations [8] and [10]

$$\sigma \equiv \frac{U}{u_\tau} = A \ln \frac{\delta}{k} + B_3 + B_2 \qquad [12]$$

where σ is a local resistance parameter. With local coefficient of resistance $C_\tau \equiv \frac{2\tau_w}{\rho U^2} = 2\left(\frac{u_\tau}{U}\right)^2 = \frac{2}{\sigma^2}$ the preceding equations appear as

$$\frac{\sqrt{2}}{\sqrt{C_\tau}} = A \ln \left[\sqrt{C_\tau}\left(\frac{U\delta}{\nu}\right)\right] + B_3 + B_1 \qquad [13]$$

or

$$\frac{\sqrt{2}}{\sqrt{C_\tau}} = A \ln \frac{\delta}{k} + B_3 + B_2 \qquad [14]$$

TYPES OF FLOW REGIMES ON ROUGH SURFACES

In Figure 1, which shows the resistance characterization for Nikuradse's sand rough pipes, it is seen that B_1 is constant for small values of k^* and that B_2 is constant for large values of k^*. For intermediate values of k^*, both B_1 and B_2 vary with k^*. From this the following classification of turbulent flow past rough surfaces can be made which is significant from the viewpoint of skin friction:

1. Quasi-smooth regime

For small values of k^* starting from zero, B_1 is constant with respect to k^* and has the same value as for a smooth flow. According to Equation [9], B_2 then plots as a straight line. Most important of all as given by Equation [13] the local coefficient of resistance C_τ is the same as for a smooth surface and is solely a function of Reynolds number $\frac{U\delta}{\nu}$.

2. General rough regime

With B_1 and B_2 both varying with k^*, C_τ is a function of Reynolds number $\frac{U\delta}{\nu}$ and relative roughness $\frac{k}{\delta}$. $\left[k^* = \sqrt{\frac{C_\tau}{2}}\left(\frac{U\delta}{\nu}\right)\left(\frac{k}{\delta}\right)\right]$.

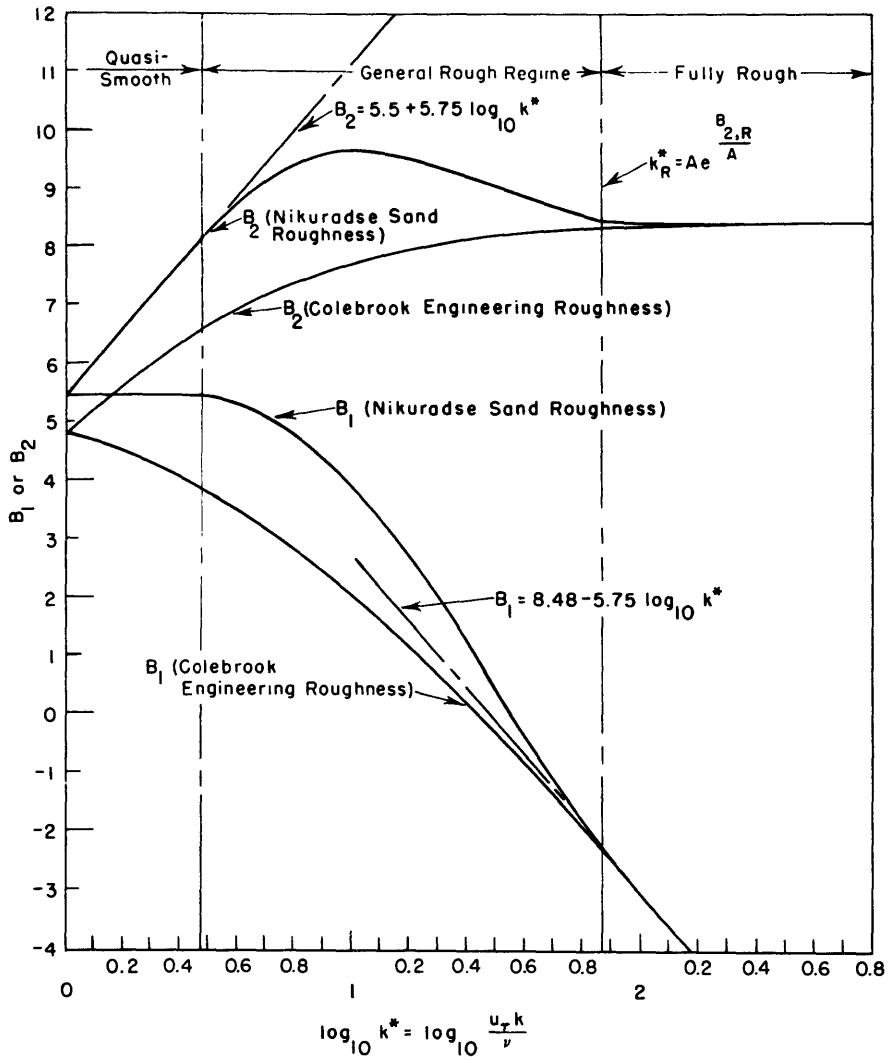

Figure 1 – Resistance Characterization for Roughness in Pipes

3. Fully rough regime

At large values of k^*, B_2 is constant with respect to k^* and B_1 plots as a straight line according to Equation [9]. C_τ is solely a function of the relative roughness $\frac{k}{\delta}$ and independent of Reynolds number.

SUBLAYERS WITHIN TURBULENT SHEAR FLOWS

Various sublayers within the turbulent shear flow can be distinguished according to the behavior of the velocity profiles as indicated schematically in Figure 2. From the wall outward the sublayers are in succession:

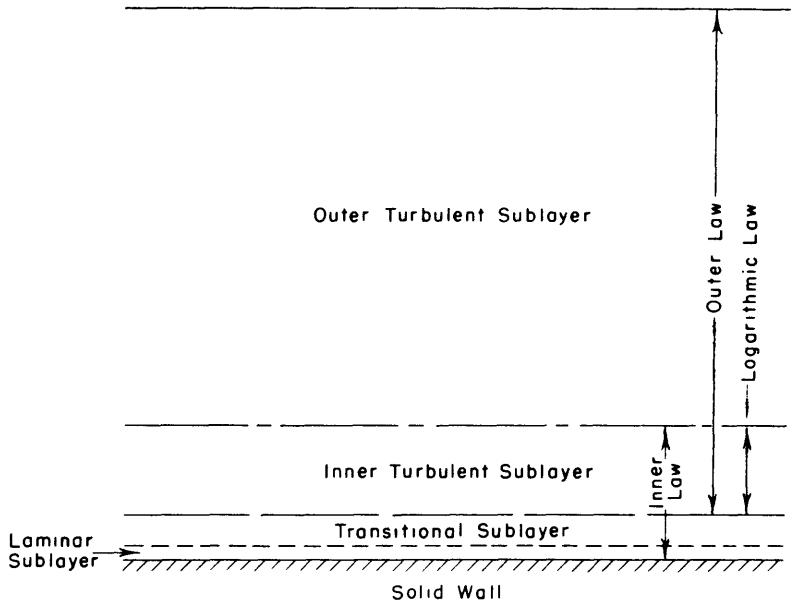

Figure 2 – Sublayers within Turbulent Shear Flows

1. Laminar sublayer

This is the very thin layer in contact with the wall where the flow is laminar and consequently the shearing stress τ is given by

$$\tau = \mu \frac{du}{dy} \qquad [15]$$

The thickness of the laminar sublayer diminishes with increasing roughness, and finally vanishes in the case of the fully rough flow regime.

2. Transitional sublayer

In this layer the effect of turbulence appears so that the shearing stress is given by

$$\tau = \mu \frac{du}{dy} - \rho \overline{u'v'} \qquad [16]$$

where $-\rho \overline{u'v'}$ is the Reynolds turbulent shearing stress.

3. Inner turbulent sublayer

The inner and outer laws overlap in this layer resulting in the logarithmic velocity law.

4. Outer turbulent sublayer

In this region the wall has no direct effect and the velocity law is the nonlogarithmic part of the outer law.

VELOCITY LAW FOR THE LAMINAR SUBLAYER

Inasmuch as the inner law Equation [2] provides only the general statement that the velocity ratio $\frac{u}{u_\tau}$ is a function of Reynolds number y^* (laminar flow is independent of roughness), other considerations must be applied in order to obtain a specific functional relationship for the laminar sublayer. If the laminar motion within the sublayer is assumed essentially a parallel flow, the shearing stress τ is given as

$$\tau = \mu \frac{du}{dy} \qquad [17]$$

Within the thin sublayer the variation of τ with y is negligible; hence

$$\tau = \tau_w \qquad [18]$$

Consequently

$$\frac{du}{dy} = \frac{\tau_w}{\mu} \qquad [19]$$

and with $u = 0$ at $y = 0$,

$$u = \frac{\tau_w}{\mu} y \qquad [20]$$

or

$$\frac{u}{u_\tau} = y^* \qquad [21]$$

Empirical verification for this relation has been obtained by Laufer[7] by measurements in a closed channel.

VELOCITY LAW FOR THE TRANSITIONAL SUBLAYER OF SMOOTH SURFACES

Expressions for the velocity law of the transitional sublayer for smooth surfaces have been derived by various authors on the basis of assumptions concerning the variation of the turbulent shearing stress with y. In general, since the shearing stress has both laminar and turbulent contributions,

$$\tau = \mu \frac{du}{dy} - \rho \overline{u'v'} \qquad [16]$$

where $-\rho\,\overline{u'v'}$ is the Reynolds stress formed by the average of the turbulent velocity fluctuations u' and v' in the x and y directions respectively. If the Boussinesq eddy viscosity ϵ is introduced where

$$\epsilon = -\frac{\overline{u'v'}}{\frac{du}{dy}} \qquad [22]$$

then

$$\tau = \rho\,(\nu + \epsilon)\frac{du}{dy} \qquad [23]$$

If it is assumed that the variation of τ with y across the thin sublayer is negligible, then $\tau = \tau_w$ and

$$u_\tau^2 = (\nu + \epsilon)\frac{du}{dy} \qquad [24]$$

Various investigators have assumed in effect different variations of ϵ with y.

Squire[6] by dimensional reasoning and by considering the turbulence effect to start just at the outer edge of the laminar sublayer uses a linear variation of ϵ across the transitional sublayer or

$$\epsilon = \kappa\, u_\tau (y - y_L) \qquad [25]$$

where κ is a constant of proportionality and y_L is the thickness of the laminar sublayer. Other authors use the mixing-length hypothesis wherein

$$\epsilon = l^2 \left|\frac{du}{dy}\right| \qquad [26]$$

l being the mixing length. Rotta[12] and Hudimoto[13] independently assume

$$l = \kappa\,(y - y_L) \qquad [27]$$

or

$$\epsilon = \kappa^2\,(y - y_L)^2 \left|\frac{du}{dy}\right| \qquad [28]$$

Hama[14] uses for both the laminar and transitional sublayers

$$l = \kappa y^2 \qquad [29]$$

or

$$\epsilon = \kappa^2 y^4 \left| \frac{du}{dy} \right| \qquad [30]$$

whereas Van Driest[15] assumes a still more complicated relation

$$l = \kappa y \left(1 - e^{-\frac{y}{\text{const}}} \right) \qquad [31]$$

or

$$\epsilon = \kappa^2 y^2 \left(1 - e^{-\frac{y}{\text{const}}} \right)^2 \left| \frac{du}{dy} \right| \qquad [32]$$

The procedures of both Squire and Rotta lead to expressions in closed form, that of Squire being much simpler. On the other hand Hama obtained a much more complicated expression involving elliptic integrals and Van Driest obtained a still more unwieldly expression requiring numerical integration. In Figure 3 where the various formulations are compared with Laufer's[16] pipe data, it is to be noted that the simple expression of Squire is an adequate approximation. The method of Squire will be extended in this report to cover the case of rough surfaces. However, full credit is due Rotta[12] for first analyzing the transitional sublayers of rough surfaces.

Squire's procedure for smooth surfaces will be now completed. Substituting Squire's ϵ, Equation [25], into Equation [24] results in

$$\frac{du}{dy} = \frac{u_\tau^2}{\nu + \kappa u_\tau (y - y_L)} \qquad [33]$$

Integrating and utilizing the initial condition that

$$\frac{u}{u_\tau} = \frac{u_\tau y_L}{\nu} \text{ at } y = y_L \qquad [34]$$

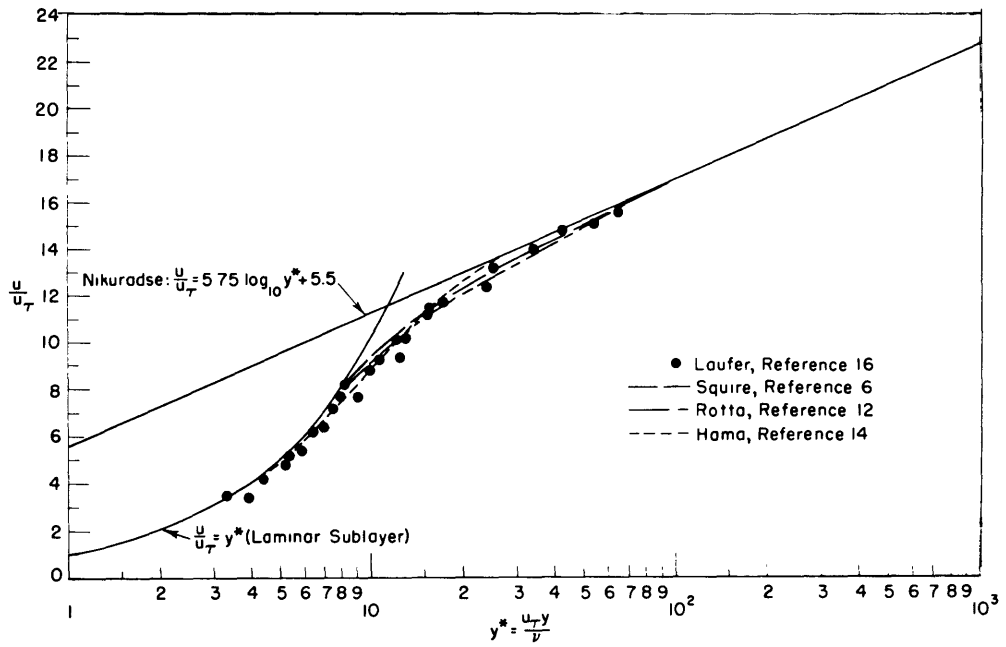

Figure 3 – Comparison of Various Formulations for Transitional Sublayer of Smooth Pipes

yields

$$\frac{u}{u_\tau} = \frac{1}{\kappa} \ln \left[\frac{u_\tau (y - y_L)}{\nu} + \frac{1}{\kappa} \right] - \frac{1}{\kappa} \ln \frac{1}{\kappa} + \frac{u_\tau y_L}{\nu} \qquad [35]$$

or

$$\frac{u}{u_\tau} = A \ln \left(y^* - B_1 + A \ln \frac{e}{A} \right) + B_1 \qquad [36]$$

where

$$A = \frac{1}{\kappa} \qquad [37]$$

$$B_1 = y_L{}^* - A \ln A \qquad [38]$$

or

$$y_L{}^* = B_1 + A \ln A \qquad [39]$$

For large y^* the velocity law asymptotically becomes

$$\frac{u}{u_\tau} = A \ln y^* + B_1 \qquad [40]$$

which is the form for the logarithmic law. Hence the A and B_1 of Equation [40] and those of the inner law Equation [7] are identical.

It is to be noted that Equation [39] gives the thickness of the laminar sublayer when values for B_1 and A are substituted in the expression.

VELOCITY LAW FOR THE TRANSITIONAL SUBLAYER OF THE GENERAL ROUGH REGIME

The velocity law for the smooth transitional sublayer, Equation [36], can be written as

$$\frac{u}{u_\tau} = A \ln (y^* - J_1) + B_1 \qquad [41]$$

where

$$J_1 = B_1 - A \ln \frac{e}{A} = y_L{}^* - A \qquad [42]$$

Since B_1 is a function of k^* for a particular roughness, then J_1 and $y_L{}^*$ are also functions of k^*. For the general rough regime J_1 will have a limiting value of $-A$ since $y_L{}^*$ is zero for the case of the fully rough regime.

The velocity law for the transitional sublayer of the general rough regime may be restated as

$$\frac{u}{u_\tau} = A \ln \left(\frac{y}{k} - J_2 \right) + B_2 \qquad [43]$$

where

$$J_2 = \frac{y_L{}^* - A}{k^*} = \frac{J_1}{k^*} = \left(\frac{y}{k} \right)_L - \frac{A}{k^*} \qquad [44]$$

Also

$$\left(\frac{y}{k} \right)_L = \frac{B_2 + A \ln \dfrac{A}{k^*}}{k^*} \qquad [45]$$

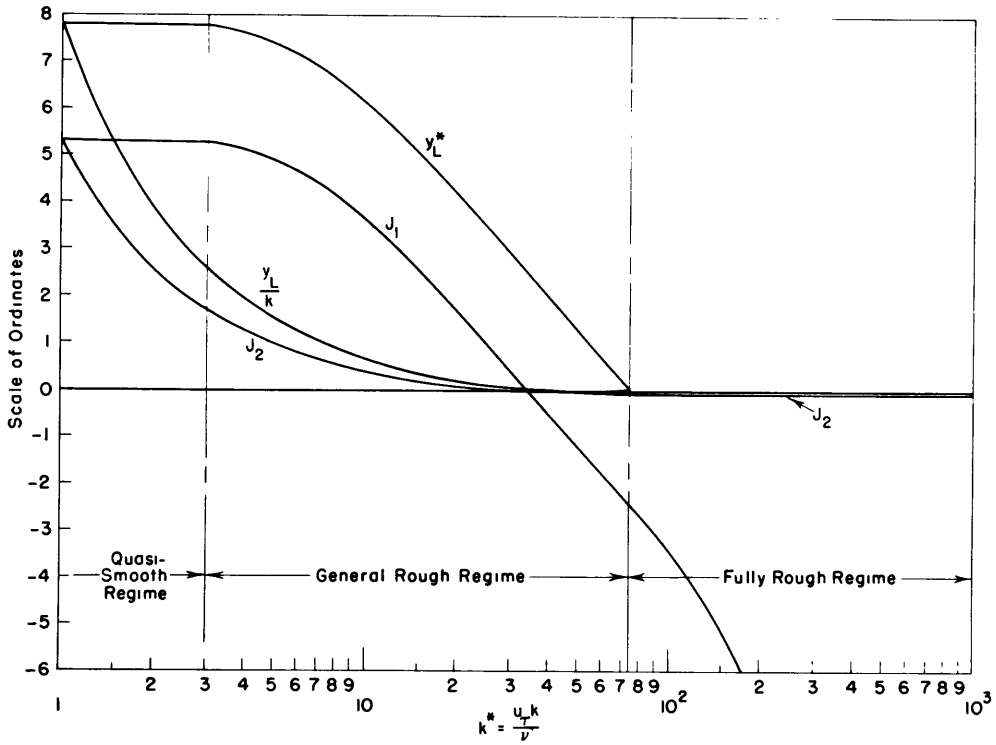

Figure 4 – Variation of Transitional Sublayer Factors for Nikuradse Sand Pough Pipes

Figure 4 shows a plot of $y_L{}^*$, J_1, and J_2 as a function of k^* calculated for Nikuradse's sand rough pipes.

VELOCITY LAW FOR THE TRANSITIONAL SUBLAYER OF THE FULLY ROUGH REGIME

In this case the laminar sublayer has vanished, $y_L = 0$, and the plane of reference for y, $y = 0$, is defined where the velocity u is zero on an average over the rough surface. There is then an initial shearing stress due to turbulence represented by ϵ_0 at the start of the transitional sublayer. Then

$$\epsilon = \epsilon_0 + \kappa u_\tau y \qquad [46]$$

and

$$\frac{du}{dy} = \frac{u_\tau^2}{\nu + \epsilon_0 + \kappa u_\tau y} \qquad [47]$$

Integrating and utilizing the initial condition $u = 0$ at $y = 0$ results in

$$\frac{u}{u_\tau} = \frac{1}{\kappa} \ln \left(1 + \frac{\kappa u_\tau y}{\nu + \epsilon_0} \right) \qquad [48]$$

or

$$\frac{u}{u_\tau} = A \ln \left(\frac{y}{k} - J_2 \right) + B_2 \qquad [49]$$

where

$$J_2 = -e^{-\frac{B_2}{A}} \qquad [50]$$

and

$$B_2 = A \ln \frac{k^*}{A \left(1 + \frac{\epsilon_0}{\nu} \right)} \qquad [51]$$

It is interesting to observe that when Equation [49] is rewritten as

$$\frac{u}{u_\tau} = A \ln \left(e^{\frac{B_2}{A}} \frac{y}{k} + 1 \right) \qquad [52]$$

it is identical to the velocity law of Prandtl and Schlichting[4] in which the unit constant had been arbitrarily added to the argument of the logarithm to make $u/u_\tau = 0$ at $y/k = 0$. The results for the velocity laws of the transitional sublayer are summarized in Table 1. Representative velocity profiles of the inner law for sand rough pipes are shown in Figures 5 and 6.

Equating Equations [44] and [50] and setting $y_L^* = 0$ specifies the start of the fully rough regime in terms of

$$k_R^* = A e^{\frac{B_2}{A}} \qquad [53]$$

The value of k_R^* determined from Equation [53] agrees well with the experimental point where B_2 becomes constant as shown in Figure 1.

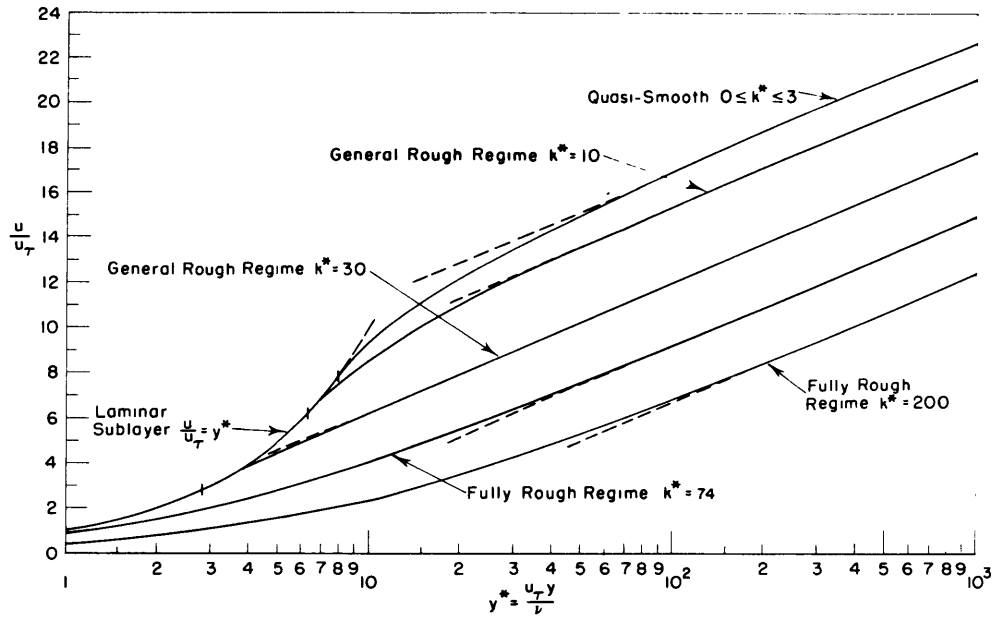

Figure 5 – Plot of Inner Law for Nikuradse's Sand Rough Pipe Data

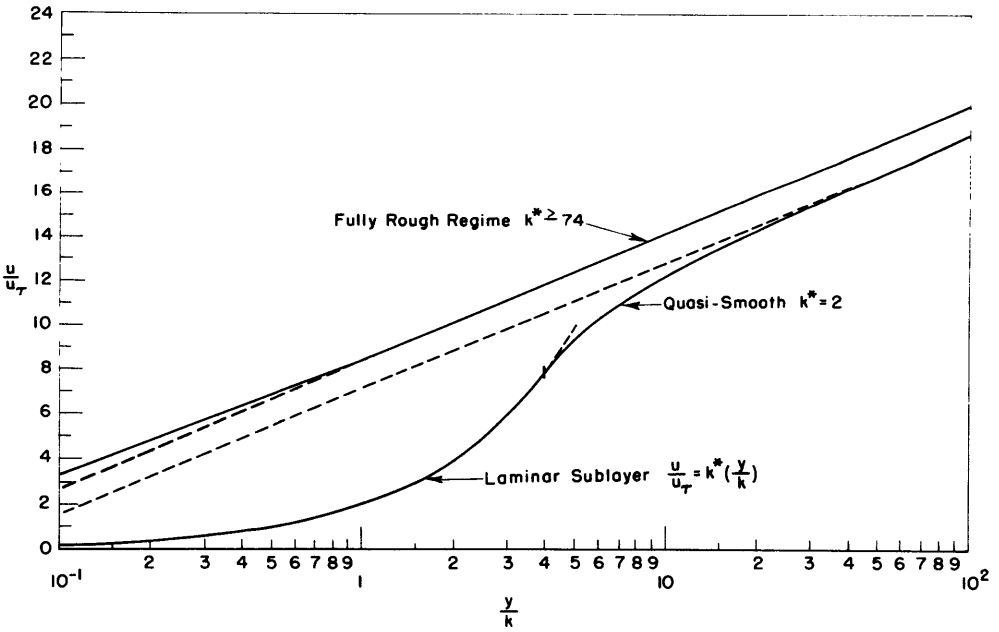

Figure 6 – Plot of Inner Law for Nikuradse's Sand Rough Pipe Data

TABLE 1

Summary of Factors for the Velocity Laws of Transitional Sublayers

Factor	Quasi-Smooth Regime $0 \le k^* \le k_s^*$	General Rough Regime $k_s^* < k^* < k_R^*$	Fully Rough Regime $k^* \ge k_R^*$
B_1	$B_{1,s}$ (const)	$f(k^*)$	$B_{2,R} - A \ln k^*$
B_2	$B_{1,s} + A \ln k^*$	$f(k^*)$	$B_{2,R}$ (const)
y_L^*	$B_{1,s} + A \ln A$	$B_1 + A \ln A =$ $B_2 + A \ln \frac{A}{k^*}$	0
J_1	$B_{1,s} - A \ln \frac{e}{A}$	$B_1 - A \ln \frac{e}{A} =$ $B_2 - A \ln \left(\frac{ek^*}{A}\right)$	$-k^* e^{-\frac{B_{2,R}}{A}}$
J_2	$\frac{1}{k^*}\left(B_{1,s} - A \ln \frac{e}{A}\right)$	$\frac{1}{k^*}\left(B_1 - A \ln \frac{e}{A}\right)$	$-e^{-\frac{B_{2,R}}{A}}$
$\frac{u}{u_\tau}$	$A \ln (y^* - J_{1,s}) + B_{1,s}$	$A \ln (y^* - J_1) + B_1 =$ $A \ln \left(\frac{y}{k} - J_2\right) + B_2$	$A \ln \left(\frac{y}{k} - J_{2,R}\right) + B_{2,R}$

The limit of quasi-smooth flow is specified by k_s^* which is the criterion for permissible roughness. The value of k_s^* may be empirically determined at the point where B_1 is substantially constant as shown in Figure 1. However, Goldstein[17] presents a method based on the flow separation induced by an individual roughness element.

BOUNDARY-LAYER PARAMETERS

In order to obtain relations for the boundary-layer parameters of displacement thickness δ^*, momentum thickness θ, and shape parameter H from the similarity laws, integrations of the velocity u with respect to y are necessary over the boundary layer. This is accomplished piecewise for each sublayer by using the appropriate velocity law as shown in Table 2.

Since the displacement thickness δ^* is defined as

$$\delta^* \equiv \int_0^\delta \left(1 - \frac{u}{U}\right) dy \qquad [54]$$

TABLE 2

Velocity Profile Integrals

on	Limits	Velocity Law	$\int \frac{u}{u_\tau} dy^*$ or $\int \frac{u}{u_\tau} d\left(\frac{y}{k}\right)$	$\int \left(\frac{u}{u_\tau}\right)^2 dy^*$ or $\int \left(\frac{u}{u_\tau}\right)^2 d\left(\frac{y}{k}\right)$
inar ayer	$0 \le y^* \le y_L{}^*$	$\frac{u}{u_\tau} = y^*$	$\frac{1}{2} y_L{}^{*2}$	$\frac{1}{3} y_L{}^{*3}$
	$0 \le \frac{y}{k} \le \left(\frac{y}{k}\right)_L$	$\frac{u}{u_\tau} = k^* \left(\frac{y}{k}\right)$	$\frac{k^*}{2} \left(\frac{y}{k}\right)_L^2$	$\frac{k^{*2}}{3} \left(\frac{y}{k}\right)_L^3$
tional er	$y_L{}^* \le y^* \le y_T{}^*$	$\frac{u}{u_\tau} = A \ln (y^* - J_1) + B_1$	$\left(y_T^* - J_1\right)\left(\frac{u}{u_\tau}\right)_T - A y_T{}^*$	$\left(y_T{}^* - J_1\right)\left[\left(\frac{u}{u_\tau}\right)_T - A\right]^2 - A(y_L^* - A)^2 + A^2 (y_T{}^* - y_L{}^*)$
	$\left(\frac{y}{k}\right)_L \le \frac{y}{k} \le \left(\frac{y}{k}\right)_T$	$\frac{u}{u_\tau} = A \ln (y^* - J_2) + B_2$	$\left[\left(\frac{y}{k}\right)_T - J_2\right]\left(\frac{u}{u_\tau}\right)_T - A\left(\frac{y}{k}\right)_T$	$\left[\left(\frac{y}{k}\right)_T - J_2\right]\left[\left(\frac{u}{u_\tau}\right)_T - A\right]^2 - \left[\left(\frac{y}{k}\right)_L - J_2\right]\left[\left(\frac{u}{u_\tau}\right)_L - A\right]^2 + I^2$
ılent yer	$y_T{}^* \le y^* \le y_G{}^*$ $y_G{}^* = \left(\frac{y}{\delta}\right)_G \left(\frac{u_\tau \delta}{\nu}\right)$	$\frac{u}{u_\tau} = A \ln y^* + B_1$	$y_G{}^*\left[\left(\frac{u}{u_\tau}\right)_G - A\right] - y_T{}^*\left[\left(\frac{u}{u_\tau}\right)_T - A\right]$	$y_G{}^*\left[\left(\frac{u}{u_\tau}\right)_G - A\right]^2 - y_T{}^*\left[\left(\frac{u}{u_\tau}\right)_T - A\right]^2 + A^2\left(y_G{}^* - y_T{}^*\right)$
	$\left(\frac{y}{k}\right)_T \le \frac{y}{k} \le \left(\frac{y}{k}\right)_G$ $\left(\frac{y}{k}\right)_G = \left(\frac{y}{\delta}\right)_G \left(\frac{\delta}{k}\right)$	$\frac{u}{u_\tau} = A \ln \frac{y}{k} + B_2$	$\left(\frac{y}{k}\right)_G\left[\left(\frac{u}{u_\tau}\right)_G - A\right] - \left(\frac{y}{k}\right)_T\left[\left(\frac{u}{u_\tau}\right)_T - A\right]$	$\left(\frac{y}{k}\right)_G\left[\left(\frac{u}{u_\tau}\right)_G - 1\right]^2 - \left(\frac{y}{k}\right)_T\left[\left(\frac{u}{u_\tau}\right)_T - A\right]^2 + A^{2\prime}\left[\left(\frac{y}{k}\right)_G - \left(\frac{y}{k}\right)_T\right]$
ılent yer	$\left(\frac{y}{\delta}\right)_G \le \frac{y}{\delta} \le 1$	$\frac{u}{u_\tau} = \sigma - F$	$\left(\frac{u_\tau \delta}{\nu}\right)\left\{\left[1 - \left(\frac{y}{\delta}\right)_G\right]\sigma - I_1\right\}$	$\left(\frac{u_\tau \delta}{\nu}\right)\left\{\left[1 - \left(\frac{y}{\delta}\right)_G\right]\sigma^2 - 2\sigma I_1 + I_2\right\}$
			$\frac{\delta}{k}\left\{\left[1 - \left(\frac{y}{\delta}\right)_G\right]\sigma - I_1\right\}$	$\frac{\delta}{k}\left\{\left[1 - \left(\frac{y}{\delta}\right)_G\right]\sigma^2 - 2\sigma I_1 + I_2\right\}$
dary	$0 \le y^* \le \frac{u_\tau \delta}{\nu}$		$\left(\frac{u_\tau \delta}{\nu}\right)(\sigma - D_1) + \alpha_1$	$\left(\frac{u_\tau \delta}{\nu}\right)\left(D_2 - 2\sigma D_1 + \sigma^2\right) + \beta_1$
	$0 \le \frac{y}{k} \le \frac{\delta}{k}$		$\frac{\delta}{k}(\sigma - D_1) + \alpha_2$	$\frac{\delta}{k}(D_2 - 2\sigma D_1 + \sigma^2) + \beta_2$

$$I_1 = \int_{\left(\frac{y}{\delta}\right)_G}^{1} F\, d\left(\frac{y}{\delta}\right), \quad I_2 = \int_{\left(\frac{y}{\delta}\right)_G}^{1} F^2\, d\left(\frac{y}{\delta}\right), \quad \alpha_1 = \frac{y_L{}^{*2}}{2} - J_1\left(\frac{u}{u_\tau}\right)_T, \quad \alpha_2 = \frac{\alpha_1}{k^*}, \quad D_1 = \left(\frac{y}{\delta}\right)_G (F_G + A) + I_1$$

$$\beta_1 = \frac{y_L{}^{*3}}{3} - J_1\left[\left(\frac{u}{u_\tau}\right)_T - A\right]^2 - A\left(y_L{}^{*2} - A y_L^* + A^2\right), \quad \beta_2 = \frac{\beta_1}{k^*}, \quad D_2 = \left(\frac{y}{\delta}\right)_G \left[(F_G + A)^2 + A^2\right] + I_2$$

then from the integrated quantities from the similarity laws in Table 2

$$\frac{\delta^*}{\delta} = \frac{1}{\sigma}\left(D_1 - \frac{\alpha_1}{\frac{u_\tau \delta}{\nu}}\right) = \frac{1}{\sigma}\left(D_1 - \frac{\alpha_2}{\frac{\delta}{k}}\right) \qquad [55]$$

$$\frac{U\delta^*}{\nu} = D_1\left(\frac{u_\tau \delta}{\nu}\right) - \alpha_1 \qquad [56]$$

and

$$\frac{\delta^*}{k} = \frac{1}{\sigma}\left(D_1 \frac{\delta}{k} - \alpha_2\right) \qquad [57]$$

In a like manner for momentum thickness θ defined as

$$\theta \equiv \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \qquad [58]$$

$$\frac{\theta}{\delta} = \frac{1}{\sigma}\left(D_1 + \frac{\alpha_1}{\frac{u_\tau \delta}{\nu}}\right) - \frac{1}{\sigma^2}\left(D_2 + \frac{\beta_1}{\frac{u_\tau \delta}{\nu}}\right) \qquad [59]$$

$$\frac{\theta}{\delta} = \frac{1}{\sigma}\left(D_1 + \frac{\alpha_2}{\frac{\delta}{k}}\right) - \frac{1}{\sigma^2}\left(D_2 + \frac{\beta_2}{\frac{\delta}{k}}\right) \qquad [60]$$

$$R_\theta \equiv \frac{U\theta}{\nu} = \frac{u_\tau \delta}{\nu}\left(D_1 - \frac{D_2}{\sigma}\right) + \alpha_1 - \frac{\beta_1}{\sigma} \qquad [61]$$

and

$$\frac{\theta}{k} = \frac{\delta}{k}\left(\frac{D_1}{\sigma} - \frac{D_2}{\sigma^2}\right) + \frac{\alpha_2}{\sigma} - \frac{\beta_2}{\sigma^2} \qquad [62]$$

Also for shape parameter H defined as

$$H \equiv \frac{\delta^*}{\theta} \qquad [63]$$

$$\frac{1}{H} = 1 - \frac{1}{\sigma}\left[\frac{D_2 + \dfrac{\beta_1 - 2\sigma\alpha_1}{\dfrac{u_\tau \delta}{\nu}}}{D_1 - \dfrac{\alpha_1}{\dfrac{u_\tau \delta}{\nu}}}\right] \qquad [64]$$

or

$$\frac{1}{H} = 1 - \frac{1}{\sigma}\left[\frac{D_2 + \dfrac{\beta_2 - 2\sigma\alpha_2}{\delta/k}}{D_1 - \dfrac{\alpha_2}{\delta/k}}\right] \qquad [65]$$

Since $\frac{u_\tau \delta}{\nu}$ and $\frac{\delta}{k}$ are given in terms of σ in Equations [11] and [12], δ^*, θ, and H are functions of σ according to the preceding relationships.

For zero pressure gradient the factors D_1 and D_2 are constant for the three types of roughness flow. However, B_1, α_1, and β_1 are constant only for the quasi-smooth regime while B_2, α_2, and β_2 are constant only for the fully rough regime. In the case of the general rough regime B_1, B_2, α_1, α_2, β_1, and β_2 are all functions of k^*.

FRICTIONAL RESISTANCE OF FLAT PLATES

GENERAL

The frictional resistance or drag of a flat plate in two-dimensional flow without pressure gradients which corresponds to a flat plate moving lengthwise in an infinite fluid is determined by the momentum thickness of the boundary-layer flow leaving the trailing edge as a wake. The Von Kármán momentum equation neglecting the small effect of the normal Reynolds stress term gives[1]

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho U^2} \qquad [66]$$

and since the resistance coefficient $C_f \equiv \dfrac{R_f}{\frac{1}{2}\rho U^2 x b}$, where R_f is the total frictional resistance on one side and b the width of the plate, is obtained from the summation of the local resistance

or

$$\frac{C_f}{2} = \frac{1}{x}\int_0^x \frac{\tau_w}{\rho U^2}\, dx \qquad [67]$$

then

$$\frac{C_f}{2} = \frac{\theta}{x} = \frac{R_\theta}{R_x} = \frac{\frac{\theta}{k}}{\frac{x}{k}} \qquad [68]$$

where $R_x \equiv \frac{Ux}{\nu}$.

Since by definition

$$\sigma^2 \equiv \frac{1}{\frac{\tau_w}{\rho U^2}} \qquad [69]$$

then Equation [66] becomes

$$dx = \sigma^2 d\theta \qquad [70]$$

or

$$dR_x = \sigma^2 dR_\theta \qquad [71]$$

or

$$d\left(\frac{x}{k}\right) = \sigma^2 d\left(\frac{\theta}{k}\right) \qquad [72]$$

For a uniform roughness where k is constant with respect to x

$$R_x = \int \sigma^2 dR_\theta = \sigma^2 R_\theta - 2\int R_\theta \sigma d\sigma + \text{const} \qquad [73]$$

or

$$\frac{x}{k} = \int \sigma^2 \, d\left(\frac{\theta}{k}\right) = \sigma^2 \frac{\theta}{k} - 2\int \frac{\theta}{k} \sigma d\sigma + \text{const} \qquad [74]$$

The general case of the frictional resistance of a rough plate is to be now considered wherein B_1 or B_2 is an unspecified function of k^*. All other cases like that of the fully rough regime or the quasi-smooth regime are then special cases.

Inserting the relation $\frac{\delta}{k}$ from Equation [12] into the relation for $\frac{\theta}{k}$ from Equation [62] and integrating by parts repeatedly results in

$$\int \frac{\theta}{k} \sigma d\sigma = \frac{\delta}{k} \left\{ AD_1 \left[1 - \frac{B_2'}{\sigma} - \frac{A}{\sigma^2} \left(B_2' + B_2'' - \frac{B_2'^2}{A} \right) + \ldots \right] - \frac{AD_2}{\sigma} \left[1 + \frac{\left(A - B_2'\right)}{\sigma} + \ldots \right] \right\}$$
$$+ \int \alpha_2 d\sigma - \int \frac{\beta_2}{\sigma} d\sigma + \text{const} \qquad [75]$$

where

$$B_2' \equiv \frac{dB_2}{d(\ln k^*)} \qquad [76]$$

and

$$B_2'' \equiv \frac{d^2 B_2}{d(\ln k^*)^2} \qquad [77]$$

Then from Equation [74]

$$\frac{x}{k} = \frac{\delta}{k} \left\{ D_1 \sigma - D_2 - 2AD_1 \left[1 - \frac{B_2'}{\sigma} - \frac{A}{\sigma^2} \left(B_2' + B_2'' - \frac{B_2'^2}{A} \right) + \ldots \right] + \frac{2AD_2}{\sigma} \left[1 + \frac{(A - B_2')}{\sigma} + \ldots \right] \right\} \qquad [78]$$
$$+ \alpha_2 \sigma - \beta_2 - 2\int \alpha_2 d\sigma + 2\int \frac{\beta_2}{\sigma} d\sigma + \text{const}$$

Also since by definition

$$\frac{x}{k} \equiv \frac{R_x}{\sigma k^*} = \frac{R_x \frac{\delta}{k}}{\sigma \frac{u_\tau \delta}{\nu}} \qquad [79]$$

Equation [78] becomes then

$$R_x = \frac{u_\tau \delta}{\nu}\left\{ D_1\sigma^2 - D_2\sigma - 2AD_1\sigma\left[1 - \frac{B_2'}{\sigma} - \frac{A}{\sigma^2}\left(B_2' + B_2'' - \frac{B_2'^2}{A}\right) + \ldots\right] + 2AD_2\left[1 + \frac{(A - B_2')}{\sigma} + \ldots\right]\right\} \qquad [80]$$

$$+ \alpha_1\sigma^2 - \beta_1\sigma - 2\int \alpha_1\sigma d\sigma + 2\int \beta_1 d\sigma + \text{const}$$

Hence from Equation [68] it is seen that C_f is a function of R_x and/or $\frac{x}{k}$ in terms of parameter σ. When σ is eliminated, logarithmic resistance formulas result. To arrive at a relation between C_f and σ, Equations [68] and [73] or [74] are combined to give after neglecting the constant of integration

$$\frac{1}{\sigma^2} = \frac{C_f}{2}\left(1 - \frac{2\int R\theta\sigma d\sigma}{\sigma^2 R_\theta}\right) \qquad [81]$$

or

$$\frac{1}{\sigma^2} = \frac{C_f}{2}\left(1 - \frac{2\int \frac{\theta}{k}\sigma d\sigma}{\sigma^2 \frac{\theta}{k}}\right) \qquad [82]$$

The expressions for $\int \frac{\theta}{k}\sigma d\sigma$ and $\frac{\theta}{k}$ from Equations [75] and [62] are inserted into Equation [82] to give with α_2 and β_2 neglected:

$$\frac{1}{\sigma^2} = \frac{C_f}{2}\left[1 - \frac{2A}{\sigma} + \frac{2AB_2'}{\sigma^2} + \frac{2A^2}{\sigma^3}\left(B_2' + B_2''' - \frac{B_2'^2}{A} + \frac{D_2}{D_1}\right) + \ldots\right] \qquad [83]$$

Through reiteration σ is replaced by C_f within the brackets so that

$$\frac{1}{\sigma^2} = \frac{C_f}{2}\left[1 - 2A\left(\frac{C_f}{2}\right)^{1/2} + 2A\left(A + B_2'\right)\left(\frac{C_f}{2}\right) + \ldots\right] \qquad [84]$$

and by the binomial expansion

$$\frac{1}{\sigma} = \sqrt{\frac{C_f}{2}}\left[1 - A\left(\frac{C_f}{2}\right)^{1/2} + A\left(\frac{A}{2} + B_2'\right)\left(\frac{C_f}{2}\right) + \ldots\right] \qquad [85]$$

and by inversion

$$\sigma = \sqrt{\frac{2}{C_f}}\left[1 + A\left(\frac{C_f}{2}\right)^{1/2} + A\left(\frac{A}{2} - B_2'\right)\left(\frac{C_f}{2}\right) + \ldots\right] \qquad [86]$$

Now, after substituting for $\frac{\delta}{k}$ from Equation [12] and eliminating α_2, β_2, and the constant of integration, $\frac{x}{k}$ in Equation [78] is written in logarithmic form as

$$\ln\frac{x}{k} = \frac{\sigma}{A} - \frac{B_3}{A} - \frac{B_2}{A} + \ln D_1 - \ln\frac{1}{\sigma} + \ln\left[1 - \left(2A + \frac{D_2}{D_1}\right)\frac{1}{\sigma} + \ldots\right] \qquad [87]$$

Substituting the appropriate expressions for σ from Equations [85] and [86] and expanding the logarithm of terms close to unity into a series results in

$$\ln\frac{x}{k}\sqrt{C_f} = \frac{\sqrt{2}}{A}\frac{1}{\sqrt{C_f}} - \frac{1}{\sqrt{2}}\left(\frac{A}{2} + \frac{D_2}{D_1} + B_2'\right)\sqrt{C_f} + 1 - \frac{B_3}{A} - \frac{B_2}{A} + \ln\sqrt{2}\,D_1 \qquad [88]$$

wherein terms of higher order than $\sqrt{C_f}$ have been neglected. Inserting the expression for $\frac{x}{k}$ from Equation [79] and $\frac{1}{\sigma}$ from Equation [85] into [88] produces

$$\ln R_x C_f = \frac{\sqrt{2}}{A}\frac{1}{\sqrt{C_f}} - \frac{1}{\sqrt{2}}\left(\frac{D_2}{D_1} - \frac{A}{2} + B_2'\right)\sqrt{C_f} + 1 - \frac{B_3}{A} - \frac{B_1}{A} + \ln 2D_1 \qquad [89]$$

Equations [88] and [89] represent logarithmic resistance formulas for the general case of constant k^* since B_1 and B_2 are functions of k^*. Separate logarithmic formulas are obtained from Equations [88] or [89] in the cases where B_1 or B_2 is analytically defined in terms of $\ln k^*$.

FULLY ROUGH REGIME

In this case the roughness effects have become strong enough to eliminate the laminar sublayer and make B_2 constant. With $B_2 = B_{2,R}$ and $B_2' = 0$, Equation [88] reduces to

$$\ln\frac{x}{k}\sqrt{C_f} = \frac{\sqrt{2}}{A}\frac{1}{\sqrt{C_f}} - \frac{1}{\sqrt{2}}\left(\frac{A}{2} + \frac{D_2}{D_1}\right)\sqrt{C_f} + 1 - \frac{B_3}{A} - \frac{B_{2,R}}{A} + \ln\sqrt{2}\,D_1 \qquad [90]$$

If the term involving $\sqrt{C_f}$ is linearized with respect to $\frac{1}{\sqrt{C_f}}$

or

$$\sqrt{C_f} = C_1 + \frac{C_2}{\sqrt{C_f}} \qquad [91]$$

then Equation [90] becomes in common logarithms

$$\log_{10} \frac{x}{k} \sqrt{C_f} = \frac{M_1}{\sqrt{C_f}} + N_1 \qquad [92]$$

where

$$M_1 = \frac{1}{2.3026} \left[\frac{\sqrt{2}}{A} - \left(\frac{A}{2} + \frac{D_2}{D_1} \right) \frac{C_2}{\sqrt{2}} \right] \qquad [93]$$

and

$$N_1 = \frac{1}{2.3026} \left[1 - \left(\frac{A}{2} + \frac{D_2}{D_1} \right) \frac{C_1}{\sqrt{2}} - \frac{(B_3 + B_{2,R})}{A} \right] + \log_{10} \sqrt{2}\, D_1 \qquad [94]$$

Furthermore, if $\log_{10} C_f$ is linearized with respect to $\frac{1}{\sqrt{C_f}}$

or

$$\log_{10} C_f = C_3 + \frac{C_4}{\sqrt{C_f}} \qquad [95]$$

then Equation [92] becomes

$$\log_{10} \frac{x}{k} = \frac{M_2}{\sqrt{C_f}} + N_2 \qquad [96]$$

or

$$C_f = \frac{M_2^2}{\left(\log_{10} \frac{x}{k} - N_2 \right)^2} \qquad [97]$$

where

$$M_2 = M_1 - \frac{C_4}{2} \qquad [98]$$

and

$$N_2 = N_1 - \frac{C_3}{2} \qquad [99]$$

QUASI-SMOOTH REGIME

Quasi-smooth flow is effectively that of a smooth surface; B_1 being constant and equal to $B_{1,s}$ and $\frac{dB_1}{d\ln k^*} = 0$.

Since, from Equation [9]

$$B_2' = \frac{dB_2}{d\ln k^*} = \frac{dB_1}{d\ln k^*} + A \qquad [100]$$

Equation [89] reduces to

$$\ln R_x C_f = \frac{\sqrt{2}}{A}\frac{1}{\sqrt{C_f}} - \frac{1}{\sqrt{2}}\left(\frac{A}{2} + \frac{D_2}{D_1}\right)\sqrt{C_f} + 1 - \frac{B_3}{A} - \frac{B_{1,s}}{A} + \ln 2D_1 \qquad [101]$$

If the term involving $\sqrt{C_f}$ is linearized with respect to $\frac{1}{\sqrt{C_f}}$ as in Equation [91], Equation [101] becomes in common logarithms

$$\log_{10} R_x C_f = \frac{M_1}{\sqrt{C_f}} + N_1' \qquad [102]$$

where M_1 is given by Equation [93] and

$$N_1' = N_1 + \frac{B_{2,R} - B_{1,s}}{2.3026A} + \log_{10}\sqrt{2} \qquad [103]$$

Equation [102] is the well-known Kármán-Schoenherr formula which is derived in Reference 20 by a somewhat different procedure.

If $\log_{10} C_f$ is linearized with respect to $\frac{1}{\sqrt{C_f}}$ as indicated in Equation [95], Equation [102] becomes

$$\log_{10} R_x = \frac{M_2'}{\sqrt{C_f}} + N_2' \qquad [104]$$

or

$$C_f = \frac{\left(M_2'\right)^2}{\left(\log_{10} R_x - N_2'\right)^2} \qquad [105]$$

where

$$M_2' = M_1 - C_4 \qquad [106]$$

and

$$N_2' = N_1' - C_3 \qquad [107]$$

ENGINEERING ROUGHNESS

From extensive tests on commercial pipes the behavior of the irregular roughness of the surfaces resulting from manufacturing processes, termed here engineering roughness, has been found to be quite different from that of the artificial sand roughness of Nikuradse.

Whereas B_2 has a maximum value for the Nikuradse roughness, B_2 has a gradual monotonic variation for engineering roughness as shown in Figure 1. Colebrook[19] with the aid of White fitted this variation of B_2 with the following formula which in the notation of this paper can be expressed as

$$B_2 = B_{2,R} - A \ln\left(1 + \frac{\lambda}{k^*}\right) \qquad [108]$$

where

$$\lambda = \exp\left[\frac{1}{A}(B_{2,R} - B_{1,s})\right] \qquad [109]$$

It is seen that for $k^* \to \infty$, $B_2 \to B_{2,R}$.

Furthermore, with Equation [9], [108] becomes

$$B_1 = B_{1,s} - A \ln\left(1 + \frac{k^*}{\lambda}\right) \qquad [110]$$

Then for $k^* \to 0$, $B_1 \to B_{1,s}$. Consequently the relation for engineering roughness is asymptotic to both the quasi-smooth regime and the fully rough regime.

To obtain a logarithmic resistance formula for engineering roughness from Equation [89], the procedure is as follows:

From Equation [108]

$$B_2' = \frac{A}{1 + \frac{k^*}{\lambda}} \qquad [111]$$

Now from Equations [79] and [85]

$$1 + \frac{k^*}{\lambda} = 1 + \frac{R_x}{\lambda \frac{x}{k}}\sqrt{\frac{C_f}{2}}\left(1 - A\sqrt{\frac{C_f}{2}} + \ldots\right) = \left(1 + \frac{R_x}{\lambda \frac{x}{k}}\right)\sqrt{\frac{C_f}{2}}\left[1 - \frac{\left(\frac{A R_x}{\lambda \frac{x}{k}}\right)\left(\frac{C_f}{2}\right)}{1 + \left(\frac{R_x}{\lambda \frac{x}{k}}\right)\left(\frac{C_f}{2}\right)^{1/2}} + \ldots\right] \qquad [112]$$

Using Equations [110], [111], and [112] in [89] results in

$$-\ln\left(\frac{1}{R_x C_f} + \frac{1}{\sqrt{2}\lambda \frac{x}{k}\sqrt{C_f}}\right) = \frac{\sqrt{2}}{A}\frac{1}{\sqrt{C_f}} - \frac{1}{\sqrt{2}}\left(\frac{A}{2} + \frac{D_2}{D_1}\right)\sqrt{C_f} + 1 - \frac{B_3}{A} - \frac{B_{1,s}}{A} + \ln 2 D_1 \qquad [113]$$

If the term containing $\sqrt{C_f}$ is linearized with respect to $\frac{1}{\sqrt{C_f}}$ as in Equation [91], Equation [113] becomes in common logarithms

$$-\log_{10}\left(\frac{1}{R_x C_f}+\frac{1}{\sqrt{2}\lambda\frac{x}{k}\sqrt{C_f}}\right)=\frac{M_1}{\sqrt{C_f}}+N_1' \qquad [114]$$

where M_1 is given by Equation [93] and N_1' by Equation [103].

Equation [114] reduces to the case for smooth flow, Equation [102], when $\frac{1}{\sqrt{2}\lambda\frac{x}{k}\sqrt{C_f}} \to 0$ and to the case for the fully rough regime, Equation [92], when $\frac{1}{R_x C_f} \to 0$.

In a less rigorous fashion an explicit expression between C_f and R_x and $\frac{x}{k}$ may be written which agrees in the limits with Equation [104] for the quasi-smooth regime and with Equation [96] for the fully rough regime or

$$-\log_{10}\left[\frac{1}{R_x}+\frac{1}{\left(\gamma\frac{x}{k}\right)^m}\right]=\frac{M_2'}{\sqrt{C_f}}+N_2' \qquad [115]$$

where M_2' and N_2' are given in Equations [106] and [107],

$$\log_{10}\gamma=\frac{N_2' M_2}{M_2'}-N_2 \qquad [116]$$

and

$$m=\frac{M_2'}{M_2} \qquad [117]$$

APPLICATION OF THE GENERAL LOGARITHMIC RESISTANCE FORMULA TO ARBITRARY ROUGHNESS

PREPARATION OF RESISTANCE DIAGRAMS FROM RESISTANCE CHARACTERIZATIONS

If the resistance characterization is known for an arbitrary roughness, e.g., B_1 or B_2 has been found empirically as a function of k^*, the procedure for preparing a resistance diagram for flat plates, C_f as a function of R_x and x/k, is as follows:

The general logarithmic resistance formula, Equation [89], is restated in common logarithms and Equation [100] substituted for B_2'

$$\log_{10} R_x C_f = \frac{\sqrt{2}}{2.3026A}\frac{1}{\sqrt{C_f}} - \frac{\sqrt{C_f}}{2.3026\sqrt{2}}\left(\frac{A}{2} + \frac{D_2}{D_1} + \frac{1}{2.3026}\frac{dB_1}{d\log_{10}k^*}\right) + \frac{1}{2.3026}\left(1 - \frac{B_3}{A} - \frac{B_1}{A}\right) + \log_{10} 2D_1 \qquad [118]$$

Since B_1 and $\frac{dB_1}{d\log_{10}k^*}$ are functions of k^* and A, B_3, D_1, and D_2 are constants, Equation [118] represents the variation of C_f with R_x for constant k^*. A rapid method of plotting Equation [118] is to relate it to smooth resistance so that for the same value of C_f the difference in R_x is

$$\log_{10} R_x - \log_{10} R_{x,s} = \frac{B_{1,s} - B_1}{2.3026A} \qquad [119]$$

where the effect due to $\frac{dB_1}{d\log_{10}k^*}$ is neglected and the subscript s refers to the smooth line. Consequently the resistance line of constant k^* for the rough case is offset from the smooth line by a constant amount in the direction of $\log_{10} R_x$.

To determine the resistance lines of constant x/k from the lines of constant k^* the following procedure is used:

Since by definition

$$R_x \equiv k^* \sigma \frac{x}{k} \qquad [120]$$

the insertion of the relation for σ from Equation [86] produces

$$R_x = k^* \frac{x}{k}\sqrt{\frac{2}{C_f}}\left(1 + A\sqrt{\frac{C_f}{2}} + \ldots\right) \qquad [121]$$

Taking the common logarithm of Equation [121] and expanding the logarithm of the series in parentheses results in

$$\log_{10} R_x = \log_{10} k^* + \log_{10}\sqrt{\frac{2}{C_f}} + \frac{A}{2.3026}\sqrt{\frac{C_f}{2}} + \log_{10}\frac{x}{k} \qquad [122]$$

wherein terms of higher order than $\sqrt{C_f}$ have been neglected. Hence for a given k^* and x/k the intersection of Equation [122] and [119] at a specified k^* gives a point on the line of constant x/k. A rapid way of plotting other lines of Equation [122] is to relate them to any plotted line of Equation [122] at constant C_f or

$$\log_{10} R_{x,2} - \log_{10} R_{x,1} = \log_{10} k_2^* - \log_{10} k_1^* + \log_{10}\left(\frac{x}{k}\right)_2 - \log_{10}\left(\frac{x}{k}\right)_1 \qquad [123]$$

where subscript 2 refers to the new line and subscript 1 to the line already plotted. Hence the various plots of Equation [122] are offset by constant amounts in the direction of $\log_{10} R_x$.

This procedure was utilized in preparing the resistance diagram for sand roughness in Figure 7 from a graphical representation of B_1 and k^*.

RESISTANCE CHARACTERIZATIONS FROM PLATE TESTS

To characterize the resistance qualities of an arbitrary roughness requires the determination of B_1 or B_2 as a function of k^*. Measurements of total resistance of flat plates with an arbitrarily rough surface can only give values of C_f against R_x for different lengths x, the roughness parameter k having only meaning for a particular roughness configuration. Hence to compare the resistance characterizations of different roughnesses from measurements of total resistance of flat plates, it is necessary to use the following procedure:

A new parameter x^* is introduced where

$$x^* \equiv \frac{u_\tau x}{\nu} \qquad [124]$$

so that by definition

$$k^* \equiv \frac{x^*}{x/k} \qquad [125]$$

or

$$\log_{10} k^* = \log_{10} x^* - \log_{10} \frac{x}{k} \qquad [126]$$

Then the single parameter variation of B_1 with k^* is replaced by a two-parameter variation of B_1 with x^* and x/k. Consequently when B_1 is plotted against $\log_{10} x^*$, a family of curves result for different values of x/k which are offset by constant amounts of the differences in $\log_{10} x/k$.

B_1 is determined from the general logarithmic resistance formula [118] and from the fact that for constant x/k

$$\frac{dB_1}{d\log_{10} k^*} = \frac{dB_1}{d\log_{10} x^*} \qquad [127]$$

or

$$B_1 = \sqrt{\frac{2}{C_f}} - \frac{A}{\sqrt{2}}\left(\frac{A}{2} + \frac{D_2}{D_1} + \frac{1}{2.3026}\,\frac{dB_1}{d\log_{10} x^*}\right)\sqrt{C_f} + A - B_3 + 2.3026A \log_{10}\left(\frac{2D_1}{R_x C_f}\right) \qquad [128]$$

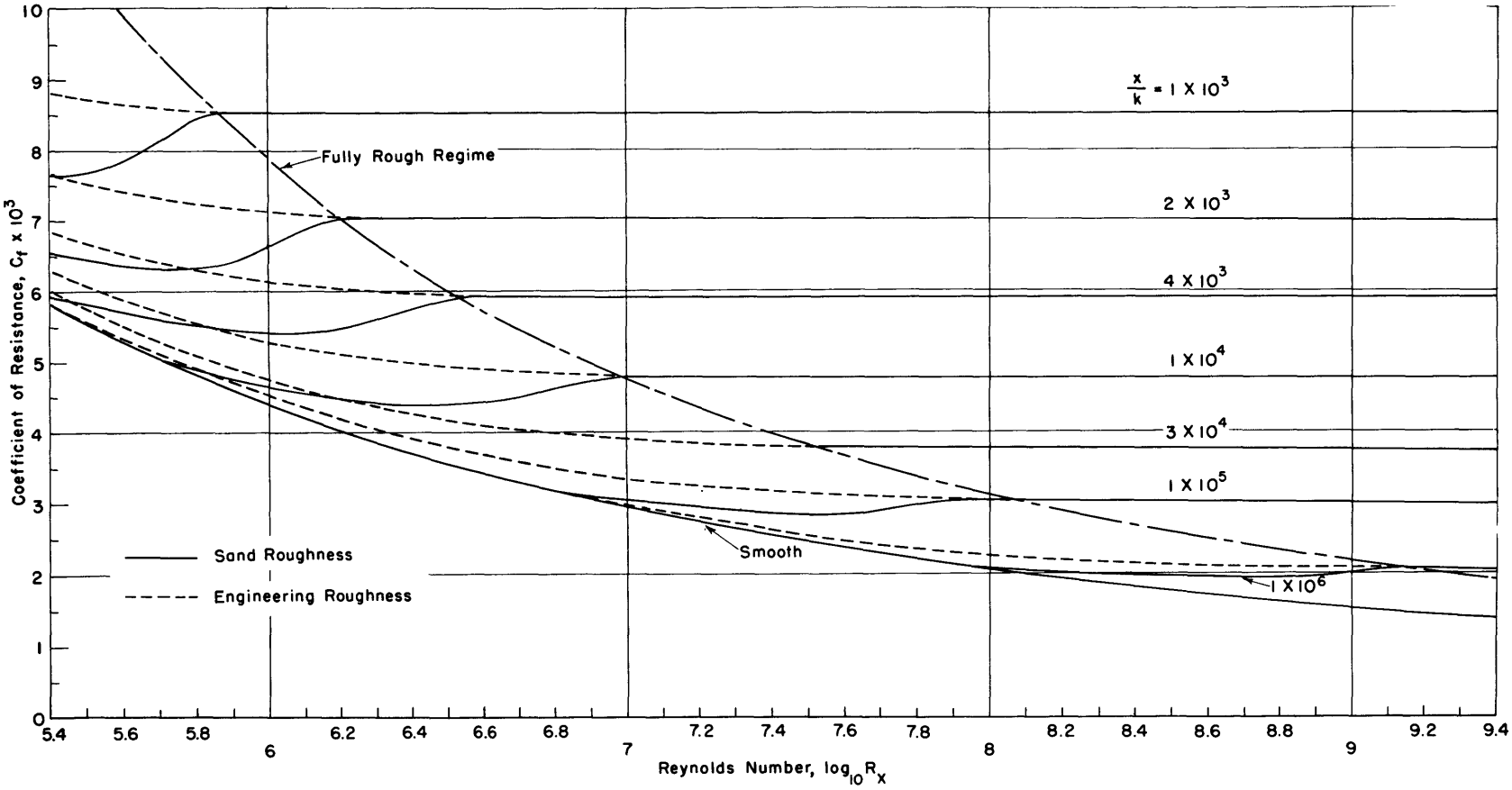

Figure 7 – Resistance Diagram for Rough Plates with Sand Roughness and Engineering Roughness

or relative to the smooth line for the same C_f

$$B_1 = B_{1,s} - \frac{A}{2.3026} \frac{dB_1}{d\log_{10} x^*} \sqrt{\frac{C_f}{2}} - 2.3026A\,(\log_{10} R_x - R_{x,s}) \qquad [129]$$

The small effect due to $\frac{dB_1}{d\log_{10} x^*}$ can be evaluated by iteration.

Since by definition

$$x^* = \frac{R_x}{\sigma} \qquad [130]$$

then substituting for $1/\sigma$ from Equation [85] and manipulating the logarithm of the series gives

$$\log_{10} x^* = \log_{10} R_x + \log_{10} \sqrt{\frac{C_f}{2}} - \frac{A}{2.3026} \sqrt{\frac{C_f}{2}} \qquad [131]$$

Hence B_1 and x^* can be obtained from test data of C_f against R_x.

For the same roughness k and different lengths of plate x, plots of B_1 against $\log_{10} x^*$ will be represented by curves offset by a constant amount given by the logarithm of the ratio of the lengths of the plates. Hence, if the similarity laws hold, the curves can be collapsed to a single line by parallel shifts of the curves in the x^* direction.

A resistance characterization was obtained from the data of 21-foot friction planes coated with Mare Island hot plastic paint as shown in Figure 8. For comparison the curve of engineering roughness, Equation [110], is plotted for $x/k = 1 \times 10^5$. The difference in curvature between the two graphs is evident.

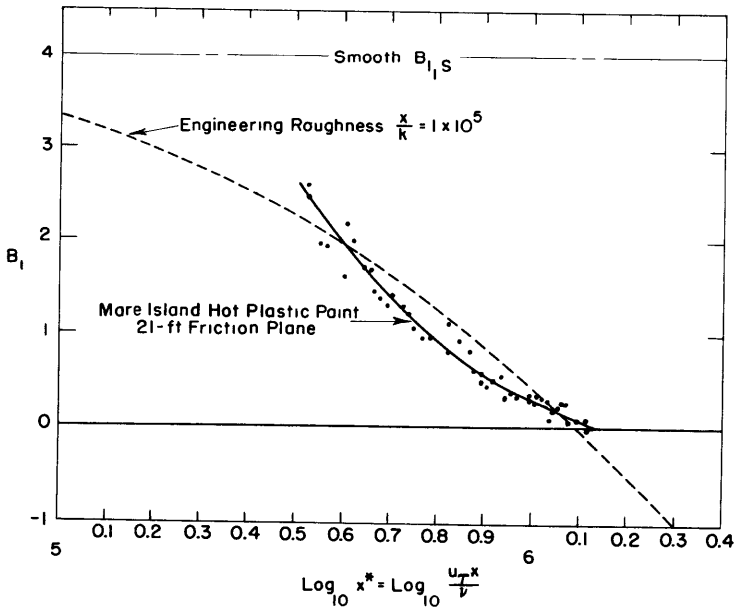

Figure 8 – Resistance Characterization of Roughness from Flat Plate Resistance Data

PREDICTION OF FULL-SCALE RESISTANCE FROM PLATE TESTS

The general logarithmic resistance law provides a means of extrapolating data from test planks with arbitrary rough surfaces to full-scale lengths. The test data are in the form of C_f against log R_x at constant x/k, k being unknown. A representative plot for Mare Island hot plastic paint[21] is shown in Figure 9. The problem is to obtain a plot of C_f against log R_x for different lengths of flat plates with the same roughness in order to predict the full-scale resistance of ships having that roughness.

According to the similarity laws there is a unique relationship between B_1 and k^* characteristic of the roughness configuration. Hence each point of the test data or faired test curve represents a value of B_1 or k^*. This is illustrated by point P_1 in Figure 9 where a line of constant B_1 or k^* is drawn through it by means of Equation [119] which simply requires that a constant distance from the smooth curve be maintained in the $\log_{10} R_x$ direction. The next step is to locate a point on this line which corresponds to the desired length of plate.

First from Equation [131] a line of C_f against R_x at constant x^* is drawn using any convenient value of x^*. From Equation [131] again it is seen that at constant C_f the difference in lines at different x^* is given by

$$\log_{10} R_{x,2} - \log_{10} R_{x,1} = \log_{10} x_2^* - \log_{10} x_1^* \qquad [132]$$

Since by definition

$$x^* \equiv k^* \frac{x}{k} \qquad [133]$$

Equation [132] becomes for constant k^* and k

$$\log_{10} R_{x,2} - \log_{10} R_{x,1} = \log_{10} \frac{x_2}{x_1} \qquad [134]$$

Consequently point P_2 for the new length x_2 is located by a line which is at a distance of $\log_{10} \frac{x_2}{x_1}$ from P_1 and drawn a constant distance from the reference line for constant x^*. This process is repeated for enough points from the test data to draw the predicted line for the desired length as illustrated in Figure 9.

If a representative sample of the rough surface can be obtained on a test plate, it is then possible to make full-scale predictions for any arbitrary roughness configuration without regard to geometrical characterization.

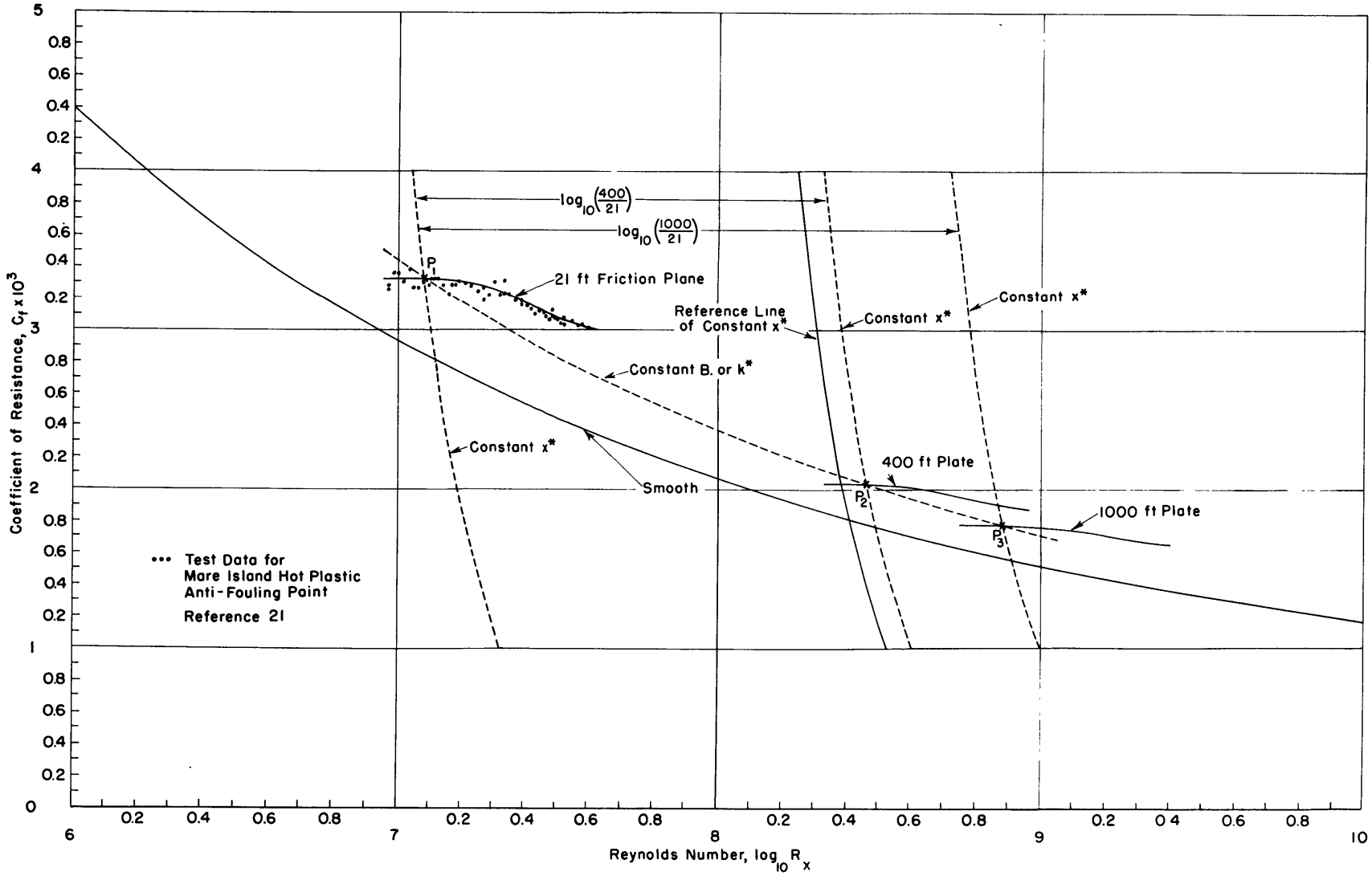

Figure 9 – Resistance Prediction Diagram for Rough Plates

LOCAL SKIN FRICTION COEFFICIENTS AND SHAPE PARAMETERS

GENERAL

In the calculation of turbulent boundary layers in pressure gradients to obtain the viscous resistance or separation point of foils and bodies of revolution,[22, 23] relations are required for the local skin friction and shape parameter as functions of momentum thickness. For rough surfaces in general the local skin friction is expressed as

$$\frac{1}{\sigma^2} \equiv \frac{C_\tau}{2} \equiv \frac{\tau_w}{\rho U^2} = f\,(R_\theta,\ \theta/k) \qquad [135]$$

and the shape parameter H as

$$H = f(R_\theta,\ \theta/k) \qquad [136]$$

From Equations [11] and [61] the local skin friction is expressed implicitly as

$$R_\theta = D_1 \left(1 - \frac{D_2}{D_1 \sigma}\right) \exp\left[\frac{1}{A}(\sigma - B_3 - B_1)\right] \qquad [137]$$

or from Equation [12] and [62]

$$\frac{\theta}{k} = \frac{D_1}{\sigma}\left(1 - \frac{D_2}{D_1 \sigma}\right) \exp\left[\frac{1}{A}(\sigma - B_3 - B_2)\right] \qquad [138]$$

where α_1, β_1, α_2, and β_2 have been neglected. Logarithmically Equation [137] becomes

$$\ln R_\theta = \frac{1}{A}\sqrt{\frac{\rho U^2}{\tau_w}} - \frac{B_3}{A} - \frac{B_1}{A} + \ln D_1 - \frac{D_2}{D_1}\sqrt{\frac{\tau_w}{\rho U^2}} \qquad [139]$$

and Equation [138] becomes

$$\ln \frac{\theta}{k} = \frac{1}{A}\sqrt{\frac{\rho U^2}{\tau_w}} - \frac{B_3}{A} - \frac{B_2}{A} + \ln D_1 + \ln\sqrt{\frac{\tau_w}{\rho U^2}} - \frac{D_2}{D_1}\sqrt{\frac{\tau_w}{\rho U^2}} \qquad [140]$$

wherein one term has been retained in the series expansion of $\ln\left(1 - \frac{D_2}{D_1 \sigma}\right)$. Equations [139] and [140] represent the general case with B_1 and B_2 as functions of k^*.

In the case of shape parameter H, Equation [64] is simplified to

$$\frac{H}{H-1} = \frac{D_1}{D_2}\sigma = \frac{D_1}{D_2}\sqrt{\frac{\rho U^2}{\tau_w}} \qquad [141]$$

by dropping α_1 and β_1.

FULLY ROUGH REGIME

Since in the case of the fully rough regime B_2 is constant, $B_2 = B_{2,R}$, Equation [140] is then

$$\ln\left(\frac{\theta}{k}\sqrt{\frac{\rho U^2}{\tau_w}}\right) = \frac{1}{A}\sqrt{\frac{\rho U^2}{\tau_w}} - \frac{B_3}{A} - \frac{B_{2,R}}{A} + \ln D_1 - \frac{D_2}{D_1}\sqrt{\frac{\tau_w}{\rho U^2}} \qquad [142]$$

If the term involving $\sqrt{\frac{\tau_w}{\rho U^2}}$ is linearized with respect to $\sqrt{\frac{\rho U^2}{\tau_w}}$ or

$$\sqrt{\frac{\tau_w}{\rho U^2}} = a_1 + a_2\sqrt{\frac{\rho U^2}{\tau_w}} \qquad [143]$$

then Equation [142] becomes in common logarithms

$$\log_{10}\left(\frac{\theta}{k}\sqrt{\frac{\rho U^2}{\tau_w}}\right) = O_1\sqrt{\frac{\rho U^2}{\tau_w}} + P_1 \qquad [144]$$

where

$$O_1 = \frac{1}{2.3026A} - \frac{a_2 D_2}{2.3026 D_1} \qquad [145]$$

and

$$P_1 = \frac{-B_3 - B_{2,R}}{2.3026A} - \frac{a_1 D_2}{2.3026 D_1} + \log_{10} D_1 \qquad [146]$$

Furthermore, if $\log_{10}\sqrt{\frac{\tau_w}{\rho U^2}}$ is linearized with respect to $\sqrt{\frac{\rho U^2}{\tau_w}}$ or

$$\log_{10}\sqrt{\frac{\tau_w}{\rho U^2}} = a_3 + a_4\sqrt{\frac{\rho U^2}{\tau_w}} \qquad [147]$$

Equation [144] becomes approximately

$$\log_{10}\frac{\theta}{k} = O_2\sqrt{\frac{\rho U^2}{\tau_w}} + P_2 \qquad [148]$$

or

$$\frac{\tau_w}{\rho U^2} = \frac{O_2^2}{\left(\log_{10}\frac{\theta}{k} - P_2\right)^2} \qquad [149]$$

where

$$O_2 = O_1 + a_4 \qquad [150]$$

and

$$P_2 = P_1 + a_3 \qquad [151]$$

In the case of H, the substitution of Equation [148] into [141] produces

$$\frac{H}{H-1} = \left(\frac{D_1}{D_2 O_2}\right)\log_{10}\frac{\theta}{k} - \frac{D_1 P_2}{D_2 O_2} \qquad [152]$$

QUASI-SMOOTH REGIME

In the case of the quasi-smooth regime B_1 is constant, $B_1 = B_{1,s}$, and Equation [139] becomes

$$\ln R_\theta = \frac{1}{A}\sqrt{\frac{\rho U^2}{\tau_w}} - \frac{B_3}{A} - \frac{B_{1,s}}{A} + \ln D_1 - \frac{D_2}{D_1}\sqrt{\frac{\tau_w}{\rho U^2}} \qquad [153]$$

If the term containing $\sqrt{\frac{\tau_w}{\rho U^2}}$ is linearized with respect to $\sqrt{\frac{\rho U^2}{\tau_w}}$ as in Equation [143], Equation [153] is then in common logarithms

$$\log_{10} R_\theta = O_1\sqrt{\frac{\rho U^2}{\tau_w}} + P_1' \qquad [154]$$

or

$$\frac{\tau_w}{\rho U^2} = \frac{O_1^2}{(\log R_\theta - P_1')^2} \qquad [155]$$

where O_1 is given by Equation [145] and

$$P_1' = P_1 + \frac{B_{2,R} - B_{1,s}}{2.3026A} \qquad [156]$$

An equation of the form of [155] was first derived by Squire and Young[24] for smooth surfaces.

The shape parameter H for quasi-smooth regime becomes after substituting Equation [154] into [141]

$$\frac{H}{H-1} = \left(\frac{D_1}{D_2 O_1}\right)\log_{10} R_\theta - \frac{D_1 P_1'}{D_2 O_1} \qquad [157]$$

ENGINEERING ROUGHNESS

In the case of engineering roughness, B_1 is analytically defined as a function of k^* by Equation [110] which, since k^* is by definition

$$k^* \equiv \frac{R_\theta}{\theta/k}\sqrt{\frac{\tau_w}{\rho U^2}} \qquad [158]$$

is written

$$B_1 = B_{1,s} - A \ln\left(1 + \frac{R_\theta}{\lambda\,\theta/k}\sqrt{\frac{\tau_w}{\rho U^2}}\right) \qquad [159]$$

For B_1 given by Equation [159], Equation [139] becomes

$$-\ln\left(\frac{1}{R_\theta} + \frac{\sqrt{\frac{\tau_w}{\rho U^2}}}{\lambda\,\theta/k}\right) = \frac{1}{A}\sqrt{\frac{\rho U^2}{\tau_w}} - \frac{B_3}{A} - \frac{B_{1,s}}{A} + \ln D_1 - \frac{D_2}{D_1}\sqrt{\frac{\tau_w}{\rho U^2}} \qquad [160]$$

If the term containing $\sqrt{\frac{\tau_w}{\rho U^2}}$ is linearized with respect to $\sqrt{\frac{\rho U^2}{\tau_w}}$ as in Equation [143], Equation [160] is then in common logarithms

$$-\log_{10}\left(\frac{1}{R_\theta} + \frac{\sqrt{\frac{\tau_w}{\rho U^2}}}{\lambda\,\theta/k}\right) = O_1\sqrt{\frac{\rho U^2}{\tau_w}} + P_1' \qquad [161]$$

where O_1 is given by Equation [145] and P_1' by [156].

Since no simple expression for H can be obtained from Equations [141] and [161], it is necessary to retain $\frac{\tau_w}{\rho U^2}$ as the intermediate parameter for calculating H.

PREPARATION OF LOCAL SKIN FRICTION COEFFICIENT AND SHAPE PARAMETER DIAGRAMS FROM RESISTANCE CHARACTERIZATIONS

Where B_1 is empirically stated as a function of k^* for any roughness configuration, a convenient method of preparing a diagram of $\frac{\tau_w}{\rho U^2}$ and H plotted against R_θ and θ/k is by comparison with $\frac{\tau_w}{\rho U^2}$ and H for the smooth case.

From Equation [139] for the same value of $\frac{\tau_w}{\rho U^2}$ and in common logarithms

$$\log_{10} R_\theta - \log_{10} R_{\theta,s} = \frac{B_{1,s} - B_1}{2.3026 A} \qquad [162]$$

Consequently, the resistance line of constant k^* or constant B_1 for the rough case is offset from the smooth line by a constant amount in the direction of $\log_{10} R_\theta$.

To determine the resistance line of constant θ/k from the lines of constant k^* the procedure is as follows:

Since by definition

$$R_\theta \equiv \frac{\theta}{k}\sqrt{\frac{\rho U^2}{\tau_w}}\, k^* \qquad [163]$$

or

$$\log_{10} R_\theta = \log_{10} \theta/k - \log_{10}\sqrt{\frac{\tau_w}{\rho U^2}} + \log_{10} k^* \qquad [164]$$

Hence for a given k^* and θ/k the intersection of Equations [162] and [164] gives the desired point. A rapid way of plotting other lines of Equation [164] is to relate them to any plotted line of [164] at constant $\frac{\tau_w}{\rho U^2}$ or

$$\log_{10} R_{\theta,2} - \log_{10} R_{\theta,1} = \log_{10} (\theta/k)_2 - \log_{10} (\theta/k)_1 + \log_{10} k_2^* - \log_{10} k_1^* \qquad [165]$$

The various plots of Equation [164] are then offset by constant amounts in the direction of $\log_{10} R_\theta$.

The procedure is the same for H since H is solely a function of $\frac{\tau_w}{\rho U^2}$ from Equation [141].

PREPARATION OF LOCAL SKIN FRICTION COEFFICIENT AND SHAPE PARAMETER DIAGRAMS FROM PLATE TESTS

To prepare diagrams of $\frac{\tau_w}{\rho U^2}$ and H from total resistance results for an arbitrary roughness configuration, the following method may be employed:

From total resistance tests of a representative sample of a particular roughness the results are given in the form of $C_f = f(R_x, x)$. The objective of the analysis is to obtain $\frac{\tau_w}{\rho U^2}, H = f(R_\theta, \theta)$. From Equation [84] approximately

$$\frac{\tau_w}{\rho U^2} = \frac{C_f}{2}\left(1 - 2A\sqrt{\frac{C_f}{2}}\right) \qquad [166]$$

and from Equation [68]

$$R_\theta = \frac{1}{2} R_x C_f \qquad [167]$$

and

$$\theta = \frac{1}{2} x C_f \qquad [168]$$

To obtain lines of constant θ, a line is first drawn at a fixed distance in the $\log R_\theta$ direction from the smooth curve for each data point which corresponds to a line of constant k^* in accordance with Equation [162]. Then to locate the point corresponding to the desired value of θ use is made of Equation [165] wherein for the same $\frac{\tau_w}{\rho U^2}$, k^*, and k

$$\log_{10} R_{\theta,2} - \log_{10} R_{\theta,1} = \log_{10} \theta_2 - \log_{10} \theta_1 \qquad [169]$$

The reference line for using the above is from Equation [164]

$$\log_{10} R_\theta = -\log_{10}\sqrt{\frac{\tau_w}{\rho U^2}} + \text{constant} \qquad [170]$$

the constant is any value convenient for plotting.

The values of H of the arbitrary roughness are calculated from Equation [141] wherein

$$H = \frac{1}{1 - \frac{D_2}{D_1}\sqrt{\frac{\tau_w}{\rho U^2}}} \qquad [171]$$

The remainder of the procedure for determining $H = f(R_\theta, \theta)$ is the same as that for $\frac{\tau_w}{\rho U^2}$.

NUMERICAL RESULTS

The experimental difficulties in boundary-layer measurements have resulted in considerable discrepancies in numerical values of constants among investigators in the field.[25] Although future tests will thus in all probability lead to somewhat different values, the numerical results gathered and calculated by Landweber[18] have been judged to be as reliable as any and have been used to evaluate the factors derived in this report. For convenience of reference, the numerical values are listed in Table 3.

TABLE 3

Numerical Values of Factors

Factor	Value	Source	Factor	Value	Source
a_1	0.0625	Calc.	m	1.076	Eq. [117]
a_2	-7.57×10^{-4}	Calc.	M_1	0.242	Eq. [93]
a_3	−1.082	Calc.	M_2	0.262	Eq. [98]
a_4	-1.306×10^{-2}	Calc.	M_2'	0.282	Eq. [106]
$2.3026A$	6	Ref. 18	N_1	−0.661	Eq. [94]
A	2.6	Ref. 18	N_1'	0.023	Eq. [103]
$B_{1,s}$	4	Ref. 18	N_2	0.215	Eq. [99]
$B_{2,R}$	7.2	Fig. 10	N_2'	1.729	Eq. [107]
B_3	2	Ref. 18	O_1	0.1689	Eq. [145]
C_1	0.105	Calc.	O_2	0.1558	Eq. [150]
C_2	-2.5×10^{-3}	Calc.	P_1	− 1.169	Eq. [146]
C_3	−1.752	Calc.	P_1'	−0.636	Eq. [156]
C_4	−0.04	Calc.	P_2	−2.251	Eq. [151]
D_1	3.499	Ref. 18	ν	24.66	Eq. [116]
D_2	23.23	Ref. 18	λ	3.42	Eq. [109]

Since the sand roughness values of Nikuradse† were obtained from pipe flow, some adjustment is needed for application to flat plates. The values of B_2 of Nikuradse were obtained by using a value of 5.75 for $2.3026A$ which differs from the value of 6 used by Landweber for flat plates. Consequently the velocity profiles of Nikuradse were replotted and new B_2 values for pipes were obtained using the slope of 6 as shown in Figure 10. The B_2 values are converted to those for flat plates by assuming that the difference in such

†Here k refers to the size of the sand grains.

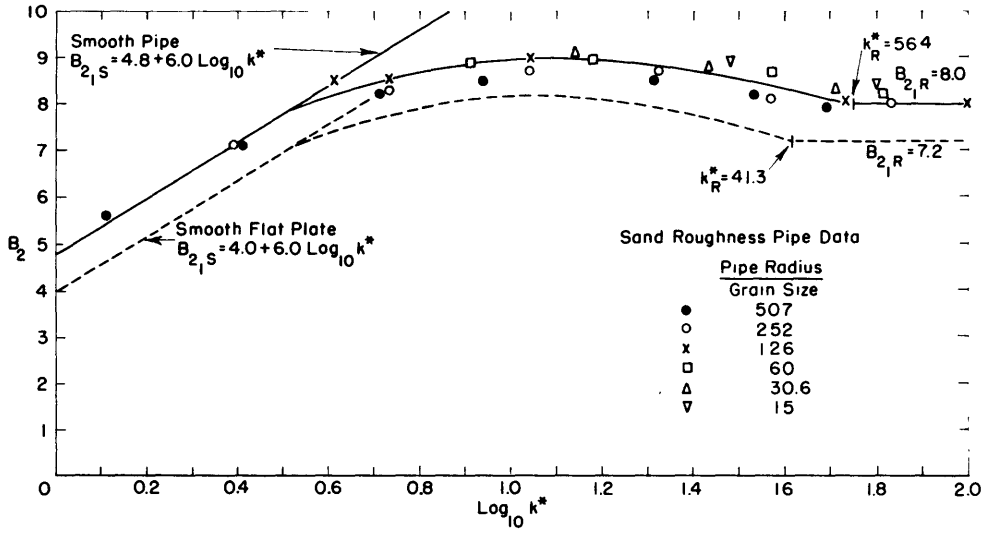

Figure 10 – Values of B_2 for Sand Roughness in Pipes and on Flat Plates

values is equal to the difference in the $B_{1,s}$ values between pipes and flat plates (here 4.8 – 4.0 = 0.8). For the fully rough regime, the $B_{2,R}$ value of 8.0 for pipes becomes 7.2 for flat plates. In order to accomodate the new k_R^* values of 41.3 for flat plates, the k^* values are also adjusted for flat plates in the general rough regime. The procedure is to assume that the ratio of laminar sublayer thickness of flat plates to that of pipes for the quasi-smooth regime is the same for the general rough regime. From Equation [39]

$$r = \frac{y_L'^*}{y_L^*} = \frac{B'_{1,s} + A \ln A}{B_{1,s} + A \ln A} \qquad [172]$$

where the primed quantities refer to flat-plate values and the unprimed quantities to pipe values. In terms of B_2 values

$$r = \frac{B_2' + A \ln A - A \ln k'^*}{B_2 + A \ln A - A \ln k^*} \qquad [173]$$

or

$$k'^* = (k^*)^r \exp\left[\frac{B_2' - B_2}{A} + (1-r) \ln A\right] \exp\left[\frac{(1-r)\, B_2}{A}\right] \qquad [174]$$

The results which are given in Figure 10 were used to prepare the resistance diagram for sand roughness of Figure 7. Compared to the original Prandtl-Schlichting diagram,[4] the C_f values are somewhat lower for the same Reynolds number; this is to be expected considering that the Prandtl-Schlichting diagram is based on pipe results. Figure 11 gives a clearer indication of this where a comparison is made for the fully rough regime.

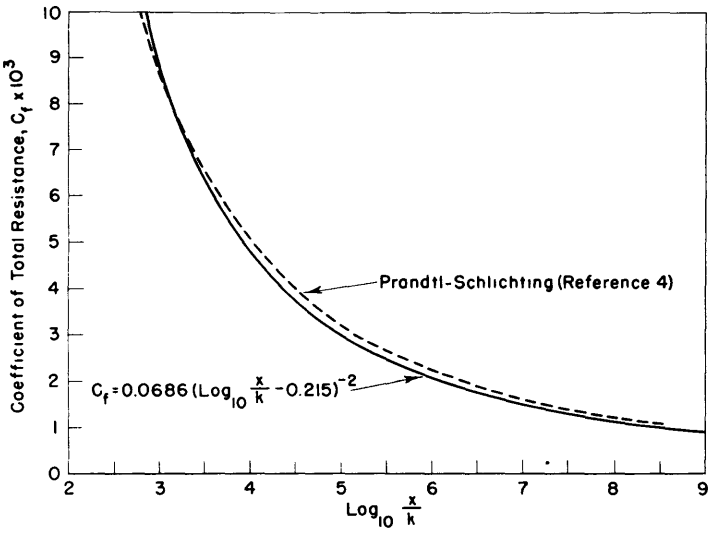

Figure 11 – Coefficients of Total Resistance for the Fully Rough Regime

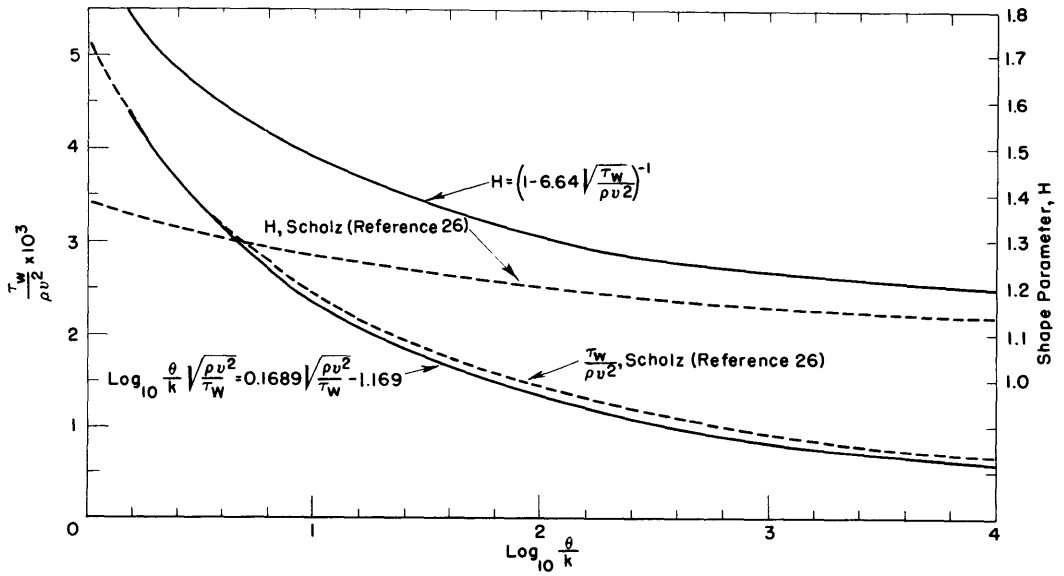

Figure 12 – Local Skin Friction and Shape Parameter for the Fully Rough Regime of Flat Plates

For the total resistance of smooth plates, the new M_1 value of 0.242 is identical to that of the Schoenherr formula and the new N_1' value of 0.023 is close to that of zero for the Schoenherr formula.

Finally, for the fully rough regime Figure 12 shows a comparison of local resistance coefficients and H values from Equations [144] and [141] to those of Scholz[26] obtained by a simple power-law analysis of the Prandtl-Schlichting formula. The local resistance coefficients compare well but the H values show a large difference which is due to the improper power-law analysis of Scholz.

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