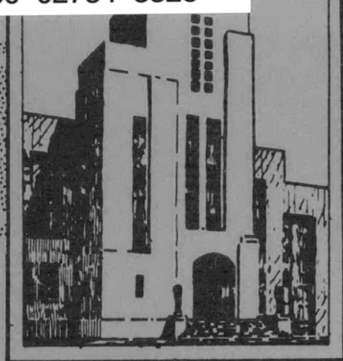

DEPARTMENT OF THE NAVY
DAVID TAYLOR MODEL BASIN

HYDROMECHANICS

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APPLIED MATHEMATICS

THE POTENTIAL FUNCTIONS FOR SINGULARITIES ASSOCIATED WITH HYDROFOILS OF FINITE SPAN MOVING WITH SUBCRITICAL AND SUPERCRITICAL VELOCITIES IN SHALLOW WATER

by

Avis Borden, Ph.D.

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HYDROMECHANICS LABORATORY

RESEARCH AND DEVELOPMENT REPORT

June 1961

Report 1351

PRNC-TMB-648 (Rev. 3-58)

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NOTATION

A_1	$\sinh Ka \cos Ka$
A_2	$\cosh Ka \sin Ka$
A_3	$\cosh Ka \cos Ka$
A_4	$\sinh Ka \sin Ka$
a	Height of hydrofoil above sea bed
b	Semispan of hydrofoil
C	Average lift coefficient of hydrofoil
C_1	$(N_1 D_1 + N_2 D_2) / (D_1^2 + D_2^2)$ *
C_2	$(N_2 D_1 - N_1 D_2) / (D_1^2 + D_2^2)$ *
c	Chord of hydrofoil
D	Subscript to designate line doublet
$D_1 + iD_2$	Complex denominator of $M_2(w, \eta)$ *
$e(\theta)$	Functions of θ occurring in $K(\eta)$ *
F	Froude number based on water depth
$f(k,a,z)$	Functions of k, a, and z occurring in $K(\eta)$
$G(k,)$	Undetermined parameter
g	Acceleration of gravity
$g(k)$	$\sinh kh \cosh kh$
$g'(k)$	$kh \cosh^2 kh$
$H(m)$	Undetermined parameter
H_1	$\sinh 2Kh \cos 2Kh$
H_2	$\cosh 2Kh \sin 2Kh$
H_3	$\cosh 2Kh \cos 2Kh$
H_4	$\sinh 2Kh \sin 2Kh$
h	Average water depth
$I(\eta,w)$	Imaginary part of kernel function $K(\eta)$ *
$J(\eta,w)$	Real part of kernel function $K(\eta)$ *

*Primed quantities are used for the supercritical range of Froude number and unprimed quantities for the subcritical range.

K	Integration variable	
$K(\eta)$	Kernel of an integral defining a potential function	*
k	Integration variable	
k_0	Pole on the real k-axis	
$L(\eta)$	Kernel of an integral defining a potential function	*
$M(w,\eta)$	Complex kernel function	
m	Integration variable	
$N_1 + iN_2$	Complex terms in the numerator of $M_2(w,\theta)$	
p	Pressure change in the fluid	
Q	Unity if $F < 1$ and $(-1)^n$ if $F > 1$	
R_1	$x^2 + S_1$	
R_2	$x^2 + S_2$	
S_1	$(\eta - y)^2 + (2nh + z - a)^2$	
S_2	$(\eta - y)^2 + (2nh + z + a)^2$	
$T(\eta)$	Kernel of an integral defining a potential function	
U	Free-stream velocity	
V	Subscript to designate semi-infinite vortex sheet	
w	Either w_+ or w_-	
w_+	$x \cos\theta + (\eta - y)\sin\theta$	
w_-	$x \cos\theta - (\eta - y)\sin\theta$	
x	Longitudinal coordinate	
y	Horizontal lateral coordinate	
Z_1	$\sinh Kz \cos Kz$	
Z_2	$\cosh Kz \sin Kz$	
Z_3	$\cosh Kz \cos Kz$	
Z_4	$\sinh Kz \sin Kz$	
z	Vertical coordinate	
$\Gamma(\eta)$	Spanwise distribution of vorticity	
ζ	Wave elevation	
η	Distance measured along the span	

θ	Angle
θ_m	$\cos^{-1}(1/F)$
λ	Shape parameter
$\mu(\eta)$	Horizontal doublet distribution along the span
ξ	Distance in the x-direction
ρ	Density of fluid
Φ	Total potential function
ϕ	Perturbation potential function *

ABSTRACT

This report shows that singularities for a semi-infinite vortex sheet and a horizontal line doublet, both of finite span, may be used to represent the vorticity and displacement produced by a finite span hydrofoil. The boundary conditions which must be satisfied when these singularities move in shallow water are derived. Expressions for the potential functions are obtained for both the subcritical and supercritical ranges of Froude number. Explicit expressions for the x-, y-, and z-derivatives of the potential functions are also presented. Methods for determining the strengths of the singularities are discussed.

INTRODUCTION

As an extension of Wu's theoretical study of finite span hydrofoils moving beneath the surface in water of infinite depth,[1] expressions are derived here for the potential functions associated with finite span hydrofoils in water of finite depth. These expressions may be obtained for any span, water depth, or depth of submergence, and for any velocity in the subcritical and supercritical ranges of Froude number. In addition to the potential functions, explicit expressions are derived for the x-, y-, and z-derivatives of these functions. With the results of Wu's report and those of other investigators,[2,3] these functions may be used to determine wave profiles, wave resistance, velocity components, pressure, and other parameters.

A few theoretical studies have been made of the wavemaking produced by singularities moving beneath a free surface in shallow water. Lunde[4] outlined the method for finding the wavemaking of a source in shallow water, and Pond[5] and DiDonato[6] have made a complete analysis of the problem for the subcritical range of Froude number. A further discussion of problems of this type, including a finite span hydrofoil, is given in Reference 7. None of these works gives a complete analysis for the supercritical range of Froude number.

When a singularity moves horizontally between a rigid bottom and a free surface, the potential function is very sensitive to the value of the Froude number based on water depth. As the velocity increases, the wavemaking builds up and becomes very severe when the Froude number approaches unity. When the Froude number exceeds unity the singularity is moving faster than the wave velocity and wavemaking effects diminish. This phenomenon is common where ships enter shallow water and the resistance suddenly decreases when the velocity exceeds the critical speed. Although most surface ships seldom exceed critical speed in normal operation, this range may be important in the operation of hydrofoil boats.

Wu has shown that the singularity which produces the same vorticity as a hydrofoil is a semi-infinite vortex sheet of finite span, extending from the position of the hydrofoil

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[1]References are listed on page 29.

to an infinite distance downstream. In addition to the vorticity, the hydrofoil produces a displacement effect. At some distance from the hydrofoil, a horizontal line doublet of finite span, whose strength is proportional to the water displaced, is adequate to represent the displacement effect. In shallow water both singularities must satisfy boundary conditions at the sea bed and at the free surface. Furthermore, the water surface far ahead of the hydrofoil is not disturbed. In this report expressions are derived for the potential functions of each of the singularities. Because the boundary conditions and methods of analysis are the same for each case, general considerations will be discussed first. Finally, a method for determining the strengths of the singularities will be discussed.

BOUNDARY CONDITIONS FOR THE THEORETICAL ANALYSIS

The hydrofoil may be considered fixed in an inertial system in which there is a uniform velocity U at $x = -\infty$ flowing in the positive x-direction. As shown in Figure 1, the hydrofoil with chord c and semispan b is fixed at $x = 0$, $z = a$ with its span extending from $y = -b$ to $y = b$. The origin of the coordinate system is at the sea bed and the plane $z = h$ is the undisturbed water surface.

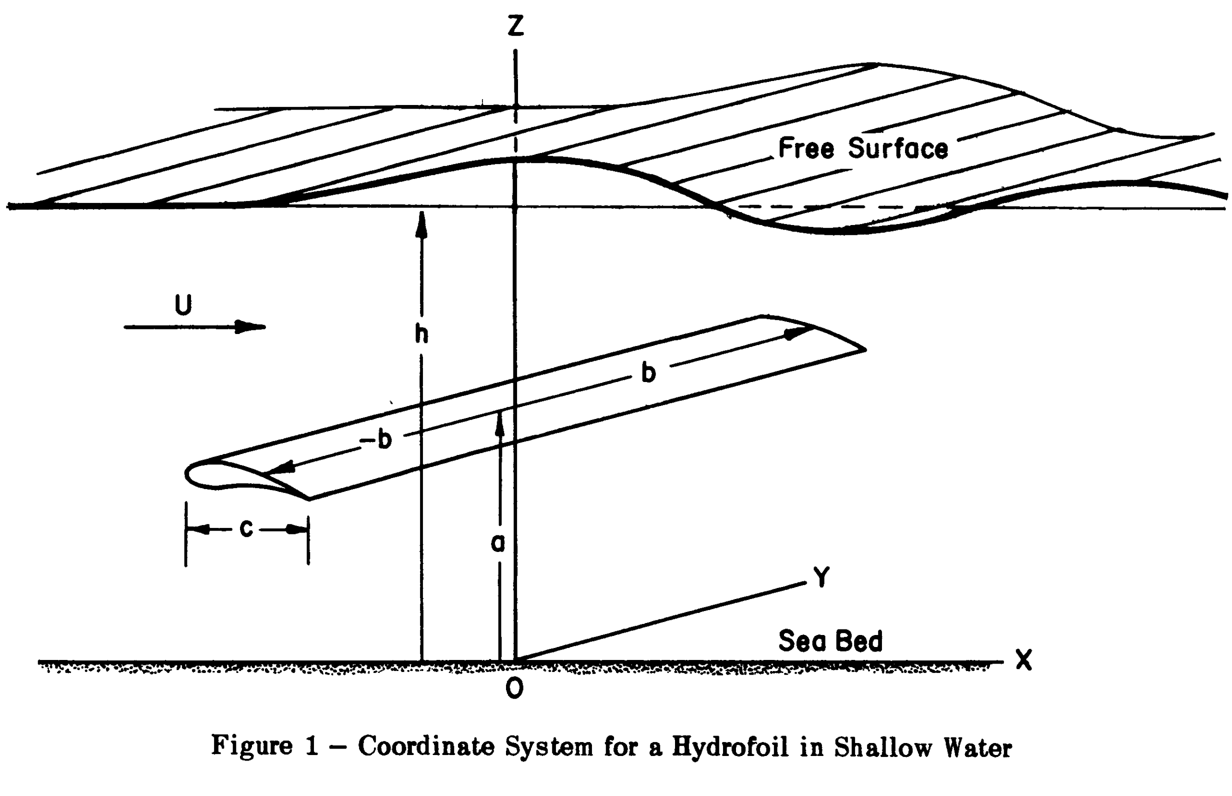

Figure 1 – Coordinate System for a Hydrofoil in Shallow Water

The potential flow about a hydrofoil may be obtained by representing the hydrofoil by several singularities extending along the span. The steady-state potential function is

$$\Phi\ (x,\ y,\ z) = U\,x + \phi\ (x,\ y,\ z) \qquad [1]$$

where ϕ is the perturbation potential. If disturbances in the flow are small, interaction terms may be neglected, and ϕ is the sum of the potential functions for each of the singularities used to represent the hydrofoil. The change in pressure p produced at any point in the fluid is obtained from Bernoulli's equation. Along any streamline

$$p\ (x,\ y,\ z) = -\,\rho\ U\ \frac{\partial \phi}{\partial x} - \rho\,g\ \ (z-h) - \ \frac{1}{2}\ \rho\ (\text{grad}\ \phi)^2 \qquad [2]$$

where ρ is the density of the fluid,

g is the acceleration of gravity, and

z is the position of the streamline in the undisturbed flow.

In the linearized theory, the last term in the Equation [2] may be neglected. Along the streamline represented by the free surface, the pressure change produced is zero and the wave elevation is given by

$$\zeta\,(x,\ y) = z - h = -\ \frac{U}{g}\ \left(\frac{\partial \phi}{\partial x}\right)_{z\,=\,h} \qquad [3]$$

If the sea bed is considered as a rigid surface, there is no flow normal to this surface and the first boundary condition is

Boundary Condition I $$\left(\frac{\partial \phi}{\partial z}\right)_{z\,=\,0} = 0 \qquad [4]$$

For disturbances at the free surface which are small compared with the wave length, the normal velocity of the surface is the same as the vertical velocity of the fluid particles. Then

$$\left(U\ \frac{\partial \zeta}{\partial x}\ -\ \frac{\partial \phi}{\partial z}\right)_{z\,=\,h} = 0 \qquad [5]$$

If Equations [3] and [5] are combined, the second boundary condition which applies at the free surface is obtained:

Boundary Condition II $$\left(\ \frac{\partial \phi}{\partial z} + F^2 h\ \frac{\partial^2 \phi}{\partial x^2}\right)_{z\,=\,h} = 0 \qquad [6]$$

where F is the Froude number based on water depth

$$F = \frac{U}{\sqrt{g h}} \qquad [7]$$

As the free surface far ahead of the hydrofoil is undisturbed, the wave elevation and hence the potential itself must vanish far upstream. Therefore the third boundary condition is

$$\text{Boundary Condition III} \qquad \text{for } x \to -\infty, \ \phi = 0 \qquad [8]$$

In addition, the potential and its derivatives are finite everywhere in the region except at the position of the singularity.

If the hydrofoil is represented as a lifting line with finite displacement and its vortex wake, two singularities are required to represent the foil in an unbounded fluid. The circulation and trailing vortex sheet are represented by a semi-infinite vortex sheet of finite span. The displacement of the foil is represented by a horizontal line doublet. As the problem is linear, the total potential is the sum of the potentials for the two singularities. Both potentials must satisfy the three boundary conditions independently. The solution for each potential is built up from the potential of that singularity in an unbounded fluid. The function ϕ_I is the potential of the singularity in an unbounded fluid plus its image in the sea bed such that Boundary Conditions I and III are satisfied. The function ϕ_{II} is determined in such a way that the potential $\phi_I + \phi_{II}$ satisfies Boundary Condition II as well as I. The potential $\phi_I + \phi_{II}$ is found to consist essentially of the sum of three terms which will be designated as ϕ_1, ϕ_2, and ϕ_3. In the subcritical range of Froude number, ϕ_1 is the total potential for zero Froude number, ϕ_2 is a short-range potential which decays rapidly to zero a short distance from the singularity, and ϕ_3 is the potential associated with wavemaking. In the supercritical range of Froude number, ϕ_1 is the total potential for infinite Froude number but the other terms have the same significance. Boundary Condition III is needed to exclude wavemaking far upstream from the hydrofoil and to determine the explicit form of ϕ_3 for positive and negative values of x. When all boundary conditions are satisfied, the total potential function is

$$\phi = \phi_I + \phi_{II} = \phi_1 + \phi_2 + \phi_3 \qquad [9]$$

POTENTIAL OF A SEMI-INFINITE VORTEX SHEET OF FINITE SPAN

The potential of a semi-infinite vortex sheet at $z = a$ and its image at $z = -a$ is obtained from Wu's report[1]

$$\phi_{V_I} = \frac{z-a}{4\pi} \int_{-b}^{b} \frac{\Gamma(\eta)\, d\eta}{(\eta-y)^2 + (z-a)^2} \left[1 + \frac{x}{[x^2 + (\eta-y)^2 + (z-a)^2]^{1/2}}\right]$$

$$- \frac{z+a}{4\pi} \int_{-b}^{b} \frac{\Gamma(\eta)\, d\eta}{(\eta-y)^2 + (z-a)^2} \left[1 + \frac{x}{[x^2 + (\eta-y)^2 + (z+a)^2]^{1/2}}\right] \qquad [10]$$

where $\Gamma(\eta)$ is the distribution of circulation along the span. It is easy to see that ϕ_{V_I} vanishes for $x = -\infty$ and its z-derivative vanishes at $z = 0$. This equation also has an integral form which is derived in Appendix A and which is the real part of

$$\phi_{V_I} = Re\left\{ \frac{1}{4\pi} \int_{-b}^{b} \Gamma(\eta)\, d\eta \int_0^\infty e^{im(\eta - y)} [\text{sgn}\,(z-a)\, e^{-m|z-a|} - e^{-m(z+a)}]\, dm \right.$$

$$\left. + \frac{1}{4\pi^2} \int_{-b}^{b} \Gamma(\eta)\, d\eta \quad ⨍_{-\pi/2}^{\pi/2} \frac{d\theta}{i\cos\theta} \int_0^\infty e^{ikw} [\text{sgn}\,(z-a)\, e^{-k|z-a|} - e^{-k(z+a)}]\, dk \right\} \qquad [11]$$

where

$$w = x\cos\theta + (\eta - y)\sin\theta \qquad [12]$$

and sgn is the signum function whose value is +1 when the argument is positive, −1 when the argument is negative, and 0 when the argument is zero. The line through the integral sign indicates that the principal value is to be taken. In this case the integral must be evaluated from $-\pi/2 + \epsilon$ to $\pi/2 - \epsilon$, where ϵ approaches zero in like manner at the two limits.

To satisfy Boundary Condition II at the free surface, an undetermined potential function $\phi_{V_{II}}$ will be added to ϕ_{V_I}. This function will be given the same x and y dependence as ϕ_{V_I} and must satisfy Boundary Condition I. If $\phi_{V_{II}}$ is written in terms of undetermined parameters $H(m)$ and $G(k,\theta)$, then the form for $z > a$ is

$$\phi_{V_I} + \phi_{V_{II}} = Re\, \frac{1}{2\pi} \int_{-b}^{b} \Gamma(\eta)\, d\eta \left\{ \int_0^\infty e^{im(\eta - y)} \sinh ma\, [e^{-mz} + H(m)\cosh mz]\, dm \right.$$

$$\left. + \frac{1}{\pi i} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\cos\theta} \int_0^\infty e^{ikw} \sinh ka\, [e^{-kz} + G(k,\theta)\cosh kz]\, dk \right\} \qquad [13]$$

The integrands of each of these integrals must vanish when they are substituted into Equation [6], the expression for Boundary Condition II. Then

$$H(m) = \frac{e^{-mh}}{\sinh mh} \qquad [14]$$

$$G(k, \theta) = -\frac{e^{-kh}\,[1 + kh\,F^2\cos^2\theta]}{\cosh kh\,[kh\,F^2\cos^2\theta - \tanh kh]} \qquad [15]$$

When the Froude number is zero, $G(k, \theta)$ becomes

$$G_0(k, \theta) = \frac{e^{-kh}}{\sinh kh} \qquad [16]$$

The potential for zero Froude number is

$$\phi_{V_1} = \frac{1}{2\pi h} \int_{-b}^{b} \Gamma(\eta) K_{V_1}(\eta)\, d\eta = \frac{1}{2\pi h} \int_{-b}^{b} \Gamma(\eta)\, [T_V(\eta) + L_{V_1}(\eta)]\, d\eta \qquad [17]$$

where h is introduced to make the kernel functions dimensionless in length. The functions $T_V(\eta)$ and $L_{V_1}(\eta)$ have the dimensionless forms

$$T_V(\eta) = Re\, \frac{h}{2} \int_0^\infty e^{im(\eta - y)} \left[\mathrm{sgn}\,(z-a)\, e^{-m|z-a|} - e^{-m(z+a)} + \frac{2e^{-mh} \sinh ma \cosh mz}{\sinh mh} \right] dm$$

[18]

$$L_{V_1}(\eta) = Re\, \frac{h}{2\pi i} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\cos\theta} \int_0^\infty e^{ikw}$$

$$\left[\mathrm{sgn}\,(z-a)\, e^{-k|z-a|} - e^{-k(z+a)} + \frac{2e^{-kh} \sinh ka \cosh kz}{\sinh kh} \right] dk \qquad [19]$$

If the relations given in Appendix A are used, these expressions have the nonintegral forms

$$T_V(\eta) = \frac{h}{2} \sum_{n=-\infty}^{\infty} \left[\frac{z + 2nh - a}{(\eta - y)^2 + (z + 2nh - a)^2} - \frac{z + 2nh + a}{(\eta - y)^2 + (z + 2nh + a)^2} \right] \qquad [20]$$

$$L_{V_1}(\eta) = \frac{h}{2} \sum_{n=-\infty}^{\infty} Q \left\{ \frac{(z + 2nh - a)\, x}{[(\eta - y)^2 + (z + 2nh - a)^2]\, [x^2 + (\eta - y)^2 + (z + 2nh - a)^2]^{1/2}} \right.$$

$$\left. - \frac{(z + 2nh + a)\, x}{[(\eta - y)^2 + (z + 2nh + a)^2]\, [x^2 + (\eta - y)^2 + (z + 2nh + a)^2]^{1/2}} \right\} \qquad [21]$$

where Q is unity. For intermediate Froude numbers in the subcritical range

$$G(k, \theta) = G_0(k, \theta) - \frac{khF^2 \cos^2 \theta}{\sinh kh \cosh kh\,[khF^2 \cos^2 \theta - \tanh kh]} \qquad [22]$$

Then

$$\phi_{V_I} + \phi_{V_{II}} = \phi_{V_1} + \frac{1}{2\pi h} \int_{-b}^{b} \Gamma(\eta)\, K_V(\eta)\, d\eta \qquad [23]$$

where

$$K_V(\eta) = -Re\, \frac{F^2 h^2}{\pi i} \int_{-\pi/2}^{\pi/2} \cos\theta d\theta \, ⨍_0^{\infty} \frac{e^{ikw} \sinh ka \cosh kz\, kdk}{\sinh kh \cosh kh\,[khF^2 \cos^2 \theta - \tanh kh]} \qquad [24]$$

For the hypothetical case of infinite Froude number

$$G_\infty(k, \theta) = -\frac{e^{-kh}}{\cosh kh} \qquad [25]$$

and

$$\phi'_{V_1} = \frac{1}{2\pi h} \int_{-b}^{b} \Gamma(\eta)\, K'_{V_1}(\eta)\, d\eta = \frac{1}{2\pi h} \int_{-b}^{b} \Gamma(\eta)\, [T_V(\eta) + L'_{V_1}(\eta)]\, d\eta \qquad [26]$$

where $T_V(\eta)$ is already defined, and $L'_{V_1}(\eta)$ is the same as $L_{V_1}(\eta)$ except for an alternating sign in the infinite sum. Therefore, $L'_{V_1}(\eta)$ is given by Equation [21] if Q is given the value

$$Q = (-1)^n \qquad [27]$$

For intermediate Froude numbers in the supercritical range

$$G(k, \theta) = G_\infty(k, \theta) + \frac{1}{\sinh kh \cosh kh} \left[1 - \frac{khF^2 \cos^2 \theta}{khF^2 \cos^2 \theta - \tanh kh} \right] \qquad [28]$$

and the potential is

$$\phi_{V_I} + \phi_{V_{II}} = \phi'_{V_1} + \frac{1}{2\pi h} \int_{-b}^{b} \Gamma(\eta)\, K'_V(\eta)\, d\eta \qquad [29]$$

where

$$K'_V(\eta) = L'_{V_2}(\eta) + K_V(\eta) \qquad [30]$$

In this expression

$$L'_{V_2}(\eta) = Re\ \frac{h}{\pi i} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\cos\theta} \int_0^{\infty} \frac{e^{ikw} \sinh ka \cosh kz dk}{\sinh kh \cosh kh} = L_{V_1}(\eta) - L'_{V_1}(\eta) \qquad [31]$$

Therefore $L'_{V_2}(\eta)$ is twice the sum of the terms of $L'_{V_1}(\eta)$ for which $Q = -1$.

Since the potentials, associated with the free surface condition, for the two ranges of Froude number are similar in form, the following discussion will apply to both functions. The range of integration over negative values of θ may be changed to positive values if w is redefined as

$$\begin{aligned} w_+ &= x \cos\theta + (\eta - y) \sin\theta \\ w_- &= x \cos\theta - (\eta - y) \sin\theta \end{aligned} \qquad [32]$$

Then $K_V(\eta)$ becomes

$$K_V(\eta) = Re\ i \int_0^{\pi/2} e(\theta)\,[M(w_+, \theta) + M(w_-, \theta)]\, d\theta \qquad [33]$$

where $M(w, \theta)$ is the complex integral

$$M(w, \theta) = \frac{h}{\pi} \int_0^{\infty} \frac{f(k,a,z)\, e^{ikw}\, dk}{g(k) \left[F^2 \cos^2\theta - \dfrac{\tanh kh}{kh} \right]} \qquad [34]$$

In this equation w is written for either w_+ or w_-. The functions $f(k,a,z)$, $e(\theta)$, and $g(k)$ are

$$\begin{aligned} f(k,a,z) &= \sinh ka \cosh kz \\ e(\theta) &= F^2 \cos\theta \\ g(k) &= \sinh kh \cosh kh \end{aligned} \qquad [35]$$

The integrand of $M(w, \theta)$ has a simple pole when $k = k_0$, where k_0 is defined by the equation

$$k_0 h\, F^2 \cos^2\theta = \tanh k_0 h \qquad [36]$$

This integral may be evaluated by means of a contour integration in the complex k-plane. The method is described in Appendix B.

The contour integration yields a value of $K_V(\eta)$ which is the sum of two terms, K_{V_2} and K_{V_3}. In the subcritical range of Froude number

$$K_{V_2}(\eta) = I_2(\eta, w_+) + I_2(\eta, w_-) \qquad [37]$$

where $I_2(\eta, w)$ is defined in Equation [15B], Appendix B, in which

$$\begin{aligned} N_1 &= A_1 Z_3 - A_2 Z_4 \\ N_2 &= A_1 Z_4 + A_2 Z_3 \end{aligned} \qquad [38]$$

The values of the terms used in N_1 and N_2 are given in Equations [17B] and [18B]. In the supercritical range of Froude number the integrals defining $K_{V_2}(\eta)$ have singularities which occur when $F \cos \theta = 1$. Therefore it is necessary to find the Cauchy principal value at the singularity.

$$K'_{V_2}(\eta) = L'_{V_2}(\eta) + K_{V_2}(\eta) \qquad [39]$$

It might appear to be more natural to combine $L'_{V_2}(\eta)$ with $K'_{V_1}(\eta)$ to give $K_{V_1}(\eta)$, the zero Froude number value. However, from the few experimental and theoretical studies which have been made with hydrofoils operating at Froude numbers slightly greater than unity, the term $K'_{V_2}(\eta)$, as given here, is a good approximation for the flow near the hydrofoil.[8] The approximation is even better for higher Froude numbers. This indicates that the other nearfield terms which are contained in $K'_{V_2}(\eta)$ are of secondary importance and probably may be neglected in many computations.

The other integral terms obtained in the contour integration are of the form $I_3(\eta, w)$, Equations [22B] and [26B] in Appendix B. As these forms depend upon regions in which w_+ and w_- are positive, they will be written explicitly for positive and negative values of x, using the results given in Table 1 of Appendix B. In the subcritical range of Froude number

$x > 0,\ \eta - y > 0$

$$K_{V_3}(\eta) = -2F^2 h \left[\int_0^{\pi/2} g(\theta) \cos k_0 w_+ d\theta + \int_0^{\pi/2-\theta_1} g(\theta) \cos k_0 w_- d\theta \right] \quad [40]$$

$x < 0,\ \eta - y < 0$

$$K_{V_3}(\eta) = -2F^2 \int_{\pi/2-\theta_1}^{\pi/2} g(\theta) \cos k_0 w_+ d\theta \quad [41]$$

where $\tan \theta_1 = \left| \frac{\eta - y}{x} \right|$, and

$$g(\theta) = \frac{k_0 h \cos \theta \sinh k_0 a \cosh k_0 z}{\sinh k_0 h \cosh k_0 h \, [F^2 \cos^2 \theta - \operatorname{sech}^2 k_0 h]} \quad [42]$$

In the supercritical range of Froude number

$x > 0,\ \eta - y > 0$

$$K'_{V_3}(\eta) = -h \left[\int_0^{\infty} f(k_0) \cos k_0 w_+ dk_0 + \int_0^{k_1} f(k_0) \cos k_0 w_- dk_0 \right] \quad [43]$$

$x < 0,\ \eta - y > 0$.

$$K'_{V_3}(\eta) = -h \int_{k_1}^{\infty} f(k_0) \cos k_0 w_+ dk_0 \quad [44]$$

where k_1 is the value of k_0 for which $\theta = \pi/2 - \theta_1$ and

$$f(k_0) = \frac{k_0 h \sinh k_0 a \cosh k_0 z}{\sinh k_0 h \cosh k_0 h \sin \theta} \quad [45]$$

For negative values of $\eta - y$, w_+ and w_- are interchanged.

The potential ϕ_{V_2} which contains the kernel $K_{V_2}(\eta)$, is a nearfield term that decays rapidly with the distance from the singularity. The potential ϕ_{V_3}, which contains the kernel $K_{V_3}(\eta)$, is a wavemaking term which accounts for the waves downstream from the hydrofoil. The waves extending upstream from the hydrofoil are of high frequency and decay rapidly with distance. Finally, the total potential function for the semi-infinite vortex sheet in shallow water is

$$\phi_V = \frac{1}{2\pi} \int_{-b}^{b} \Gamma(\eta) \left[K_{V_1}(\eta) + K_{V_2}(\eta) + K_{V_3}(\eta)\right] d\eta \qquad [46]$$

The x–, y–, and z–derivatives of ϕ_V are readily obtained from the derivatives of the kernel functions. The three derivatives of K_{V_1} are obtained from Equations [20] and [21]. If R_1, R_2, S_1, and S_2 are introduced for

$$\begin{aligned} R_1^2 &= x^2 + (\eta - y)^2 + (z + 2nh - a)^2 \\ R_2^2 &= x^2 + (\eta - y)^2 + (z + 2nh + a)^2 \end{aligned} \qquad [47]$$

$$\begin{aligned} S_1^2 &= (\eta - y)^2 + (z + 2nh - a)^2 \\ S_2^2 &= (\eta - y)^2 + (z + 2nh + a)^2 \end{aligned} \qquad [48]$$

the derivatives of $T_V + L_{V_1}$ are

$$h \frac{\partial (T_V + L_{V_1})}{\partial x} = \frac{h^2}{2} \sum_{n=-\infty}^{\infty} Q \left[\frac{z + 2nh - a}{R_1^3} - \frac{z + 2nh + a}{R_2^3}\right] \qquad [49]$$

$$h \frac{\partial (T_V + L_{V_1})}{\partial y} = \frac{h^2}{2} (\eta - y) \sum_{n=-\infty}^{\infty} \left\{ \frac{2(z + 2nh - a)}{S_1^4} \left[1 + \frac{Qx}{R_1}\right] \right.$$

$$\left. - \frac{2(z + 2nh + a)}{S_2^4} \left[1 + \frac{Qx}{R_2}\right] + \frac{Qx\,(z + 2nh - a)}{S_1^2 R_1^3} - \frac{Qx\,(z + 2nh + a)}{S_2^2 R_2^3} \right\} \qquad [50]$$

$$h \frac{\partial (T_V + L_{V_1})}{\partial z} = \frac{h^2}{2} \sum_{n=-\infty}^{\infty} \left\{ \frac{(\eta - y)^2 - (z + 2nh - a)^2}{S_1^4} \left[1 + \frac{Qx}{R_1}\right] \right.$$

$$\left. - \frac{(\eta - y)^2 - (z + 2nh + a)^2}{S_2^4} \left[1 + \frac{Qx}{R_2}\right] - \frac{Qx\,(z + 2nh - a)^2}{S_1^2\, R_1^3} + \frac{Qx\,(z + 2nh + a)^2}{S_2^2\, R_2^3} \right\} \qquad [51]$$

The derivatives of the other kernel functions are obtained from the derivatives of $K_V(\eta)$ in Equation [33]. The x-derivative is the real part of

$$h \frac{\partial K_V}{\partial x} = -J(\eta, w_+) - J(\eta, w_-) = -\int_0^{\pi/2} e(\theta)\,[M(w_+, \theta) + M(w_-, \theta)]\, d\theta \qquad [52]$$

where $e(\theta)$ is the product of cos θ and its former value. The y-derivative is

$$h \frac{\partial K_V}{\partial y} = J(\eta, w_+) - J(\eta, w_-) = \int_0^{\pi/2} e(\theta)\,[M(w_+, \theta) - M(w_-, \theta)]\, d\theta \qquad [53]$$

where $e(\theta)$ is the product of sin θ and its former value. In both cases the new value of $f(k,a,z)$ is

$$f(k,a,z) = kh \sinh ka \cosh kz \qquad [54]$$

Expressions for the functions $J_2(\eta, w)$ and $J_3(\eta, w)$ may be found in Appendix B. The functions N_1 and N_2 needed for $J_2(\eta, w)$ are obtained by replacing k by $K(1+i)$ in Equation [54]. Then the real and imaginary parts are

$$\begin{aligned} N_1 &= Kh\,[A_1 Z_3 - A_2 Z_4 - A_1 Z_4 - A_2 Z_3] \\ N_2 &= Kh\,[A_1 Z_3 - A_2 Z_4 + A_1 Z_4 + A_2 Z_3] \end{aligned} \qquad [55]$$

The z-derivative of $K_V(\eta)$ is

$$h \frac{\partial K_V}{\partial z} = I(\eta, w_+) + I(\eta, w_-) \qquad [56]$$

The function $e(\theta)$ is unchanged but $f(k,a,z)$ becomes:

$$f(k,a,z) = kh \sinh ka \sinh kz \qquad [57]$$

Expressions for $I_2(\eta, w)$ and $I_3(\eta, w)$ may be found in Appendix B. The functions N_1 and N_2 required for $I_2(\eta, w)$ are

$$\begin{aligned} N_1 &= Kh\,[A_1 Z_1 - A_2 Z_2 - A_1 Z_2 - A_2 Z_1] \\ N_2 &= Kh\,[A_1 Z_1 - A_2 Z_2 + A_1 Z_2 + A_2 Z_1] \end{aligned} \qquad [58]$$

Explicit forms for the three derivatives of $K_{V_3}(\eta)$ for positive and negative values of x are readily obtained from the derivatives of the functions given in Equations [40] through [45].

POTENTIAL OF A HORIZONTAL LINE DOUBLET OF FINITE SPAN

The potential of a horizontal line doublet of finite span at $z = a$ and its image in the sea bed at $z = -a$ is

$$\phi_{D_I} = xh \int_{-b}^{b} \mu(\eta) \left[\frac{1}{[x^2 + (\eta - y)^2 + (z-a)^2]^{3/2}} + \frac{1}{[x^2 + (\eta - y)^2 + (z+a)^2]^{3/2}} \right] d\eta \qquad [59]$$

where $\mu(\eta)$ is the doublet strength per unit length. This potential vanishes for large values of x and its z-derivative vanishes at the sea bed. The integral form for ϕ_{D_I} is

$$\phi_{D_I} = Re\, \frac{h}{\pi i} \int_{-b}^{b} \mu(\eta)\, d\eta \int_{-\pi/2}^{\pi/2} \cos\theta d\theta \int_0^{\infty} e^{ikw}\, [e^{-k|z-a|} + e^{-k(z+a)}]\, kdk \qquad [60]$$

where w is defined in Equation [12].

To satisfy the free surface boundary condition of Equation [6], it is necessary to add a potential $\phi_{D_{II}}$ which contains an undetermined parameter $G(k, \theta)$. As in the case of the semi-infinite vortex sheet, $G(k, \theta)$ must have the same x and y dependence as ϕ_{D_I} and must satisfy Boundary Condition I. For points near the surface where z is greater than a, the potential is

$$\phi_{D_I} + \phi_{D_{II}} = Re\, \frac{2h}{\pi i} \int_{-b}^{b} \mu(\eta)\, d\eta \int_{-\pi/2}^{\pi/2} \cos\theta d\theta \int_0^{\infty} e^{ikw} \cosh ka\, [e^{-kz} + G(\theta, k) \cosh kz]\, kdk$$

[61]

When Boundary Condition II is applied, $G(k, \theta)$ has the same form it had for the case of the semi-infinite vortex sheet, Equations [15] through [28]. It is therefore possible to write down the solution at once.

$$\phi_D = \frac{2}{h} \int_{-b}^{b} \mu(\eta) \, [K_{D_1}(\eta) + K_{D_2}(\eta) + K_{D_3}(\eta)] \, d\eta \qquad [62]$$

In this equation the values of $K_{D_1}(\eta)$ for the two ranges of Froude number are

$$K_{D_1}(\eta) = Re \, \frac{h^2}{2\pi i} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta \int_0^\infty e^{ikw} \left[e^{-k|z-a|} + e^{-k(z+a)} + \frac{2 e^{-kh} \cosh ka \cosh kz}{\sinh kh} \right] kdk \qquad [63]$$

$$K_{D_1}'(\eta) = Re \, \frac{h^2}{2\pi i} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta \int_0^\infty e^{ikw} \left[e^{-k|z-a|} + e^{-k(z+a)} - \frac{2 e^{-kh} \cosh ka \cosh kz}{\cosh kh} \right] kdk \qquad [64]$$

These functions have the nonintegral form

$$K_{D_1}(\eta) = \frac{h^2}{2} \sum_{n=-\infty}^{\infty} Q \left\{ \frac{x}{[x^2 + (\eta - y)^2 + (z + 2nh - a)^2]^{3/2}} + \frac{x}{[x^2 + (\eta - y)^2 + (z + 2nh + a)^2]^{3/2}} \right\} \qquad [65]$$

where Q is unity in the subcritical case but has the value $(-1)^n$ in the supercritical case.

The functions $K_{D_2}(\eta)$ and $K_{D_3}(\eta)$ are obtained from the real part of the double integral

$$K_D(\eta) = I(w_+, \eta) + I(w_-, \eta) = Re \; i \int_0^{\pi/2} e(\theta) \, [M(w_+, \theta) + M(w_-, \theta)] \, d\theta \qquad [66]$$

where $M(w, \theta)$ is the complex function defined in Equation [34]. The functions $f(k,a,z)$, $e(\theta)$, and $g(k)$ are

$$f(k,a,z) = kh \cosh ka \cosh kz$$
$$e(\theta) = F^2 \cos^3 \theta \qquad [67]$$
$$g(k) = \cosh kh \sinh kh$$

The functions $I_2(\eta, w)$ and $I_3(\eta, w)$ are defined in Appendix B. The functions N_1 and N_2 needed for $I_2(\eta, w)$ are

$$N_1 = Kh\,[A_3Z_3 - A_4Z_4 - A_3Z_4 - A_4Z_3]$$
$$N_2 = Kh\,[A_3Z_3 - A_4Z_4 + A_3Z_4 + A_4Z_3] \qquad [68]$$

In the supercritical range of Froude number it is necessary to determine the Cauchy principal value at the singularity which occurs when $F \cos \theta = 1$. Explicit forms of $K_{D_3}(\eta)$ are the same as those for $K_{V_3}(\eta)$, Equations [40] through [44], if the functions $g(\theta)$ and $f(k_0)$ are given the values

$$g(\theta) = \frac{(k_0h)^2 \cos^3 \theta \cosh k_0a \cosh k_0z}{\sinh k_0h \cosh k_0h\,[F^2 \cos^2 \theta - \operatorname{sech}^2 k_0h]} \qquad [69]$$

$$f(k_0) = \frac{(k_0h)^2 \cos^2 \theta \cosh k_0a \cosh k_0z}{\sinh k_0h \cosh k_0h \sin \theta} \qquad [70]$$

The derivatives of ϕ_D are obtained from the derivatives of the kernel functions. The three derivatives of $K_{D_1}(\eta)$ in Equation [65] are

$$h\,\frac{\partial K_{D_1}}{\partial x} = \frac{h^3}{2} \sum_{n=-\infty}^{\infty} Q \left[\frac{1 - 3x^2/R_1^2}{R_1^3} + \frac{1 - 3x^2/R_2^2}{R_2^3} \right] \qquad [71]$$

$$h\,\frac{\partial K_{D_1}}{\partial y} = \frac{3}{2}\, h^3 x(\eta - y) \sum_{n=-\infty}^{\infty} Q \left[\frac{1}{R_1^5} + \frac{1}{R_2^5} \right] \qquad [72]$$

$$h\,\frac{\partial K_{D_1}}{\partial z} = -\frac{3}{2}\, h^3 x \sum_{n=-\infty}^{\infty} Q \left[\frac{z + 2nh - a}{R_1^5} + \frac{z + 2nh + a}{R_2^5} \right] \qquad [73]$$

where R_1 and R_2 are as defined in Equation [47]. The derivatives of the other kernel functions are obtained from the derivatives of $K_D(\eta)$ in Equation [66]. The x-derivative is the real part of

$$h \frac{\partial K_D}{\partial x} = -J(\eta, w_+) - J(\eta, w_-) \tag{74}$$

where $e(\theta)$ is the product of cos θ and its former value. The y-derivative is

$$h \frac{\partial K_D}{\partial y} = J(\eta, w_+) - J(\eta, w_-) \tag{75}$$

where $e(\theta)$ is the product of sin θ and its former value. In both cases the new value of $f(k,a,z)$ is

$$f(k,a,z) = (kh)^2 \cosh ka \cosh kz \tag{76}$$

Expressions for the functions $J_2(\eta, w)$ and $J_3(\eta, w)$ may be found in Appendix B. The functions N_1 and N_2 needed for $J_2(\eta, w)$ are

$$\begin{aligned} N_1 &= -2(Kh)^2 [A_3 Z_4 + A_4 Z_3] \\ N_2 &= 2(Kh)^2 [A_3 Z_3 - A_4 Z_4] \end{aligned} \tag{77}$$

The z-derivative of $K_D(\eta)$ is

$$h \frac{\partial K_D}{\partial z} = I(\eta, w_+) + I(\eta, w_-) \tag{78}$$

The function $e(\theta)$ is unchanged but $f(k,a,z)$ becomes

$$f(k,a,z) = (kh)^2 \cosh ka \sinh kz \tag{79}$$

The functions N_1 and N_2 needed for $I_2(\eta, w)$ are

$$\begin{aligned} N_1 &= -2(Kh)^2 [A_3 Z_2 + A_4 Z_1] \\ N_2 &= 2(Kh)^2 [A_3 Z_1 - A_4 Z_2] \end{aligned} \tag{80}$$

STRENGTHS OF THE SINGULARITIES USED TO REPRESENT A HYDROFOIL

The strength of vorticity along the span of a hydrofoil is a function of the chord and lift developed at any section and the free-stream velocity. If the vorticity is assumed to have an elliptic distribution, the vorticity per unit length $\Gamma(\eta)$ in the equations for ϕ_V has the form[9]

$$\Gamma(\eta) = \frac{2L}{\pi\rho U h b} \sqrt{1 - \frac{\eta^2}{b^2}} = \frac{2}{\pi} \frac{c C_L U}{h} \sqrt{1 - \frac{\eta^2}{b^2}} \qquad [81]$$

In this expression L is the total lift and C_L is the average lift coefficient of the foil.

The strength per unit length of the horizontal line doublet, used to represent the displacement effect of the hydrofoil, may be written in terms of a shape parameter λ as follows

$$\mu(\eta) = \frac{1}{8h^2} \lambda\, t^2(\eta)\, U \qquad [82]$$

where $t(\eta)$ is the maximum width of the hydrofoil section. If the hydrofoil had a circular cross section, λ would be unity and $\mu(\eta)$ would be the usual strength of a line doublet Lock[10] has computed λ for a number of different geometric cross sections. For typical hydrofoil sections λ is approximately

$$\lambda \approx 0.42\,(1 + c/t) \qquad [83]$$

where c/t is the length-width ratio of the section.

ACKNOWLEDGMENTS

The author is grateful to Mr. T. F. Ogilvie for valuable discussions of the problem and for checking the report. The algebraic expressions were checked for accuracy by Mr. T. J. Langan.

APPENDIX A

INTEGRAL RELATIONS USED IN THE POTENTIAL FUNCTIONS

The integral relations used for the potential functions in the foregoing analysis are discussed in Wu's report[1] and are summarized here. If the relations are known for one of the singularities, the others may be obtained by differentiation or integration. If the basic integral relation is given for the source, the doublet relations are obtained by differentiating this function with respect to one of the coordinates. The integral relation for the semi-infinite vortex sheet is obtained by integrating the expression for the vertical doublet from the position of the hydrofoil downstream to infinity.

As the potential of a source is proportional to the reciprocal of the distance, the first integral relation is

$$\frac{1}{\sqrt{x^2 + y^2 + z^2}} = \int_0^\infty e^{-k|z|} \, J_0(k\sqrt{x^2 + y^2}) \, dk \qquad [1A]$$

where the zeroth-order Bessel function $J_0 \,(k \sqrt{x^2 + y^2})$ may be defined by the integral

$$J_0 \,(k \sqrt{x^2 + y^2}) = \frac{2}{\pi} \int_0^{\pi/2} \cos\,(kx \cos\,\theta) \cos\,(ky \sin\,\theta) \, d\theta$$

$$= \frac{1}{\pi} \int_0^{\pi/2} [\cos\,kw_+ + \cos\,kw_-] \, d\theta \qquad [2A]$$

where

$$w_+ = x \cos\,\theta + y \sin\,\theta$$

$$w_- = x \cos\,\theta - y \sin\,\theta \qquad [3A]$$

Equation [2A] may also be written as the real part of

$$J_0 \,(k \sqrt{x^2 + y^2}) = Re \, \frac{1}{\pi} \int_0^{\pi/2} [e^{ikw_+} + e^{ikw_-}] \, d\theta = Re \, \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{ikw_+} \, d\theta \qquad [4A]$$

If Equations [1A] and [4A] are combined, the potential for the source is

$$\frac{1}{\sqrt{x^2 + y^2 + z^2}} = Re \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} d\theta \int_0^{\infty} e^{-k|z|} e^{ikw_+} dk \qquad [5A]$$

The potential for the horizontal doublet is obtained by finding the x-derivative of Equation [5A].

$$\frac{x}{[x^2 + y^2 + z^2]^{3/2}} = Re \frac{1}{\pi i} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta \int_0^{\infty} e^{-k|z|} e^{ikw_+} kdk \qquad [6A]$$

The potential of a vertical doublet is obtained by finding the z-derivative of Equation [5A].

$$\frac{z}{[x^2 + y^2 + z^2]^{3/2}} = Re \frac{\text{sgn } z}{\pi} \int_{-\pi/2}^{\pi/2} d\theta \int_0^{\infty} e^{-k|z|} e^{ikw_+} kdk \qquad [7A]$$

If x in this expression is replaced by $x - \xi$, the potential of a semi-infinite distribution of doublets is obtained by integrating both sides of the equation over ξ from zero to infinity.

$$\frac{z}{y^2 + z^2} \left[1 + \frac{x}{\sqrt{x^2 + y^2 + z^2}} \right] =$$

$$- Re \lim_{\xi_m = \infty} \frac{\text{sgn } z}{\pi i} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\cos \theta} \int_0^{\infty} e^{-k|z|} e^{iky \sin \theta} \left[e^{ik(x-\xi)\cos \theta} \right]_{\xi=0}^{\xi=\xi_m} dk \qquad [8A]$$

Near the upper limit where ξ_m is very much larger than x, the integral over θ for any k becomes

$$\frac{2}{\pi} \int_0^{\pi/2} \frac{\sin (k \xi_m \cos \theta) \cos (ky \sin \theta)}{\cos \theta} d\theta = \frac{2}{\pi} \int_0^{k \xi_m} \frac{\sin u \cos (ky\sqrt{1 - (u/k\xi_m)^2})\, du}{u\sqrt{1 - (u/k\, \xi_m)^2}}$$

[9A]

In the limit as ξ_m approaches infinity, this integral becomes

$$\frac{2}{\pi} \cos ky \int_0^\infty \frac{\sin u}{u}\, du = \cos ky \qquad [10A]$$

Therefore, the two integral relations for the semi-infinite line distribution of doublets are the real parts of

$$\frac{z}{y^2 + z^2} = Re \ \mathrm{sgn}\ z \int_0^\infty e^{-k|z|}\, e^{iky}\, dk \qquad [11A]$$

$$\frac{z}{y^2 + z^2} \ \frac{x}{\sqrt{x^2 + y^2 + z^2}} = Re \ \frac{\mathrm{sgn}\ z}{\pi i} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\cos\theta} \int_0^\infty e^{-k|z|}\, e^{ikw_+}\, dk \qquad [12A]$$

When Boundary Condition II is applied in the case of the semi-infinite line distribution of doublets, the kernel function $L_{V_1}(\eta)$ from Equation [19] is

$$L_{V_1}(\eta) = Re \ \frac{h}{2\pi i} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\cos\theta} \int_0^\infty e^{ikw_+}\, F(k,a,z)\, dk \qquad [13A]$$

For subcritical Froude number

$$F(k,a,z) = \mathrm{sgn}\,(z-a)\, e^{-k|z-a|} - e^{-k(z+a)} + e^{-kh}\, \frac{\sinh ka \cosh kz}{\sinh kh} \qquad [14A]$$

If the hyperbolic functions are written in their exponential forms

$$F(k,a,z) = \sum_{n=0}^{\infty} \mathrm{sgn}\,(2nh + z - a)\, e^{-k|2nh+z-a|} - \sum_{n=1}^{\infty} e^{-k(2nh - z + a)}$$

$$- \sum_{n=0}^{\infty} e^{-k(2nh + z + a)} + \sum_{n=1}^{\infty} e^{-k(2nh - z - a)} \qquad [15A]$$

If the sign of n is changed in the second and fourth sums, the terms combine to give

$$F(k,a,z) = \sum_{n=-\infty}^{\infty} \operatorname{sgn}(2nh+z-a)\, e^{-k|2nh+z-a|} - \sum_{n=-\infty}^{\infty} \operatorname{sgn}(2nh+z+a)\, e^{-k|2nh+z+a|} \qquad [16A]$$

If the integral relation of Equation [12A] is used

$$L_{V_1}(\eta) = \frac{h}{2} \sum_{n=-\infty}^{\infty} \left\{ \frac{2nh+z-a}{(\eta-y)^2+(2nh+z-a)^2} \, \frac{x}{\sqrt{x^2+(\eta-y)^2+(2nh+z-a)^2}} - \frac{2nh+z+a}{(\eta-y)^2+(2nh+z+a)^2} \, \frac{x}{\sqrt{x^2+(\eta-y)^2+(2nh+z+a)^2}} \right\} \qquad [17A]$$

In the supercritical range of Froude number, the last term in the expression for $F(k,a,z)$ is $-\cosh kh$ instead of $\sinh kh$; see Equations [25] and [26]. Then in the kernel function $L_{V_1'}(\eta)$ the terms in the infinite sums alternate in sign.

APPENDIX B

EVALUATION OF THE INTEGRALS BY CONTOUR INTEGRATION

The wavemaking potentials derived in this report are obtained from the real or imaginary part of an integral of the type

$$L(\eta) = J(\eta, w_+) + J(\eta, w_-) - i\,[I(\eta, w_+) + I(\eta, w_-)] \qquad [1B]$$

Whereas the kernel functions of the potentials $K(\eta)$ and their z-derivatives are of the type $I(\eta, w)$, the x- and y-derivatives are of the type $J(\eta, w)$. In this expression $J(\eta, w)$ and $I(\eta, w)$ are the real and imaginary parts of the complex integral

$$J(\eta, w) - i\,I(\eta, w) = \int_0^{\pi/2} e(\theta)\, M(w, \theta)\, d\theta \qquad [2B]$$

where $M(w, \theta)$ is the principal value of the integral

$$M(w, \theta) = \frac{h}{\pi} \int_0^{\infty} \frac{f(k,a,z)\, e^{ikw}\, dk}{g(k)\left[F^2 \cos^2\theta - \dfrac{\tanh kh}{kh}\right]} \qquad [3B]$$

In these equations w_+ and w_- are as defined in Equation [32], and w is written for either w_+ or w_-. The functions $e(\theta)$ and $f(k,a,z)$ depend upon the type of singularity, and $g(k)$ is

$$g(k) = \sinh kh \cosh kh \qquad [4B]$$

The integrals are not defined for $F = 1$.

The integrals in Equations [2B] and [3B] have singularities in the integrands and must be evaluated by means of a contour integration. In the integral defining $M(w, \theta)$ there are no singularities in the quotient $f(k,a,z)/g(k)$. There is a simple pole in the integrand, however, when $k = k_0$, where k_0 is defined by the equation

$$k_0 h\, F^2 \cos^2\theta = \tanh k_0 h \qquad [5B]$$

In the complex k-plane there are infinitely many singularities on the imaginary axis, and the contour of integration must be chosen in such a way that this axis is excluded and the integrand remains finite. If w is positive, Contour I of Figure 2, which is in the upper part of

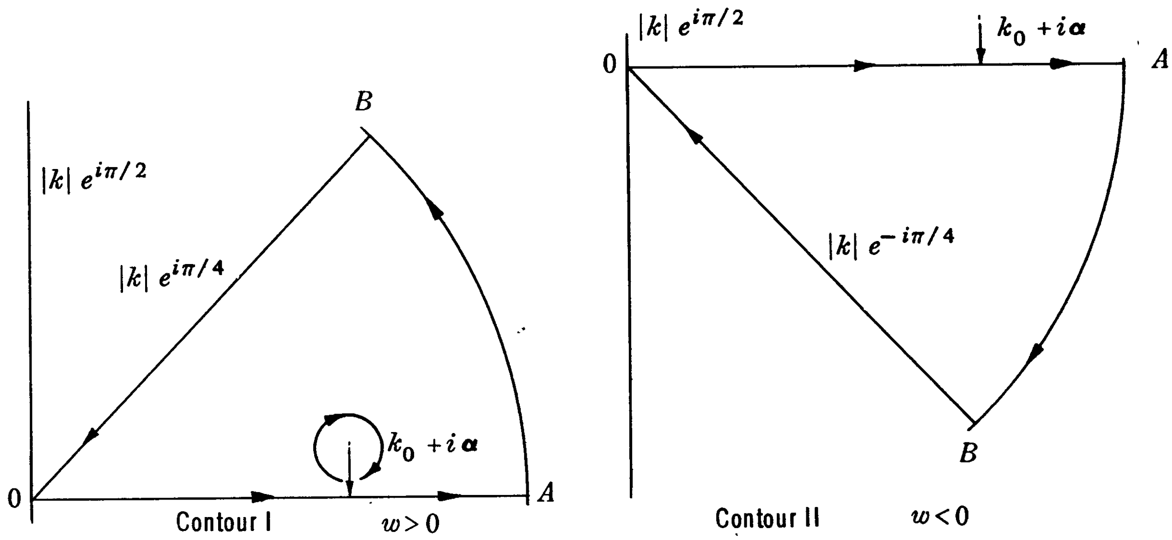

Figure 2 – Contours in the Complex k-Plane for Evaluating the Integrals

the plane, may be used. If w is negative, Contour II in the lower part of the plane may be used. The signs of w_+ and w_- for different ranges of x and θ are shown in Table 1 for positive values of $\eta - y$. For negative values of $\eta - y$, w_+ and w_- are interchanged. In this table

$$\tan\theta_1 = \left|\frac{\eta - y}{x}\right| \qquad [6B]$$

TABLE 1

Signs of w_+ and w_- for Different Ranges of x and θ

$x > 0$	$w_+ > 0$	$0 < \theta < \pi/2$	Contour I
	$w_- > 0$	$0 < \theta < \pi/2 - \theta_1$	Contour I
	$w_- < 0$	$\pi/2 - \theta_1 < \theta < \pi/2$	Contour II
$x < 0$	$w_- < 0$	$0 < \theta < \pi/2$	Contour II
	$w_+ < 0$	$0 < \theta < \pi/2 - \theta_1$	Contour II
	$w_+ > 0$	$\pi/2 - \theta_1 < \theta < \pi/2$	Contour I

Figure 2 shows the pole within Contour I at $k_0 + i\alpha$, where α approaches the limit zero. Therefore in the integration about Contour I, there is a residue term which provides wavemaking; but there is no residue or wavemaking associated with Contour II. For large positive values of x, θ_1 is very small and w_+ and w_- are positive over most of the range of θ. For small negative values of x, there is wavemaking ahead of the hydrofoil which decreases as x increases and vanishes as x approaches minus infinity. Therefore, Boundary Condition III is satisfied. If the pole were exactly on the axis and shared equally by both contours, there would be as much wavemaking ahead of the hydrofoil as downstream from it. This clearly contradicts Boundary Condition III, and Figure 2 gives the correct location of the pole. This analysis is equivalent to the use of a fictitious viscosity to locate the singularity.

When w is positive, the integration around Contour I gives

$$\oint_A^B [\quad]\, dk = -\int_A^B [\quad]\, k e^{i\theta}\, d\theta + \int_0^B [\quad]\, e^{i\pi/4}\, d\,|k| + 2\pi i\, Res(k_0) \qquad [7B]$$

In the limit, when A and B become infinite, the integral over the arc AB vanishes. Along OB

$$|k|\, e^{i\pi/4} = K(1+i) \qquad [8B]$$

When w is negative, the integration around Contour II gives

$$\oint_A^B [\quad]\, dk = \int_0^B [\quad]\, (1-i)\, dK \qquad [9B]$$

If the term for the integration along OB is given the subscript 2 and the term for the residue, subscript 3, then M is given by the sum of M_2 and M_3, where

$$M_2(w,\theta) = \frac{h}{\pi}\int_0^\infty \frac{f(K(1+i\,\mathrm{sgn}\,w),\, a, z)\, e^{-K|w|}\, e^{iKw}(1+i\,\mathrm{sgn}\,w)\, dK}{g(K(1+i\,\mathrm{sgn}\,w))\left[F^2\cos^2\theta - \dfrac{\tanh(Kh(1+i\,\mathrm{sgn}\,w))}{Kh(1+i\,\mathrm{sgn}\,w)}\right]}$$

$$= i\,\mathrm{sgn}\,w\,\frac{2h}{\pi}\int_0^\infty \frac{f(K(1+i\,\mathrm{sgn}\,w),\, a, z)\, e^{-K|w|}\, e^{iKw}\, dK}{\dfrac{g(K(1+i\,\mathrm{sgn}\,w))}{Kh}\,[Kh(1+i\,\mathrm{sgn}\,w)\, F^2\cos^2\theta - \tanh(Kh(1+i\,\mathrm{sgn}\,w))]} \qquad [10B]$$

$$M_3(w,\theta) = \frac{i(1+\mathrm{sgn}\,w)\, f(k_0, a, z)\, e^{ik_0 w}}{g(k_0)\,[F^2\cos^2\theta - \mathrm{sech}^2\, k_0 h]} \qquad [11B]$$

To obtain the real and imaginary parts of $M_2(w, \theta)$, let

$$N_1 + i \operatorname{sgn} w \, N_2 = f(K(1+i \operatorname{sgn} w), a, z) \qquad [12B]$$

$$D_1 + i \operatorname{sgn} w \, D_2 = \frac{g(K(1+i\operatorname{sgn} w))}{Kh} \left[Kh(1+i\operatorname{sgn} w) F^2 \cos^2\theta - \tanh(Kh(1+i\operatorname{sgn} w))\right] \qquad [13B]$$

The real and imaginary parts of $K_2(\eta)$ are

$$J_2(\eta, w) = -\frac{2h}{\pi} \int_0^{\pi/2} e(\theta)\, d\theta \int_0^{\infty} e^{-K|w|} \{C_1 \operatorname{sgn} w \sin Kw + C_2 \cos Kw\}\, dK \qquad [14B]$$

$$I_2(\eta, w) = -\frac{2h}{\pi} \int_0^{\pi/2} e(\theta)\, d\theta \int_0^{\infty} e^{-K|w|} \{C_1 \operatorname{sgn} w \cos Kw - C_2 \sin Kw\}\, dK \qquad [15B]$$

where

$$C_1 = \frac{N_1 D_1 + N_2 D_2}{D_1^2 + D_2^2}; \quad C_2 = \frac{N_2 D_1 - N_1 D_2}{D_1^2 + D_2^2} \qquad [16B]$$

The values of N_1 and N_2 must be determined for the particular singularity under consideration. These are functions of the following quantities:

$$\left.\begin{aligned} A_1 &= \sinh Ka \cos Ka & A_2 &= \cosh Ka \sin Ka \\ A_3 &= \cosh Ka \cos Ka & A_4 &= \sinh Ka \sin Ka \end{aligned}\right\} \qquad [17B]$$

$$\left.\begin{aligned} Z_1 &= \sinh Kz \cos Kz & Z_2 &= \cosh Kz \sin Kz \\ Z_3 &= \cosh Kz \cos Kz & Z_4 &= \sinh Kz \sin Kz \end{aligned}\right\} \qquad [18B]$$

Then the expressions for D_1 and D_2 are

$$\begin{aligned} D_1 &= \frac{1}{2Kh}\left[Kh\, F^2 \cos^2\theta\, (H_1 - H_2) - H_3 + 1\right] \\ D_2 &= \frac{1}{2Kh}\left[Kh\, F^2 \cos^2\theta\, (H_1 + H_2) - H_4\right] \end{aligned} \qquad [19B]$$

where

$$H_1 = \sinh 2Kh \cos 2Kh \qquad H_2 = \cosh 2Kh \sin 2Kh$$
$$H_3 = \cosh 2Kh \cos 2Kh \qquad H_4 = \sinh 2Kh \sin 2Kh \tag{20B}$$

As K becomes small N_1 and N_2 in Equation [16B] are of magnitude $(Kh)^n$, where n is equal to or greater than unity for all the singularities considered. In the denominator D_1 and D_2 have magnitudes $(Kh)^3$ and $Kh\,[F^2 \cos^2 \theta - 1]$, respectively. Therefore, C_1 and C_2 are defined for all values of K and θ in the subcritical range of Froude number. In the supercritical range of Froude number, $F \cos \theta$ may also be greater than unity and becomes equal to unity when $\theta = \theta_0$, defined by

$$\cos \theta_0 = 1/F \tag{21B}$$

When the value of the integrand is investigated as θ approaches θ_0 from both sides of the singularity, the Cauchy principal value is found to exist and the integrals in Equations [14B] and [15B] are defined.

The real and imaginary parts of the functions defining $K_3(\eta)$ for the subcritical range of Froude number are

$$J_3(\eta, w) = -h \int_0^{\pi/2} \frac{e(\theta)\, f(k_0, a, z)\,(1 + \operatorname{sgn} w) \sin k_0 w \; k_0 d\theta}{\sinh k_0 h \cosh k_0 h \,[F^2 \cos^2 \theta - \operatorname{sech}^2 k_0 h]} \tag{22B}$$

$$I_3(\eta, w) = -h \int_0^{\pi/2} \frac{e(\theta)\, f(k_0, a, z)\,(1 + \operatorname{sgn} w) \cos k_0 w \; k_0 d\theta}{\sinh k_0 h \cosh k_0 h \,[F^2 \cos^2 \theta - \operatorname{sech}^2 k_0 h]} \tag{23B}$$

As long as the Froude number is less than unity, the smallest value k_0 can assume is always greater than zero and these expressions are defined for all values of k_0 and θ.[11]

In the supercritical range of Froude number, the expressions for $J_3(\eta, w)$ and $I_3(\eta, w)$ are the same except for the lower limit of the integral which is θ_0 instead of zero. As k_0 is zero when $\theta = \theta_0$, these expressions appear to have singularities for these values of the parameters. The difficulty is avoided, however, if the integration is evaluated over k_0 instead of θ. If Equation [5B] is differentiated

$$\frac{d\theta}{F^2 \cos^2 \theta - \operatorname{sech}^2 k_0 h} = \frac{h dk_0}{2F^2 \cos \theta \sin \theta} \tag{24B}$$

then

$$J_3'(\eta, w) = -\frac{h^2}{2F^2} \int_0^\infty \frac{e(\theta)\, f(k_0,a,z)\,(1+\operatorname{sgn} w)\, \sin k_0 w \; k_0 dk_0}{\sinh k_0 h \cosh k_0 h \cos\theta \sin\theta} \qquad [25B]$$

$$I_3'(\eta, w) = -\frac{h^2}{2F^2} \int_0^\infty \frac{e(\theta)\, f(k_0,a,z)\,(1+\operatorname{sgn} w)\, \cos k_0 w \; k_0 dk_0}{\sinh k_0 h \cosh k_0 h \cos\theta \sin\theta} \qquad [26B]$$

The integrands of these integrals have no singularities and converge to zero as k_0 approaches infinity and θ approaches $\pi/2$.

The derivatives of $L(\eta)$ with respect to x, y, and z are

$$h \frac{\partial L}{\partial x} = I(\eta, w_+) + I(\eta, w_-) - i\,[J(\eta, w_+) + J(\eta, w_-)] \qquad [27B]$$

where $e(\theta)$ in Equation [2B] is replaced by the product of $\cos\theta$ and its former value.

$$h \frac{\partial L}{\partial y} = -I(\eta, w_+) + I(\eta, w_-) + i\,[J(\eta, w_+) - J(\eta, w_-)] \qquad [28B]$$

where $e(\theta)$ is replaced by the product of $\sin\theta$ and its former value. In both cases $f(k,a,z)$ becomes the product of k and its former value.

$$h \frac{\partial L}{\partial z} = J(\eta, w_+) + J(\eta, w_-) - i\,[I(\eta, w_+) + I(\eta, w_-)] \qquad [29B]$$

where $e(\theta)$ is unchanged but $f(k,a,z)$ is replaced by the z-derivative of its former value.

REFERENCES

1. Wu, Y.T., "A Theory of Hydrofoils of Finite Span," Hydrodynamics Laboratory, California Institute of Technology Report 26-8 (1953).

2. Kaplan, P., et al., "Evaluation of the Theory for the Flow Pattern of a Hydrofoil of Finite Span," Experimental Towing Tank, Stevens Institute of Technology Report 561 (1955).

3. Breslin, J.P., "Application of Ship Wave Theory to the Hydrofoil of Finite Span," Experimental Towing Tank, Stevens Institute of Technology Note 380 (1956).

4. Lunde, J.K., "On the Linearized Theory of Wave Resistance for Displacement Ships in Steady and Accelerated Motion," Transactions of Society of Naval Architects and Marine Engineers, Vol. 59, pp. 25–76 (1951).

5. Pond, H.L., TMB SECRET Report C-743 (1957).

6. DiDonato, A.R., "The Calculation on NORC of Source Functions for Ship Waves," U. S. Naval Proving Ground Report 1588 (1958).

7. Inui, T. and Maruo, H., "Advances in Calculation of Wave-Making Resistance of Ships," Society of Naval Architects of Japan, 60th Anniversary Series, Vol. 2, Tokyo (1957).

8. Borden, A. and Power, J.L., TMB SECRET Report C-990 (1959).

9. Glauert, H., "The Elements of Airfoil and Airscrew Theory," Second Edition, University Press, Cambridge (1947).

10. Lock, C.N.H., "The Interference of a Wind Tunnel on a Symmetrical Body," Aeronautical Research Council London, R & M 1275 (Asr 421) (1929).

11. Cuthill, E.H. and Wrench, J.W., Jr., TMB SECRET Report C-832 (1957).

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POTENTIAL FUNCTIONS FOR SINGULARITIES ASSOCIATED WITH HYDROFOILS OF FINITE SPAN MOVING WITH SUBCRITICAL AND SUPERCRITICAL VELOCITIES IN SHALLOW WATER, by Avis Borden. June 1961. v, 31p. refs.

UNCLASSIFIED

This report shows that singularities for a semi-infinite vortex sheet and a horizontal line doublet, both of finite span, may be used to represent the vorticity and displacement produced by a finite span hydrofoil. The boundary conditions which must be satisfied when these singularities move in shallow water are derived. Expressions for the potential functions are obtained for both the subcritical and supercritical ranges of Froude number. Explicit expressions for the x-, y-, and z-derivatives of the potential functions are also presented. Methods for determining the strengths of the singularities are discussed.

1. Hydrofoils--Finite span
2. Potential flow--Shallow water effects--Mathematical analysis

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