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THEORY OF RUDDER-DIVING PLANE-SHIP VIBRATIONS AND FLUTTER, INCLUDING METHODS OF SOLUTION

by

E. H. Kennard

and

Ralph C. Leibowitz

STRUCTURAL MECHANICS LABORATORY

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NOTATION

A	Cross-sectional area of rudder stock
A_i	Amplification factor of triode (i = 1, 2, 3)
a_i	Positive conversion factors (i = 1, 2 . . . 6) (Sections 4.3, 4.4, and 5.3)
B	Lift constant (Section 8.2) (also, position of bearing of rudder or diving plane)
b	z-coordinate of effective point of attachment of rudder to rudder stock (using rudder x-axis)
b_i	Positive conversion factors (i = 1, 2 . . . 6) (Sections 4.3, 4.4, and 5.3)
b_L	Distance in z-direction above rudder x-axis of center of pressure on rudder
C, c, c_i, c_{ij}	Damping constants (Section 8 and Appendix A)
C_i	Capacitance (i = 1, 2 . . .)
C_L, C_p	Center of pressure
d	Distance of neutral axis of ship cross section above rudder stock bearing
E	Young's modulus of elasticity
e	Horizontal distance from centerplane of ship to centerplane of rudder or to bearing of diving plane
F_L	Transverse lift force on rudder at center of pressure
G	Shear modulus of elasticity
g	Distance of bearing of upper or lower rudder from horizontal plane of symmetry of ship (Section 7.3)
h	x-coordinate of effective center of attachment of rudder to rudder stock (using rudder x-axis)
h_b	Height of rudder bearing above rudder x-axis (equal to $\ell + b$)
I	Area moment of inertia of cross section of rudder stock relative to a diameter through the centroid

I_x, I_y, I_z	Moments of inertia of combined rudder and virtual mass about x-, y-, and z-axes with origin at the effective center of mass of the rudder in water
I_{xy}, I_{yz}, I_{zx}	Products of inertia corresponding to I_x, I_y, I_z
J_e	Polar moment of inertia of cross section of rudder stock about a perpendicular axis through the centroid
K	Shear warping constant or shear-flexibility factor; this numerical factor depends on shape of cross section of rudder stock (3/4 for circular and 2/3 for rectangular cross sections)
k, k_s	See Equations [12a,b]
L	Horizontal distance between rudder stock and center of pressure on rudder
L_i	Inductance (i = 1, 2, 3)
ℓ, ℓ_T	Effective length of rudder stock for computing bending or torsional flexibility, respectively
M	Bending moment on bottom of stock (Section 2), positive when it tends to produce positive θ; or on rudder at center of mass about x-axis positive from y toward z
M_b	Similar moment about x-axis acting on ship at rudder bearing
m	Mass of rudder
m_x, m_y, m_z	Effective mass of rudder including its virtual mass for $\ddot{u}$, $\ddot{v}$, and $\ddot{w}$ motions, respectively
m_{xy}, m_{xz}	Cross-inertial constants associated with m_x, m_y, and m_z
n, n'	Station numbers (Section 4.2)
P	Shear force on bottom of stock (Section 2); external force on the ship, positive toward positive y (Section 4.2)
P_v	Similar force on ship when ship's y-axis is vertical
p	Positive conversion factor equal to $\frac{t}{t'}$ (Section 4.3)

Q	Moment of force on rudder (or starboard plane) about rudder (or plane) y-axis positive from x toward z; similar external moment on the ship about the ship's z-axis positive from ship x-axis to ship y-axis (Sections 5.1, 5.2, and Reference 3)
Q_b	Moment on ship at bearing in rudder (or diving plane) xz-plane, positive from x toward z
q	Conversion factor (positive) for energy (Appendix B)
R_i	Resistance ($i = 1, 2, 3$)
r	Transformer turns ratio
r_s	A ratio defined following Equation [7]
S	Ship's forward speed relative to water
s	A fraction, $0 \leqq s < 1$; $s\Delta x$ is the distance from station $n - \frac{1}{2}$ to rudder bearing
s'	A fraction, $0 \leqq s' < 1$; $s'\Delta x$ is the distance from station n' to rudder bearing (Section 4.2)
T	Torsional moment in rudder stock; moment of force about z-axis on rudder at center of mass, positive from x toward y
T_b	Similar moment of force on ship at effective point of attachment of rudder stock for torsion
T_k	Kinetic energy of rudder in v, γ, α motion
t	Time (for mechanical system)
U	External torsional moment acting on ship in yz-plane (Section 4.2)
U_p	Potential energy of rudder in v, γ, α motion
u, v, w	Small translations of effective center of mass of rudder in rudder x-, y-, z-directions, respectively
u_b, v_b, w_b	Corresponding translations of top of stock
X, Y, Z	Forces on rudder acting at its effective center of mass positive in rudder x-, y-, z-directions, respectively

X_b, Y_b, Z_b	Corresponding forces on ship at top of stock
x, y, z	Rectangular coordinates with x-axis always parallel to the ship axis. In rudder theory (except as otherwise specified for an upper rudder), the origin is at the effective center of mass of the rudder and the z-axis is vertical and positive upward. In the theory of ship vibrations,[3] when dealing with horizontal-torsional hull vibrations, the ship y- and z-axes are drawn in the same directions as for a rudder but usually with a different origin; for vertical hull vibrations, the ship y-axis is vertical and positive upward so that the ship z-axis is horizontal and positive in the opposite direction from positive rudder y. In certain cases other axial positions are used temporarily (see Section 7).
y_h	Horizontal displacement of point of ship initially on its x-axis
y_v	Vertical displacement of ship
z_b	Height of rudder stock bearing above x-axis as drawn in ship theory (see Figure 6)
α, β, γ	Small rotations of rudder about rudder x-, y- and z-axes, respectively; α is positive from y toward z, β from x toward z, γ from x toward y
α_b, β_b, γ_b	Corresponding small rotations of top of stock
Δx	Length of element (distance between stations)
θ	See Equations [3] to [7b]
λ	Conversion factor defined in Equation [21a]
$\bar{\lambda}$	Conversion factor defined below Equation [31]
ν	Poisson's ratio
ρ_1, ρ_2	Conversion factors defined by Equations [21b,c]
$\bar{\rho}_1$, $\bar{\rho}_2$	Conversion factors defined below Equation [31]
ϕ	Torsional angle of rotation about stock axis of rudder relative to ship; torsional angle of rotation of hull about ship x-axis, positive from positive ship y-axis toward the upward vertical z-axis

NOTATION (continued)

ω	Natural circular frequency of vibration of mechanical system
'	Electrical quantities analagous to mechancial quantities are denoted by a primed exponent such as V_n' and t' corresponding to V_n and t, respectively, except where otherwise noted.

ABSTRACT

A theory is advanced for treating the vibration characteristics of a control surface (e.g., rudder or diving plane)-hull system subject to hydrodynamic forces on the control surface. The control surface may have 6 degrees of freedom whereas the flexible hull itself may have additional sprung bodies, representing machinery, cargo, or superstructures, with 1 or 2 degrees of freedom elastically attached to it at various locations. The purpose of this report is to more adequately represent a ship in forward motion and its appendages as a mass-hydroelastic system subject to vibrations, including flutter. Analytical, digital, and electrical-analog methods are devised to determine the natural frequencies, mode shapes, critical flutter speeds, and damping of this system and/or parts of this system.

1. INTRODUCTION

In the theory of ship vibrations a control surface, such as a rudder or diving plane, is usually treated merely as a mass or sprung mass[1,2,3*] added to the ship, and in most cases this procedure is quite adequate. In some cases, however, vibration of a control surface considered as a body having 6 degrees of freedom relative to the ship assumes practical importance. The question is sometimes raised as to the influence of such control surface vibrations upon the frequencies and mode patterns of the ship vibrations themselves. In attacking such questions, the control surface may be idealized as a rigid body mounted on the end of a flexible control surface stock. The control surface will then be a dynamical system with 6 degrees of freedom. The amplitude of vibration is assumed to be small. Equations of motion for a rudder alone, with the ship stationary, were used in calculations of rudder frequencies for USS ALBACORE (AGSS 569).[4] The purpose of the present report is to extend the theory to include reactions on the ship control surface and water causing control surface flutter

*References are listed on page 119.

and to present a design for an analog network representing the various control surfaces subject to hydrodynamic forces, which may also be attached in a suitable manner to the network representing the ship,[3] for a study of the vibrations of the combined system. In addition, finite difference equations of motion for the control surface-hull system subject to hydrodynamic forces on the control surfaces have been devised and coded on the IBM 7090.*

2. ELASTICITY OF THE RUDDER STOCK

The rudder itself can be assumed to be rigid without serious error but the stock by which the rudder is attached to the ship exhibits appreciable flexibility. Let ℓ_T denote the effective length of the stock for torsion, including, perhaps, additions to its actual length to allow for local deformation of the ship structure at its top and of the rudder at its bottom. At an intermediate point the rudder is restrained by a bearing. It will be assumed to twist freely in this bearing but to be effectively constrained by it against translation or rotation about a horizontal axis. The effective length of the stock ℓ for bending may be taken as the actual distance between the bearing and the point of attachment to the rudder, or the length may be increased somewhat to allow for local flexibilities in the bearing or the rudder. These effective lengths were used in computations for ALBACORE.[4]

Regarded as a uniform beam, the stock has four elastic constants:

a. Extensional stiffness EA

b. Torsional stiffness GJ_e

c. Bending rigidity EI

d. Shear rigidity or shear-slope constant KAG.

Here E is Young's modulus and G is the shear modulus, or $G = E/[2(1 + \nu)]$ with ν denoting Poisson's ratio; A is the cross-sectional area of the stock;

*Application of this theory and the effects of a ship maneuver on flutter will be given in two separate reports presently in preparation.

I is the areal moment (or "moment of inertia") of the cross section about an axis drawn in it through its centroid; and J_e is its "polar moment" or areal moment about a perpendicular axis drawn through the centroid. K might be called the shear-warping constant.

The mode of allowing for the effects of shear warping requires a brief discussion. In a pin-ended beam loaded only at the ends, attention need not be given to shear warping, and KAG does not appear in the formulas; i.e., since V = constant, $\frac{\partial V}{\partial x} = 0$, so that $M = EI\left(\frac{d^2y}{dx^2} + \frac{d}{dx}\frac{V}{KAG}\right) = EI\frac{d^2y}{dx^2}$ (see Reference 3 and Appendix A2). The significance of a "pin end" is that the end does not retain its shape but rather is free to distort locally at will. If, however, either end is fastened to a rigid base, the shape of the end is retained but the position of this end is subject to a constraint. If a shear force P exists in the beam at this end, the associated warping is hindered, and the effect of the resulting local distortion is to rotate the beam relative to the base in the direction of P through an angle $\frac{P}{KAG}$. (See Figure 1.) The value of K depends on the shape of the

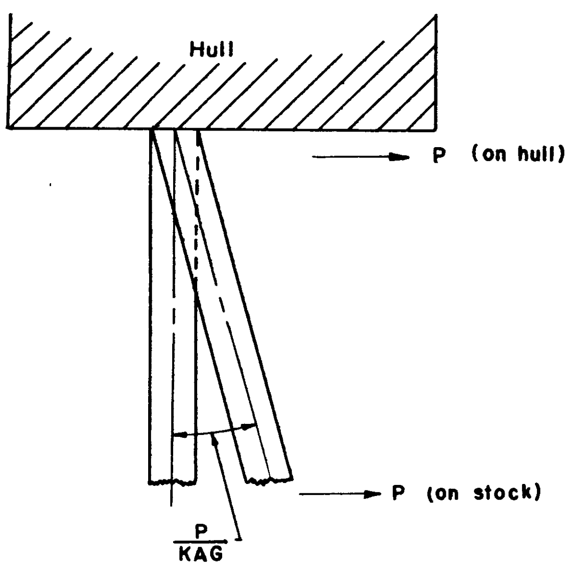

Figure 1 - Deflection of Rudder Stock Due to Shear Force P

cross section and is probably always less than unity; for a round uniform cross section, the value K = 0.75 is commonly used. This shear slope at a clamped end constitutes a modified boundary condition in the solution of problems. (It is not certain that the same value of K should be used to correct for clamping effects as is used in the Timoshenko beam-vibration equations, although this assumption has often been made.[3])

The rudder mounted on the stock will then have five elastic coefficients, expressible in terms of the constants of the stock, as follows:

a. If the rudder moves upward a distance w relative to the ship, a compressive force F is developed in the stock of magnitude

$$F = \frac{EAw}{\ell} \qquad [1]$$

b. If the rudder rotates about the axis of the stock through an angle ϕ relative to the ship, a torsional torque T is developed by the stock of magnitude

$$T = GJ_e \frac{\phi}{\ell_T} \qquad [2]$$

Hence $\frac{EA}{\ell}$ and $\frac{GJ_e}{\ell_T}$ are two of the five elastic coefficients.

c,d,e. Lateral deflections due to translation or rotation require a more complicated analysis. Assume for simplicity that the <u>ship is at rest</u>, and draw the z-axis downward from the ship along the axis of the stock in its unstrained position. Let $y'(z)$ denote the displacement curve of the stock in a certain plane drawn through the z-axis, caused by a translation y of the rudder in this plane together with a rotation θ about a perpendicular axis, θ being positive when it tends to give a positive value to dy'/dz; see Figure 2. Let P be the force and M the moment that the rudder then exerts on the lower end of the stock, these being positive in the y- and θ-directions, respectively. An equal force P and moment M then act on the ship also, and there will be equal shear slopes of magnitude P/KAG at the bearing level and at the bottom of the stock. See Figure 2, where all quantities shown are positive.

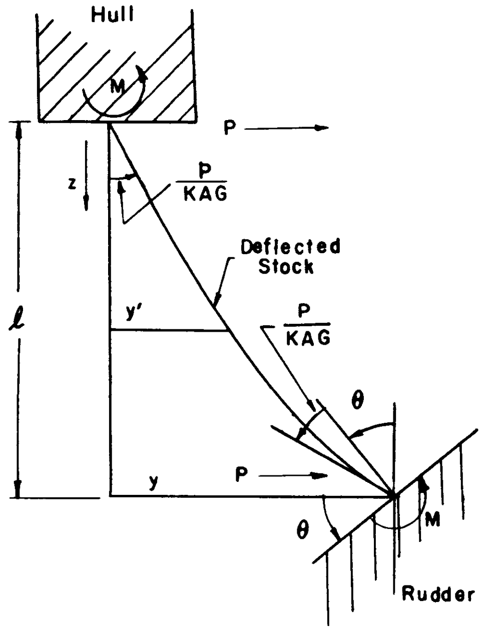

Figure 2 - Forces and Moments Acting on Rudder and Rudder Stock
Stock has bending and shearing flexibility.

To find the relation between P, M, y, and θ, we solve the bending equation for the stock. At any point z, to preserve equilibrium, the bending moment must equal $M + (\ell - z)P$ with ℓ denoting the length of the stock; hence:

$$EI\,\frac{d^2y'}{dz^2} = M + (\ell - z)P \qquad [3]$$

The boundary conditions are:

$$\text{At } z = 0: \quad y' = 0, \quad \frac{dy'}{dz} = \frac{P}{KAG};$$

$$\text{At } z = \ell: \quad y' = y, \quad \frac{dy'}{dz} = \theta + \frac{P}{KAG}.$$

Integrating Equation [3] once and choosing the constant of integration so as to satisfy the second boundary condition at $z = 0$ gives

$$EI \frac{dy'}{dz} = Mz + (\ell z - \frac{1}{2} z^2) P + \frac{EI}{KAG} P \qquad [4]$$

Taking $z = \ell$ in Equation [4] and using the fourth boundary condition, we obtain

$$EI \left(\theta + \frac{P}{KAG}\right) = \ell M + \frac{1}{2} \ell^2 P + \frac{EI}{KAG} P$$

or

$$\ell M + \frac{1}{2} \ell^2 P = EI\theta \qquad [5]$$

Also, integrating Equation [4] and then making $y' = 0$ at $z = 0$ to satisfy the first boundary condition yields

$$EIy' = \frac{1}{2} Mz^2 + (\frac{1}{2} \ell z^2 - \frac{1}{6} z^3) P + \frac{EI}{KAG} Pz \qquad [6]$$

Taking $z = \ell$ in this equation and noting that then $y' = y$ by the third boundary condition, we obtain

$$\frac{1}{2} \ell^2 M + (\frac{1}{3} \ell^3 + \frac{EI}{KAG} \ell) P = EIy \qquad [7]$$

This last equation and Equation [5] can now be solved for M and P. For convenience we write, as a shear-reduction ratio for forces,[4]

$$r_s = \frac{1}{1 + \frac{12EI}{KAG\ell^2}}$$

Then we find that

$$P = 12r_s \frac{EI}{\ell^3} y - 6r_s \frac{EI}{\ell^2} \theta \qquad [7a]$$

$$M = -6r_s \frac{EI}{\ell^2} y + (3r_s + 1) \frac{EI}{\ell} \theta \qquad [7b]$$

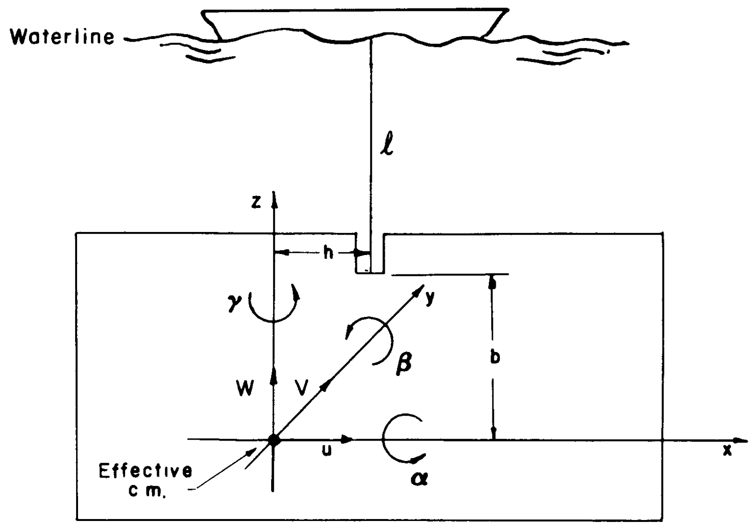

Figure 3 - Sign Convention for Coordinates and Displacements of Rudder

These equations exhibit the three lateral-stiffness coefficients $12r_s \frac{EI}{\ell^3}$, $-6r_s \frac{EI}{\ell^2}$, and $(3r_s + 1) \frac{EI}{\ell}$ for the mounted stock in terms of the elastic stock parameters EI and r_s.

3. RUDDER COORDINATES AND PARAMETERS

Let the right-hand x-, y-, z-axes be taken with the xz-plane passing through the undisplaced position of the axis of the rudder stock, and with the x-axis parallel to the ship axis, the z-axis being, therefore, vertical; see Figure 3. Let any small displacement of the rudder be resolved into translations u, v, w in the x-, y-, z-directions, respectively, plus small rotations α, β, γ about the x-, y-, z-axes, respectively. α will be taken positive from y toward z and γ from x toward y but β from x toward z, in order to harmonize better with the usual theory of the ship motion.* Let the origin of the coordinates be at the effective center of

*In Reference 4, β was taken positive from z toward x.

mass of the rudder which will be defined presently. Let the effective point of attachment of the rudder to the stock be at the point $x = h$, $z = b$, $y = 0$. If the ship moves, let its motion consist, for the present only, of small oscillations including as special cases small rigid-body displacements.

Since the rudder is symmetrical with respect to the xz-plane as drawn, a little thought shows that, when the ship is at rest, the reactions of the stock on the rudder caused by v, γ, α displacements of the rudder have no tendency to excite u, w, or β motions, and vice versa.[4] In each of these two types of displacement the stock moves only in a certain plane: in a transverse plane during v, γ, α or "transverse" motions; in the xz-plane during u, w, β or "longitudinal" motions. Furthermore, there is no inertial coupling of mechanical origin between these two types of motion, since the two products of inertia $\bar{I}_{xy}$ and $\bar{I}_{yz}$ vanish because of the symmetry (see Reference 4, Tables 1 and 5), and consequently, rotational velocity $\dot{\alpha}$ does not contribute to angular momentum about the y-axis and rotational velocity $\dot{\beta}$ does not contribute to angular momentum about the z-axis, and vice versa.

The same lack of inertial coupling persists when the rudder is immersed in water provided any objects nearby, such as a skeg, have surfaces symmetrical relative to the xz-plane. A little thought shows that, at any two points on the rudder surface that are mirror images of each other in the xz-plane, an acceleration $\ddot{v}$, $\ddot{\gamma}$, or $\ddot{\alpha}$ evokes water pressures of equal and opposite sign; because of this fact and because of the relation between the slopes of the surface at mirror-image points, these pressures give rise to no net force in either the x- or z-direction and also to no net moment about the y-axis; see Figure 4. Conversely, accelerations $\ddot{u}$, $\ddot{w}$, or $\ddot{\beta}$ give rise to equal pressures at mirror-image points and these give rise to zero net y force and zero net moment about the x- or z-axes.

Since the transverse and the longitudinal motions of the rudder are thus independent of each other, they can be treated separately. The complexity of the problem is thus greatly reduced. The more important case, the transverse motion, is considered first.

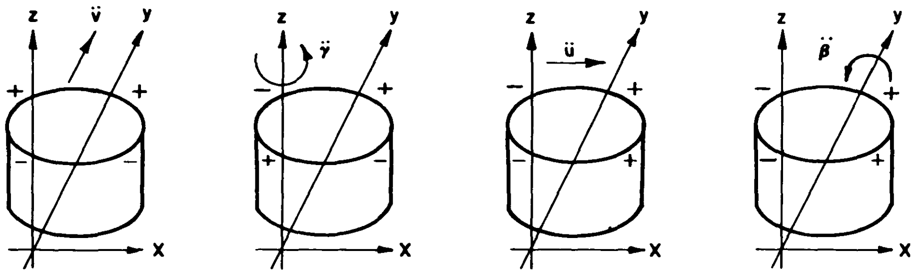

Figure 4 - Pressure at Mirror Image Points of Rudder Surface Due to $\ddot{v}$, $\ddot{\gamma}$, $\ddot{u}$, and $\ddot{\beta}$, Respectively

4. RUDDER-HULL MOTIONS AND CORRESPONDING MOBILITY ANALOGS

4.1 TRANSVERSE OR v, γ, α MOTION OF THE RUDDER

When the rudder is given a translational acceleration $\ddot{v}$, the water reactions on it are equivalent to a single force acting along a line parallel to y and meeting the xz-plane in a certain point C_p called the center of pressure. The magnitude of this force is $-\bar{\bar{m}}_y \ddot{v}$, where $\bar{\bar{m}}_y$ is the virtual mass due to the water for y motion of the rudder. If the rudder also undergoes angular accelerations $\ddot{\gamma}$ and $\ddot{\alpha}$ about axes meeting at C_p and drawn parallel to z and x, respectively, these accelerations will give rise to no further net force on the rudder, as can be shown from the conservation of energy. They may give rise, however, to couples expressible in terms of virtual moments and a virtual product of inertia, $\bar{\bar{I}}_{x''}$, $\bar{\bar{I}}_{z''}$, and $\bar{\bar{I}}_{x''z''}$, defined with respect to axes through C_p. The reactions on the rudder are the same as if the water were replaced by a rigid body attached to the rudder and having the same inertial parameters; hence such a replacement may be supposed made in dealing dynamically with the v, γ, α motion of the rudder.

The rudder itself has corresponding mechanical parameters m, $\bar{I}_{x'}$, $\bar{I}_{z'}$, and $\bar{I}_{x'z'}$. The rudder and the virtual-mass body taken together as a combined body will have a center of mass whose position can be calculated.[4] This will be called the effective center of mass of the rudder. Hereafter the origin of the x-, y-, z-axes will be taken at this effective center of

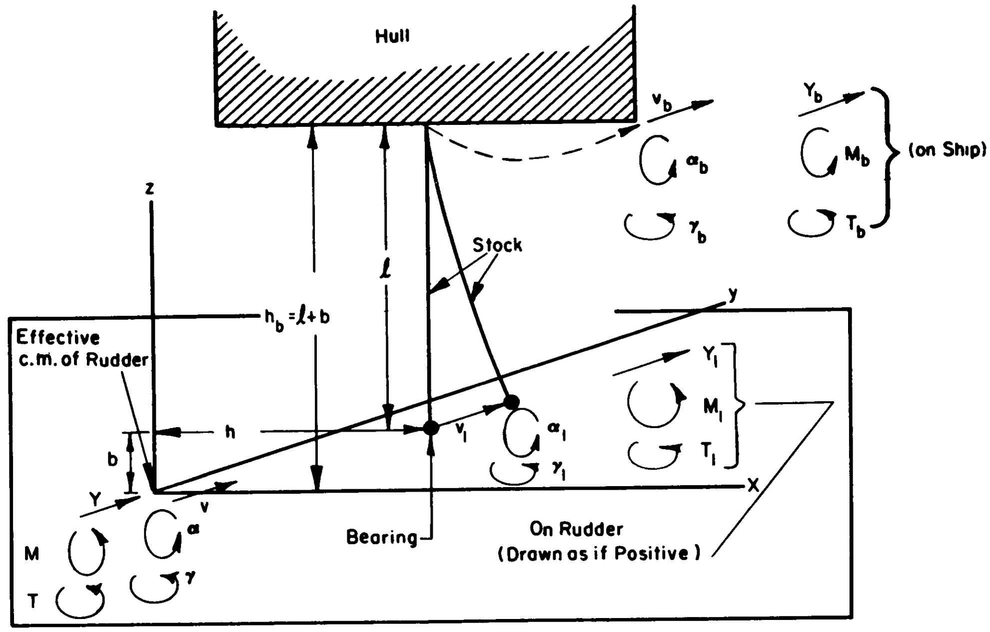

Figure 5 - Positive Forces and Moments Acting on Rudder and Rudder Stock for Transverse or v, γ, α Motion of Rudder Parallel to yz-Plane

mass of the combined body. The combined mass, $m + \bar{\bar{m}}_y$ will be denoted by m_y; the moments and a product of inertia I_x, I_z, I_{xz} for the combined body, defined with respect to the x-, y-, z-axes in their new position, can be calculated by the usual formulas for rigid bodies.[4]

These inertial constants should include the inertial effects of the stock. It may be sufficiently accurate to treat the stock as rigid in this connection. Or, approximate corrections can be made for the differential motion between stock and rudder by a process that will not be considered further here.

The displacement of the (effective) center of mass is then v, whereas γ and α represent rotations about axes drawn through the effective center of mass; see Figure 5, which shows positive directions.

Forces and moments due to the elasticity of the stock must be considered next. The small displacements v, γ, α produce corresponding displacements v_1, γ_1, α_1 of the rudder at the bottom of the stock of magnitude

$$v_1 = v - b\alpha + h\gamma \; ; \quad \gamma_1 = \gamma \; ; \quad \alpha_1 = \alpha \qquad [8a,b,c]$$

The forces and moments acting on the rudder at this point, denoted by Y_1, T_1, M_1 will be, respectively, equal to -P, -T, -M, where P, T, M are given by Equations [7a,b] and [2] with "y" = v_1, "θ" = α_1, "ϕ" = γ_1; see Figure 2. Hence (when the ship is at rest)

$$Y_1 = - 12r_s \frac{EI}{\ell^3} v_1 + 6r_s \frac{EI}{\ell^2} \alpha$$

$$T_1 = - \frac{GJ_e}{\ell_T} \gamma$$

$$M_1 = 6r_s \frac{EI}{\ell^2} v_1 - (3r_s + 1) \frac{EI}{\ell} \alpha$$

When the ship also moves, these equations require generalization. Let the rudder stock bearing undergo displacements v_b, γ_b, α_b due to the ship motion, defined in the same way as v, γ, α for the rudder; see Figure 5. (The subscript "b" denotes "bearing.") Then, if rudder and stock were to move as a rigid system attached to the ship, the bottom of the stock would undergo displacements v_{1b}, γ_{1b}, α_{1b} of magnitudes

$$v_{1b} = v_b + \ell\alpha_b \; ; \quad \gamma_{1b} = \gamma_b \; ; \quad \alpha_{1b} = \alpha_b$$

and Y_1, T_1, M_1 would all be zero. Otherwise, Y_1, T_1, M_1 will have values determined by the differences $v_1 - v_{1b}$, $\gamma - \gamma_{1b}$, $\alpha - \alpha_{1b}$, or, in general,

$$Y_1 = - 12r_s \frac{EI}{\ell^3} (v_1 - v_b - \ell\alpha_b) + 6r_s \frac{EI}{\ell^2} (\alpha - \alpha_b) \qquad [9a]$$

$$T_1 = - \frac{GJ_e}{\ell_T} (\gamma - \gamma_b) \qquad [9b]$$

$$M_1 = 6r_s \frac{EI}{\ell^2} (v_1 - v_b - \ell\alpha_b) - (3r_s + 1) \frac{EI}{\ell} (\alpha - \alpha_b) \qquad [9c]$$

The net force Y on the rudder acting at the effective center of mass and the total moments of force T and M about the z- and x-axes drawn through the effective center of mass of the rudder are

$$Y = Y_1 \; ; \quad T = T_1 + hY_1 \; ; \quad M = M_1 - bY_1 \qquad [10a,b,c]$$

where b and h are the z- and x-coordinates of the effective center of attachment of rudder to rudder stock, respectively. The angular moments about the x- and z-axes are, respectively,

$$I_x\dot{\alpha} - I_{xz}\dot{\gamma} \; ; \quad I_z\dot{\gamma} - I_{xz}\dot{\alpha}$$

Hence the equations of motion of the rudder for the v, γ, α motion are, respectively,

$$m_y\ddot{v} = Y \; ; \quad I_z\ddot{\gamma} - I_{xz}\ddot{\alpha} = T \; ; \quad I_x\ddot{\alpha} - I_{xz}\ddot{\gamma} = M \qquad [11a,b,c]$$

To simplify the notation, we write

$$k = \frac{EI}{\ell^3} \quad ; \; k_s = r_s \frac{EI}{\ell^3} \qquad [12a,b]$$

Then, after substituting from Equations [10a,b,c], [9a,b,c], and [8a,b,c] and collecting terms, Equations [11a,b,c] become

$$m_y\ddot{v} = -\,12k_s \left[v + h\gamma - \frac{1}{2}(\ell + 2b)\,\alpha - v_b - \frac{1}{2}\,\ell\alpha_b\right] \qquad [13a]$$

$$\begin{aligned} I_z\ddot{\gamma} - I_{xz}\ddot{\alpha} = & -12k_s h\,(v - v_b) + 6k_s h\,(\ell + 2b)\,\alpha \\ & - \left[\frac{GJ_e}{\ell_T} + 12k_s h^2\right]\gamma + 6k_s h\,\ell\alpha_b + \frac{GJ_e}{\ell_T}\,\gamma_b \end{aligned} \qquad [13b]$$

$$\begin{aligned} I_x\ddot{\alpha} - I_{xz}\ddot{\gamma} = \; & 6k_s\,(\ell + 2b)(v - v_b) - [3k_s\,(\ell + 2b)^2 + k\ell^2]\,\alpha \\ & + 6k_s h\,(\ell + 2b)\,\gamma - [3k_s\,\ell\,(\ell + 2b) - k\ell^2]\,\alpha_b \end{aligned} \qquad [13c]$$

in which the right-hand members are expressions for Y, T, and M, respectively.

Equations [13a,b,c] have a useful symmetry. Let v be regarded as the leading variable in [13a], γ in [13b], and α in [13c]; and call the equations the v, γ, and α equations, respectively. Then note that terms <u>not</u> containing a leading variable have symmetrical coefficients; that is, the α and γ terms in the v equation have the same coefficients as the v term has in the α and γ equations, respectively; the term $-I_{xz}\ddot{\gamma}$ in the α equation is matched by $-I_{xz}\ddot{\alpha}$ in the γ equation; and similarly, for γ and α on the right. If the equations of motion for an elastic mechanical system do not already possess such symmetry, it can always be introduced by multiplying the equations, if necessary, by suitable constants. Furthermore, the coefficients of the leading variables can be made to have, as here, positive values on the left and negative values on the right.* (See Appendix A.)

4.2 RELATIONS WITH HORIZONTAL-TORSIONAL SHIP MOTION

Equations [13a,b,c] constitute equations of motion for the rudder alone and can be solved to determine its v, γ, α motion when the ship is entirely at rest so that $v_b = \gamma_b = \alpha_b = 0$. When the ship moves, however, v_b, γ_b, and α_b are all functions of the time, expressible in terms of the displacements of the ship.

Coupled horizontal bending and torsion are the types of ship vibration in which only displacements of the type of v_b, γ_b, and α_b occur. In the approximate theory of such vibrations as formulated at TMB, axes are drawn in the same direction as those drawn here for the transverse motion of the rudder but with the x-axis for the ship drawn at a height convenient for the ship theory. Also, the displacement of any cross section of the

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*The coefficients of v, γ, and α in Equations [13a,b,c] are expressions for some of the elastic constants denoted by K_{ij}'s in References 4, 9, 10. The independence of the v, γ, α and u, w, β motions corresponds to the vanishing of K_{ij}, if i = v, γ, or α and j = u, w, or β. For v, γ, α motion, $K_{vv} = 12k_s$, $K_{v\gamma} = K_{\gamma v} = 12k_s h$, $K_{v\alpha} = K_{\alpha v} = -6k_s(\ell + 2b)$; $K_{\alpha\gamma} = K_{\gamma\alpha} = -6k_s h(\ell + 2b)$, $K_{\gamma\gamma} = 12k_s h^2 + GJ_e/\ell_T$, $K_{\alpha\alpha} = 3k_s(\ell + 2b)^2 + k\ell^2$.

ship is specified in terms of the horizontal displacement "y" of the point that is initially on the x-axis, an equivalent rotation "γ" of the cross section about an axis through its centroid parallel to z, and a rotation ϕ about the x-axis, taken positive from y toward z.* To avoid confusion with the other case of ship motion, y_h and γ_h will be written in this report for "y" and "γ", respectively, to refer to horizontal bending. The variables then correspond in direction to the rudder variables in this way:

Ship:	y_h	γ_h	ϕ
Rudder:	v	γ	α

Both ship and rudder will be assumed to have the xz-plane as a plane of symmetry. The special case of paired rudders offset from the median plane of the ship will be discussed later in the report.

Let the rudder stock bearing be at a height z_b above the x-axis, as drawn in the ship theory.[3] Then (see Figure 6) the displacements of the rudder stock bearing v_b, γ_b, and α_b are related to the ship displacements as follows:

$$v_b = y_h - z_b\phi \quad ; \quad \gamma_b = \gamma_h \quad ; \quad \alpha_b = \phi \qquad [14a,b,c]$$

where y_h, γ_h, and ϕ refer to the ship cross section that contains the stock. Let the forces on the ship due to the stock be equivalent to a force Y_b in the y-direction acting at the level of the bearing together with couples about axes parallel to z and x of magnitude T_b and M_b. The reactions on the stock will then be $-Y_b$, $-T_b$, and $-M_b$; these reactions must be statically equivalent to force Y and moments T and M acting on the center of mass of the rudder, since the rudder is treated as rigid and the mass of the stock is either ignored or allowed for by a correction to the rudder mass.**

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*Thus the rudder and ship do not necessarily have a common x-axis.

**In Equation [10c] $M_1 = -M_b - (\ell)(-Y_b)$, $-Y_b = Y_1 = Y$; hence Equation [15c].

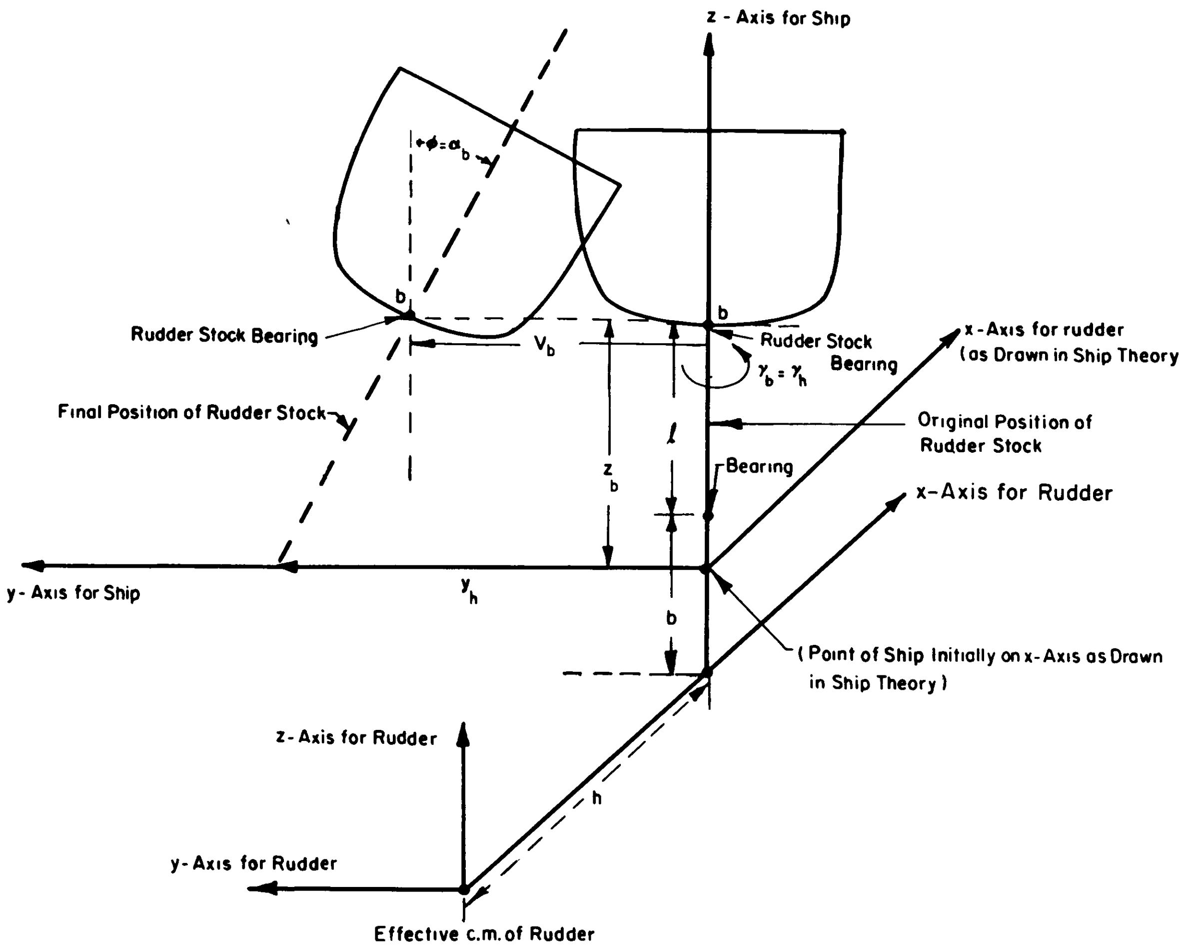

Figure 6 - Correspondence between Displacements of Rudder and Hull for Rudder Motion Associated with Coupled Torsion-Horizontal-Bending Hull Motion

The effective center of mass of rudder is shown undisplaced. Actually, the center of mass is displaced when the ship vibrates.

Hence

$$Y = -Y_b \quad ; \quad T = -T_b + h(-Y_b) \quad ; \quad M = -M_b - (\ell + b)(-Y_b)$$

or [15a,b,c]

$$Y_b = -Y \quad ; \quad T_b = -T + hY \quad ; \quad M_b = -M - (\ell + b)\,Y$$

By substituting here for Y, T, M the right-hand members of [13a,b,c], respectively, we can express Y_b, T_b, and M_b in terms of the rudder coordinates v, γ, α and v_b, γ_b, α_b.

In practice, however, difference equations are employed in representing the ship. In the system now preferred at TMB,[3] values labeled $\gamma_{h,n}$ of γ_h are chosen at stations Δx apart and values $y_{h,n+\alpha}$ and $\phi_{n+\alpha}$ of y_h and ϕ, at points midway between these stations, where n = 0,1,2 ... and $\alpha = \frac{1}{2}$. (Ordinarily, of course, the subscript h is not used.) For some purposes it may be sufficiently accurate to use in the formulas for v_b and α_b, the values of $y_{h,n+\alpha}$ and $\phi_{n+\alpha}$ at the nearest midstation, and for γ_b, the value of $\gamma_{h,n}$ at the nearest station on the ship; also to assume that Y_b and M_b act at the midstation and T_b at the station thus selected. Y_b, M_b, and T_b may then be identified with the terms in the ship equations that represent external force, torsional and bending moment, actions on the ship, such as $P_{n+\alpha}$, $U_{n+\alpha}$, and Q_n in Equations [2.42] through [2.49] in Reference 3. In Equation [2.46],[3] also, h is to be replaced by z_b.* There is, of course, a certain inconsistency in this procedure, but the resulting error may be tolerable.

Otherwise, an interpolation procedure may be used; see Reference 3. Suppose that the point of attachment of the stock is located at a distance $s\Delta x$ toward positive x from the station labeled $n - \alpha$, where $0 \leqq s < 1$. Then it will also be at a distance $s'\Delta x$ from a certain station n' with $0 \leqq s' < 1$, where

$$\text{If} \quad s \leqq \tfrac{1}{2} \quad : \quad n' = n - 1 \quad ; \quad s' = s + \tfrac{1}{2}$$

$$\text{If} \quad s > \tfrac{1}{2} \quad : \quad n' = n \quad ; \quad s' = s - \tfrac{1}{2}$$

(See Figure 7, which is drawn for $s < \frac{1}{2}$.) Good approximations are then as follows:

$$v_b = (1 - s)\left[y_{h,n-\alpha} - z_b\,\phi_{n-\alpha}\right] + s\left[y_{h,n+\alpha} - z_b\,\phi_{n+\alpha}\right] \qquad [16a]$$

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*In Reference 3, h was defined as the height of the rudder bearing above the x-axis and is, therefore, replaced by z_b to avoid confusion with h as defined in the present report.

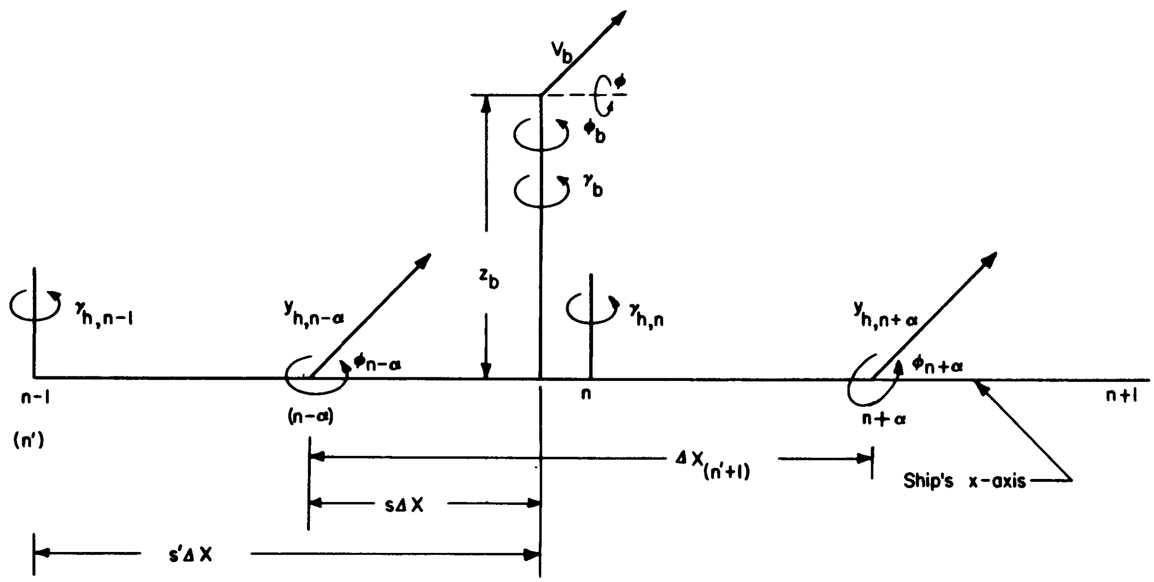

Figure 7 - Method of Interpolating Displacements

$$\gamma_b = (1 - s')\,\gamma_{h,n'} + s'\,\gamma_{h,n'+1} \qquad [16b]$$

$$\alpha_b = (1 - s)\,\phi_{n-\alpha} + s\,\phi_{n+\alpha} \qquad [16c]$$

Similarly, Y_b may be replaced by two parallel forces, $P_{n-\alpha}$ acting on the ship at $n - \alpha$ and $P_{n+\alpha}$ at $n + \alpha$ and at a height z_b above the ship x-axis, and similarly M_b by $U_{n-\alpha}$ at $n - \alpha$ and $U_{n+\alpha}$ at $n + \alpha$, and T_b by $Q_{n'}$ at n' and $Q_{n'+1}$ at $n' + 1$, where

$$P_{n-\alpha} = (1 - s)\,Y_b \quad ; \quad P_{n+\alpha} = sY_b \qquad [17a,b]$$

$$U_{n-\alpha} = (1 - s)\,M_b \quad ; \quad U_{n+\alpha} = sM_b \qquad [17c,d]$$

$$Q_{n'} = (1 - s')\,T_b \quad ; \quad Q_{n'+1} = s'T_b \qquad [17e,f]$$

Here the forces $P_{n\pm\alpha}$ act at a height z_b above the ship's x-axis.

It is shown that conservation of energy is preserved if both displacements and forces are split in this manner.

Sufficient materials are now assembled to serve as a basis for a numerical calculation of the normal frequencies and mode patterns of either the rudder alone ($v_b = \gamma_b = \alpha_b = 0$) or the ship-rudder system. In the latter case Equations [2.42] through [2.49] in Reference 3 might be used with the values given by Equations [17a-f] substituted for $P_{n\pm\alpha}$, $U_{n\pm\alpha}$, $Q_{n'}$, or $Q_{n'+1}$ in the equations referring to certain values n (and with h replaced by z_b). In vibration at a definite circular frequency ω, $\ddot{v} = -\omega^2 v$, $\ddot{\gamma} = -\omega^2\gamma$, and $\ddot{\alpha} = -\omega^2\alpha$, so that the three Equations [13a,b,c] can be solved for v, γ, α in terms of v_b, γ_b, α_b; for the latter the expressions given in [16a,b,c] may be substituted. Then from Equations [11a,b,c] Y, T, and M can be calculated as $Y = -\omega^2 m_y v$, $T = -\omega^2 I_z\gamma + \omega^2 I_{xz}\alpha$, and $M = -\omega^2 I_x\alpha + \omega^2 I_{xz}\gamma$; from these equations and from Equations [15a,b,c] Y_b, M_b, and T_b can be found for use in Equations [17a-f]. Thereby, everything in the ship equations is finally expressed in terms of ω and certain ship variables. The set of equations thus obtained can then be solved step by step, although not without some trouble. <u>The finite difference formulation and solution of these equations by means of a digital computer is found in Reference 5.</u>

In the present report only the design of a representative analog network will be considered further.

4.3 MOBILITY ANALOG FOR TRANSVERSE MOTION OF THE RUDDER

In an analog network, nonleading terms in the equations of motion usually require the use of transformers. In the present case, each equation of Equations [13] contains in its right-hand member all three of the variables v, γ, α; for this reason design of the analog is facilitated if the terms are grouped in a certain way. Let the terms on the right in [13b,c] be rearranged so that v occurs in them only in the same combination with γ and α as it does in the v equation, [13a], where v is the leading variable. The equations then read:

$$m_y\ddot{v} = -12\,k_s\left[v + h\gamma - \left(\frac{1}{2}\ell + b\right)\alpha - v_b - \frac{1}{2}\ell\alpha_b\right] \qquad [18a]$$

$$I_z\ddot{\gamma} - I_{xz}\ddot{\alpha} = -12\ k_s h\ \left[v + h\gamma - (\tfrac{1}{2}\ell + b)\alpha - v_b - \tfrac{1}{2}\ell\alpha_b\right] - \frac{GJ_e}{\ell_T}(\gamma - \gamma_b)$$

[18b]

$$I_x\ddot{\alpha} - I_{xz}\ddot{\gamma} = 12\ k_s\ (\tfrac{1}{2}\ell + b)\ \left[v + h\gamma - (\tfrac{1}{2}\ell + b)\alpha - v_b - \tfrac{1}{2}\ell\alpha_b\right] - k\ell^2(\alpha - \alpha_b)$$

[18c]

A similar grouping procedure could be used for the left-hand members of Equations [18b,c] also. The general rule is that the grouping procedure should be continued, on the left-hand and the right-hand members separately, until in each member there remains only one ungrouped term and this term contains only the leading variable for the equation in which it occurs (γ or α in [18b,c]). It can be shown that, if the original equations represent a stable elastic system and are written with the symmetry and the sign characteristics previously described, then, after grouping, any term in an equation that contains only the leading variable for that equation or a group beginning with that variable must have a positive coefficient if it is in the left-hand member of the equation but a negative coefficient if it is in the right-hand member. (Examples are $-k\ell^2$ in [18c], $-12\,k_s$ in [18a], and I_x in [18c].) See Appendix A for proof. These properties of the coefficients guarantee that a representative network can be constructed from passive elements.

For members or groups containing only two variables, however, a special procedure is preferable because it opens the way to a useful freedom in the choice of the transformer ratio. This procedure is described presently.

Design of the analog may now proceed. In a mobility analog, velocities may be represented by voltages above ground at certain nodal points, whereas the equations of motion are represented by current summations.[3] In the present case, let $\dot{v}$, $\dot{\gamma}$, and $\dot{\alpha}$ be represented by voltages $\dot{v}'$, $\dot{\gamma}'$, and $\dot{\alpha}'$ at three nodal points. The variables v, γ, and α themselves, being equal to $\int \dot{v}\,dt$, $\int \dot{\gamma}\,dt$, and $\int \dot{\alpha}\,dt$, will then be represented by voltage impulses at these nodes. If the ship remains entirely at rest, $v_b = \gamma_b = \alpha_b = 0$, and these quantities in the equations will be represented by connections to ground. When the ship moves, on the other hand, v_b, γ_b, and α_b will be represented by voltage impulses occurring either at certain points on the network that represent the ship or at points connected to this network in a suitable manner, as will be explained later.

Denote by t' the time in the electrical network, which need not advance at the same rate as time in the mechanical system. Then the correspondence will be such that

$$\dot{v} = b_1 \dot{v}' \quad ; \quad \dot{\gamma} = b_2 \dot{\gamma}' \quad ; \quad \dot{\alpha} = b_3 \dot{\alpha}' \quad ; \quad t = pt'$$

where b_1, b_2, b_3, and p denote fixed conversion factors, which will be assumed to be positive. Note that the time differentiation indicated in $\dot{v}$, for example, is included in the symbol $\dot{v}'$. However,

$$\ddot{v} = \frac{d}{dt}\dot{v} = \frac{b_1}{p}\frac{d}{dt'}\dot{v}' \quad ; \quad v = \int \dot{v}\,dt = pb_1 \int \dot{v}'dt'$$

and similarly for $\ddot{\gamma}$, $\ddot{\alpha}$, γ, and α. In particular

$$v = pb_1 v' \quad , \quad \gamma = pb_2 \gamma' \quad , \quad \alpha = pb_3 \alpha'$$

where v', γ', and α' denote $\int \dot{v}'\,dt'$, $\int \dot{\gamma}'dt'$, $\int \dot{\alpha}'\,dt'$, respectively, or the voltage impulses that represent v, γ, and α. Also, similarly, assume for the variables that refer to the displacements of the ship at the level of the rudder bearing:

$$\dot{v}_b = b_1 \dot{v}_b' \quad ; \quad \dot{\gamma}_b = b_2 \dot{\gamma}_b' \quad ; \quad \dot{\alpha}_b = b_3 \dot{\alpha}_b'$$

$$v_b = pb_1 v_b' \; ; \; \gamma_b = pb_2 \gamma_b' \; ; \; \alpha_b = pb_3 \alpha_b'$$

Substitution into Equations [18a,b,c] and division of these equations by positive numbers a_1, a_2, a_3, respectively, which remain to be determined, gives as electrical equations:

$$\frac{b_1 m_y}{pa_1}\frac{d}{dt'}\dot{v}' + 12\frac{b_1}{a_1}pk_s\left[v' + \frac{b_2}{b_1}h\gamma' - \frac{b_3}{b_1}(b + \frac{1}{2}\ell)\alpha' - v_b' - \frac{b_3}{b_1}\frac{\ell}{2}\alpha_b'\right] = 0 \qquad [19a]$$

$$\frac{b_2}{pa_2} I_z \frac{d}{dt'} \dot{\gamma}' - \frac{b_3}{pa_2} I_{xz} \frac{d}{dt'} \dot{\alpha}' + 12 \frac{pb_1}{a_2} k_s h \left[v' + \frac{b_2}{b_1} h\gamma' - \frac{b_3}{b_1} (b + \frac{1}{2} \ell)\alpha' \right.$$

$$\left. - v_b' - \frac{b_3}{b_1} \frac{\ell}{2} \alpha_b' \right] + \frac{pb_2}{a_2} \frac{GJ_e}{\ell_T} (\gamma' - \gamma_b') = 0 \qquad [19b]$$

$$\frac{b_3}{pa_3} I_x \frac{d}{dt'} \dot{\alpha}' - \frac{b_2}{pa_3} I_{xz} \frac{d}{dt'} \dot{\gamma}' - 6 \frac{pb_1}{a_3} k_s (\ell + 2b) \left[v' + \frac{b_2}{b_1} h\gamma' \right. \qquad [19c]$$

$$\left. - \frac{b_3}{b_1} (b + \frac{1}{2} \ell)\alpha' - v_b' - \frac{b_3}{b_1} \frac{\ell}{2} \alpha_b' \right] + \frac{pb_3}{a_3} k\ell^2 (\alpha' - \alpha_b') = 0$$

In Equation [19a], the first term may be the value of a current flowing to ground from the $\dot{v}'$ node through a capacitance of magnitude C_1 equal to $b_1 m_y / pa_1$; see Figure 8. Similarly, the second term may denote a current leaving $\dot{v}'$ through an inductance L_1 of magnitude such that $L_1^{-1} = 12 b_1 p k_s / a_1$, on whose terminals has acted a voltage impulse of magnitude equal to the quantity in brackets. One of these currents, of course, must be negative.

The subtraction of a voltage impulse $\frac{b_3}{b_1} (b + \frac{\ell}{2}) \alpha'$ from v', as required in the terms in the bracket in Equation [19a], can be effected by connecting an ideal transformer, as shown at the upper left in Figure 8, provided the transformer ratio for <u>voltage</u> r_3 has a value of magnitude

$$r_3 = \left| \frac{b_3}{b_1} (\frac{\ell}{2} + b) \right|$$

In the figure the ends of the windings that become positive or negative simultaneously (relative to the other end) are shown by plus (+) marks, on the assumption that the quantity $b_3 (\frac{1}{2} \ell + b)/b_1$ is positive; if this quantity is in reality negative, one plus mark must be shifted to the other end of the winding to correct the diagram.

Every transformer, however, relates two currents to each other as

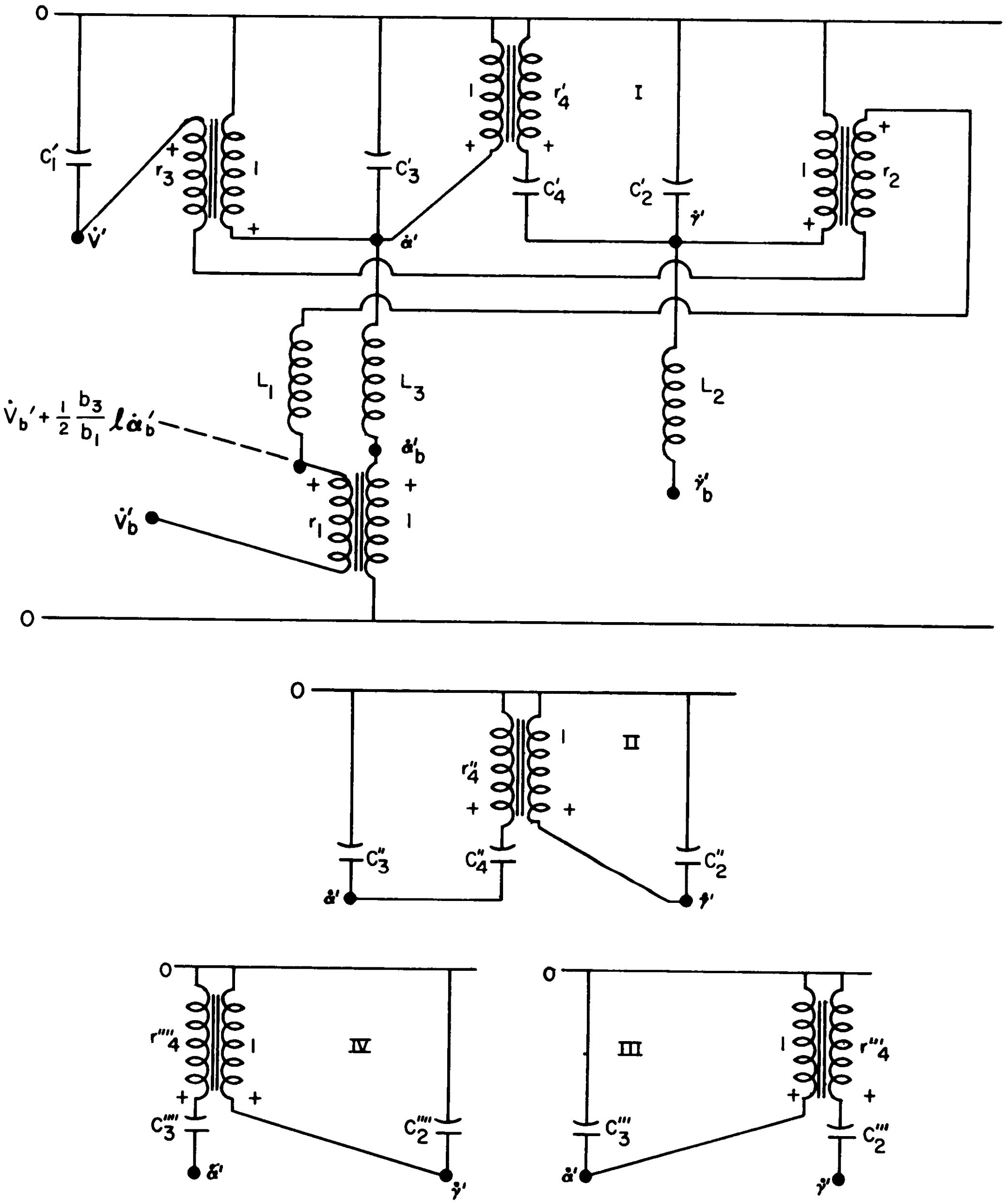

Figure 8 - Mobility Analog of Rudder for Transverse or v, γ, α Motion of Rudder

well as two voltages.* In the present case, a current will also be caused to <u>arrive</u> at the $\dot{\alpha}'$ node of a magnitude equal to r_3 times the current represented by the second term of Equation [19a]. This current will supply also that which is required to arrive at $\dot{\alpha}'$ by the third term of [19c] if

$$r_3\, L_1^{-1} = \left[\frac{b_3}{b_1}\left|\frac{\ell}{2} + b\right|\right]\left(12\,\frac{b_1}{a_1}\,pk_s\right) = 12\,\frac{pb_1}{a_3}\,k_s\left|\frac{1}{2}\,\ell + b\right|$$

or if $a_1b_1 = a_3b_3$. It is readily seen that the usual relative reversal of the currents in the two windings of a transformer relative to the directions of the voltages provides the negative sign in [19c], whatever the sign of $\ell + 2b$.

Similar treatment of the $h\gamma'$ term in [19a] and of the third term in [19b], with use of another transformer, leads to the requirement that

$$r_2\, L_1^{-1} = \left[\left|\frac{b_2}{b_1}\,h\right|\left(12\,\frac{b_1}{a_1}\,pk_s\right)\right] = \left|12\,p\,\frac{b_1}{a_2}\,k_s h\right|$$

or $a_1b_1 = a_2b_2$. Thus it is necessary that

$$a_1b_1 = a_2b_2 = a_3b_3 \qquad [20]$$

The voltage $v_b' + b_3\,\ell\alpha_b'/2b_1$, in which $b_3/b_1 = a_1/a_3$, for subtraction from v' in the brackets, is easily built up out of v_b' and α_b' with the help of another transformer, as shown in Figure 8; note $r_1 = \frac{b_3}{b_1}\,\frac{\ell}{2}$.

The first two terms in each of Equations [19b] and [19c] must then

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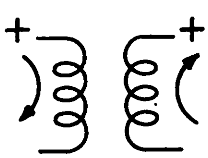
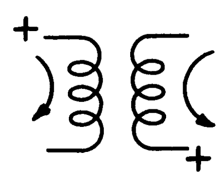
*To preserve the conservation of energy the ratios of the current and voltage drops in the transformer will both be positive only if one of the voltages is taken positive in opposite direction to a positive current;

i.e.,  or  . In other branches of the network the direction of positive current flow corresponds to the direction of positive voltage drop.

represent currents leaving the $\dot{\gamma}'$ and $\dot{\alpha}'$ nodes through additional capacitances. Application of the general grouping method previously described leads to two alternative arrangements; in each the transformer ratio is required to have a certain fixed value. However, if worthwhile, a range of choices for this ratio can be secured by using an additional capacitance; and again different arrangements are possible. One of these latter arrangements, labeled I, is shown in the main diagram of Figure 8; the three alternatives, labeled II, III, and IV, are shown in the figure as supplementary diagrams. The validity of these alternative arrangements is discussed presently. In all cases the plus marks on transformers are positioned on the assumption that the quantity whose absolute value is taken as the transformer ratio is itself positive; if this quantity is in reality negative, one plus sign is to be moved to the other end of the winding.

Before the necessary magnitudes of the network elements are written, it is convenient to consider the conversion factors further and to introduce a more convenient notation. Suppose, first, that the ship is at rest, so that $v_b = \gamma_b = \alpha_b = 0$. Then p can be chosen freely, and, since the six conversion factors a_1, a_2, a_3, b_1, b_2, b_3 are subject only to the two restrictions implied by the double Equation [20], four of these conversion factors can also be chosen arbitrarily. However, if all six factors are multiplied by any common factor, none of the coefficients in Equations [19a,b,c] are altered. It is readily seen, in fact, that only the ratio of the amplitudes of mechanical and electrical vibration is affected, and this ratio is of little interest. Thus it suffices to fix only the ratios of the conversion factors; and, since assigning the value of one factor together with the values of three independent ratios fixes the values of all factors, only three independent ratios can be chosen arbitrarily. When the ship moves, the choice of conversion factors is further restricted, as is explained presently.

It seems to be most convenient in practice to choose arbitrarily, besides p, the three quantities λ, ρ_1, and ρ_2, defined as

$$\lambda = \frac{b_1}{pa_1} \quad ; \quad \rho_1 = \frac{b_2}{b_1} = \frac{a_1}{a_2} \quad ; \quad \rho_2 = \frac{b_3}{b_1} = \frac{a_1}{a_3} \qquad [21a,b,c]$$

Then $C_1 = \lambda m_y$. The ρ's have simple relations to transformer ratios; it is convenient to be able to choose these with some freedom. The following further relations are easily verified by using Equation [20]:

$$\frac{b_2}{pa_2} = \rho_1^2 \lambda \quad ; \quad \frac{b_3}{pa_3} = \rho_2^2 \lambda \quad ; \quad \frac{b_3}{pa_2} = \rho_1 \rho_2 \lambda$$

$$\frac{b_2}{pa_1} = \frac{b_1}{pa_2} = \rho_1 \lambda \quad ; \quad \frac{b_3}{pa_1} = \frac{b_1}{pa_3} = \rho_2 \lambda \quad ; \quad \frac{b_3}{b_2} = \frac{a_2}{a_3} = \frac{\rho_2}{\rho_1}$$

In terms of λ, ρ_1, and ρ_2, Equations [19a,b,c] become, whether the ship moves or not:

$$\lambda m_y \frac{d}{dt'} \dot{v}' + 12\, p^2 \lambda k_s \left[v' + \rho_1 h \gamma' - \rho_2 \left(\frac{\ell}{2} + b\right)\alpha' - v_b' - \rho_2 \frac{\ell}{2} \alpha_b' \right] = 0 \qquad [22a]$$

$$\rho_1^2 \lambda I_z \frac{d}{dt'} \dot{\gamma}' - \rho_1 \rho_2 \lambda I_{xz} \frac{d}{dt'} \dot{\alpha}' + 12\, p^2 \rho_1 \lambda k_s h \left[v' + \rho_1 h \gamma' - \rho_2 \left(\frac{\ell}{2} + b\right)\alpha' - v_b' - \rho_2 \frac{\ell}{2} \alpha_b' \right] + p^2 \rho_1^2 \lambda \frac{GJ_e}{\ell_T} (\gamma' - \gamma_b') = 0 \qquad [22b]$$

$$\rho_2^2 \lambda I_x \frac{d}{dt'} \dot{\alpha}' - \rho_1 \rho_2 \lambda I_{xz} \frac{d}{dt'} \dot{\gamma}' - 6\, p^2 \rho_2 \lambda k_s (\ell + 2b) \left[v' + \rho_1 h \gamma' - \rho_2 \left(\frac{\ell}{2} + b\right)\alpha' - v_b' - \rho_2 \frac{\ell}{2} \alpha_b' \right] + p^2 \rho_2^2 \lambda k \ell^2 (\alpha' - \alpha_b') = 0 \qquad [22c]$$

From these equations the following values can be read off or verified as correct for the elements required in the network shown in Figure 8. Note that b and h may be either positive or negative, whereas ℓ is necessarily positive.

$$C_1 = \lambda m_y$$

$$L_1^{-1} = 12\, p^2 \lambda k_s \quad ; \quad L_2^{-1} = p^2 \rho_1^2 \lambda GJ_e / \ell_T \quad ; \quad L_3^{-1} = p^2 \rho_2^2 \lambda k \ell^2$$

$$r_1 = \rho_2 \frac{\ell}{2} \quad ; \quad r_2 = \rho_1 |h| \quad ; \quad r_3 = \rho_2 \left| b + \frac{1}{2} \ell \right|$$

I

$$\begin{cases} \dfrac{\rho_2}{\rho_1}\dfrac{|I_{xz}|}{I_z} \leqq r_4' \leqq \dfrac{\rho_2}{\rho_1}\dfrac{I_x}{|I_{xz}|} & ; \quad C_4' = \rho_1\rho_2\lambda\dfrac{|I_{xz}|}{r_4'} \\ C_2' = \rho_1^2\lambda I_z - C_4' & ; \quad C_3' = \rho_2^2\lambda I_x - (r_4')^2 C_4' \end{cases}$$

II

$$\begin{cases} \dfrac{\rho_1}{\rho_2}\dfrac{|I_{xz}|}{I_x} \leqq r_4'' \leqq \dfrac{\rho_1}{\rho_2}\dfrac{I_z}{|I_{xz}|} & ; \quad C_4'' = \rho_1\rho_2\lambda\dfrac{|I_{xz}|}{r_4''} \\ C_2'' = \rho_1^2\lambda I_z - (r_4'')^2 C_4'' & ; \quad C_3'' = \rho_2^2\lambda I_x - C_4'' \end{cases}$$

III

$$\begin{cases} r_4''' = \dfrac{\rho_2}{\rho_1}\dfrac{|I_{xz}|}{I_z} & \\ C_2''' = \rho_1^2\lambda I_z & ; \quad C_3''' = \rho_2^2\lambda\left(I_x - \dfrac{I_{xz}^2}{I_z}\right) \end{cases}$$

IV

$$\begin{cases} r_4'''' = \dfrac{\rho_1}{\rho_2}\dfrac{|I_{xz}|}{I_x} & \\ C_2'''' = \rho_1^2\lambda\left(I_z - \dfrac{I_{xz}^2}{I_x}\right) & ; \quad C_3'''' = \rho_2^2\lambda I_x \end{cases}$$

The validity of alternative arrangements I and II in representing the left-hand members of Equations [22b,c], although perhaps not intuitively obvious, is easily verified. In arrangement I, for example, the current i through C_4' taken positive upward, provided $I_{xz} \gtreqless 0$, is of magnitude

$$i = C_4' \frac{d}{dt'}(\dot{\gamma}' - r_4'\dot{\alpha}')$$

Thus the total current leaving the $\dot{\alpha}'$ node, except through L_1 and L_3, is

$$C_3' \frac{d}{dt'} \dot{\alpha}' - r_4' i = \left[C_3' + (r_4')^2 C_4' \right] \frac{d}{dt'} \dot{\alpha}' - r_4' C_4' \frac{d}{dt'} \dot{\gamma}'$$

$$= \rho_2^2 \lambda I_x \frac{d}{dt'} \dot{\alpha}' - \rho_1 \rho_2 \lambda I_{xz} \frac{d}{dt'} \dot{\gamma}'$$

in agreement with Equation [22c]. Similarly, the additional current leaving $\dot{\gamma}'$ is

$$C_2' \frac{d}{dt'} \dot{\gamma}' + i = (C_2' + C_4') \frac{d}{dt'} \dot{\gamma}' - r_4' C_4' \frac{d}{dt'} \dot{\alpha}'$$

$$= \rho_1^2 \lambda I_z \frac{d}{dt'} \dot{\gamma}' - \rho_1 \rho_2 \lambda I_{xz} \frac{d}{dt'} \dot{\alpha}'$$

in agreement with Equation [22b]. If $I_{xz} < 0$, so that $|I_{xz}| = -I_{xz}$, the same results are obtained after reversing the connections to one winding of the transformer. The transformer ratio r_4' may be given any convenient value between the limits indicated; these limits are necessary to keep C_2' and C_3' positive or zero. A value of r_4' between these limits can always be found, since always $I_x I_z \geqq I_{xz}^2$, so that

$$\frac{|I_{xz}|}{I_z} \leqq \frac{I_x}{|I_{xz}|}$$

Arrangement II is verified similarly.

In some cases it may be preferable to eliminate one capacitance by setting r_4' or r_4'' equal to its least permissible value. Then either C_2' or C_3'' is zero and the two resulting alternative arrangements obtained from I and II, respectively, are shown as III and IV, with subscripts changed and C_4' or C_4'' relabeled as C_2''' or C_3'''', respectively.

If $I_{xz} = 0$, then $C_4' = C_4'' = 0$, and the corresponding capacitance and the associated transformer may be omitted. If $0 < I_{xz} \leqq I_x$ and also $I_{xz} \leqq I_z$, then it is possible to choose $r_4' = 1$ so that this transformer can be omitted, C_4' being connected directly between the $\dot{\alpha}'$ and $\dot{\gamma}'$ nodes.

An alternative (energy) method of deriving the mobility analog for the rudder is given in Appendix B.

4.4 CONNECTION TO THE SHIP NETWORK

In the ship network the velocities $\dot{y}_{h,n+\alpha}$, $\dot{\gamma}_{h,n}$, and $\dot{\phi}_{n+\alpha}$ are represented by voltages at nodal points. When the rudder network is connected to the ship network in order to study the vibrations of the combined system, the network representing the ship must be connected to the nodes representing $\dot{v}_b$, $\dot{\gamma}_b$, and $\dot{\alpha}_b$, shown in Figure 8, so as to represent Equations [16a,b,c]. This connection is facilitated if the same conversion factors are used for both networks; that is, if p is the same for both and if the factors b_1, b_2, and b_3 used for $\dot{v}'_b$, $\dot{\gamma}'_b$, and $\dot{\alpha}'_b$, respectively, are the same as the factors used for $\dot{y}'_{h,n+\alpha}$, $\dot{\gamma}'_{h,n}$, and $\dot{\phi}'_{n+\alpha}$ in ship theory. These factors are commonly denoted by the same symbols.[3] Then p, b_1, b_2, and b_3 are all fixed when their values have been chosen for the ship network.

One of the boundary conditions at the juncture of two networks, the requirement of a common voltage at common points, is thereby satisfied. A current balance is also necessary, however; the (algebraic) currents entering the networks at any common point must be equal and opposite. This requirement corresponds to the law of action and reaction in mechanics, just as the identity of voltages at a common point represents the existence there of a single velocity.

In ship theory the force and moments $P_{h,n+\alpha}$, Q_n, and $U_{n+\alpha}$ are represented by currents defined in terms of conversion factors a_1, a_2, and a_3. In rudder theory analogous conversion factors have not been introduced explicitly, but they are easily discovered. The right-hand members of Equations [13a,b,c] or [18a,b,c] represent values of Y, T, and M, respectively. They still represent such values after substitution for v, γ, and α in terms of v', γ', and α'; after division by a_1, a_2, and a_3, respectively; however, they have become values of currents or current sums. Thus the a's already introduced constitute conversion factors from force or moment to current. Hence

$$Y = a_1 Y' \quad ; \quad T = a_2 T' \quad ; \quad M = a_3 M'$$

In view of Equations [15a,b,c] and [17a-f], the conversion factors a_1, a_2, and a_3 for Y, T, and M must be the same as the conversion factors for $P_{h,n\pm\alpha}$, Q_n, and $U_{n\pm\alpha}$, respectively. Thus a_1, a_2, and a_3, also, are <u>determined for the rudder by the values chosen for them in the ship theory</u>.

The relations, Equations [16a,b,c] become the electrical requirement (since $b_3/b_1 = a_1/a_3$) that

$$\dot{v}'_b = (1 - s)\left[\dot{y}'_{h,n-\alpha} - \frac{a_1}{a_3} z_b \dot{\phi}'_{n-\alpha}\right] + s\left[\dot{y}'_{h,n+\alpha} - \frac{a_1}{a_3} z_b \dot{\phi}'_{n+\alpha}\right]$$

$$\dot{\gamma}'_b = (1 - s')\,\dot{\gamma}'_{h,n'} + s'\,\dot{\gamma}'_{h,n'+1} = (1 - s')\,\dot{\gamma}'_{h,n-1} + s'\,\dot{\gamma}'_{h,n}$$

$$\dot{\alpha}'_b = (1 - s)\,\dot{\phi}'_{n-\alpha} + s\,\dot{\phi}'_{n+\alpha}$$

Such combinations can be made by means of transformers, as shown in Figure 9. The nodes labeled $\dot{y}'_{h,n\pm\alpha}$, $\dot{\phi}'_{n\pm\alpha}$, $\dot{\gamma}'_{h,n-1}$, or $\dot{\gamma}'_{h,n}$ are nodes in the ship network; points labeled $\dot{v}'_b$, $\dot{\gamma}'_b$, $\dot{\alpha}'_b$ are those so labeled in Figure 8 for the rudder network. Figure 9 is labeled for $z_b > 0$; if $z_b < 0$, one plus sign is to be moved to the other end of the winding for each transformer; i.e., connections to one winding of both transformers having voltage ratio r:1 are to be reversed from those shown in the figure.

Equations [17a-f] become current relations, such as

$$P'_{h,n-\alpha} = (1 - s)\,Y'_b \quad ; \quad P'_{h,n+\alpha} = s\,Y'_b$$

in which $P'_{h,n\pm\alpha}$ denotes currents entering the ship network at $n\pm\alpha$, and Y'_b denotes the current leaving the rudder network at the $\dot{v}'_b$ node. It is easily verified that these relations are also satisfied. The currents entering the $\dot{\phi}'_{n\pm\alpha}$ ship nodes through the r transformers represent torsional moments $-z_b\, P_{n\pm\alpha}$ about the ship's x-axis and are of magnitude $-\frac{a_1}{a_3} z_b\, P'_{n+\alpha}$.

An alternative (<u>energy</u>) method of deriving the mobility analog for the rudder-hull system is given in Appendix B.

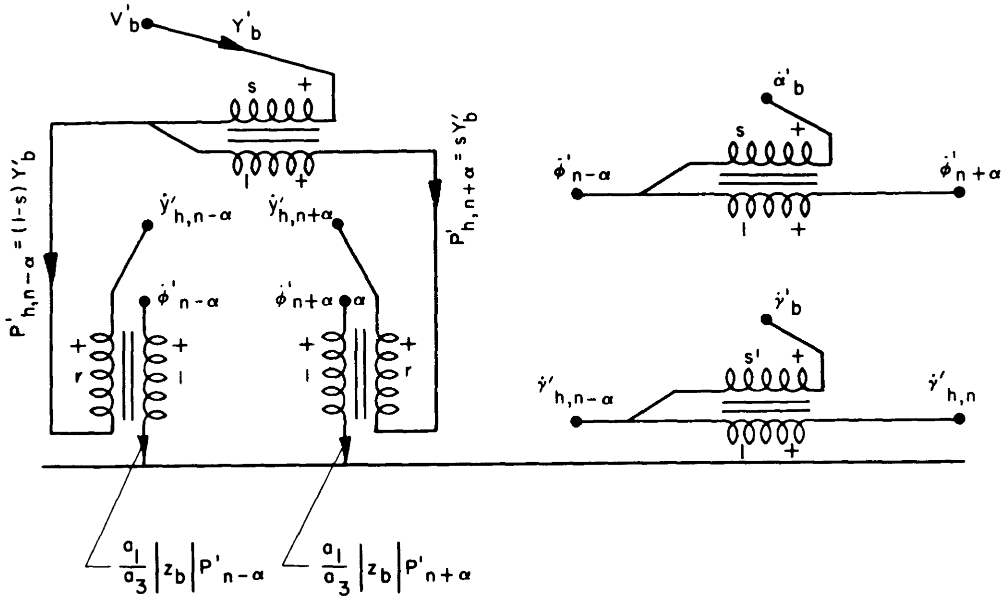


Figure 9 - Mobility Analog for Connection of Rudder
Shown in Figure 8 to Hull

$$r = \frac{a_1}{a_3} |z_b|$$

For s, s', see Equations [16a,b,c].
The connections shown are for $z_b > 0$.

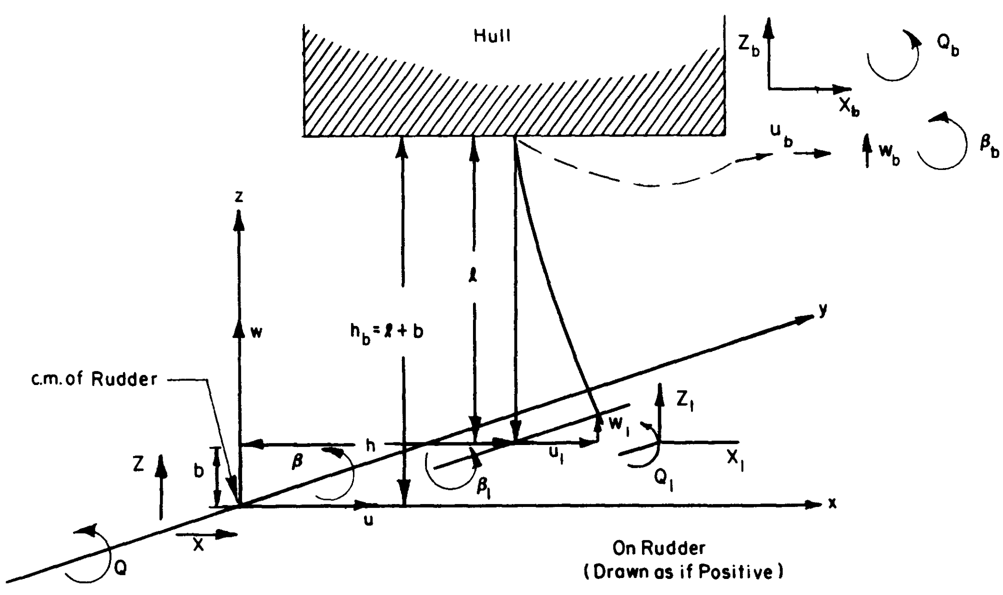


Figure 10 - Positive Forces and Moments Acting on Rudder
and Rudder Stock for Longitudinal (xz-Plane)
or u, w, β motion of Rudder

5. LONGITUDINAL RUDDER-HULL MOTIONS AND CORRESPONDING MOBILITY ANALOGS

5.1 THE LONGITUDINAL OR u, w, β MOTION OF THE RUDDER

The treatment of the longitudinal motion of the rudder largely parallels that of its transverse motion and may be stated more concisely, especially since it is of less interest. The problem will be simplified for the present by assuming that the rudder lies in the median longitudinal plane of the ship.

The displacements of the rudder to be considered are the two translations u and w parallel to the rudder x- and z-axes, respectively, and the rotation β about the rudder y-axis through the effective center of mass, taken positive from x toward z; see Figure 10. The relevant inertial parameters of the rudder will be only its mass and its moment of inertia about the y-axis. There will probably be different virtual masses for accelerations $\ddot{u}$ and $\ddot{w}$. Furthermore, $\ddot{u}$ will probably produce a z-component of reaction by the water on the rudder, and $\ddot{w}$ a corresponding x reaction. This cross reaction will usually be relatively small, however, and is difficult to estimate; furthermore, the virtual masses for $\ddot{u}$ and $\ddot{w}$ acceleration are themselves much smaller than that for $\ddot{v}$ acceleration. Hence, the cross reaction will be ignored. The inertial parameters for rudder plus water will then be only total masses m_x and m_z for $\ddot{u}$ and $\ddot{w}$ acceleration and an equivalent moment of inertia I_y, including water inertia, about the y-axis.*

The displacements of the bottom of the stock will be, in this case, u_1, w_1, and β_1, where

$$u_1 = u - b\beta \quad ; \quad w_1 = w + h\beta \quad ; \quad \beta_1 = \beta \qquad [23a,b,c]$$

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*The cross effect could easily be included in the equations of motion. The left-hand members of Equations [26a,b] would become, respectively,

$$m_x\ddot{u} + m_{xy}\ddot{w} \quad ; \quad m_z\ddot{w} + m_{xz}\ddot{u}$$

with m_{xz} denoting a cross-inertial constant that may be either positive or negative. It is always possible to make $m_{xz} = 0$ by rotating the axes, but in the present case this would be inconvenient.

The reactions of the stock on the rudder at the bottom of the stock will be forces X_1 and Z_1 in the x- and z-directions, respectively, and a moment Q_1 in the xz-plane, positive from x toward z. When the top of the stock is fixed, the magnitude of X_1 is -P as given by Equation [7a] with "y" = u_1 and $\theta = \beta_1$; see Figure 2. Z_1 equals -F as given by Equation [1] with "w" = w_1 ; and Q_1 = -M as given by Equation [7b] with "y" = u_1 and "θ" = β_1 . Thus, changing notation, as shown in Equations [12a,b], when the ship is at rest:

$$X_1 = -12\, k_s\, u_1 + 6\, k_s \ell\, \beta_1 \quad ; \quad Z_1 = -\frac{EAw_1}{\ell} \quad ;$$

$$Q_1 = 6\, k_s\, \ell u_1 - (3\, k_s + k)\, \ell^2 \beta_1$$

If now the ship moves and the top of the stock (at the level of the bearing) undergoes displacements u_b, w_b, and β_b, then, if the stock were to move as if rigid, its lower end would undergo displacements u_{1b}, w_{1b}, and β_{1b} of magnitude

$$u_{1b} = u_b + \ell\beta_b \quad ; \quad w_{1b} = w_b \quad ; \quad \beta_{1b} = \beta_b$$

and in this case $X_1 = Z_1 = Q_1 = 0$. Hence, in general,

$$X_1 = -12\, k_s\, (u_1 - u_b - \ell\beta_b) + 6\, k_s \ell\, (\beta_1 - \beta_b) \qquad [24a]$$

$$Z_1 = -\frac{EA}{\ell}\, (w_1 - w_b) \qquad [24b]$$

$$Q_1 = 6\, k_s \ell\, (u_1 - u_b - \ell\beta_b) - (3\, k_s + k)\, \ell^2\, (\beta_1 - \beta_b) \qquad [24c]$$

The corresponding forces X and Z on the center of mass and the moment Q about the rudder y-axis are

$$X = X_1 \quad ; \quad Z = Z_1 \quad ; \quad Q = Q_1 - bX_1 + hZ_1 \qquad [25a,b,c]$$

The angular momentum of the rudder is $I_y \dot{\beta}$. Hence

$$m_x \ddot{u} = X \quad ; \quad m_z \ddot{w} = Z \quad ; \quad I_y \ddot{\beta} = Q$$

Thus, in terms of u, w, β, the equations of motion for the u, w, β motion of the rudder are

$$m_x \ddot{u} = -12\, k_s\, (u - b\beta - u_b - \ell\beta_b) + 6\, k_s \ell\, (\beta - \beta_b) \qquad [26a]$$

$$m_z \ddot{w} = -\frac{EA}{\ell}\, (w + h\beta - w_b) \qquad [26b]$$

$$I_y \ddot{\beta} = 6\, k_s\, (\ell + 2b)(u - b\beta - u_b - \ell\beta_b) - \frac{EAh}{\ell}\, (w + h\beta - w_b) - [3\, k_s\, \ell\, (\ell + 2b) + k\ell^2]\, (\beta - \beta_b) \qquad [26c]$$

The right-hand members of these equations are expressions for X, Z, and Q, respectively.

Note that if $h = 0$, the w motion is independent of the other motions; it represents a simple vertical translational vibration of the rudder. If EA is relatively large, the frequency of this vibration (where $w_b = 0$) is high. Even if $h \neq 0$, there is likely to be one mode of vibration at relatively high frequency in which compressional elasticity plays a dominant role.

There appears to be an advantage in concentrating the coefficient EA in one equation by returning to w_1 as a variable instead of to w. We have $w_1 = w + h\beta$, therefore $\ddot{w} = \ddot{w}_1 - h\ddot{\beta}$. Direct substitution now destroys the symmetry of the coefficients, but this can be restored by subtracting h times the new form of the second equation from the third equation. The design is also facilitated if the terms are grouped as in Equations [18a,b,c]. The result is the following alternative set of equations of motion in terms of u, w_1, and β:

$$m_x \ddot{u} = -12\, k_s \left[u - \frac{1}{2}\, (\ell + 2b)\, \beta - u_b - \frac{\ell}{2}\, \beta_b \right] \qquad [27a]$$

$$m_z \ddot{w}_1 - h\, m_z\, \ddot{\beta} = -\frac{EA}{\ell}\, (w_1 - w_b) \qquad [27b]$$

$$(I_y + h^2 m_z)\, \ddot{\beta} - h\, m_z\, \ddot{w}_1 = 6\, k_s\, (\ell + 2b) \left[u - \frac{1}{2}\, (\ell + 2b)\, \beta - u_b - \frac{\ell}{2}\, \beta_b \right] - k\ell^2\, (\beta - \beta_b) \qquad [27c]$$

The right-hand members of these equations are equal to X, Z, and Q - hZ, respectively.

If EA is large enough, it may be reasonable to ignore the high-frequency mode of vibration mentioned previously by assuming as an approximation that $w_1 = w_b$ and then using only the first and third equations. As a check, the difference $w_1 - w_b$ as given by Equation [27b] can then be calculated to see if this difference is sufficiently small to justify dropping Equation [27b].

The vibrational frequencies of the rudder when the stock is clamped at the bearing level can be determined by solving Equations [26a,b,c] or [27a,b,c] with $u_b = w_b = \beta_b = 0$.

5.2 RELATIONS OF LONGITUDINAL SHIP MOTION WITH VERTICAL SHIP MOTION

In the usual theory of the vertical vibrations of ships,[3] the vertical displacement of any cross section of the hull is commonly denoted by "y", the y-axis being drawn vertical in this case instead of horizontal.* Then there is an (equivalent) rotation γ of the cross section about its horizontal neutral axis. These displacements do not evoke a net compressive or tensile longitudinal force in the ship. Such a force might result from longitudinal vibration, but such vibrations are of relatively high frequency and are not usually considered. In this report, y and γ for the vertical motion of the ship will be replaced by y_v and γ_v to avoid confusion. Thus, the correspondence of displacements in the present case is as follows; see Figure 11:

$$\text{Ship:} \quad (u),\ y_v,\ \gamma_v$$

$$\text{Rudder:} \quad u,\ w,\ \beta$$

The rotation γ_v will displace the stock bearing if the latter does not lie on the neutral axis of the ship cross section to which the stock

*For the case of vertical vibrations the y- and z-axes for the ship, i.e., y_v and z_v, are rotated through 90 deg so that the y_v-axis is vertical and the z_v-axis is horizontal; see Reference 3.

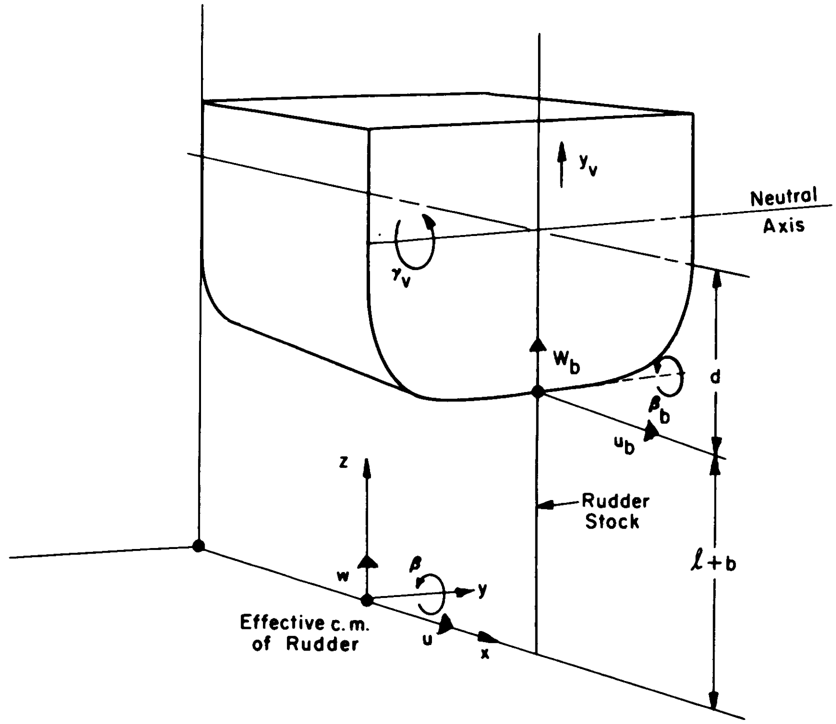

Figure 11 - Correspondence between Displacements in Rudder and Hull for Rudder Motion u, w, β Associated with Vertical Ship Motion

is attached. Let the neutral axis lie a distance d above the stock bearing. Then the displacements of the bearing will be, omitting that displacement due to longitudinal vibration of the ship

$$u_b = d\,\gamma_v \quad ; \quad w_b = y_v \quad ; \quad \beta_b = \gamma_v \qquad [28a,b,c]$$

These values substituted for u_b, w_b, and β_b in Equations [26a,b,c] or [27a,b,c] furnish equations for the rudder which, together with the ship equations, make possible a calculation of the vertical vibration of the ship as modified by motion of the rudder.

If finite-difference equations are employed, the displacement of the ship may be represented by values $y_{v,n+\alpha}$ at midstations labeled $n + \alpha$ and by $\gamma_{v,n}$ at stations labeled $n\ (\alpha = \frac{1}{2})$. The connection of these values with u_b, w_b, and β_b can be established by the interpolation procedure described just before Equations [16a,b,c],

$$u_b = d\,[(1 - s')\,\gamma_{v,n'} + s'\gamma_{v,n'+1}] = d\,\beta_b \tag{29a}$$

$$w_b = (1 - s)\,y_{v,n-\alpha} + s\,y_{v,n+\alpha} \tag{29b}$$

$$\beta_b = (1 - s')\,\gamma_{v,n'} + s'\gamma_{v,n'+1} \tag{29c}$$

Here $0 \leqq s < 1$ and $0 \leqq s' < 1$; the rudder stock is located $s\Delta x$ from point $n - \alpha$ and simultaneously, $s'\Delta x$ from point n'.

The equivalent forces X_b and Z_b and couple Q_b acting on the ship, assumed to act at the stock bearing, will have the values

$$X_b = -X \quad ; \quad Z_b = -Z \quad ; \quad Q_b = -Q - (\ell + b)X + hZ = -Q_s - (\ell + b)X \tag{30a,b,c}$$

where X, Z, and Q are equal, respectively, to the right-hand members of Equations [26a,b,c], or Q_s is the right-hand member of Equation [27c]; i.e., $Q_s = Q - hZ$. The values of Z_b and Q_b thus found may now be substituted for Y_b and T_b, respectively, in Equations [17a,b,e,f] to obtain values of the external force $P_{n\pm\alpha}$, and external moment $Q_{n'}$ or $Q_{n'+1}$ for use in ship equations representing vertical vibration. Ship equations given in Reference 3 may be used by adding $P_{n+\alpha}$ on the right in Equations [2.5] or [2.18] and $-Q_n$ on the right in Equations [2.7] or [2.20], in analogy with Equations [2.42] and [2.44] for horizontal vibration. For Q_n, either $Q_{n'}$ or $Q_{n'+1}$ will be used according to the location of the rudder, as was explained following Equations [17e,f] in the present report.*

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*Note that logically n has different meanings. In Equations [17a,b], n is such that the rudder lies between $n - \alpha$ and $n + \alpha$, whereas for ship theory in Equations [2.5] or [2.18] of Reference 3, n may have any value within the specified range. The latter equations refer to station $n + \alpha$ only, but by varying n all stations are covered; e.g., suppose the rudder lies between stations $15 - \alpha$ and $15 + \alpha$, then rudder n = 15. Now watch the calculator: $P_{n+\alpha} = 0$ for n = 0 . . . 13, 16 . . . 20. $P_{n+\alpha} \neq 0$ for n = 14, 15. Similarly for $Q_{n'}$ and $Q_{n'+1}$.

The ship equations thus modified may then be solved in conjunction with Equations [29a,b,c] and either Equations [26a,b,c] or [27a,b,c], noting that in the latter $w_1 = w + h\beta$.

The force X_b on the ship is ignored simply because longitudinal vibrations of the ship are not being considered. The finite difference formulation and solution of these equations by means of a digital computer is found in Reference 5.

5.3 MOBILITY ANALOG FOR LONGITUDINAL OR u, w, β MOTION

The procedure for designing an analog is similar to that used for the transverse motion. Write

$$\dot{u} = b_6\, \dot{u}' \quad ; \quad \dot{w} = b_4\, \dot{w}' \quad ; \quad \dot{\beta} = b_5\, \dot{\beta}' \quad ; \quad t = pt'$$

$$u = b_6 p\, u' \quad ; \quad w = b_4 p\, w' \quad ; \quad \beta = b_5 p\, \beta'$$

$$u_b = b_6\, p\, u_b' \quad ; \quad w_b = b_6 p\, w_b' \quad ; \quad \beta_b = b_5\, p\, \beta_b'$$

After substitution in Equations [27a,b,c] and division of these equations by a_6, a_4, a_5, respectively, we see that it is advantageous to require that

$$a_4 b_4 = a_5 b_5 = a_6 b_6 \qquad [31]$$

It is found also that a_6 and a_4 are the conversion factors from x force (X or X_b) or z force (Z or Z_b) to current, and a_5 similarly from y moment (Q or Q_b) to current. Again the six factors a_4, a_5, a_6, b_4, b_5, and b_6 may be taken as positive, and three of their five independent ratios may be chosen arbitrarily so long as $u_b = w_b = \beta_b = 0$. When, however, the ship moves, these ratios must agree with the choice made for the ship. In the latter case, a_4 and β_4 are the same factors as those denoted by a_1 and b_1 in the ship theory, and a_5 and β_5 are equal similarly to a_2 and b_2.[3] Since longitudinal motion of the ship itself is being ignored, the ratio b_6/a_6 is arbitrary.

For greater convenience, write, in analogy with Equations [21a,b,c]:

$$\bar{\lambda} = \frac{b_4}{pa_4} \quad ; \quad \bar{\rho}_1 = \frac{b_5}{b_4} = \frac{a_4}{a_5} \quad ; \quad \bar{\rho}_2 = \frac{b_6}{b_4} = \frac{a_4}{a_6}$$

Then

$$\frac{b_6}{b_5} = \frac{a_5}{a_6} = \frac{\bar{\rho}_2}{\bar{\rho}_1} \quad ; \quad \frac{b_5}{pa_4} = \frac{b_4}{pa_5} = \bar{\rho}_1 \bar{\lambda} \quad ; \quad \frac{b_6}{pa_4} = \frac{b_4}{pa_6} = \bar{\rho}_2 \bar{\lambda}$$

$$\frac{b_5}{pa_5} = \bar{\rho}_1^2 \bar{\lambda} \quad ; \quad \frac{b_6}{pa_6} = \bar{\rho}_2^2 \bar{\lambda} \quad ; \quad \frac{b_5}{pa_6} = \frac{b_6}{pa_5} = \bar{\rho}_1 \bar{\rho}_2 \bar{\lambda}$$

Conversion of the mechanical Equations [27a,b,c] by substitution then yields the electrical equations:

$$\bar{\rho}_2^2 \bar{\lambda} m_x \frac{d}{dt'} \dot{u}' + 12 p^2 \bar{\rho}_2^2 \bar{\lambda} k_s \left[u' - \frac{\bar{\rho}_1}{\bar{\rho}_2} (\tfrac{1}{2}\ell + b) \beta' - u_b' - \frac{\bar{\rho}_1}{\bar{\rho}_2} \frac{\ell}{2} \beta_b' \right] = 0 \qquad [32a]$$

$$\bar{\lambda} m_z \frac{d}{dt'} \dot{w}_1' - \bar{\rho}_1 \bar{\lambda} h m_z \frac{d}{dt'} \dot{\beta}' + p^2 \bar{\lambda} \frac{EA}{\ell} (w_1' - w_b') = 0 \qquad [32b]$$

$$\bar{\rho}_1^2 \bar{\lambda} (I_y + h^2 m_z) \frac{d}{dt'} \dot{\beta}' - \bar{\rho}_1 \bar{\lambda} h m_z \frac{d}{dt'} \dot{w}_1' - 6 p^2 \bar{\rho}_1 \bar{\rho}_2 \bar{\lambda} k_s (\ell + 2b) \qquad [32c]$$

$$\left[u' - \frac{\bar{\rho}_1}{\bar{\rho}_2} (\tfrac{1}{2}\ell + b) \beta' - u_b' - \frac{\bar{\rho}_1}{\bar{\rho}_2} \frac{\ell}{2} \beta_b' \right] + p^2 \bar{\rho}_1^2 \bar{\lambda} k \ell^2 (\beta' - \beta_b') = 0$$

One form of the corresponding network is shown in Figure 12; the values of the electrical elements are collected in the following list. The auxiliary connections needed to represent Equations [29a,b,c] are also shown in Figure 12; the nodes labeled similarly in the various diagrams being the same nodes in all three diagrams.

$$C_4 = \bar{\lambda} m_z (1 - r_4 \bar{\rho}_1 |h|)$$

$$C_5 = \bar{\rho}_1 \bar{\lambda} \left[\bar{\rho}_1 (I_y + h^2 m_z) - \frac{m_z |h|}{r_4} \right]$$

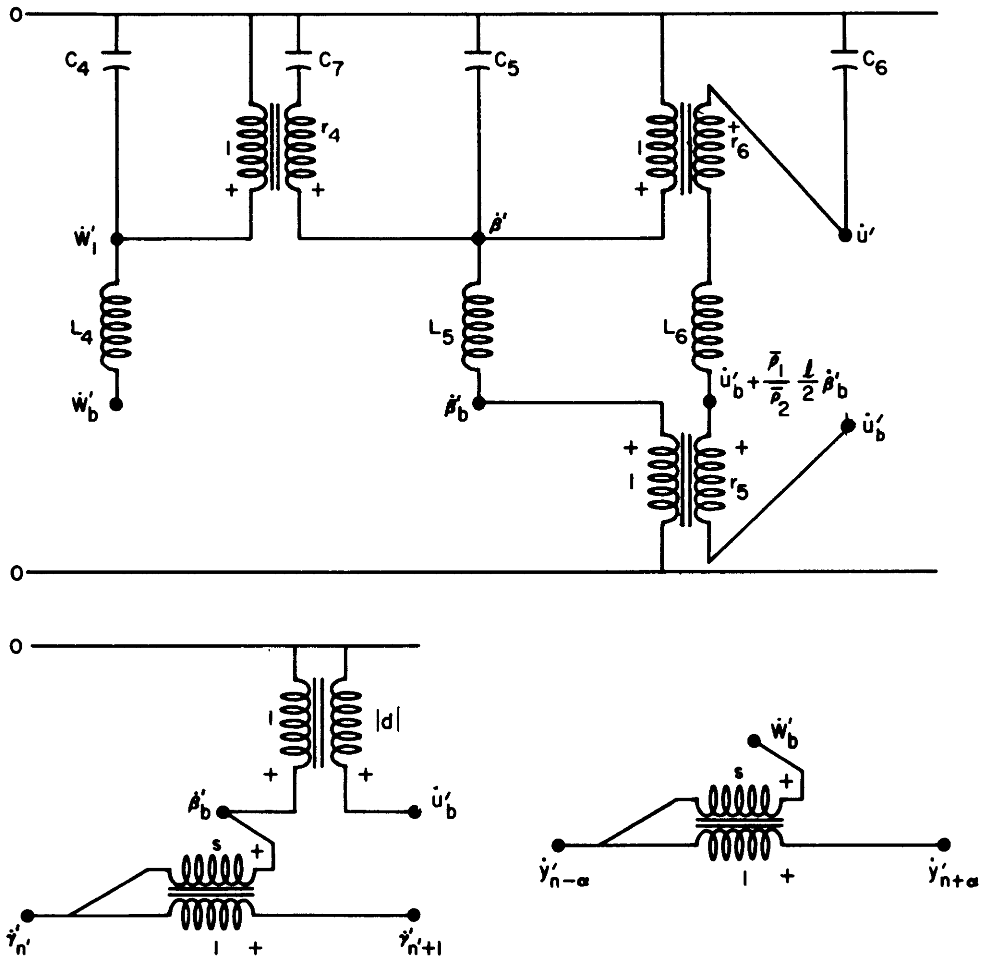

Figure 12 - Mobility Analog of Rudder for Longitudinal or u, w, β Motion of Rudder and Connection of Rudder to Hull

The rotation γ is about a horizontal axis; connections to the $|d|$ transformer are for $d > 0$.

$$C_6 = \bar{\rho}_2^2 \bar{\lambda} m_x$$

$$C_7 = \bar{\rho}_1 \bar{\lambda} m_z \frac{|h|}{r_4}$$

$$L_4^{-1} = p^2 \bar{\lambda} \frac{EA}{\ell} \quad ; \quad L_5^{-1} = p^2 \bar{\rho}_1^2 \bar{\lambda} k \ell^2 \quad ; \quad L_6^{-1} = 12 p^2 \bar{\rho}_2^2 \bar{\lambda} k_s$$

$$\frac{m_z |h|}{\bar{\rho}_1 (I_y + h^2 m_z)} \leqq r_4 \leqq \frac{1}{\bar{\rho}_1 |h|}$$

$$r_5 = \frac{\bar{\rho}_1}{\bar{\rho}_2} \frac{\ell}{2} \quad ; \quad r_6 = \frac{\bar{\rho}_1}{\bar{\rho}_2} \left| \frac{1}{2} \ell + b \right|$$

As usual, if the quantity whose absolute value appears in the formula for a transformer ratio is negative, connections to one winding of that transformer must be reversed from those shown in the figure. The value of r_4 is arbitrary within the limits shown.

As an alternative, the C_7 capacitor can be connected through the transformer to the $\dot{w}'$ node, as in changing from I to II in Figure 8; then C_4, C_5, C_7, and r_4 all have different values.

If $h = 0$, then $C_7 = 0$, and both this capacitor and the r_4 transformer are to be omitted. If $h > 0$ but is small enough to make r_4 equal to unity, then again the r_4 transformer can be omitted, C_7 being connected directly between the $\dot{w}_1'$ and $\dot{\beta}'$ nodes.

6. PAIRED RUDDERS AND CORRESPONDING MOBILITY ANALOG

Previously, a single rudder has been assumed to be located in the median plane of the ship. If a single rudder, although parallel to the median plane, is offset from it a distance $\underline{e}$ toward positive y, the theory is more complicated. Let a line drawn through the center of the rudder bearing and perpendicular to the median plane meet this plane at 0; see Figure 13. This line will be at a height z_b above the x-axis, as drawn in the ship theory, and the displacements of the ship at 0 will be the same as those of the rudder bearing when the rudder is in the median plane with its bearing at 0, so that the displacements at 0 will be those given by

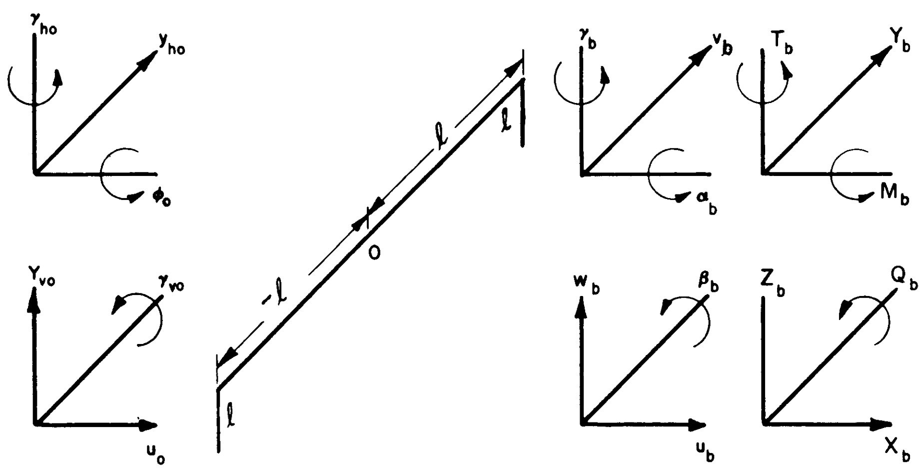

Figure 13 - Positive Directions for Hull Displacement at 0 (Intersection of a Line through Bearings and Hull Medium Plane) and for Reactions on Hull at Bearings

Offset of rudder "e" is toward positive y.

Equations [14a,b,c] and [28a,b,c]. In addition, however, the offset rudder bearing has two other displacements, so that its total displacements due to the ship motion, distinguished by the usual subscript <u>b</u>, are as follows:

$$u_b = d\gamma_v - e\gamma_h \quad ; \quad v_b = y_h - z_b\phi \quad ; \quad w_b = y_v + e\phi \qquad [33a,b,c]$$

$$\alpha_b = \phi \quad ; \quad \beta_b = \gamma_v \quad ; \quad \gamma_b = \gamma_h \qquad [33d,e,f]$$

When the ship vibrates in torsion, the term $e\phi$ in Equation [33c] gives rise to vertical motion of the rudder, and the resulting reactions on the ship will excite vertical vibration of the ship. For this and other reasons, the horizontal-torsional and the vertical vibrations of the ship are <u>coupled</u> <u>together</u> by the presence of a single offset rudder.

Usually, however, offset rudders occur symmetrically in pairs, and only this simpler case will be considered further in this report.

Let there be a pair of similar rudders with offsets from the median plane $\pm e$, where $e > 0$; see Figure 13. It is shown that such a pair does <u>not</u> couple the two types of ship vibration and that the vibrations of the combined ship-rudder system fall into the following three distinct classes:

a. Vertical ship vibration accompanied by equal u, w, β vibrations of the two rudders in the same phase.

b. Horizontal-torsional ship vibration accompanied by equal v, γ, α

rudder vibrations in the same phase and also equal u, w, β vibrations in opposite phases.

c. Equal v, γ, α rudder vibrations in opposite phases, with the ship stationary.

These three types are considered in order:

a. Vertical Ship Vibration. Suppose that the ship is vibrating vertically, with $y_h = \gamma_h = \phi = 0$. Then the e terms disappear from Equations [33a-f]. The vertical ship motion will excite the same u, w, β motion in both rudders, causing equal X_b, Z_b, Q_b reactions on the ship (at the two rudder bearings). The two equal X_b forces, acting together in the same direction, have no tendency to excite γ_h rotation of the ship about a vertical axis; similarly, the two Z_b forces do not excite ϕ rotation; and nothing excites y_h motion. Thus no y_h, γ_h, ϕ motion of the ship is excited, and it becomes clear that a possible type of vibration of the rudder-ship system consists of vertical vibration of the ship accompanied by equal u, w, β vibrations of the rudders in the same phase.

The combined effect of the two rudders on the vertical vibration of the ship is the same as that of a single rudder in the median plane with bearing at 0, constructed exactly like each of the actual rudders except that the inertial and elastic constants m_x, m_z, I_y, k, k_s, and EAh are all doubled. The equations of motion for u, w, β motion, Equations [26a,b,c] or [27a,b,c], show that for given values of u_b, w_b, and β_b the single rudder so designed will execute the same motion as each rudder of the actual pair. The forces on the ship due to the single rudder, however, will all be twice as great as those due to either of the actual pair. Hence, the effect of the single rudder on the ship motion will be the same as the effect of the two actual rudders, whose combined reactions on the ship are equivalent to single forces $2X_b$ and $2Z_b$ acting at 0 in the midplane, plus a torque $2Q_b$ about y.

This type of vibration may be treated, therefore, by using the single-rudder equations, [23a,b,c] through [30a,b,c], modified by the doubling of constants just described. The values of u_b, w_b, β_b to be used in the equations will be the same as those given in Equations [33a,c,e] for the actual rudders. The reactions of the single rudder on the ship equivalent to those of the actual rudders can be written from Equations [30a,b,c]:

$$2X_b = -2X \quad ; \quad 2Z_b = -2Z$$

$$2Q_b = -2Q - (\ell + b)(2X) + h\,(2Z) = -2Q_s - (\ell + b)(2X)$$

where 2X, 2Z, and 2Q now represent the respective right-hand members of the modified Equations [26a,b,c], or $2Q_s$ represents those of the modified Equation [27c]. For greater clarity, X_b, Z_b, and Q_b in these equations may be replaced by the symbols $\overline{X}_b$, $\overline{Z}_b$, and $\overline{Q}_b$, representing the reactions due to the equivalent single rudder and, hence, also the equivalent reactions due to the paired rudders acting in the median plane and at 0. The force $\overline{X}_b$ acts at the height of the actual rudder bearings.

The procedure in calculations for the vibrations of the rudder-ship system will now be the same as the one for the single midplane rudder after Equations [30a,b,c] at the end of Section 5.2, except that the modified equations just described will be used, and the values just given for $2Z_b$ and $2Q_b$ will be substituted for Y_b and T_b, respectively, in Equations [17a,b,e,f] in obtaining values of $P_{n\pm\alpha}$, $Q_{n'}$, and $Q_{n'+1}$.

b. Horizontal-Torsional Ship Vibration. Suppose, on the other hand, that the ship vibrates in horizontal bending and torsion, with y_h, γ_h, and ϕ active but with $y_v = \gamma_v = 0$. Then Equations [33a-f] become, after substituting for γ_h and ϕ from Equations [33f,d] in the e terms and rearranging:

$$v_b = y_h - z_b\phi \quad ; \quad \gamma_b = \gamma_h \quad ; \quad \alpha_b = \phi \qquad [34a,b,c]$$

$$u_b = \mp e\,\gamma_b \quad ; \quad w_b = \pm e\,\alpha_b \quad ; \quad \beta_b = 0 \qquad [34d,e,f]$$

Here the upper signs (-, +) refer to the rudder whose offset is +e and the lower signs (+, -) to the other rudder.

Thus, in addition to the v, γ, α motions of the rudders that would obviously be excited by horizontal-torsional motion of the ship, u, w, β motions of the rudders are also excited. There is, however, no tendency to excite vertical ship vibration since the contrasting signs in Equations [34d,e] cause the two rudders to move oppositely, vibrating in their u, w, β motions, with equal amplitudes but in opposite phases. The resulting two X_b forces on the ship are also equal and opposite, and they act at the same height above the xy-plane, whereas the two Y_b forces, also equal and opposite, act along the same line. The two torques Q_b cancel because of opposite phases. The two Z_b forces are also equal and opposite. Thus none of these reactions excite x or z translation of the ship or rotation about a horizontal axis parallel to y. It follows that the paired rudders allow the ship to vibrate in horizontal bending and torsion without simultaneous vertical vibration of the ship.

Equal v, γ, α motions of the rudders are excited by the equal v_b, γ_b, α_b displacements of their bearings as given by Equations [34a,b,c]. Since these motions are in phase, the actual rudders may again be replaced in the calculation by a single rudder in the median plane with bearing at 0, with doubled inertia and elasticity. The equations of motion for this equivalent rudder will be Equations [13a,b,c] with m_y, I_x, I_z, I_{xz}, k, k_s, and GJ_e all twice as large as for one of the actual rudders. Given values of v_b, γ_b, α_b will cause the single rudder to execute the same v, γ, α motion as that of each of the actual rudders, but its reactions on the ship will be twice as great as the reactions due to one of the actual rudders, thus simulating correctly the combined effect of the paired rudders.

The relations between the single rudder and the ship will be those expressed by Equations [14a,b,c] and [15a,b,c]. Here, for clarity, Y_b, T_b, and M_b may be replaced by the symbols $\overline{Y}_b$, $\overline{T}_b$, and $\overline{M}_b$, respectively, since they now represent twice the reactions on the ship due to one of the actual rudders. The quantities Y, T, M, in Equations [15a,b,c] are equal to the right-hand members of the modified Equations [13a,b,c].

For the u, w, β rudder motions that also occur in this case, it suffices to calculate the motion of one rudder; for this the rudder with offset +e will be chosen. The motion of the other rudder will then be the same but in opposite phase. The equations to be used for the chosen rudder are Equations [26a,b,c] or [27a,b,c]. In these equations u_b, w_b, β_b are to be given the values specified for this rudder in Equations [34d,e,f], or

$$u_b = -e\,\gamma_b \quad ; \quad w_b = e\,\alpha_b \quad ; \quad \beta_b = 0 \qquad [35a,b,c]$$

The reactions acting on the ship at the bearing of the chosen rudder due to its u, w, β motion will be X_b, Z_b, Q_b, as given by Equations [30a,b,c]. The torque Q, however, is balanced out by an opposite torque due to the u, w, β motion of the other rudder in the opposite phase. The X_b and Z_b forces due to the two rudders, acting in opposite directions along parallel lines 2e apart, have zero force resultants but are equivalent to a torque $-2eX_b$ about the z-axis and a torque $2eZ_b$ about the x-axis.

Addition of the reactions due to both types of rudder motion then gives as the total equivalent reactions on the ship, expressed in the notation of the ship theory, a force P acting toward positive y along a line through the rudder bearings and torques Q about the vertical z-axis and U about the x-axis of magnitudes:

$$P = \overline{Y}_b \quad ; \quad Q = \overline{T}_b - 2eX_b \quad ; \quad U = \overline{M}_b + 2eZ_b \qquad [36a,b,c]$$

Here $\overline{Y}_b$, $\overline{T}_b$, and $\overline{M}_b$ arise from the v, γ, α motions of the rudders described previously. The method of introducing such external reactions into the ship equations is described in Reference 3 and also briefly following Equations [15a,b,c] in the present report.

The equations to be used simultaneously in calculating frequencies of the rudder-ship system when the ship moves horizontally torsionally may be summarized as follows:

(1) For the in-phase v, γ, α rudder motions:
Equations [14a,b,c] or [16a,b,c] to give v_b, γ_b, α_b in terms of y_h, γ_h, ϕ; Equations [13a,b,c] with constants doubled; Equations [15a,b,c] multiplied through by 2 to give reactions $2Y_b$, $2T_b$, $2M_b$, on ship due to this motion of the two rudders.

(2) For the opposite-phased u, w, β motions:
Equations [35a,b,c] to give u_b, w_b, β_b in terms of γ_b and α_b, and thus from Equations [14a,b,c] indirectly in terms of y_h, γ_h, ϕ; Equations [26a,b,c] or [27a,b,c] as equations of motion for rudder with offset +e; Equations [30a,b] for reactions X_b and Z_b on ship due to this rudder alone.

(3) Equations [36a,b,c] for net reactions P, Q, and U on ship due to both motions (v, γ, α and u, w, β) of the two rudders.

(4) Ship equations such as Equations [2.42] through [2.49] in Reference 3, including forcing terms representing P, Q, and U. The way to introduce such terms into difference equations for the ship is explained in Reference 3 and also briefly following Equations [15a,b,c] in this report. The expressions for P, Q, U given by Equations [36a,b,c] may be substituted for Y_b, T_b, and M_b, respectively, in Equations [17a-f] of the present report to obtain values of $P_{n\pm\alpha}$, $Q_{n'}$, $Q_{n'+1}$, and $U_{n\pm\alpha}$ for use in difference equations for the ship.

c. Ship Stationary. Assume that the rudders execute equal v, γ, α motions but in opposite phases. Then, of the reactions cited in Equations [15a,b,c], the two Y_b forces on the ship act in opposite directions and along the same line, while the two T_b and M_b torques cancel. Thus, there is no net reaction on the ship at all. It follows that there exists a type of vibration of the rudder-ship system in which the rudders vibrate in equal but opposite v, γ, α motions while the ship, although free to move, stands still.

The frequencies of vibration and the associated motions may be found by solving Equations [13a,b,c] for one rudder, with $v_b = \gamma_b = \alpha_b = 0$.

The question may arise of whether still other types of vibration of the rudder-ship system are possible (without appreciable axial vibration of the ship). Always, in undamped vibration of an elastically coupled system, all parts of the system come to rest periodically at the same instant. It follows that in any type of free vibration of the rudder-ship system any vibratory motions of the two rudders must occur either in the same or in opposite phases. The alternative as to phases will hold separately for the v, γ, α and u, w, β motions of the rudders, since they are not coupled together; thus four combinations of rudder motions are possible. Since these four combinations, together with the ship motions that each tends to excite, have all appeared in the treatment just given, and since both of the possible modes of ship vibration have been considered, it is clear that no other type of vibration of the system (besides those types that have been discussed) is possible.

Each of the three classes of vibrations consists of a sequence of modes of increasing natural frequency. In classes a and b, the frequency and the mode of motion of the ship must be nearly the same as those calculated on the assumption of rigid rudders, unless this frequency happens to lie close to a natural frequency for the rudders when vibrating with fixed bearings. In the latter case, relatively large amplitudes of rudder vibration may occur, accompanied by a considerable distortion of the ship motion. The theory should hold even in such a case, provided the simplifying assumptions just made remain sufficiently valid.

Simplified formulas which may serve as a measure of the resonance effect on a rudder when the frequency of ship vibration approaches the natural frequency of a rudder are given in Appendix C. The significance of these approximate formulas is that they give the conditions for which large amplitudes of rudder vibration and corresponding damage to the rudder system become possible.

6.1 ANALOG NETWORK FOR TWO RUDDERS

For vibrations of class c, in which the ship remains stationary while the rudders execute similar v, γ, α motions in opposite phases, it suffices to represent one rudder by the network described in Section 4.3 and shown in Figure 8. For the present purpose, however, the nodes labeled $\dot{v}_b' + \frac{1}{2} (b_3/b_1) \ell \dot{\alpha}_b'$, $\dot{\alpha}_b'$, and $\dot{\gamma}_b'$ in Figure 8 are to be grounded, since the rudder bearing is here stationary. The r_1 transformer, therefore, is not needed.

For vibrations of class a, in which the ship vibrates vertically and the rudders have similar u, w, β motions, the equivalent median rudder described previously for this case may be represented by the network shown in the upper part of Figure 12, but with all capacitances made twice as large and all inductances half as large as they would be to represent one of the actual rudders. The lower part of Figure 12 shows the additional elements necessary to connect the $\dot{u}_b'$, $\dot{w}_b'$, and $\dot{\beta}_b'$ nodes of the rudder network to the network representing the ship in vertical vibration, as is required by Equations [33a,c,e] with the e terms omitted.

For class b, involving horizontal-torsional motion of the ship, rudder networks of both forms must be connected simultaneously to the ship.

To represent the equal and in-phase v, γ, α motions of the rudders, the equivalent median rudder described previously for this motion may be represented by the network shown in Figure 8 with all capacitances made twice as great and all inductances half as great as they would be for one of the actual rudders. The connections to the ship as required to represent Equations [33b,f,d] are shown in Figure 9.

For the u, w, β motions, the rudder having an offset +e might be represented by a network designed as shown in the upper part of Figure 12, with no change in the elements. The connections to the ship, however, must be different from those shown in the lower part of Figure 12, since the connecting equations to be represented are now Equations [33a,c,e] with $\gamma_v = y_v = 0$, or, in electrical form,

$$\dot{u}_b' = -e\dot{\gamma}_b' \quad ; \quad \dot{w}_b' = e\dot{\alpha}_b' \quad ; \quad \dot{\beta}_b' = 0$$

The node labeled $\dot{\beta}_b'$ in the upper part of Figure 12, therefore, is to be connected to ground so that the r_5 transformer is superfluous and the node shown at the bottom of L_6 may be relabeled $\dot{u}_b'$. The nodes $\dot{u}_b'$ and $\dot{w}_b'$ may then be connected to the $\dot{\gamma}_b'$ and $\dot{\alpha}_b'$ nodes of the v, γ, α network already set up (Figure 8) through e transformers, as shown in Figure 14.

Still another network might then be added to represent the other rudder. It is simpler, however, to include the effect of this rudder by doubling all capacitances and halving all inductances in the network representing the rudder with offset +e. This change does not alter the potentials occurring in the network for given bearing potentials $\dot{u}_b'$ and $\dot{w}_b'$, but it doubles all currents, including those that flow from the $\dot{u}_b'$ and $\dot{w}_b'$ nodes into the ship network. The latter currents, therefore, will represent correctly the doubled terms $-2eX_b$ and $2eZ_b$ in Equations [36b,c].

This change in the rudder network corresponds to doubling the elastic and inertial parameters of the rudder itself. Such a physical alteration of the rudder having offset e, with removal of the other rudder, would indeed provide correct values of the resultant torques $-2eX_b$ and $2eZ_b$. However, the rudder would also react on the ship with x and z forces and a y torque, which have no place in the resultant reactions due to the

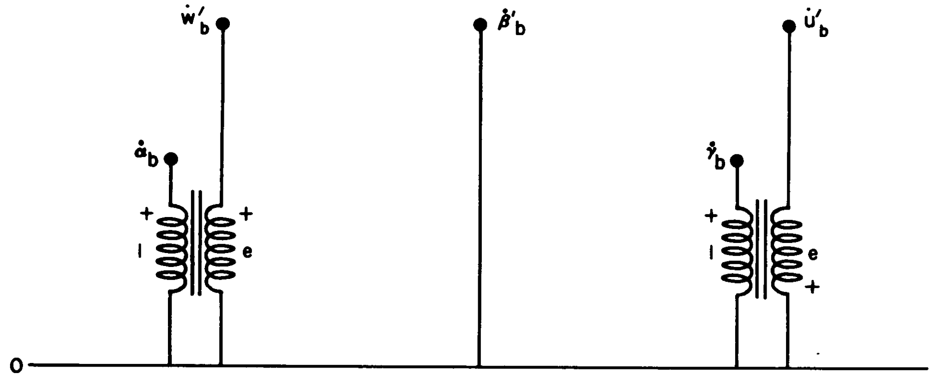

Figure 14 - Mobility Analog for Connection of Rudder in Longitudinal or u, w, β Motion to Hull in Torsion-Horizontal-Bending Motion

actual paired rudders. In calculations such spurious reactions can simply be ignored. They are also ignored automatically by the modified network just described, which delivers no corresponding currents to the ship. Perhaps this becomes more plausible if it is noted that nothing would be changed if x and z translation and y rotation of the ship at 0 were prevented by an external support without interfering with other components of motion. Then the spurious reactions would fall upon the supporting structure instead of on the ship, and the correctness of the modified network becomes obvious.

Note that currents coming from both rudder networks simultaneously enter the ship network at the $\dot{u}'_b$ and $\dot{w}'_b$ nodes. This feature corresponds to the occurrence of two terms on the right in Equations [36b,c].

7. COPLANAR CONTROL FOILS AND CORRESPONDING MOBILITY ANALOGS

A "foil" is considered here as an appendage thin enough to be treated satisfactorily as a two-dimensional body attached to the ship in a single location. If several control foils are present, they can always be treated by establishing separate equations for each one and adding up the reactions on the ship or by connecting individual foils to the ship network. In some cases, however, foils occur symmetrically; then it is possible to represent the vibrations of both foils of a pair and their combined reactions on the ship by a single set of equations or a single analog network. Besides the horizontally paired rudders already considered, important examples are paired diving planes and, perhaps, upper and lower rudders, as on a submarine.

Paired diving planes are discussed, special equations are derived for an upper rudder, and finally symmetrical upper and lower rudders and continuous shafts are considered.

7.1 PAIRED DIVING PLANES

The diving planes will be assumed to be symmetrical so that each is the mirror image of the other in the midplane of the ship, and for the present they will be assumed to be attached independently.

As with paired rudders, it is sufficient to write explicit equations for only one plane, for which the starboard plane will be chosen. The horizontal z-axis for this plane will be taken positive toward the bearing at B, whereas y is positive downward, the origin being at the effective center of mass of the plane.

The relations of these axes to the starboard plane are then the same as those between the axis and a single rudder, as previously defined. Also, by adopting the same notation, it will be possible to use the rudder equations as previously established, with the proper values inserted for the constants. The significance of ℓ and ℓ_T in these equations will be discussed later. The distance "h" will be positive as usual toward positive x but "b" will not be positive toward the ship, as drawn in Figures 15 and 16.

An <u>analog</u> to represent the motion of the starboard plane may be the same as if the plane were a rudder, since the equations to be represented are the same (only the y and z spatial directions being different).

To treat the port plane, note that the starboard plane can be converted into the port plane by a rotation through 180 deg about an axis parallel to x. By imagining the displacements as vectors to share in this rotation, without change of magnitude (so that the relation of the displacements to the plane is not changed), we see that each motion of the starboard plane is converted by the rotation into a possible motion of the port plane. Thus, separate calculations for the port plane are not necessary. Nor, in an electric analog, is it necessary to provide a separate network for the port plane.

In dealing with vibrations of the ship-plane system, it turns out to be useful to pair off the possible motions of the two planes in certain ways, in analogy with the treatment of paired rudders.

Note first that rotation of any starboard motion through 180 deg about an axis parallel to x generates a motion of the port plane in which v, γ, w, β displacements are reversed in space while α and u remain unchanged. Positive displacements of the starboard plane are shown in the lower parts of Figures 15 and 16; the rotation changes these into displacements of the port plane as shown at the upper right in the same figures. They are distinguished by a bar over each symbol.

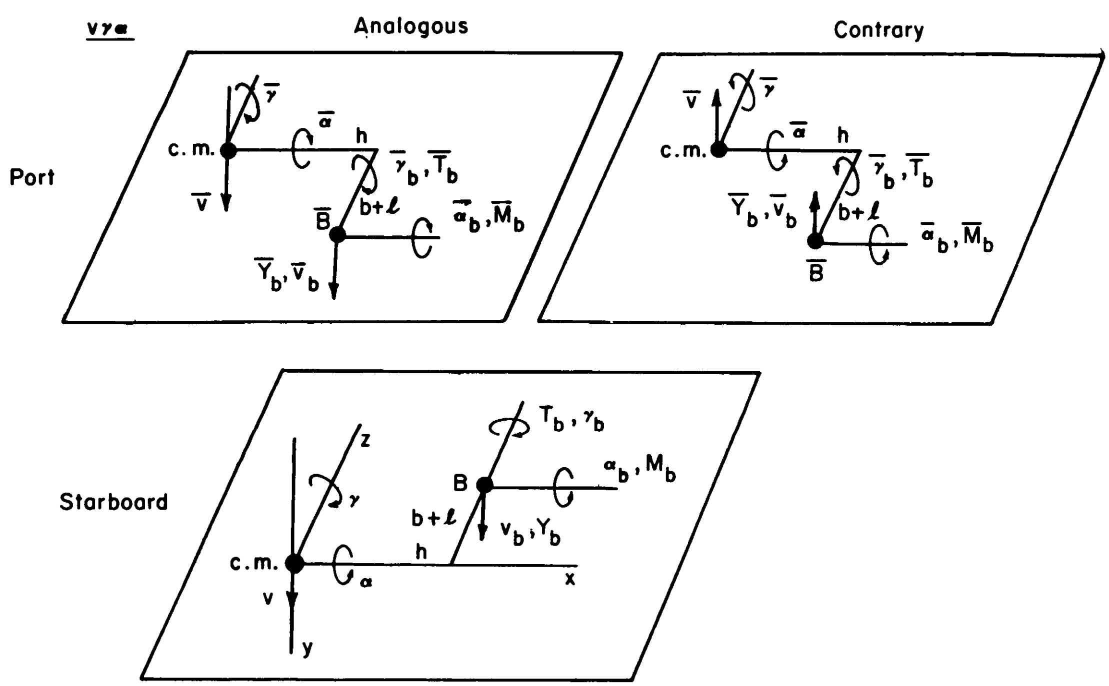

Figure 15 - Diagram for Paired Diving Planes Showing for v, γ, α Motion, Positive Directions of Displacements and Reactions on Ship at Bearing for Starboard Plane, and Accompanying Directions for Port Plane and in Each Type of Associated Motion

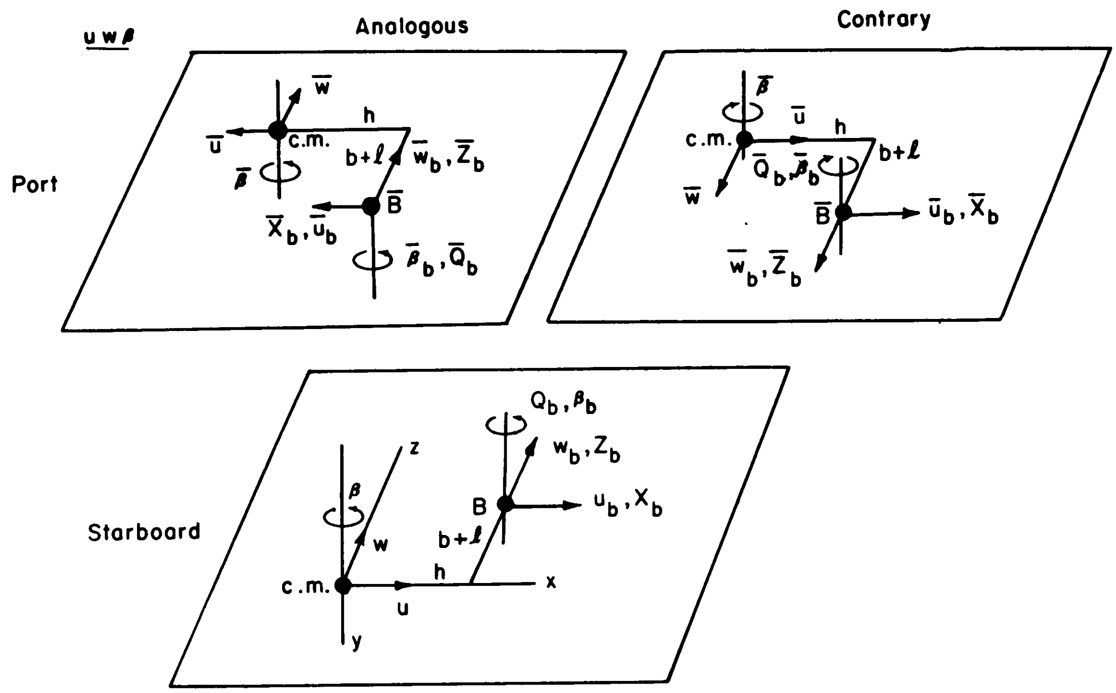

Figure 16 - Diagram for Paired Diving Planes Showing for u, w, β Motion, Positive Directions of Displacements and Reactions on Ship at Bearing for Starboard Plane, and Accompanying Directions for Port Plane and in Each Type of Associated Motion

Let the starboard and port motions that are thus associated together be called contrary motions of the two planes, since in the two motions four of the six displacements have opposite directions in space.

Every motion of a plane, however, can also occur with all displacements reversed. If the port motion just described is reversed, the result is the new motion shown at the upper left in Figures 15 and 16. Since the v, γ, w, β displacements in this new motion of the port plane are geometrically the same as those in the initial starboard motion, with only α and u being reversed, it is convenient to call this port motion and the initial starboard motion with which it is associated analogous motions of the two planes. By reversing the port motion, any pair of analogous motions can be converted into a contrary pair of motions, and vice versa.

These two ways of pairing off motions of the two planes correspond roughly to motions of two paired rudders, and their utility will become evident as the theory is developed. It will turn out that only paired motions of these two sorts need be considered in combination with vibrations of the ship. If the ship is held stationary, of course, each plane moves independently of the other and paired motions are not likely to occur.

In all cases the directions of forces and torques undergo the same changes as do the directions of the corresponding displacements, without any changes in magnitude. Directions for these reactions are shown in Figures 15 and 16 also, as well as displacements and reactions on the ship at the two points of attachment to the stock labeled, respectively, B and $\overline{B}$. The geometrical relationships thus defined are collected for convenience in the following table:

Mode	Analogous Motion		Contrary Motion	
	Same	Reversed	Same	Reversed
v, γ, α	v, γ, Y_b, T_b	α, M_b	α, M_b	v, γ, Y_b, T_b
u, w, β	w, β, Z_b, Q_b	u, X_b	u, X_b	w, β, Z_b, Q_b

Expressions representing any motion of the starboard plane alone can be converted into expressions for an associated motion of the port plane by adding a bar over the displacement symbols and assuming the appropriate changes in the geometrical directions, as determined from the table or from a comparison of the upper and lower parts of Figures 15 and 16.

Discussions of vibrations of the plane-ship system now parallel closely the treatment of a ship with paired rudders.

a. <u>Ship vibrating vertically with analogous v, γ, α motion of the planes</u>.

Vertical ship vibration causes equal v_b and γ_b displacements of the points of attachment B and $\overline{B}$ of the two planes, thereby exciting the v, γ, α type of plane vibrations; see Figure 17. Conversely, in view of the relative directions of the reactions on the ship in analogous motions, the resulting torques M_b and $\overline{M}_b$ balance out, but T_b and $\overline{T}_b$ result in a total torque equal to $2T_b$, whereas the equal forces Y_b and $\overline{Y}_b$ are equivalent to

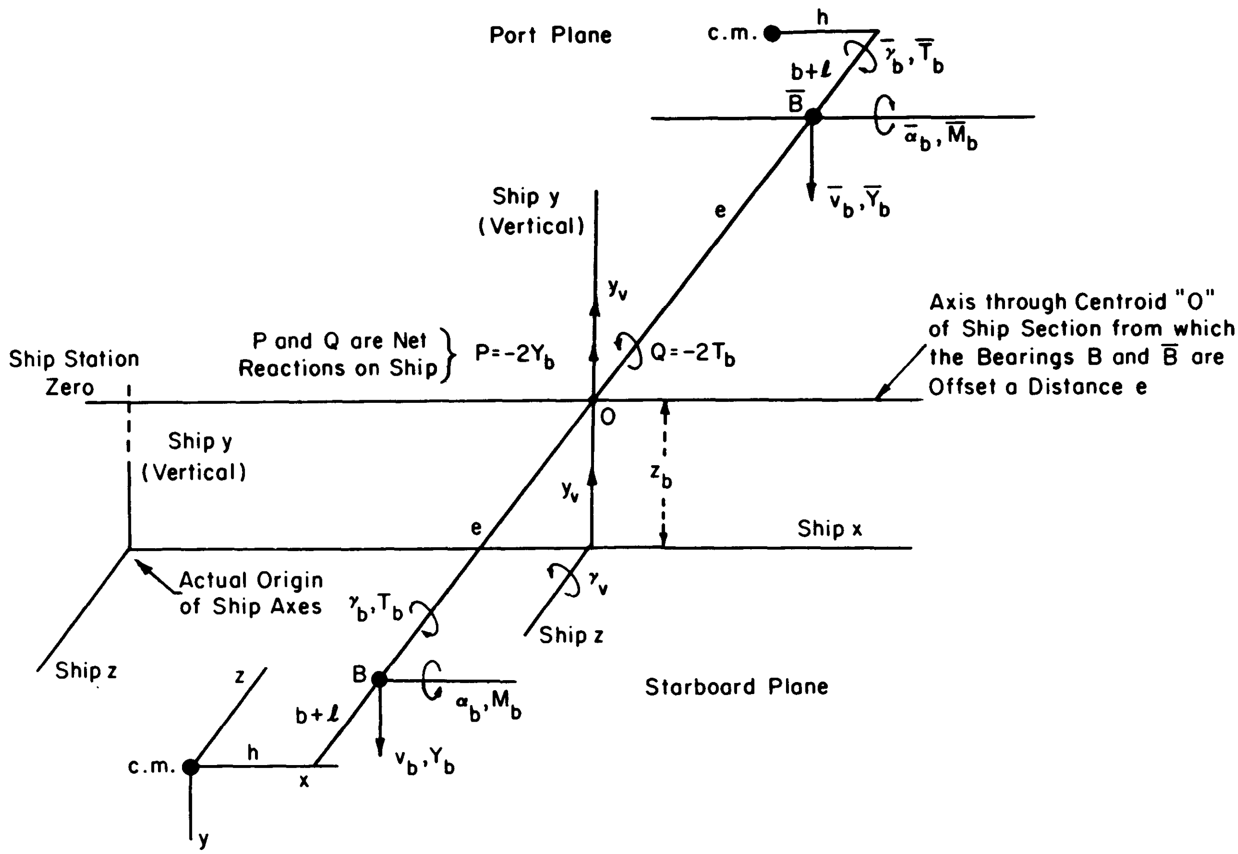

Figure 17 - Relations for Paired Diving Planes in Analogous v, γ, α Motion with Ship Vibrating Vertically

Positive directions are shown for ship and starboard plane, associated directions for port plane. The ship y- and z-axes, which intersect at distance z below centroid, are axes parallel to ship y- and z-axes intersecting at actual origin.

a single force $2Y_b$ acting in the midplane. Thus, simultaneous analogous v, γ, α motions of the two planes tend to excite only vertical vibration of the ship.

Equations [13a,b,c] and [15a,b,c] may be used for the starboard plane. It will be simpler, however, to include the dynamic effects of the port plane by doubling the constants m_y, I_x, I_z, I_{xz}, $k\ell^2$, k_s, GJ_e in Equations [13a,b,c]. This change does not affect the values of v, γ, α as determined by the equations for given v_b, γ_b, α_b, but it does double the values of both members of the equations. Hence, the right-hand members of the modified equations represent, respectively, 2Y, 2T, and 2M, and Equations [15a,b,c] multiplied by 2 give as reactions on the ship:

$$2Y_b = -2Y \quad ; \quad 2T_b = -2T + h\,(2Y)$$

The equations give also a resultant torque $2M_b$ but this may be ignored, since in reality the M_b and $\overline{M}_b$ torques cancel each other.

In the <u>ship theory</u> for vertical vibration, as cited in Section 5.2, the ship displacements are y_v upward and a rotation γ_v about a horizontal axis, positive from positive x toward the upward vertical. If the axis for γ_v does not lie on a transverse line through B and $\overline{B}$, the γ_v rotation will cause a slight fore-and-aft motion of B and $\overline{B}$. Such a motion gives rise to analogous u, w, β motion of the planes. This slightly complicates matters and will be ignored as unimportant, together with all other longitudinal motion of the ship. Hence, in the present case

$$v_b = -y_v \; ; \; \gamma_b = -\gamma_v \quad ; \quad \alpha_b = 0 \qquad [37a,b,c]$$

In the notation used in the ship theory, the reactions on the ship are a force P acting in the midplane and taken positive upward together with a moment Q about a horizontal axis, positive from x toward the upward vertical. In view of the differences in positive directions

$$P = -2Y_b \quad ; \quad Q = -2T_b \quad ; \quad U = 0 \qquad [38a,b,c]$$

Similarly, the analogs of Equations [29b,c] and more remote analogs of [17a,b,e,f] will be

$$v_b = -\left[(1-s)\,y_{v,n-\alpha} + s y_{v,n+\alpha}\right] \qquad [39a]$$

$$\gamma_b = -\left[(1-s')\,\gamma_{v,n'} + s'\,\gamma_{v,n'+1}\right] \qquad [39b]$$

$$P_{n-\alpha} = -(1-s)(2Y_b) \quad , \quad P_{n+\alpha} = -s\,(2Y_b) \qquad [40a]$$

$$Q_{n'} = -(1-s')(2T_b) \; , \; Q_{n'+1} = -s'(2T_b) \qquad [40b]$$

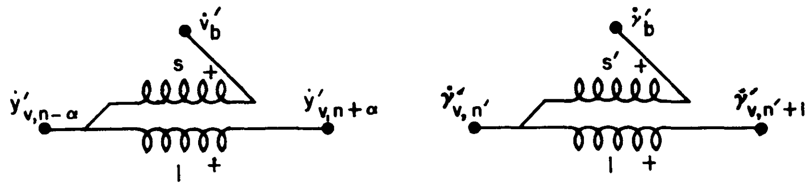

Figure 18 - Mobility Analog for Connection of Diving Plane in v, γ, α Motion with Ship Vibrating Vertically

In calculations, however, the minus signs in Equations [37a] through [40b] may all be dropped (except, of course, in -s or -s'). The result is to reverse the values of v_b and γ_b inserted in the modified Equations [13a,b,c] and, hence, to reverse also the calculated values of v, γ, α; however, then the calculated values of $2Y_b$ and $2T_b$ are also reversed in sign, so that, if $2Y_b$ and $2T_b$ stand for values thus calculated, P and Q are correctly given by Equations [38a,b] with the minus signs erased. If a correct description of the actual motions of the planes is also desired, the calculated values of v, γ, α must be reversed in sign.

The network analog for the modified Equations [13a,b,c] is that shown in Figure 8 except that all capacitances are to be twice as great as they would be in representing the starboard plane alone,* and the $\dot{\alpha}_b'$ node is here to be grounded, the r_1 transformer being omitted. Connection of the $\dot{v}_b'$ and $\dot{\gamma}_b'$ nodes to appropriate points in the ship network, as required by Equations [39a,b] with the minus signs before the brackets omitted, may be made as in Figure 18. The voltages and currents in the plane network will all be of the wrong sign to correspond to the equations as written, but this is harmless. Inclusion of the minus signs would require two extra 1:1 transformers. The whole difficulty arises from the arbitrary choice

*The port plane analog is not discussed because the modified starboard analog does the work of both planes. The motions of the port plane with ship fixed are just like those of the starboard plane, only rotated about x through 180 deg; hence, the port analog is the same as the starboard analog and has the same frequencies.

of opposite positive directions for certain plane variables and for the analogous ship variables; actually, omission of the minus signs leads to a closer resemblance to the geometrical situation.

b. Ship in horizontal bending and torsion vibration with planes in both analogous u, w, β motion and contrary v, γ, α motion.

Let y_{h0} denote the horizontal displacement of 0, the common projection of B and $\overline{B}$ on the midplane. Then y_{h0} will be related to y_h and ϕ by the following equation, analogous to Equation [14a]:

$$y_{h0} = y_h - z_b\phi \qquad [41]$$

where z_b denotes the height, positive or negative, of the points of attachment B and $\overline{B}$ of the planes above the x-axis as drawn for the ship theory. In addition, there are ship rotations γ_h and ϕ.

Let e denote, as for paired rudders, the distance from B or $\overline{B}$ to the midplane. Then the displacements of B and $\overline{B}$ are; see Figure 19:

$$v_b = e\phi \; ; \; \gamma_b = 0 \; ; \; \alpha_b = \phi \; ; \; u_b = e\gamma_h \; ; \; w_b = y_{h0} \; ; \; \beta_b = \gamma_h \qquad [42a\text{-}f]$$

$$\overline{v}_b = -e\phi \; ; \; \overline{\gamma}_b = 0 \; ; \; \overline{\alpha}_b = \phi \; ; \; \overline{u}_b = -e\gamma_h \; ; \; \overline{w}_b = y_{h0} \; ; \; \overline{\beta}_b = \gamma_h \qquad [42g\text{-}\ell]$$

Comparison with Figures 15 and 16 or with the table on page 53 shows that these bearing displacements will excite analogous u, w, β motions of the planes as well as contrary v, γ, α motions.

For the u, w, β motion of the starboard plane, Equations [26a,b,c] or [27a,b,c] may be used with u_b, w_b, and β_b given the values required by Equations [42d,e,f]. (A 90-deg rotation of the axis for a lower rudder brings the axes into a suitable position for the starboard plane.) The analogous u, w, β motion of the port plane produces a force $\overline{Z}_b$ which combines with Z_b to give a resultant horizontal force $2Z_b$ on the ship (Figures 16 and 19, and $\overline{Q}_b$ and Q_b give a total torque $2Q_b$ about the downward vertical. Also, $\overline{X}_b$ and X_b, acting in opposite directions, combine into a further torque $2eX_b$ about the vertical (positive as usual from x toward z). Thus the net reactions on the ship due to both planes are a horizontal

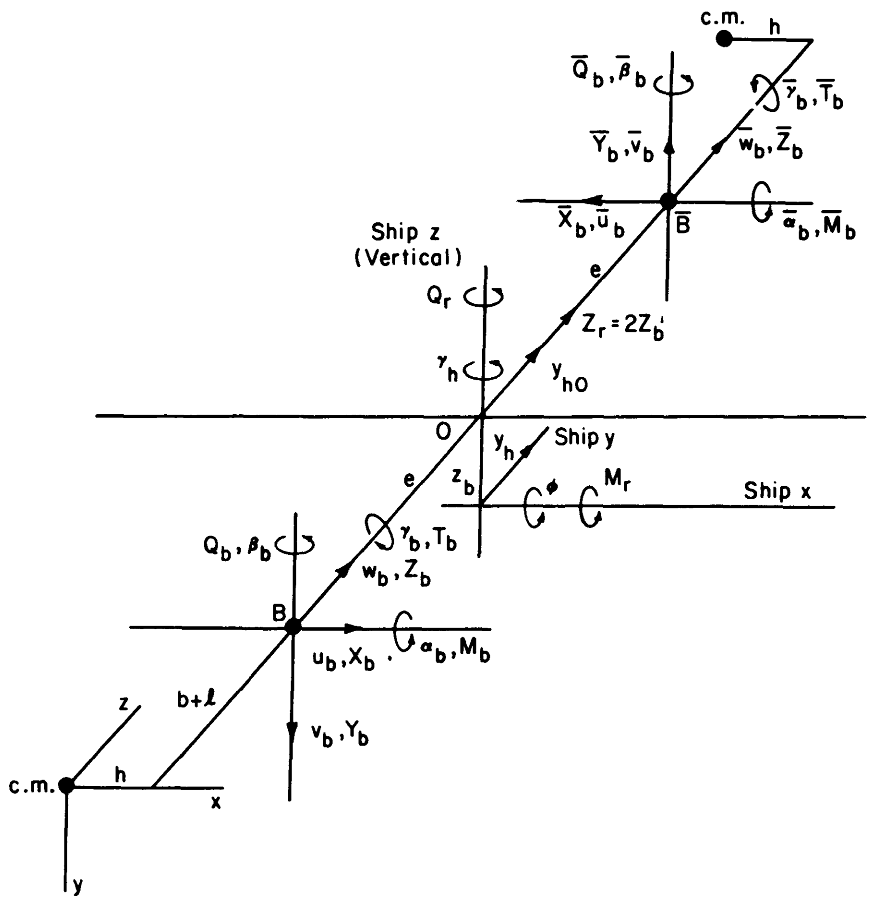

Figure 19 - Relations for Paired Diving Planes in Contrary v, γ, α and Analogous u, w, β Motions with Ship Vibrating Horizontally Torsionally

force Z_r and a torque Q_r of magnitudes

$$Z_r = 2Z_b \quad ; \quad Q_r = 2Q_b + 2eX_b \qquad [43a,b]$$

The factor 2 may again be provided by doubling m_x, m_z, I_y, $k\ell^2$, k_s, and EA/ℓ in Equations [26a,b,c] or [27a,b,c]; then Equations [30a,b,c] give

$$2X_b = -\,2X \quad , \quad 2Z_b = -\,2Z$$

$$2Q_b = -\,2Q - (\ell + b)(2X) + 2hZ = -\,2Q_s - (\ell + b)(2X)$$

where 2X, 2Z, and 2Q are equal, respectively, to the right-hand members of the modified Equations [26a,b,c] or $2Q_s$, to that of the modified Equation [27c].

For the v, γ, α motion of the starboard plane, Equations [13a,b,c] may be used; to include the effect of the port plane, let m_y, I_x, I_z, I_{xz}, $k\ell^2$, k_s, GJ_e/ℓ_T be doubled. The torques T_b and $\overline{T}_b$ cancel (See Figure 15, contrary motion, and Figure 19), but M_b and $\overline{M}_b$ add to a resultant torque $2M_b$ about x. The vertical forces Y_b and $\overline{Y}_b$ acting in opposite directions along lines 2e apart are equivalent to an additional x torque $2eY_b$. The resultant reaction on the ship due to this motion is thus only a torque M_r about x of magnitude

$$M_r = 2M_b + 2e\,Y_b \qquad [44]$$

Here $2M_b$ and $2Y_b$ are given by Equations [15a,c] multiplied through by 2, 2Y, and 2M then representing the respective right-hand members of Equations [13a,c] as modified by the doubling of constants.

The combined reactions on the ship due to both types of plane motion are equivalent to a horizontal force P along a line through B and $\overline{B}$, together with torques U and Q acting in transverse planes through B and $\overline{B}$, of magnitudes $P = Z_r$, $Q = Q_r$, and $U = M_r$. Obviously, such reactions do not tend to excite vertical vibration of the ship. The theory of horizontal-torsional motion was discussed briefly in Section 4.2. In Equation [2.46] of Reference 3, h is to be replaced here by z_b. Values of $P_{n\pm\alpha}$, $U_{n\pm\alpha}$, $Q_{n'}$, and $Q_{n'+1}$ for substitution in difference equations for the ship may be obtained from Equations [17a-f] with Y_b, M_b, and T_b replaced by Z_r, M_r, and Q_r, respectively.

A <u>mobility analog</u> for the present case must include both types of network for the starboard plane (i.e., u, w, β and v, γ, α), connected simultaneously to the ship network.

For the u, w, β motion, the network may be that shown in the upper part of Figure 12 and described in Section 5.3, with all capacitances twice as great and all inductances half as great as they would be for the starboard plane so as to represent the modified Equations [27a,b,c]. Connection to the ship network to represent Equations [42d,e,f] may be as shown

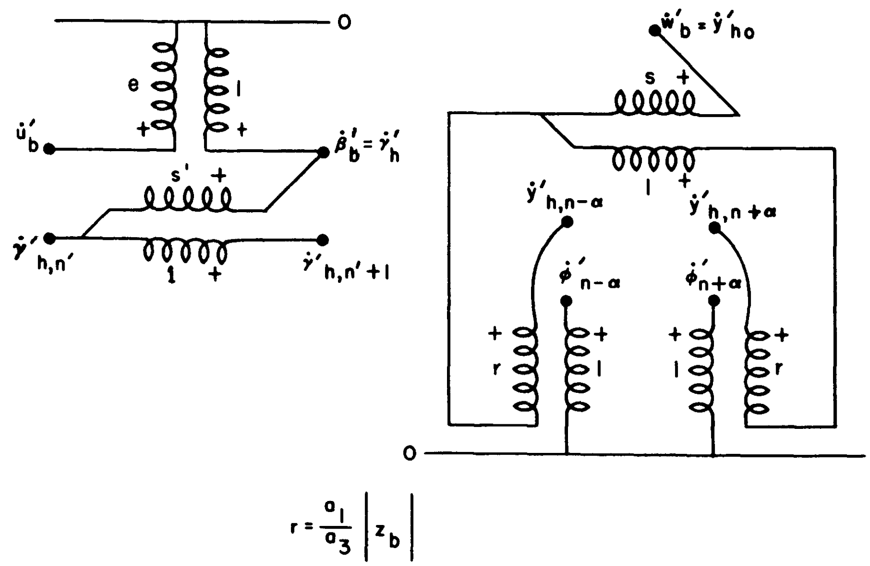

Figure 20 - Mobility Analog for Connection of Diving Plane in u, w, β Motion to Ship in Torsion and Horizontal Bending

The connections shown are for $z_b > 0$.

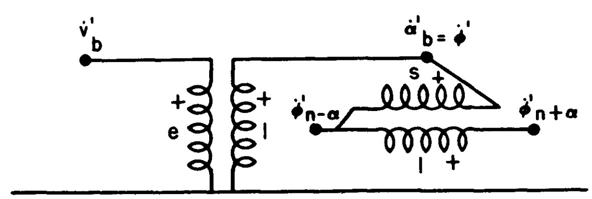

Figure 21 - Mobility Analog for Connection of Diving Plane in v, γ, α Motion to Ship in Torsion and Horizontal Bending

in Figure 20, drawn with reference to Equations [16a,b,c] and Figure 9.

For the v, γ, α motion, the network, as in Case a, may be that shown in Figure 8 and described in Section 4.3, with all capacitances doubled and all inductances halved as they would be to represent the starboard plane alone. Here, however, the $\dot{\gamma}_b'$ node is to be grounded. The connections to represent Equations [42a,c] in analogy with part of Figure 9 may be as shown in Figure 21.

c. Ship stationary, planes in contrary u, w, β motion.

In this case, $\overline{Z}_b$ and $\overline{Q}_b$ balance Z_b and Q_b; $\overline{X}_b$ and X_b add up but the result is only to excite slight axial vibration of the ship, which may be ignored as usual. Thus the ship is not disturbed.

The starboard plane will move according to Equations [26a,b,c] or [27a,b,c]. An analog network to represent it may be the network described in Section 5.3 and shown in Figure 12.

It may be noticed that, because of the geometrical relations of ship and plane, neither type of ship vibration is associated with plane motions of both analogous v, γ, α and contrary u, w, β types. Such a combination of plane motions may be due to other causes, of course, and also to many other combinations. The difference between the analogous and the contrary motions of the port plane that may be associated in different cases with a given starboard motion can be regarded as merely a difference in phase of half a period between the two alternative port motions. Any other phase difference is possible; in that case the given motions may be regarded as due to superposition of analogous and contrary paired motions of certain amplitudes. Any component of analogous v, γ, α motions will then be associated with vertical ship vibration and any component of contrary v, γ, α or analogous u, w, β motions with horizontal-torsional motion of the ship.

7.2 EQUATIONS FOR AN UPPER RUDDER

The most significant difference between an upper and a lower rudder is that the upper rudder is attached to its stock at a point above the bearing or other effective point of attachment to the ship. The effect of this difference on the equations of motion can be discovered by reviewing the derivation of the equations, but it is perhaps more easily found by the following maneuver:

Let axes for the upper rudder be drawn with origin at its effective center of mass but, temporarily, with the z-axis pointing downward and the y-axis, therefore, in the opposite direction from that for a lower rudder; see Figure 22. Then the axes will be related to the rudder and to its stock in the same way as were the rudder axes employed in Sections 4.1 and 5.1, and the equations there derived can be used; namely, Equations [13a,b,c], [15a,b,c], [26a,b,c], or [27a,b,c] and [30a,b,c].

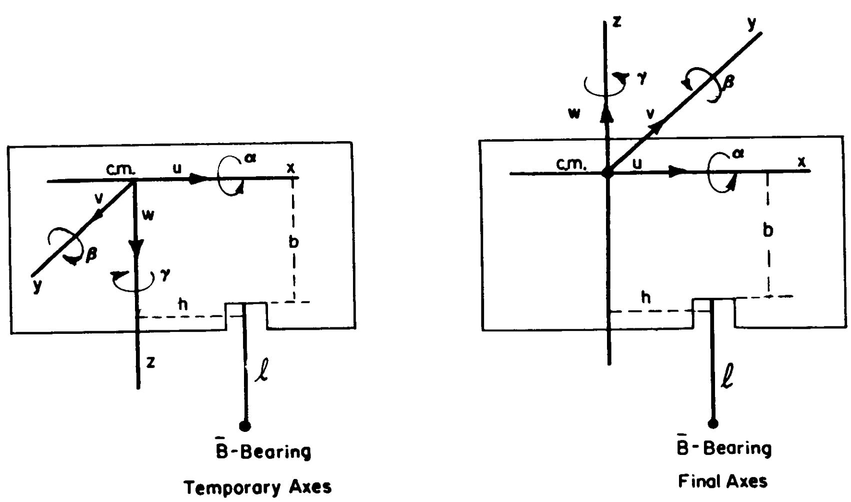

Figure 22 - Temporary and Final Axes for an Upper Rudder

Positive directions of u, v, w, α, β, γ are shown; α, β, γ are positive, respectively, from y to z, x to z, x to y.

It is inconvenient, however, in dealing with the interaction between the rudder and ship, to have their z-axes pointing in opposite directions. Hence, let the axes be rotated through 180 deg about x so that y and z come into their usual positions, with z pointing upward; see Figure 22. The positive directions are thereby reversed for all displacements and reactions associated with y or z so that, if the equations just specified are to be used, <u>all variables in the equations related to y or z (but not x) must be replaced by their negatives</u>.

Accordingly, let the following changes be made in succession in all these equations, the last two changes being added merely for convenience:

a. Reverse the signs of all terms containing v, γ, w, w_1, β, v_b, γ_b, w_b, β_b, Y_b, T_b, and Q_b in the equations specified.

b. Multiply Equations [13a,b], [15a,b], [26b,c], [27b,c], and [30b,c] throughout by -1 in order to restore the convenient initial plus signs on the left.

c. Redefine Y, T, Z, and Q or Q_s to represent the values of the right-hand members of Equations [13a,b] or [26b,c], or [27b,c] after change b. The terms containing Y, T, Z, and Q or Q_s in Equations [15a,b,c] or [30c] must then be reversed in sign (many terms are thus reversed in sign and so return to their original sign; e.g., Y_b term is reversed by change a and again by change b).

A careful check now reveals the following simple rule for adapting all specified equations to the upper rudder, with the z-axis drawn upward as usual: replace I_{xz} by $-I_{xz}$, ℓ by $-\ell$ and b by -b throughout, except ℓ in the coefficient EA/ℓ. (Thus ℓ^2 and $b\ell$ are not changed.)

Note that b is assumed positive when the attachment of the stock to the upper rudder is <u>above</u> the bearing or other attachment to the ship.

7.3 SYMMETRICAL UPPER AND LOWER RUDDERS

Any case of upper and lower rudders in the midplane of the ship can be treated by using the special equations just derived for the upper rudder together with the usual equations for the lower rudder. An analog network may then contain a separate network for each rudder.

If symmetry is present, however, a simpler method can be used, as with paired offset rudders or diving planes. Suppose that the upper and lower rudders are mirror images of each other in a horizontal plane of symmetry. It is assumed that this plane contains the transverse axis for the rotation γ_v of the ship cross sections. This assumption should be sufficiently valid at least for submarines; if the assumption is not made, the theory becomes considerably more complicated.

Motions of the upper and lower rudders can now usefully be paired off in the same way as was done in Section 7.1 for paired diving planes. In fact, the treatment of the diving planes can be taken over bodily if axes are drawn as usual for the lower rudder (y horizontal, z upward, origin at the effective center of mass). The axes so drawn for the lower rudder are then related to the rudder in the same way as the axes defined in Section 7.1 for the starboard plane are related to this plane, and every statement in Section 7.1 will hold if "lower" and "upper rudder" are substituted for "starboard" and "port plane," respectively. Of course, the spatial directions of the axes and the relations to the ship motions will be different in the case of the rudders. Also, let e be replaced by g, so that the distance is 2g between B and $\overline{B}$, the respective points of attachment of the lower and upper rudders. For generality, assume that the ship's x-axis is drawn a distance z_s below the plane of symmetry.

Analogous and contrary motions of the two diving planes now become analogous and contrary paired motions of the lower and upper rudders. The corresponding relations between displacements and bearing reactions on the ship may be read from Figures 15 and 16 or from the table in Section 7.1. To produce correct directions in space for the rudders, imagine that Figures 15 and 16 are rotated through 90 deg so as to make y horizontal and z vertically upward, as they have been drawn for rudders.

a. Vertical ship vibration with analogous u, w, β rudder motions.

From Equations [28a,b,c], in which d = g (the axis for γ_v having been assumed to lie in the plane of symmetry), the displacements of B or $\overline{B}$ are:

$$\text{B:} \quad u_b = g\gamma_v \; ; \quad w_b = y_v \; ; \quad \beta_b = \gamma_v \qquad [45a,b,c]$$

$$\bar{B}: \quad \bar{u}_b = -g\gamma_v \quad ; \quad \bar{w}_b = y_v \quad ; \quad \bar{\beta}_b = \gamma_v \qquad [45d,e,f]$$

In stating values in the equations, the same positive directions are assumed for both rudders (and planes). In Equations [45a,d] $u_b = g\gamma_v$ but $\bar{u}_b = -g\gamma_v$, hence if u_b is positive as shown in Figure 23, $\bar{u}_b$ is negative and so has the opposite direction.

Clearly, only analogous u, w, β rudder motions are excited (w_b and β_b the same, u_b reversed for upper and lower rudders); see Figure 23.

The analysis for the rudder motions is the same as that given in Section 7.1(b) for analogous u, w, β motions of the planes but with e replaced here by g and Equations [42d,e,f] by Equations [45a,b,c]. The bearing reactions on the ship are

$$X_r = 0 \quad ; \quad Z_r = 2Z_b \quad ; \quad Q_r = 2Q_b + 2gX_b \qquad [46a,b,c]$$

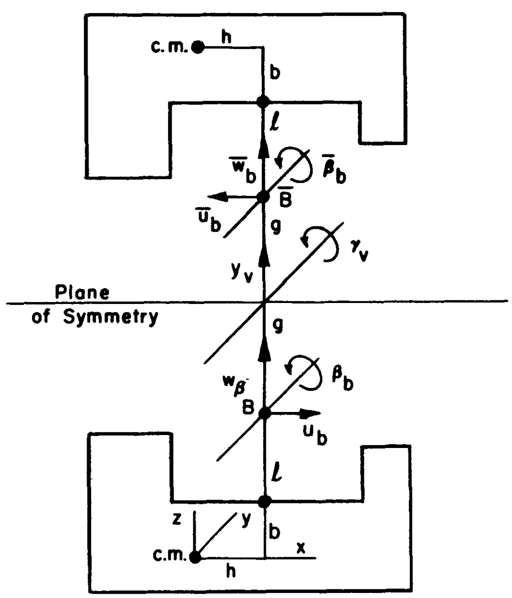

Figure 23 - Upper and Lower Rudders in Analogous u, w, β
Motion with Ship Vibrating Vertically

Ship axis and Z_s are not shown.

with Z_r and Q_r copied from Equation [43a,b]. The values of X_b, Z_b, and Q_b are explained following Equations [43a,b]. Here, however, Z_b and Z_r are vertically upward. According to the explanation given at the end of Section 4.2, Z_r and Q_r may be substituted for Y_b and T_b, respectively, in Equations [17a,b,e,f] to obtain $P_{n\pm\alpha}$, $Q_{n'}$, and $Q_{n'+1}$ for use in difference equations for the ship. (Here $X_b = 0$.)

An analog network for the lower rudder must represent Equations [26a,b,c] or [27a,b,c], but with m_x, m_z, I_y, $k\ell^2$, k_s, and EA/ℓ all doubled. (See Equations [32a,b,c].) Connections to the ship network for vertical motion to represent Equations [45a,b,c] may be as shown in Figure 20 if this figure is modified by changing e to g, $\dot{\gamma}_h'$ to $\dot{\gamma}_v'$, and $\dot{y}_{h0}'$ to $\dot{y}_v'$; also, the connections shown below $\dot{y}_{h0}'$ are to be omitted.

b. Horizontal-torsional ship vibration and v, γ, α rudder motion.

Ship displacements y_h, γ_h, and ϕ produce at B and $\bar{B}$ the following displacements; see Figure 24:

$$v_b = y_h - (z_s - g)\phi \quad ; \quad \bar{v}_b = y_h - (z_s + g)\phi$$

$$\gamma_b = \bar{\gamma}_b = \gamma_h \quad ; \quad \alpha_b = \bar{\alpha}_b = \phi$$

These displacements are neither analogous nor contrary. In this case it might be preferable to use separate sets of equations for the two rudders. As an alternative, the procedure followed previously can be used by resolving the displacements into two superposed sets defined as follows:

$$v_{b1} = \bar{v}_{b1} = y_h - z_s\phi \quad ; \quad \gamma_{b1} = \bar{\gamma}_{b1} = \gamma_h \quad ; \quad \alpha_{b1} = \bar{\alpha}_{b1} = 0$$

$$v_{b2} = g\phi \quad ; \quad \bar{v}_{b2} = -g\phi \quad ; \quad \gamma_{b2} = \bar{\gamma}_{b2} = 0 \quad ; \quad \alpha_{b2} = \bar{\alpha}_{b2} = 0$$

The first of these sets of partial displacements, in which B and $\bar{B}$ have equal v, γ displacements, will generate analogous v, γ, α motions of the rudders; whereas the second set, similar in form to the displacements in Equations [42a,b,c] for planes, will excite contrary v, γ, α motions. These two sets of v, γ, α motions occur superposed.

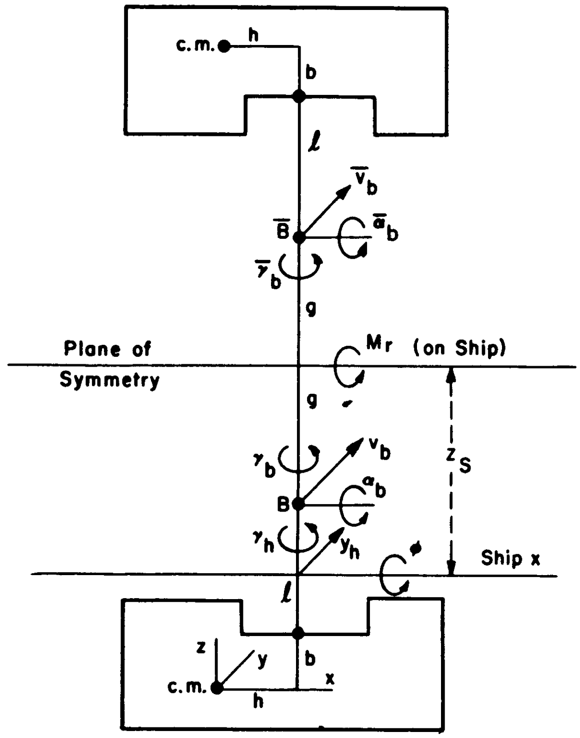

Figure 24 - Upper and Lower Rudders in v, γ, α Motion with Ship Vibrating Horizontally Torsionally

Wherever two rudders (or planes) are drawn, the directions shown are positive for lower rudder (or starboard plane). However, for upper rudder (or port plane) the actual directions are shown that accompany positive quantities for lower rudder (or starboard plane) in the type of associated motion shown (analogous or contrary). This enables the reader to see the directions in space that go together.

For each partial motion, Equations [13a,b,c] may be used for the lower rudder with the constants m_y, I_x, I_z, I_{xz}, $k\ell^2$, k_s, and GJ_e/ℓ_T all doubled (as for the planes) and with the proper values substituted for v_b, γ_b, and α_b .

For the first or analogous v, γ, α motions, the method of Section 7.1(a) is available. In the modified Equations [13a,b,c], v_{b1}, γ_{b1}, and α_{b1} are to be inserted for v_b, γ_b, and α_b . The resultant reactions on the ship are given by Equations [15a,b] (lower row) multiplied through by 2, where 2Y and 2T then represent, respectively, the right-hand members of the modified Equations [13a,b]. Let these reactions be relabeled $2Y_{b1}$ and $2T_{b1}$. $2Y_{b1}$ is a force acting in the y-direction and in the plane of symmetry, while the moment $2T_{b1}$ acts about vertical z.

For the contrary v, γ, α rudder motions, v_{b2}, γ_{b2}, and α_{b2} are to be inserted for v_b, γ_b, and α_b. The argument used in the v, γ, α part of Section 7.1(b) then shows that the resultant reaction on the ship is only an x moment M_r of magnitude (corresponding to Equation [44])

$$M_r = 2M_{b2} + 2g\,Y_{b2}$$

where $2Y_{b2}$ and $2M_{b2}$ are found by substituting them for $2Y_b$ and $2M_{b2}$ in Equations [15a,c] multiplied through by 2; 2Y and 2M then represent the respective right-hand members of the modified Equations [13a,c].

Thus the combined reactions on the ship due to both types of rudder motions are:

$$\begin{aligned} Y_r &= 2Y_{b1} = -\,2Y_1 \\ T_r &= 2T_{b1} = -\,2T_1 + h(2Y_1) \\ M_r &= 2M_{b2} + 2g\,Y_{b2} = -\,2M - (\ell + b)(2Y_2) - 2g\,Y_2 \end{aligned} \qquad [47a,b,c]$$

Equation [47c] is physically reasonable because Y is being shifted from the center of mass up to bearing $(\ell + b)$, then up to the plane of symmetry.

The values of $P_{n\pm\alpha}$, $U_{n\pm\alpha}$, $Q_{n'}$, and $Q_{n'+1}$ to be used in the ship equations to represent the combined reactions on the ship due to both types of v, γ, α rudder motions may then be found by substituting $2Y_{b1}$, $2T_{b1}$, and $2M_{b2} + 2g\,Y_{b2}$ into Equations [17a-f] for Y_b, T_b, and M_b, respectively.

In an analog network, separate rudder networks must be used to represent the two v, γ, α motions. Internally they may be duplicates of each other, like one of the alternatives described in Section 4.3 and shown in Figure 8, except that all capacitances are to be doubled and all inductances halved. Their connections to the ship network must be different, however.

The $\dot{v}'_{b1}$ and $\dot{\gamma}'_{b1}$ nodes of the first rudder network may be connected as shown in Figure 9 except that z_b is to be replaced here by z_s. The $\dot{\alpha}'_{b1}$

node is to be grounded.

The $\dot{v}'_{b2}$ and $\dot{\bar{v}}'_{b2}$ nodes of the second rudder network may be connected as shown in Figure 21 with e changed to g and b to b_2. The $\dot{\gamma}'_{b2}$ node is to be grounded.

c. Contrary u, w, β motions of the two rudders.

Contrary u, w, β motions of the rudders may occur without appreciable disturbance of the ship, corresponding to Section 7.1(c) for diving planes.

Remarks similar to those at the end of Section 7.1 may also be made concerning the motions of symmetrical upper and lower rudders.

For convenience, the relative directions in space of the various sets of axes used up to this point, together with corresponding displacements and principal reactions, are shown in Figure 25. This figure does not show the relative locations of the center of mass, bearings, centroid, ship axes, etc.

7.4 CONTINUOUS SHAFTS

Frequently two control foils are mounted on a continuous shaft extending across the ship. In such cases interaction between the two foils may occur by way of the shaft. If the foils are significantly unlike, it may be necessary to treat them together as a many-dimensional system. When the foils are symmetrical, however, the method of paired motions described in Sections 7.1 and 7.3 becomes available. Simple relations then exist between the distortions of the two halves of the shaft so that, just as it was necessary to write equations of motion for only one of the foils, it suffices to analyze the distortion of only the corresponding half of the shaft. It is found that allowance for the effect of the simultaneous displacement of the other foil can be made by assuming suitable boundary conditions for the half shaft at 0, the midpoint of the shaft in the plane of symmetry.

Torsion of the shaft will be considered first. In contrary v, γ, α motion of the foils, the opposite rotations γ_b and $\bar{\gamma}_b$ (Figure 15) of the ends of the shaft leave undisplaced its middle cross section at 0. Hence, in Equation [13b] ℓ_T runs to the middle of the shaft. In analogous v, γ,

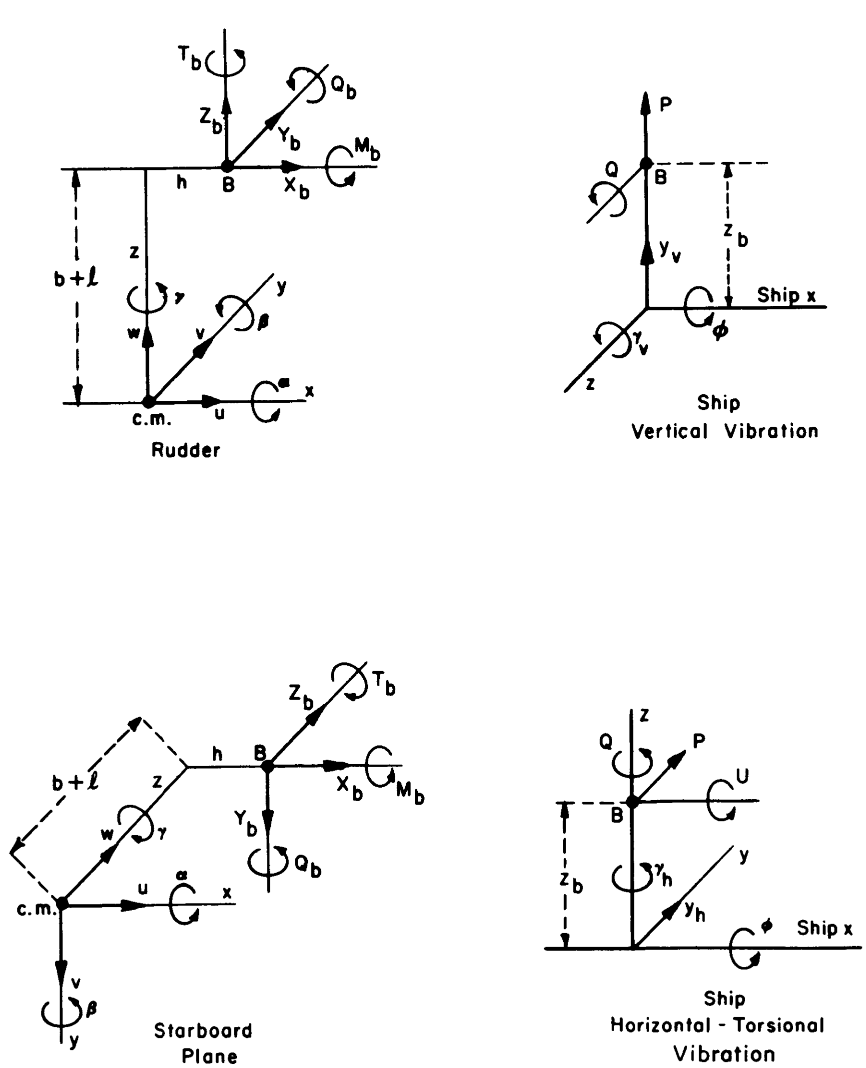

Figure 25 - Sign Conventions for Rudder-Diving Plane-Ship Coordinate Systems, Motions, and Reactions

Check list of positive directions.

Rotations and moments are positive y toward z, x toward z, or x toward y.

u_b, w_b, β_b, v_b, γ_b, α_b have directions of u, w, β, v, γ, α, respectively.

Relative locations of center of mass, bearings, centroid, ship axes, etc., are not shown here.

α motion, however, the reactions to the moments T_b and $\bar{T}_b$ rotate the ends of the shaft in the same direction. The effect of this depends upon the elasticity of the structure by which the shaft is attached to the ship at 0. If this structure may be assumed to be rigid, ℓ_T runs again to 0. Otherwise allowance for the effect of elastic yielding at 0 may be made by a suitable increase in the value assumed for ℓ_T.

Bending of the shaft, which may occur with either type of foil motion, presents a much more complicated problem. In practical cases a simple rough correction may suffice. Since, however, the correction varies with the type of shaft mounting, an exact analysis will be attempted here.

In analogous $\underline{v}$, $\underline{\gamma}$, $\underline{\alpha}$ or contrary $\underline{u}$, $\underline{w}$, $\underline{\beta}$ motion of the foils, the similar displacements v_b and $\bar{v}_b$ or u_b and $\bar{u}_b$ of the bearings, and the opposite rotations α_b and $\bar{\alpha}_b$ or β_b and $\bar{\beta}_b$, all tend to bend the inner section of the shaft like a bow, with zero slope at 0; see Figure 26. On the other

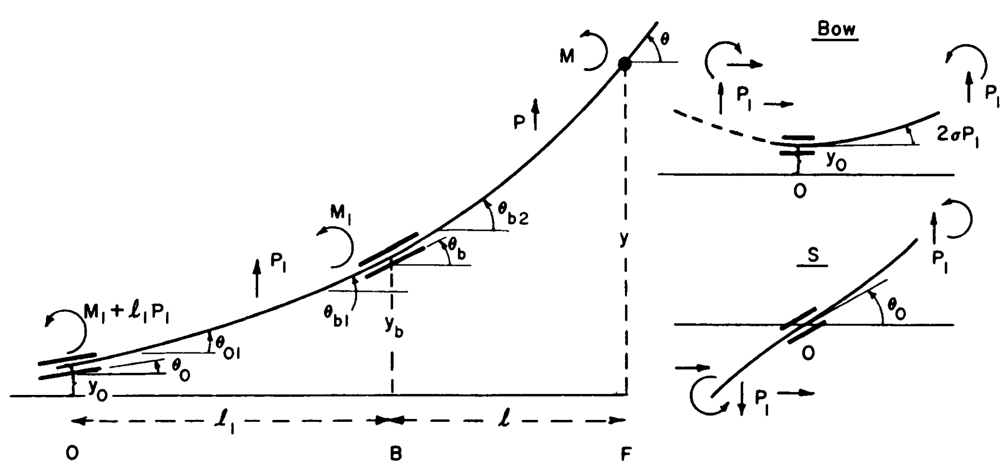

Figure 26 - Positive Forces and Moments Acting on a Continuous Shaft in Flexure

The shear force and bending moment shown are those transmitted toward 0.

hand, in contrary $\underline{v}$, $\underline{\gamma}$, $\underline{\alpha}$ or analogous $\underline{u}$, $\underline{w}$, $\underline{\beta}$ motion, v_b and $\bar{v}_b$ are oppositely directed, as are u_b and $\bar{u}_b$, whereas α_b and $\bar{\alpha}_b$ are in the same direction, as are β_b and $\bar{\beta}_b$. The effect is to bend the shaft in an S-shape with zero displacement at 0; see Figure 26.

In analyzing one-half the shaft (the starboard half for diving planes or the lower half for vertically paired rudders), the effect of the other half is partly equivalent to a geometrical boundary condition at 0. A second condition concerning the shear force or bending moment will be stated presently. Otherwise, the other half of the shaft may be ignored.

The relevant quantities are shown in Figure 26. The length of the section of shaft between 0 and bearing B is denoted by ℓ_1, that between B and foil attachment F by ℓ, as usual. The mounting at 0 may be displaced transversely a distance y_0, the bearing B by y_b, the attachment to the foil by y; corresponding rotations are θ_0, θ_b, θ. The cross section of the shaft at 0 is also displaced y_0; the slope of the shaft just outside the mounting, which may be altered by shear warping, is denoted by θ_{01}. Similarly, the shaft slope is θ_{b1} on the ℓ_1 side of B and θ_{b2} on the ℓ side of B.

The shear force (transmitted toward 0) is denoted by P in the ℓ section, by P_1 in ℓ_1. The associated bending moment is M at F, $M + \ell P$ on the ℓ side of B, M_1 on the ℓ_1 side, and $M_1 + \ell_1 P_1$ at 0.

In bending like a bow the shaft cross section at 0 is unrotated, as if the shaft were rigidly held here, hence the usual shear slope should occur; thus $\theta_{01} = \theta_0 + P_1/KAG$. Also, the net force on the mounting at 0 due to both halves of the shaft is $2P_1$, hence $y_0 = 2P_1/D_0$, with $2D_0$ denoting the (total) rigidity of the mounting at 0 against lateral displacement. (If $2D_0 = 0$, $P_1 = 0$ necessarily.)

When the bending is of the S-type, the mounting may be rotated through an angle $\theta_0 = 2(M_1 + \ell_1 P_1)/2H_0$, the rotational rigidity at 0 being denoted by $2H_0$. In this case the shear force in the shaft is practically constant through the mounting, and it is assumed that the mounting does not interfere with shear warping in the shaft. Hence there is no shear correction, and $\theta_{01} = \theta_0$.

At the bearing B it may happen that $P_1 = P$; then, as at 0 in S-bending, the shear force is uniform throughout the bearing, hence there will be no shear effect and $\theta_{b1} = \theta_b = \theta_{b2}$. Or it may happen that $P_1 = -P$. Then the forces on the ends of the section of shaft inside the bearing are similarly directed, being $-P_1$ or P at one end and P at the other. Hence in this case shear slopes occur of magnitude P/KAG at the right end of the bearing and -P/KAG at the end toward 0.

Any given values of P_1 and P can be resolved into two superposed sets of these two types, thus:

$$\frac{1}{2}(P + P_1) - \frac{1}{2}(P - P_1) = P_1 \;;\quad \frac{1}{2}(P + P_1) + \frac{1}{2}(P - P_1) = P$$

The $P + P_1$ pair, being related as are P_1 and P, will cause no shear effect; whereas the $P - P_1$ set will cause shear slopes as do P_1 and P when $P_1 = -P$. Hence, in general,

$$\theta_{b1} = \theta_b - \sigma(P - P_1) \;;\quad \theta_{b2} = \theta_b + \sigma(P - P_1) \;;\quad \sigma = \frac{1}{2\ KAG} \qquad [48a,b,c]$$

Amplification of this analysis is given in Appendix D.

Also, the net reactions on the bearing itself being $P - P_1$ and $M + \ell P - M_1$,

$$P - P_1 = D_b\, y_b \;;\quad M + \ell P - M_1 = H_b\, \theta_b \qquad [49a,b]$$

where D_b and H_b denote rigidities of the bearing in transverse displacement and in rotation, respectively.

a. $\underline{\ell_1 \text{ section of shaft.}}$

For this section it is convenient to use as a basis Equations [5] and [7] with KAG $\rightarrow \infty$, or

$$\bar{y} = q_{11}\, P_1 + q_{12}\, M_1 \;;\quad \bar{\theta} = q_{12}\, P_1 + q_{22}\, M_1 \qquad [50a,b]$$

$$q_{11} = \frac{\ell_1^3}{3\ EI} \;;\quad q_{12} = \frac{\ell_1^2}{2\ EI} \;;\quad q_{22} = \frac{\ell_1}{EI} \qquad [51a,b,c]$$

Here $\bar{y}$, $\bar{\theta}$ represent displacement and shaft slope at the B end, those at the 0 end being assumed to be zero.

(1) Bow-like bending.

Here, as explained previously,

$$\theta_0 = 0 \quad ; \quad \theta_{01} = 2\sigma P_1 \quad ; \quad P_1 = D_0 y_0 \qquad [52a,b,c]$$

To fit this case, the shaft, deformed according to Equations [50a,b], must be given an additional translation y_0 and a rotation θ_{01} about 0. Then

$$y_b = \bar{y} + y_0 + \ell_1 \theta_{01} \quad ; \quad \theta_{b1} = \bar{\theta} + \theta_{01}$$

It seems better, however, to introduce θ_{b2} here instead of θ_{b1}. From Equations [48a,b] and [49a]:

$$\theta_b = \theta_{b2} - \sigma D_b y_b \quad ; \quad \theta_{b1} = \theta_{b2} - 2\sigma D_b y_b \qquad [53a,b]$$

Then, from Equations [50a,b] and [52b,c] also:

$$y_b = y_0 + 2\sigma \ell_1 P_1 + q_{11} P_1 + q_{12} M_1$$

$$\theta_{b2} = 2\sigma P_1 + 2\sigma D_b y_b + q_{12} P_1 + q_{22} M_1$$

Now multiply this y_b equation through by D_0, substitute $D_0 y_0 = P_1$ from Equation [52c], and then also, in both equations, from [49a,b] and [53a]:

$$P_1 = P - D_b y_b \quad ; \quad M_1 = \ell P + M - H_b (\theta_{b2} - \sigma D_b y_b)$$

The resulting equations, rearranged, are:

$$\left[D_0 + D_b + (q_{11} + 2\sigma \ell_1) D_0 D_b - \sigma q_{12} D_0 D_b H_b \right] y_b + q_{12} D_0 H_b \theta_{b2}$$

$$= \left[1 + (q_{11} + \ell q_{12} + 2\sigma \ell_1) D_0 \right] P + q_{12} D_0 M \qquad [54a]$$

$$(q_{12} - \sigma q_{22} H_b) D_b y_b + (1 + q_{22} H_b) \theta_{b2} = (q_{12} + \ell q_{22} + 2\sigma) P + q_{22} M \qquad [54b]$$

These two equations may now be solved for y_b and θ_{b2} in terms of P and M. If, as is likely, D_b is large, it may be preferable to rearrange the first term of [54a] and solve for $D_b\, y_b$ and θ_{b2}. Then y_b itself is small but $D_b\, y_b$ need not be.

(2) S-shape bending.

Here

$$y_0 = 0 \quad ; \quad \theta_{01} = \theta_0 \quad ; \quad M_1 + \ell_1 P_1 = H_0 \theta_0 \qquad [55a,b,c]$$

In using Equations [50a,b], a rotation θ_0 about 0 must be added, so that

$$y_b = \bar{y} + \ell_1 \theta_0 \quad ; \quad \theta_{b1} = \bar{\theta} + \theta_0$$

Or, using Equation [53b] also:

$$y_b = \ell_1 \theta_0 + q_{11} P_1 + q_{12} M_1$$

$$\theta_{b2} = \theta_0 + 2\sigma D_b y_b + q_{12} P_1 + q_{22} M_1$$

Multiply both equations by H_0, substitute for $H_0 \theta_0$ from Equation [55c], and then for P_1 and M_1 from [49a,b], and use [53a] for θ_b. The result, rearranged, is:

$$\left[H_0 + (\ell_1^2 + q_{11} H_0 - \sigma \ell_1 H_b - \sigma q_{12} H_0 H_b) D_b \right] y_b + (\ell_1 + q_{12} H_0) H_b \theta_{b2}$$
$$= \left[\ell_1 (\ell_1 + \ell) + (q_{11} + \ell q_{12}) H_0 \right] P + (\ell_1 + q_{12} H_0) M \qquad [56a]$$

$$(\ell_1 + q_{12} H_0 - 2\sigma H_0 - \sigma H_b - \sigma q_{22} H_0 H_b) D_b y_b + (H_0 + H_b + q_{22} H_0 H_b)\, \theta_{b2}$$
$$= \left[\ell_1 + \ell + (q_{12} + \ell q_{22}) H_0 \right] P + (1 + q_{22} H_0) M \qquad [56b]$$

These two equations may now be solved for y_b and θ_{b2} in terms of P and M. As in the bow case, if D_b is large, it may be best to solve for $D_b\, y_b$ and θ_{b2} rather than for y_b and θ_{b2}.

b. ℓ section.

For the ℓ section, Equations [7a,b] and [12a] may be used with KAG $\rightarrow \infty$:

$$P = 12\,k\bar{y} - 6\,\ell k\bar{\theta} \quad ; \quad M = -6\,\ell k\bar{y} + 4\,\ell^2 k\bar{\theta} \quad ; \quad k = \frac{EI}{\ell^3} \qquad [57a,b,c]$$

Here $\bar{y}$ and $\bar{\theta}$ denote displacement and slope of attachment to the foil, there being no displacement at the other end.

If θ is the rotation of the foil itself, $\theta = \bar{\theta} - 2\,\sigma P$ (Figure 2). Also, to fit conditions at B, a translation y_b and a rotation θ_{b2} about B must be added. Hence, if y is the displacement of the foil,

$$y = \bar{y} + y_b + \ell\,\theta_{b2}. \quad ; \quad \theta = \bar{\theta} - 2\,\sigma P + \theta_{b2}$$

c. Final equations.

Either Equations [54a,b] or [56a,b] may be solved for y_b and θ_{b2}. The result may be written in the form

$$y_b = a_{11}\,P + a_{12}\,M \quad ; \quad \theta_{b2} = a_{21}\,P + a_{22}\,M$$

The a's represent net flexibilities at the bearing B and are different for bending of the bow or S types.

Insertion of these values of y_b and θ_{b2} into the expressions for y and θ and then substitution from the resulting expressions for $\bar{y}$ and $\bar{\theta}$ in Equations [57a,b], gives the equations:

$$\left[1 + 6\,k\,(2a_{11} + \ell a_{21} + 2\,\sigma\ell)\right] P + 6k\,(2a_{12} + \ell a_{22})M = 12\,ky - 6\,\ell k\theta$$

$$-2\,\ell k\,(3a_{11} + \ell a_{21} + 4\,\sigma\ell)P + \left[1 - 2\,\ell k\,(3a_{12} + \ell a_{22})\right] M = -6\,\ell ky + 4\,\ell^2 k\theta$$

The solutions of these equations for P and M may be written in the form

$$P = 12\, r_{11}\, k y - 6\, r_{12}\, \ell k \theta$$

$$M = -6\, r_{21}\, \ell k y + 4\, r_{22}\, \ell^2 k \theta \qquad [58a,b]$$

Here the r's represent correction factors (compare with Equations [7a,b,c] for yielding of the bearing and for shear slope there), and are different for the bow and for the S types of shaft bending.

d. Use in equations of motion.

Equations [58a,b] may be used instead of Equations [7a,b] in deriving equations of motion for the chosen foil. A possible inequality of r_{12} and r_{21} may result in some complexities. However, such an inequality can only be caused by the σ terms (involving KAG). A similar inequality is easily shown to exist for a cantilever forced by P and M at the free end; apparently it is prevented from violating the conservation of energy through additional work done by M when the shear warping varies (see Appendix E).

The difference between r_{12} and r_{21} must be relatively small, however, and it may be sufficiently accurate to ignore it in practice by replacing both r_{12} and r_{21} by $\overline{r} = (r_{12} + r_{21})/2$. If this is done, Equations [58a,b] may be rewritten in a form to resemble Equations [7a,b] more closely, as follows:

$$P = 12\, k_s' y - 6\, \ell' k_s' \theta \quad ; \quad M = -6\, \ell' k_s' y + \left[(\ell')^2 (3k_s' + k')\right]\theta \qquad [59a,b]$$

where

$$k_s' = r_{11} k \quad ; \quad \ell' = \frac{\overline{r}}{r_{11}}\, \ell \quad ; \quad k' = k_s' \left(4\, \frac{r_{11} r_{22}}{\overline{r}^2} - 3\right) \qquad [60a,b,c]$$

In the equations of motion, however, the factor ℓ enters not only from Equations [7a,b] but also because a rotation α_b or β_b produces a displacement $\ell\alpha_b$ or $\ell\beta_b$. A careful check shows that use of Equations [59a,b] instead of [7a,b] changes the results given in Sections 4 and 5 only in the following ways: k_s is replaced by k_s' and k by k'; also, ℓ becomes ℓ' except that ℓ is to be kept in EA/ℓ and in the $-\ell\alpha_b$ term in

Equations [9a,c] and the $-\ell\beta_b$ term in Equations [24a,b,c] and [26a,b,c], whereas ℓ is to be replaced by $(2\ell - \ell')$ in the formulas for r_1 and r_5; also where it immediately precedes α_b in Equations [13a,b,c] and [18a,b,c], or α_b' in [19a,b,c] and [22a,b,c], or β_b in [27a,c] or β_b' in [32a,c], and in the factor $3k_s\ell$ in [13c], which becomes $3k_s(2\ell - \ell')$.

8. RUDDER DAMPING AND LIFT AND CORRESPONDING MOBILITY ANALOGS

8.1 RUDDER DAMPING

Two possible features of the actual situation have not yet been considered; namely, damping of the rudder motion and the effect of forward motion of the ship. Damping will be considered first.

Damping forces proportional to velocities can be easily introduced into the equations of motion. For example, in the right-hand members of Equations [13a,b,c], which are the equations of motion for the transverse or v, γ, α motion of the rudder, the following respective expressions may be added:

$$- c_1\dot{v} - c_{12}\dot{\gamma} - c_{13}\dot{\alpha}$$

$$- c_{21}\dot{v} - c_2\dot{\gamma} - c_{23}\dot{\alpha}$$

$$- c_{31}\dot{v} - c_{32}\dot{\gamma} - c_3\dot{\alpha}$$

It can be shown that, since "damping" forces by definition must have the effect of dissipating energy during any motion whatever, necessarily $c_1 \geqq 0$, $c_2 \geqq 0$, $c_3 \geqq 0$, and $c_{21} = c_{12}$, $c_{23} = c_{32}$, $c_{31} = c_{13}$.* (See Appendix A.)

Since the c's thus have the same characteristics as to sign and symmetry as the coefficients representing elastic reactions, the additional

*At least the c's have these characteristics in the equations as obtained by direct substitution in Lagrange's equations of motion. See Reference 6, Section 81 of Vol. I.

network to be connected to the $\ddot{v}'$, $\dot{\gamma}'$, $\dot{\alpha}'$ nodes in order to represent the damping terms has the same general form as that already used to represent the right-hand members of Equations [13a,b,c] or [18a,b,c], except that resistances are now used instead of inductances and all values of parameters are expressed in terms of the c's. The same grouping process may be used here as for Equations [18a,b,c], for example:

$$- c_1\left(\dot{v} + \frac{c_{12}}{c_1}\dot{\gamma} + \frac{c_{13}}{c_1}\dot{\alpha}\right);$$

$$- c_{12}\left(\ddot{v} + \frac{c_{12}}{c_1}\dot{\gamma} + \frac{c_{13}}{c_1}\dot{\alpha}\right) - \left(c_2 - \frac{c_{12}^2}{c_1}\right)\dot{\gamma} - \left(c_{23} - \frac{c_{12}c_{13}}{c_1}\right)\dot{\alpha} \quad;$$

$$- c_{13}\left(\dot{v} + \frac{c_{13}}{c_1}\dot{\gamma} + \frac{c_{23}}{c_1}\dot{\alpha}\right) - \left(c_{23} - \frac{c_{13}^2}{c_1}\right)\dot{\gamma} - \left(c_3 - \frac{c_{13}c_{23}}{c_1}\right)\dot{\alpha}$$

A simple example will be shown in detail presently.

A second set of damping terms in which $\dot{v}$, $\dot{\gamma}$, $\dot{\alpha}$ are replaced, respectively, by $\dot{v} - \dot{v}_b$, $\dot{\gamma} - \dot{\gamma}_b$, $\dot{\alpha} - \dot{\alpha}_b$ may be added or substituted for those just shown.

8.2 EFFECT OF SHIP'S FORWARD MOTION

The ship so far has been assumed to be dead in the water, its only motion being vibratory. If it is moving toward positive x at a steady speed S relative to the water, new characteristics are encountered. When the rudder is either displaced or moved, the flow of water past it gives rise to lift forces and moments acting on the rudder, analogous to the lift on the wings of an airplane.* The complete theory of these forces is complicated. To illustrate the general nature of their effect, the same simple assumption will be made here as was made by McGoldrick and Jewell

*Free surface effects are not considered, since the rudder is assumed to be deeply submerged.

in TMB Report 1222.[7*]

The dynamical effect of the flow is assumed to be a simple horizontal, transverse lift force F_L acting on the rudder along a line at a distance L ahead of the axis of the rudder stock and also at a distance b_L above the x-axis (in the z-direction). L is usually relatively small, if not zero, and may be negative. The line of action of F_L meets the median plane of the rudder in a point C_L called the center of pressure.

First, let the rudder be stationary relative to the ship and in such an angular position that the lift force vanishes; in this position a single rudder in the median plane of the ship points dead astern. If the rudder is turned from this position about the stock through an angle γ and, hence, by Equation [8b] about the center of mass by the same angle γ, the magnitude of the lift force on it can be assumed, with sufficient accuracy, to be $BS^2\gamma$ in terms of a positive constant B.

If, now, because of <u>vibratory motion</u>, the rudder has a transverse translational velocity $\dot{v}$, its velocity relative to the general mass of water is inclined at an angle $\dot{v}/S$ to the axis of the ship (Figure 27), hence the flow relative to the rudder is inclined at this same angle and the "angle of attack"[**] that determines the lift force is changed from γ to $\gamma - (\dot{v}/S)$. The rudder may also have rotational velocities, but if rotation occurs about an axis through the center of pressure C_L, its effect on the lift force may reasonably be assumed to be, if not zero, at least negligible. If, however, small angular velocities $\dot{\gamma}$ and $\dot{\alpha}$ exist about the z- and x-axes, respectively, these velocities are equivalent to equal angular velocities about parallel axes through C_L plus a translational velocity

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*As indicated in Reference 7, the validity of the equations for the lift forces and moments is still to be determined. Nevertheless, it is likely that the <u>method of solution</u> of ship problems involving some revised set of equations would be similar to the method presented in this report. Hence, it is the analytical and the corresponding analog or digital treatment rather than the correctness of the analytical description of the hydrodynamic forces and moments that is stressed in this report (see, however, references given in footnote on page 82).

**The angle of attack is the angle between the direction of motion and the chord line or centerplane of the rudder.

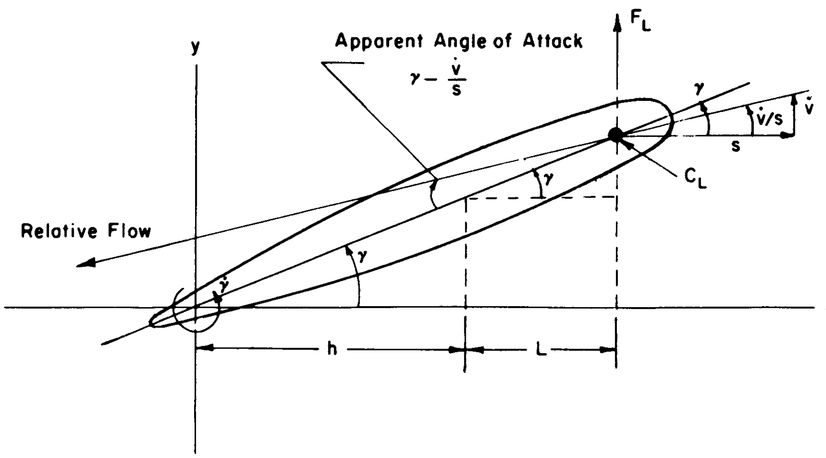

Figure 27 - Lift Force and Angle of Attack for Rudder with Forward Motion S and Transverse Translational Velocity $\dot{v}$

equal to the velocity of C_L, or $(h + L)\dot\gamma - b_L\dot\alpha$. These terms must be added to $\dot{v}$ to represent the angle of attack. Thus the complete formula for F_L during rudder vibrations is*

$$F_L = BS^2\left[\gamma - \left(\frac{\dot v + (h+L)\dot\gamma - b_L\dot\alpha}{S}\right)\right] = BS^2\gamma - BS\left[\dot v + (h+L)\dot\gamma - b_L\dot\alpha\right] \quad [61]$$

---

*The equations corresponding to the extended simplified analysis given in Appendix C of Reference 7 are a special case of Equations [61] through [62c]. To compare the equations, the origin of the coordinate system of Reference 7 is translated to the center of mass of the rudder by letting $Y = Y_{c.g.} + h'\theta$. Then $\dot Y = \dot Y_{c.g.} + h'\dot\theta$ and $\ddot Y = \ddot Y_{c.g.} + h'\ddot\theta$, the symbols Y, $Y_{c.g.}$, h', and θ being defined in Reference 7. Since the definition for h in Reference 7 is different from the definition here, the symbol h in Reference 7 is replaced by h'.

The equation for the lift force F_L corresponding to the modified Theodorsen analysis given in Appendix C of Reference 7 has the same form as Equation [61] for F_L. This is shown as follows. In Reference 7 let

$$\dot Y = \dot Y_{c.g.} + h'\dot\theta$$

$$C_K = \frac{1}{2}$$

(Footnote continued on page 82.)

The u, w, β motion of the rudder is not affected by such a lift force, at least to the first order in γ, $\dot{\gamma}$ and the other rudder displacements or velocities. Hence, only the transverse or v, γ, α motion will be considered here.

The effect of the lift force on the rudder motions may be represented by adding F_L on the right in the first equation of motion, Equation [13a], and also a term $(h + L)F_L$ in the right-hand member of the second equation, [13b], which is intended to represent the total moment of force about the z-axis, and a corresponding term $-b_L F_L$ in the right-hand member of the third equation, [13c], which represents the total moment about x. An example of such additions follows.

Then, in Reference 7

$$F_L = \frac{A}{2}(S^2\theta) - \frac{A}{2}(S\dot{Y}_{c.g.}) - \left(\frac{Ah'}{2} - Ab\right)(S\dot{\theta})$$

Therefore, certain terms in Equation [61] are replaced as follows (and the values of the components in the corresponding analog are easily changed to agree with the revised value of the coefficients):

$$\gamma \rightarrow \theta$$

$$\dot{v} \rightarrow \dot{Y}_{c.g.}$$

$$B \rightarrow \frac{A}{2}$$

$$B(h + L) \rightarrow A\left(\frac{h'}{2} + b\right)$$

$$b_L \rightarrow 0$$

Depending upon the form of the expression for M_θ in Reference 7, the analog for M_θ is easily obtained from the new expression and analog for F_L.

Recent results[8,9] indicate that the Modified Theodorsen Analysis appears to be the most suitable analysis for yielding good predictions of:

a. The damping ratio and frequency for a given speed.
b. The critical flutter speed.

Finally, it is of interest to note that if Theodorsen's lag function C_K is taken to be a complex quantity, it can be represented by linear transfer functions to within an accuracy of 2 percent for the complete frequency spectrum (0 to ∞). For a more restricted range of frequencies, greater accuracy is achievable; see Figure 24 of Reference 10.

8.3 AN EXAMPLE OF DAMPING AND LIFT

The general treatment of linear damping and lift is illustrated by writing out equations of motion and designing the analog for a case in which there is a lift force F_L of the kind just described and also, as in Reference 7, a damping term $-C\dot{v}$ such as might arise from water resistance to transverse motion and a term $-c(\dot{\gamma} - \dot{\gamma}_b)$ representing a damping couple such as might arise from viscous friction in the rudder bearings. Here $C > 0$, $c > 0$. It is assumed that the damping force $-C\dot{v}$ acts through the center of mass of the combined rudder-water system, so that it has no turning effect about the z- and x-axes.

When these additional forces are included, the equations of motion for the rudder become

$$m_y\ddot{v} = \ldots\ldots - C\dot{v} + BS^2\gamma - BS\left[\dot{v} + (h + L)\dot{\gamma} - b_L\dot{\alpha}\right] \qquad [62a]$$

$$I_z\ddot{\gamma} - I_{xz}\ddot{\alpha} = \ldots\ldots - c(\dot{\gamma} - \dot{\gamma}_b) + (h + L)BS^2\gamma - (h + L)BS \left[\dot{v} + (h + L)\dot{\gamma} - b_L\dot{\alpha}\right] \qquad [62b]$$

$$I_x\ddot{\alpha} - I_{xz}\ddot{\gamma} = \ldots\ldots - b_L BS^2\gamma + b_L BS\left[\dot{v} + (h + L)\dot{\gamma} - b_L\dot{\alpha}\right] \qquad [62c]$$

where ..... stands for the right-hand member of Equations [13a,b,c] or [18a,b,c], respectively.*

An additional point must be noted, however, if these equations are used in calculating the combined vibratory motion of rudder and ship. The reactions to the $C\dot{v}$ damping force and to the lift act on the water, not on the ship. Hence, in using expressions [15a,b,c] for the reactions Y_b, T_b, M_b on the ship, only the term $-c(\dot{\gamma} - \dot{\gamma}_b)$ is to be added to the right-hand

*It is important to note that in solving the combined rudder (or diving plane)-hull equations, normally by means of an analog or digital computer, a term representing hull damping must be included in the equations for the ship. Such a term, based on experimental data, is included in Reference 5.

member of Equation [13b] to obtain T, and Y and M are still just the right-hand members of [13a] or [13c] with no additions.

The terms in C, c, or BS all have a damping effect.* This is easily seen from the origin of these terms. It also can be shown, as a check on the equations, by writing an expression for the time rate of change of $T_K + U_P$, the sum of the kinetic and potential energies of the rudder. Expressions for $T_K + U_P$ are given in Equations [B.1] and [B.4] of Appendix B. Multiplication of Equations [49a,b,c] by $\dot{v}$, $\dot{\gamma}$, and $\dot{\alpha}$, respectively, gives, when $v_b = \gamma_b = \alpha_b = 0$,

$$\frac{d}{dt}(T_K + U_P) = -C\dot{v}^2 - c\dot{\gamma}^2 - BS\left[\dot{v} + (h+L)\dot{\gamma} - b_L\dot{\alpha}\right]^2$$

$$+ BS^2\left[\dot{v}\gamma + (h+L)\dot{\gamma}\gamma - b_L\dot{\alpha}\gamma\right]$$

Thus the C, c, and BS terms in Equations [62a,b,c] act continually to decrease the energy and, as a result, have a positive damping effect.

The $BS^2\gamma$ term in Equation [62b] is equivalent in part to a simple change in the torsional stiffness of the stock. To see this clearly, subtract h times Equation [62a] from [62b], obtaining, if $v_b = \gamma_b = \alpha_b = 0$,

$$I_z\ddot{\gamma} - I_{xz}\ddot{\alpha} - hm_y\ddot{v} = -\left(\frac{GJ_e}{\ell_T} - BS^2\right)\gamma + hC\dot{v} - c\dot{\gamma} - LBS\left[\dot{v} + (h+L)\dot{\gamma} - b_L\dot{\alpha}\right]$$

The same equation would be obtained if all BS^2 terms were omitted and the torsional constants were changed to be numerically equal to $\left(\frac{GJ_e}{\ell_T} - BS^2\right)$. If S is so large that the latter expression is zero, a static solution of Equations [62a,b,c] is possible in which γ has an arbitrary constant value accompanied by constant values of v and α, as determined by Equations [62a,c]. In design such a lack of static stability is made impossible at speeds within or near the operating range.

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*The BS part of the lift force F_L acts at the center of pressure and is always oppositely directed to the velocity there, hence it necessarily dissipates energy.

The other effects of the BS^2 terms are more subtle. Study of several simplified problems suggests that, even if the C, c, and BS terms are omitted from the equations, the BS^2 terms may merely alter the frequencies of vibration, may result in solutions exponential in time, or may introduce positive or negative damping. A damping effect due to the BS^2 term in Equation [62a] is facilitated if other damping terms are present, because then v and γ are likely to differ in phase so that in a vibration γ contains a part proportional to $\dot{v}$.

When damping terms are present, such as the C, c, and BS terms in Equations [62a,b,c], then if S is small enough, terms containing S^2 may be neglected, and positive damping will exist. As S increases, the damping may decrease until it vanishes at a speed S_F, so that at this speed a steady harmonic vibration may occur. As S increases above S_F, the damping becomes negative and the amplitude of vibration increases without limit (according to the linearized equations).* Such a self-excited vibration is called flutter, and the speed S_F is called the (critical) flutter speed. At this speed the energy lost through damping forces is just offset by energy supplied out of the water by means of the lift force; see Reference 7.

The numerical solution of Equations [62a,b,c] and determination of S_F (if such a speed exists) presents a complicated problem, conveniently soluble only with the help of a digital automatic computer. Alternatively, an analog network may be used. In general, solutions are sought for S_F, ω, and the degree of damping as a function of speed; see Reference 7, page 39.

8.4 NETWORK REPRESENTATION OF DAMPING AND LIFT TERMS

The substitution of electrical quantities and the division of the equations by a_1, a_2, a_3, respectively, which resulted previously in the conversion of Equations [13a,b,c] or [18a,b,c] into Equations [22a,b,c], give

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*In practice the increase would lead to structural damage unless it were brought to a halt by nonlinear effects that are not represented in the linear equations.

as the electrical equivalent of Equations [36a,b,c]:

$$\lambda m_y \frac{d}{dt'} \dot{v}' + (\ldots\ldots) + p\lambda C\dot{v}' - p^2\rho_1 \lambda BS^2\gamma' + p\lambda BS\left[\dot{v}' + \rho_1(h + L)\dot{\gamma}' - \rho_2 b_L\dot{\alpha}'\right] = 0 \quad [63a]$$

$$\rho_1^2\lambda I_z \frac{d}{dt'} \dot{\gamma}' - \rho_1\rho_2\lambda I_{xz} \frac{d}{dt'} \dot{\alpha}' + (\ldots\ldots) + p\rho_1^2\lambda c\,(\dot{\gamma}' - \dot{\gamma}_b') - p^2\rho_1^2\lambda\,(h + L)BS^2\gamma' + p\rho_1\lambda\,(h + L)\ BS\left[\dot{v}' + \rho_1(h + L)\dot{\gamma}' - \rho_2 b_L\dot{\alpha}'\right] = 0 \quad [63b]$$

$$\rho_2^2\lambda I_x \frac{d}{dt'} \dot{\alpha}' - \rho_1\rho_2\lambda I_{xz} \frac{d}{dt'} \dot{\gamma}' + (\ldots\ldots) + p^2\rho_1\rho_2\lambda b_L BS^2\gamma' - p\rho_2\lambda b_L BS\left[\dot{v}' + \rho_1(h + L)\dot{\gamma}' - \rho_2 b_L\dot{\alpha}'\right] = 0 \quad [63c]$$

where $(\ldots\ldots)$ now stands for the collection of k_s or k and GJ_e terms in the corresponding term of Equations [22a,b,c].

The added damping and lift terms require only appropriate additions to the network already designed and shown in Figure 8. Thus if connections of the rudder network to the ship network are made, they need not be altered by inclusion of damping and lift networks.

The term $p\lambda C\dot{v}'$ in Equation [63a] requires a resistor of resistance $(p\lambda C)^{-1}$ connected between the $\dot{v}'$ node and ground; the term $p\rho_1^2\lambda c(\dot{\gamma}' - \dot{\gamma}_b')$ in Equation [63b] requires, similarly, a resistance $(p\rho_1^2\lambda c)^{-1}$ between $\dot{\gamma}'$ and $\dot{\gamma}_b'$ (or $\dot{\gamma}'$ and ground if $\dot{\gamma}_b' = 0$). The terms in BS require a resistance of magnitude $(p\lambda BS)^{-1}$ between ground and a point at which, by means of two transformers and suitable connections, a voltage is maintained equal to that specified by the quantity in brackets; all three of the BS terms are then seen to be represented.

The BS^2 terms, however, present a new problem. Because these terms are not symmetrical in Equations [63a,b,c], they cannot be represented entirely by passive elements, but require the use of a nonpassive device such as an amplifier.

An ideal linear amplifier is a device that accepts as input any voltage above ground unaccompanied by current, and delivers as output a voltage above ground equal to the input voltage reversed in sign and multiplied by a constant amplification factor, together with whatever current the load requires. (The amplification factor is a positive number.) The output current is supplied from a separate ground connection, so that an amplifier is really a 3-terminal element. A fairly satisfactory approximation to the ideal amplifier can be obtained by use of a 3-electrode electronic tube.

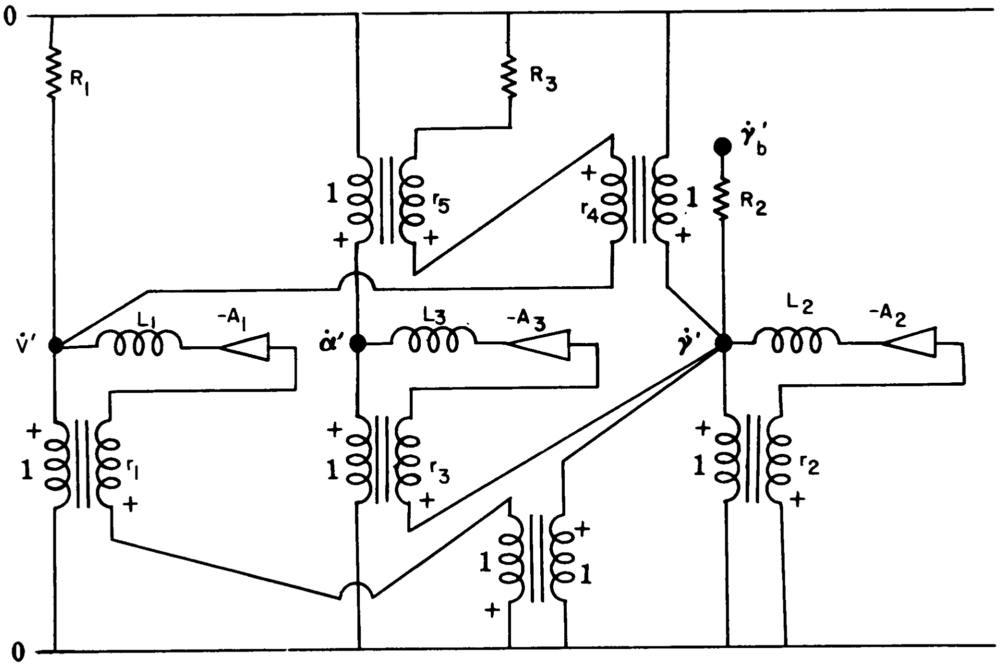
The term $-p^2\rho_1 \lambda BS^2\gamma'$ in Equation [63a] requires that a current <u>enter</u> the $\dot{v}'$ node proportional to the voltage impulse γ' at the $\dot{\gamma}'$ node, but without disturbing the current balance at the $\dot{\gamma}'$ node itself. Reversing the $\dot{\gamma}'$ voltage with a 1:1 transformer and then delivering it to the input of an amplifier produces an output voltage impulse that is positively proportional to γ', and does not draw any current from the $\dot{\gamma}'$ node. If the output terminal of the amplifier is then connected to the $\dot{v}'$ node through an inductance of suitable magnitude, the desired current will be entering the $\dot{v}'$ node whenever $\dot{v}'$ is zero. Disturbance by a nonzero $\dot{v}'$ can be suppressed by using another transformer to add a voltage drop proportioned to $\dot{v}'$ in the input to the amplifier; the transformer should be of such magnitude as to add a voltage equal to $\dot{v}'$ at the output terminal. The arrangement thus invented is shown in Figure 28.

The $BS^2\gamma'$ term in Equation [63c] can be represented similarly except that here the $\dot{\gamma}'$ voltage need not be reversed. The $BS^2\gamma'$ term in the γ-equation [63b] presents a simpler problem. If $h + L < 0$, so that the current is to <u>leave</u> the $\dot{\gamma}'$ node when $\gamma' > 0$, a simple inductance of magnitude $[p^2\rho_1^2\lambda\ (h + L) BS^2]^{-1}$, connected between $\dot{\gamma}'$ and ground, will be sufficient. An alternative when $\dot{\gamma}_b' = 0$, provided h + L is not too large a positive number (as cannot really happen), is to combine this term with the last one in Equation [22b], thus:

$$+ p^2\rho_1^2\lambda\left[\frac{GJ_e}{\ell_T} - (h + L)\,BS^2\right]\gamma'$$

This expression requires only an inductance of magnitude

$$\left\{p^2\rho_1^2\lambda\left[\frac{GJ_e}{\ell_T} - (h + L)\,BS^2\right]\right\}^{-1}$$



$R_1^{-1} = p\lambda C$

$R_2^{-1} = p\rho_1^2 \lambda c$

$R_3^{-1} = \rho \lambda BS$

$L_1^{-1} = p^2 \rho_1 \lambda BS^2/A_1$

$L_2^{-1} = \dfrac{p^2 \rho_1^2 \lambda (h+L) BS^2}{r_2 A_2 - 1}$

$L_3^{-1} = p^2 \rho_1 \rho_2 \lambda b_L BS^2/A_3$

$r_1 A_1 = 1$

$r_3 A_3 = 1$

$r_4 = \rho_1 (h+L)$

$r_5 = \rho_2 b_L$

Figure 28 - Mobility Analog for Addition of Damping and Lift Forces on Rudder in Transverse or v, γ, α Motion to Rudder Network Shown in Figure 8

The ratios r_1/A_1, r_3/A_3, and the product r_2A_2 are free. For alternatives to the A_2 amplifier, see text. The triangle is the usual symbol for an amplifier, with input to the right and output to the left. "A_i" denotes the numerical amplification factor; the minus sign before it is a reminder of the voltage reversal.

between $\dot{\gamma}'$ and ground. As a last resort, when $h + L > 0$, a simplified form of the device used for the other BS^2 terms can be employed.

The additions to the rudder network required by the damping and lift terms are shown in Figure 28, where the nodes labeled $\dot{v}'$, $\dot{\gamma}'$, $\dot{\alpha}'$ are simply the nodes so labeled in Figure 8. However, only the last of the three alternatives for the $BS^2\gamma'$ term in Equation [63b] is shown. Magnitudes are stated below Figure 28.

Note that the added current $-c(\dot{\gamma}' - \dot{\gamma}_b')$ that enters the $\dot{\gamma}'$ node comes from the $\dot{\gamma}_b'$ node and correctly represents there the added reaction on the ship. The currents representing the C and the S or S^2 terms come from ground and have no direct effect on the ship network.

9. SUMMARY

A theory has been advanced for determining the vibrations, including flutter of a control surface-hull system. The control surface may have 6 degrees of freedom whereas the hull may have additional sprung bodies with 1 or 2 degrees of freedom elastically attached to it at various locations.

The transverse and longitudinal motions of the control surface and their coupled relations with the hull motions have been treated; the control surfaces include single rudders, horizontally paired rudders, upper and lower rudders, paired diving planes, and foils mounted on a continuous shaft.

Equations of motion derived for flutter analysis based on the Modified Theodorsen Analysis include structural damping and lift force terms.

Analytical, digital, and electric-analog methods have been devised to determine the natural frequencies, mode shapes, critical flutter speeds, and damping of this control surface-hull system and/or parts of this system.

The theory and methods of solution established here permit a more adequate representation of a ship in forward motion and its appendages (rudders, diving planes, machinery, cargo, superstructures, nuclear reactors, boilers, radar masts, etc.) as a mass-hydroelastic system subject to vibrations and flutter.

The theory may be used to predict the vibrations and/or flutter characteristics of a hydroelastic system and to design such a system (or its components) to prevent excessive vibrations or flutter. General application of the theory for an existing or contemplated system requires evaluation of a specific set or variable sets of hydroelastic parameters, respectively, for use as data in the digital and analog solutions of the equations of motion.[14,15,2] Based on the solutions obtained for a range of parameters, graphs and/or nomographs may be devised to aid in the design of an optimum system with respect to minimum vibrations and avoidance of flutter for a given speed range.

The theory, while having some degree of verification,[9] requires additional validation through further comparison with experiment. In particular, it is important to establish the conditions (range of parameters) for which it is valid.

APPENDIX A

FEASIBILITY OF MOBILITY ANALOGS FOR ELASTIC SYSTEMS INCLUDING DAMPING

Consider any elastic mechanical system that is at least not unstable. If it is attached to other bodies, let the points or areas of attachment be immovable. The kinetic energy T_K of the elastic system can be expressed as a homogeneous quadratic function of the velocities of the coordinates of the system,[6] assumed finite in number and denoted by q_1, q_2 ..... q_n; thus:

$$T_K = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{q}_i^2 + \sum_{i=1}^{n} \sum_{j>i}^{n} m_{ij} \dot{q}_i \dot{q}_j \qquad [A.1]$$

The potential energy U_P, assumed zero in the position of equilibrium, is a similar function of the coordinates themselves:

$$U_P = \frac{1}{2} \sum_{i=1}^{n} k_i q_i^2 + \sum_{i=1}^{n} \sum_{j>i}^{n} k_{ij} q_i q_j \qquad [A.2]$$

The coefficients m_i, m_{ij}, k_i, and k_{ij} stand for constants such that neither T_K nor U_P can be negative for any values of the q's or $\dot{q}^2$'s. Obviously, all $m_i \geqq 0$ and all $k_i \geqq 0$.

The equations of motion for a conservative system having n degrees of freedom as obtained from Lagrange's equations,[6] or

$$\frac{d}{dt} \frac{\partial T_K}{\partial \dot{q}_i} - \frac{\partial T_K}{\partial q_i} = - \frac{\partial U_P}{\partial q_i} \qquad [A.3]$$

can be written in the convenient form

$$m_i \ddot{q}_i + \sum_{j \neq i} m_{ij} \ddot{q}_j = - k_i q_i - \sum_{j \neq i} k_{ij} q_j \qquad [A.4]$$

where i = 1, 2 ..... n, and it is to be understood that by definition

$m_{ji} = m_{ij}$, $k_{ji} = k_{ij}$. (Note that in the expressions as written for T_K and U_P, $j > i$ always, whereas here this is no longer the case. Note too that $\frac{\partial T_K}{\partial q_i} = 0$, since T_K is not a function of the coordinates but only of the velocities of the coordinates.)

The equations in this form obviously have the symmetry described for Equations [13a,b,c]. In each equation q_i appears as the leading variable, and nonleading variables occur symmetrically in the equations since $m_{ji} = m_{ij}$, $k_{ji} = k_{ij}$. Also, the coefficients m_i of all $\ddot{q}_i$'s on the left are nonnegative and those of all q_i's on the right are all nonpositive. This symmetry and these signs may have been destroyed in the equations of motion as encountered in a given case, but these features can always be recovered by multiplying certain equations throughout by the proper constants.

To point out that a passive analog network can always be set up for such a system in spite of the fact that some of the coefficients m_{ij} and k_{ij} may be negative, we show first that both T_K and U_P can be expressed as sums of squares with positive coefficients. This is possible because T_K and U_P can never be negative.

Let any one of the q's be chosen and labeled q_1, then collect all terms containing q_1 or $\dot{q}_1$ into a square term thus:

$$T_K = \frac{1}{2} m_1 \left(\dot{q}_1 + \sum_{j>1} \frac{m_{1j}}{m_1} \dot{q}_j\right)^2 + T_2 (\dot{q}_2 \;\cdots\cdots\; \dot{q}_n) \qquad [A.5]$$

$$U_P = \frac{1}{2} k_1 \left(q_1 + \sum_{j>1} \frac{k_{1j}}{k_1} q_j\right)^2 + U_2 (q_2 \;\cdots\cdots\; q_n) \qquad [A.6]$$

Here T_2 and U_2 are quadratic expressions not containing q_1 or $\dot{q}_1$; they contain, however, the following terms added to those already appearing in T_K or U_P in order to correct for the unwanted terms introduced by the first term as written, namely:

$$-\sum_{j>1} \frac{1}{2} \frac{m_{1j}^2}{m_1} \dot{q}_j^2 \text{ in } T_K \quad ; \quad -\sum_{j>1} \frac{1}{2} \frac{k_{1j}^2}{k_1} q_j^2 \text{ in } U_P \qquad [A.7]$$

Now, whatever values q_2 ..... q_n or $\dot{q}_2$ ..... $\dot{q}_n$ may have, $\dot{q}_1$ and q_1 can be so chosen as to cause the square terms in T_K and U_P as just written to vanish. But T_K and U_P as a whole can never be negative; hence T_2 and U_2 themselves cannot be negative for any values of q_2 ..... q_n, $\dot{q}_2$ ..... $\dot{q}_n$. It follows then that the new coefficients of $\dot{q}_j^2$ in T_K and of q_j^2 in U_P cannot be negative. Thus T_2 and U_2 have the same properties as T_K and U_P themselves.

Obviously, this process can be repeated until all q's and $\dot{q}$'s have been included in squares with coefficients that cannot be negative.

A mobility analog network can then be constructed containing n nodes at which the voltages above ground are proportional to the $(\dot{q}')$'s; the ground may be regarded as an (n + 1) phantom node. A term such as $\frac{1}{2}\, m_i'(\dot{q}_i')^2$ can at once be represented by connecting a capacitor between $\dot{q}_i'$ and ground, and a term such as $\frac{1}{2}\, k_i'(\dot{q}_i')^2$ by similarly connecting an inductor. The combinations occurring in other squares can be produced by suitable connections involving ideal transformers.

It can be shown that with a suitable choice of conversion factors, the same network is obtained in this way as by representing the equations of motion. Alternatively, the q's can be made proportional to successive voltage drops between points or lines within the network. An example of this latter procedure is shown in Appendix B.

If the damping forces acting on the system are all proportional to either the velocity of the system at a certain point or the difference between the velocities at two points, then the equations of motion become[6]

$$m_i \ddot{q}_i + \sum_{j \neq i} m_{ij} \ddot{q}_j = - k_i q_i - \sum_{j \neq i} k_{ij} q_j - c_i \dot{q}_i - \sum_{j \neq i} c_{ij} \dot{q}_j \qquad [A.8]$$

where c_i and c_{ij} denote damping constants. By definition a damping force is always in such a direction that it decreases the energy of the system (that is, the sum of the kinetic and potential energies). It follows from this that all $c_i \geqq 0$ and $c_{ij} = c_{ji}$. (See Reference 6, page 102.)

APPENDIX B

ENERGY METHOD OF DERIVING MOBILITY ANALOG FOR TRANSVERSE OR v, γ, α MOTION OF RUDDER ATTACHED THROUGH FLEXIBLE RUDDER STOCK TO FLEXIBLE HULL

The energy method for deriving an analog, described briefly in Appendix A, is especially convenient in dealing with an elastic system that has only a few degrees of freedom and for which energy expressions can be written down without knowledge of the differential equations of motion. In designing the ship network, on the other hand, it is more convenient to work from the equations of motion; also, if the rudder motions are to be included, it seems less confusing to use the same method in designing the rudder network. This has been done in the present report.

Since, however, the energy method has found wide application, its use will be shown here for the v, γ, α motion of the rudder. Expressions for the energy are found easily from materials already assembled.

The kinetic energy T_K of the rudder moving as a rigid body is equal to the work required to set it moving by means of external forces, hence

$$\begin{aligned} T_K &= \int (Y\dot{v} + M\dot{\alpha} + T\dot{\gamma})\, dt \\ &= \int [\, m_y \ddot{v}\dot{v} + I_x \ddot{\alpha}\dot{\alpha} + I_z \ddot{\gamma}\dot{\gamma} - I_{xz}(\ddot{\gamma}\dot{\alpha} + \ddot{\alpha}\dot{\gamma})]\, dt \end{aligned}$$

by Equations [11a,b,c]. (See also Figure 5.) Thus

$$T_K = \frac{1}{2} m_y \dot{v}^2 + \frac{1}{2} I_x \dot{\alpha}^2 + \frac{1}{2} I_z \dot{\gamma}^2 - I_{xz}\dot{\alpha}\dot{\gamma} \qquad [B.1]$$

This expression can be converted into a sum of squares with positive coefficients in various ways. The general procedure described in Appendix A gives, as one form,

$$T_K = \frac{1}{2} m_y \dot{v}^2 + \frac{1}{2} I_x \left(\dot{\alpha} - \frac{I_{xz}}{I_x} \dot{\gamma} \right)^2 + \frac{1}{2} I_z \left(1 - \frac{I_{xz}^2}{I_x I_z} \right) \dot{\gamma}^2 \qquad [B.2]$$

For analog purposes, however, some freedom in the choice of transformer ratios may be desirable. Hence the following form is preferred here:

$$T_K = \frac{1}{2} m_y \dot{v}^2 + \frac{1}{2}\left(I_x - \frac{I_{xz}}{\bar{r}}\right)\dot{\alpha}^2 + \frac{1}{2}(I_z - \bar{r}\, I_{xz})\dot{\gamma}^2 + \frac{1}{2}\frac{I_{xz}}{\bar{r}}(\dot{\alpha} - \bar{r}\,\dot{\gamma})^2 \quad [B.3]$$

where, to keep all coefficients positive, $\bar{r}$ has the same sign as I_{xz} and is to be chosen so that

$$\frac{|I_{xz}|}{I_x} \leqq |\bar{r}| \leqq \frac{I_z}{|I_{xz}|} \quad [B.3a]$$

Since $I_{xz}^2 \leqq I_x I_z$, at least one allowed value of $\bar{r}$ exists. Equation [B.2] results from the choice: $\bar{r} = I_{xz}/I_x$. The limits on $|\bar{r}| = r$ are necessary to keep both $(I_x - I_{xz}/r)$ or $(I_z - rI_{xz})$ from being negative.

The potential energy stored in the rudder stock at any moment is equal to the work that would have to be done in producing the elastic deformation that exists at that moment. If the stock remained undistorted while its top acquired displacements v_b, γ_b, and α_b, the bottom of the stock would receive displacements (see page 11):

$$v_{1b} = v_b + \ell\,\alpha_b \quad ; \quad \alpha_{1b} = \alpha_b \quad ; \quad \gamma_{1b} \doteq \gamma_b$$

As these displacements change further to v_1, α_1, and γ_1, the reactions on the bottom of the stock increase from 0 to $-Y_1$, $-T_1$, and $-M_1$. Hence, the potential energy U_P has the value

$$U_P = -\frac{1}{2}\left[Y_1(v_1 - v_b - \ell\,\alpha_b) + M_1(\alpha_1 - \alpha_b) + T_1(\gamma_1 - \gamma_b)\right]$$

Substituting here Y_1, M_1, and T_1 from Equations [9a,b,c] and v_1, α_1, γ_1 from Equations [8a,b,c], and also using the notation defined in Equations [12a,b], we obtain:

$$U_P = 6k_s\,(v - b\alpha + h\gamma - v_b - \ell\alpha_b)^2$$

$$- 6k_s\ell\,(v - b\alpha + h\gamma - v_b - \ell\alpha_b)(\alpha - \alpha_b) \qquad [B.4]$$

$$+ \frac{1}{2}(3k_s + k)\,\ell^2\,(\alpha - \alpha_b)^2 + \frac{1}{2}\frac{GJ_e}{\ell_T}(\gamma - \gamma_b)^2$$

or converting into a sum of squares with positive coefficients by the usual procedure, we obtain:

$$U_P = 6k_s\left[v - (b + \frac{1}{2}\ell)\alpha + h\gamma - v_b - \frac{1}{2}\ell\alpha_b\right]^2$$

$$+ \frac{1}{2}k\ell^2(\alpha - \alpha_b)^2 + \frac{1}{2}\frac{GJ_e}{\ell_T}(\gamma - \gamma_b)^2 \qquad [B.5]$$

For a mobility analog, use the following conversion relations:

$$\dot{v} = b_1\dot{v}' \qquad \dot{\alpha} = b_3\dot{\alpha}' \qquad \dot{\gamma} = b_2\,\dot{\gamma}'$$

$$v = pb_1v' \qquad \alpha = pb_3\alpha' \qquad \gamma = pb_2\gamma'$$

$$t = pt' \qquad T_K = q\,T_K' \qquad U_P = q\,U_P'$$

Here T_K' and U_P' are electrical energies representing T_K and U_P. Let b_1, b_2, b_3, p, and q all be positive. Substitution in Equations [B.3] and [B.5] gives

$$T_K' = \frac{1}{2}m_y'(\dot{v}')^2 + \frac{1}{2}I_x''(\dot{\alpha}')^2 + \frac{1}{2}I_z''(\dot{\gamma}')^2 + \frac{1}{2}I_{xz}(\dot{\alpha}' - r\dot{\gamma}')^2 \qquad [B.6]$$

$$U_P' = \frac{1}{2}(12k_s)'\left[v' - (b + \frac{1}{2}\ell)'\alpha' + h'\gamma' - v_b' - (\frac{1}{2}\ell)'\alpha_b'\right]^2$$

$$+ \frac{1}{2}(k\ell^2)'(\alpha' - \alpha_b')^2 + \frac{1}{2}\left(\frac{GJ_e}{\ell_T}\right)'(\gamma' - \gamma_b')^2 \qquad [B.7]$$

where

$$q\,m_y' = m_y b_1^2 \;;\quad q I_x'' = \left(I_x - \frac{I_{xz}}{\bar{r}}\right) b_3^2 \;;\quad q I_z'' = (I_z - \bar{r} I_{xz}) b_2^2 \;;$$

$$q\, I_{xz}'' = \frac{I_{xz}}{\bar{r}}\, b_3^2$$

$$q\,(12\,k_s)' = 12\,k_s\, p^2\, b_1^2 \;;\quad q\,(k\,\ell^2)' = k\,\ell^2\, p^2\, b_3^2 \;;\quad q\left(\frac{GJ_e}{\ell_T}\right)' = \frac{GJ_e}{\ell_T}\, p^2\, b_2^2$$

$$r = |\bar{r}|\,\frac{b_2}{b_3} \;;\quad (b + \tfrac{1}{2}\,\ell)' = (b + \tfrac{1}{2}\,\ell)\,\frac{b_3}{b_1} \;;\quad h' = |h|\,\frac{b_2}{b_1} \;;$$

$$\left|\tfrac{1}{2}\,\ell\right|' = \tfrac{1}{2}\,|\ell|\,\frac{b_3}{b_1}$$

Here, for convenience, the transformer ratios have also been stated.

In the usual procedure, $\dot{v}'$, $\dot{\alpha}'$, and $\dot{\gamma}'$ would now be assumed to be voltages above ground. In ship theory this method is the most convenient. For a system involving only a few variables, however, a slightly neater result is obtained by using different reference lines for the separate variables.

Let $\dot{v}'$, $\dot{\alpha}'$, and $\dot{\gamma}'$ be <u>voltage drops</u> between successive pairs in a system of four nodes or lines, numbered 1 to 4 in Figure 29. The first term of T_K' may then represent the energy in a capacitance m_y' subjected to the voltage $\dot{v}'$, and similarly for the other terms of T_K'. The combination of $\dot{\gamma}'$ and $\dot{\alpha}'$ in the last term of Equation [B.3] requires a transformer. Analogously the terms of Equation [B.7] require inductances.

In a term such as $\frac{1}{2}\,(12k_s)'\,(v')^2$, for example, $v' = \int \dot{v}'\,dt'$ so that v' is a voltage impulse; if $\dot{v}'$ is the voltage drop across an inductance L, the current through the inductance is $\frac{1}{L}\int \dot{v}'\,dt' = \frac{1}{L}\,v'$, and the energy stored in the inductance is $\frac{1}{2}\,\frac{1}{L}\,(v')^2$. Hence, to represent the term just cited, $L = [(12k_s)']^{-1}$. In U_P' a similar term occurs with v' replaced by a combination that must be obtained with the use of transformers.

The terms v_b', α_b', and γ_b' may be represented by voltage impulses

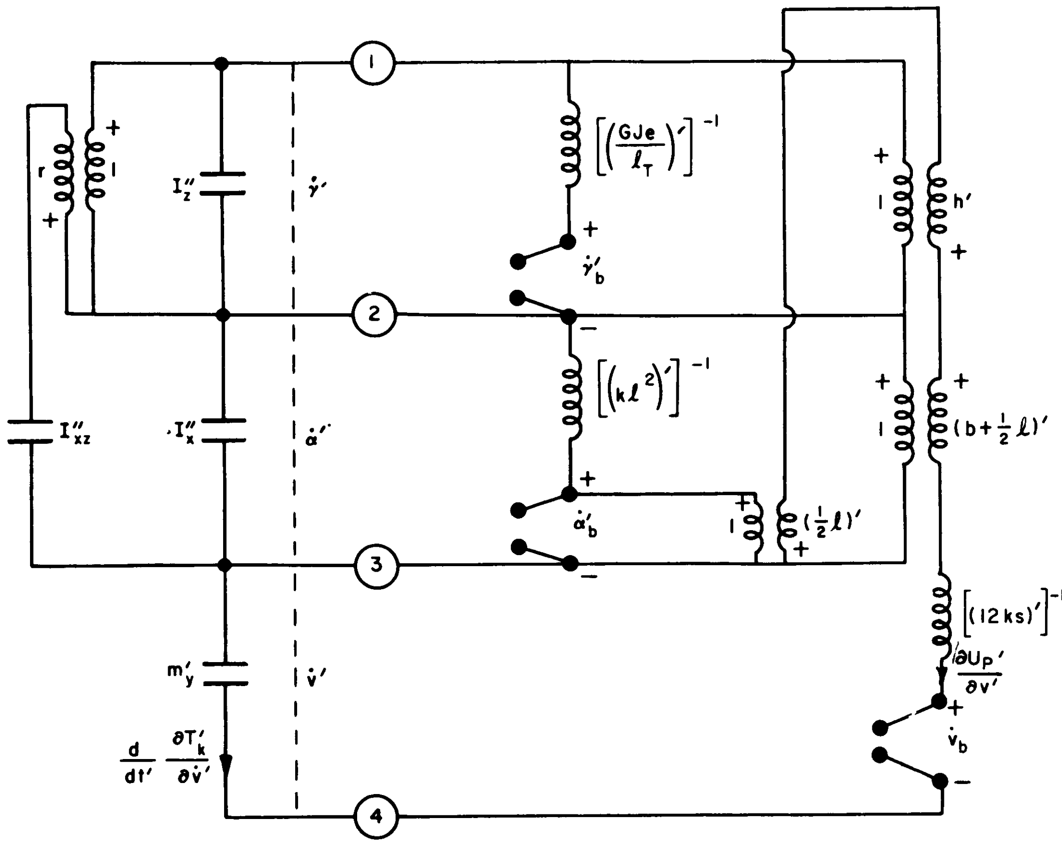


Figure 29 - Alternative Mobility Analog for Rudder in Transverse or v, γ, α Motion Attached to Moving Hull

Transformer signs are shown for positive $\overline{r}$, b + $\frac{1}{2}\ell$, h, and ℓ.

across three gaps located as shown in Figure 29. To add $(\frac{1}{2}\ell)'\alpha_b'$ to v_b' requires a transformer.

The network thus designed is shown in Figure 29. Transformer connections are shown for positive values of $\overline{r}$, b + $\frac{1}{2}\ell$, h, and ℓ; if any one of these quantities is negative, one + mark is to be moved to the other end of the winding on the corresponding transformer.

Note that the left and right halves of the network, representing, respectively, T_K' and U_P', must be connected, thus representing the fact that only the sum $T_K' + U_P'$ is constant.

It remains to be shown, however, that a correspondence thus established at one moment between a mechanical and an electrical vibration will persist.* In an isolated network, i.e., network not connected electrically to anything else, the electrical energy must remain constant, as does the mechanical energy of an isolated system. This condition does not determine the mechanical motion, however; it is necessary that Lagrange's equation be satisfied:**

$$\frac{d}{dt}\frac{\partial T_K}{\partial \dot{q}_j} + \frac{\partial U_P}{\partial q_j} = 0$$

where $q_j = v$, α, or γ. This equation holds even when v_b, α_b, and γ_b vary with time.

The corresponding electrical equation, obtained by substituting the conversion relations in Lagrange's equation and then canceling out q, p, and the b's is:

$$\frac{d}{dt'}\frac{\partial T'_K}{\partial \dot{q}'_j} + \frac{\partial U'_P}{\partial q'_j} = 0$$

---

*Consider any given motion of the mechanical system with total energy $T_K + U_P$. At a given time t_1, we may start the network with voltages and currents satisfying $\dot{v} = b_1\dot{v}'$, etc., and with total energy $T'_K + U'_P = \frac{(T_K + U_P)}{q}$. Then this relation between the <u>total energies</u> will persist, since both systems obey the conservation of energy (hence $T_K + U_P$ does not change, nor does $T'_K + U'_P$). Now is it possible for the currents and voltages at time $t > t_1$ to wander off so that the ratios $\dot{v}/\dot{v}'$ change, but of course in such a way that $T'_K + U'_P$ does not change? The proof given is supposed to show that the voltages and currents will <u>not</u> wander off in this way — the correspondence once established (in terms of certain values of b, etc.) will persist.

If the analog is set up from the <u>equations of motion</u>, this proof is not necessary; $m\frac{d\dot{v}}{dt}$ is matched by $C\frac{d\dot{v}'}{dt'}$, etc., so that electrical quantities obviously change with time at the right rate.

**The correctness of U_P and T_K can be checked by comparing the equations of motion previously derived by using Newton's laws with the equations of motion derived from U_P and T_K by using Lagrange's equation.

where $q_j' = v'$, α', or γ'. If this equation holds for electrical vibrations in the network, the correspondence between v and $\dot{v}'$ and v and v', etc., will continue to hold. Now from Equations [B.6] and [B.7] and Figure 29, we can see that $\frac{d}{dt'}\frac{\partial T_K'}{\partial \dot{v}'} + \frac{\partial U_P'}{\partial v'}$ is the sum of the two currents flowing downward from Line 3 into Line 4 from which there is no outlet. It can be reasoned next that the sum of the currents flowing downward from Line 2 must vanish, and similarly for Line 1. Hence the electrical Lagrange equation will hold in the network.

It should be noted that the network shown in Figure 29 does not differ essentially from that shown as choice II in Figure 8. In fact, if $q = b_1^2/\lambda$, as in Equation [B.8], and if ρ_1 and ρ_2, that is, b_2/b_1 and b_3/b_1, have the same values in the two cases, then the only difference is that in Figure 29 the interior Line 3 above the $\dot{v}'$ drop serves, in effect, as a "ground or reference line" for $\dot{\alpha}'$, and Line 2 serves similarly for $\dot{\gamma}'$. This difference affects some of the actual voltages in the network but not the voltage differences that control the currents.

That the elements and transformer ratios are the same is easily verified by comparing the values given for Figure 29 with those given at the end of Section 4.3 for Figure 8II. For example, after substituting (see equations following Equation [B.7]) $|\bar{r}| = \frac{b_3 r'}{b_2} = \frac{\rho_2 r'}{\rho_1}$, Equation [B.3a] becomes

$$\frac{|I_{xz}|}{I_x} \leqq \frac{\rho_2}{\rho_1}\, r' \leqq \frac{I_z}{|I_{xz}|}$$

so that the range of choice for r' is the same as for r_4'' in Figure 8II. Also

$$I_{xz}'' = \frac{1}{q}\, b^3 \frac{1}{\bar{r}_4}\, |I_{xz}| = \frac{\lambda}{b_1^2}\, b_3^2 \frac{b_2}{b_3 r'}\, |I_{xz}| = \lambda \rho_1 \rho_2 \frac{|I_{xz}|}{r'}$$

so that $I_{xz}'' = C_4''$ if r' is chosen equal to r_4''. Again,

$$m_y' = \lambda\, m_y = C_1 \quad ; \quad (12\, k_s)' = \lambda\, (12\, k_s\, p^2) = L_1^{-1}$$

Choice I in Figure 8 would have been obtained if Equation [B.2] had been so written that the last term contained $(\dot{\gamma} - \bar{r}\dot{\alpha})^2$, $\bar{r}$ being different in valué. Choices III and IV result merely from choosing an extreme value for the transformer ratio.

An increase in q decreases all capacitances and increases all inductances in proportion to q, thereby decreasing the network currents caused by given voltages; however, there is no effect on the natural frequencies of vibration or on the mode patterns.

If the rudder bearing is fixed so that $v_b = \alpha_b = \gamma_b = 0$, the gaps shown in Figure 29 are to be closed. In this case, it is simplest to choose $q = 1$. Such a choice does not restrict the range of possible values for the elements, since the same changes in the elements can be made either by changing q in a certain ratio or by changing b_1^2, b_2^2, and b_3^2 in the inverse ratio. The correspondence between mechanical and electrical vibrations is altered by such changes, but this is of little interest because the amplitude of vibration is arbitrary in any case. If, on the other hand, the bearing is forced to vibrate in a certain way, corresponding voltage drops are to be impressed upon the $\dot{v}_b'$, $\dot{\alpha}_b'$, and $\dot{\gamma}_b'$ gaps in the network. In this case, a choice for q other than unity may be preferable.

Connections to the ship network can be made provided two precautions are observed. Let the ship network be extended as in Figure 9 to provide $\dot{v}_b'$, $\dot{\alpha}_b'$, and $\dot{\gamma}_b'$ nodes. Then Line 4 of the rudder network may be connected directly to the ground line for the ship network and the top of the gap labeled $\dot{v}_b'$, to the $\dot{v}_b'$ node just specified. The other two gaps, however, require 1:1 transformers in order to impress the voltage drops, between $\dot{\alpha}_b'$ and $\dot{\gamma}_b'$ and ground in the ship extension upon the corresponding gaps provided in the rudder network.

Furthermore, the conversion factor for energy must be the same for both networks. This requirement may be regarded as arising from the fact that in both the mechanical and the electrical systems energy lost by the rudder is gained by the ship and vice versa. In Section 4.4, however, it was required that, besides b_1, b_2, b_3, the conversion factors a_1, a_2, a_3

from force or torque to current must be the same for rudder and ship, but this is easily shown to be equivalent to identity of the energy factors. Consideration of any element described in Section 4.3 and shown in Figure 8 leads to the conclusion that

$$q = \frac{b_1^2}{\lambda} = p\,a_1\,b_1 \qquad [B.8]$$

For example, the electrical energy in the m_y' capacitor is

$$\frac{1}{2}\,m_y'\,(\dot{v}')^2 \quad \text{or} \quad \frac{1}{2}\,(\lambda m_y)\left(\frac{\dot{v}}{b_1}\right)^2 \quad \text{or} \quad \frac{\lambda}{b_1^2}\,(\frac{1}{2}\,m_y\,\dot{v}^2)$$

so that the mechanical energy is b_1^2/λ times the electrical energy. Since p, a_1, and b_1 were all required to have identical values in rudder and ship network, the same was true of q.

APPENDIX C

RUDDER RESONANCE

When a natural frequency of the rudder lies close to a frequency of ship vibration, large amplitudes of rudder vibration become possible. In such cases it may be important to make sure that any vibration of the rudder-ship system that may occur in practice will not cause damage to the rudder structure. For this purpose calculations can be made by the methods described in Sections 4 and 5 of the present report, either numerically or with the use of electrical analogs. Near rudder resonance, however, useful simplified formulas can be obtained.

1. Transverse or v, γ, α Rudder Motion

Transverse motion of the rudder will be considered first. To shorten the notation, write

$$\frac{m_y}{12k_s} = \zeta \quad ; \quad \frac{GJ_e}{\ell_T} = k_T \quad ; \quad \frac{1}{2}(\ell + 2b) = b_o$$

Assume harmonic motion so that $\ddot{v} = -\omega^2 v$, etc., and simplify Equations [13a,b,c] for calculation by dividing Equation [13a] through by k_s, subtracting the original Equation [13a] multiplied by h from [13b], and adding the original Equation [13a] multiplied by $(b + \frac{1}{2}\ell)$ to Equation [13c]. The modified equations thus obtained can be written, for harmonic motion:

$$(1 - \zeta\omega^2)v + h\gamma - b_o\alpha = v_b + \frac{1}{2}\ell\alpha_b \qquad [C.1a]$$

$$hm_y\lambda^2 v + (k_T - I_z\omega^2)\gamma + I_{xz}\omega^2\alpha = k_T\gamma_b \qquad [C.1b]$$

$$-b_o m_y\omega^2 v + I_{xz}\omega^2\gamma + (k\ell^2 - I_x\omega^2)\alpha = k\ell^2\alpha_b \qquad [C.1c]$$

The natural frequencies for v, γ, α motion of the rudder alone may be found by setting $v_b = \gamma_b = \alpha_b = 0$ and equating to zero the determinant

D of the coefficients of the v, γ, α terms. The equation thus obtained has the form

$$D = -\zeta I_x I_z [\omega^6 + \omega^4 (\ldots) + \omega^2 (\ldots) + \ldots] = 0$$

This cubic equation in ω^2 has three roots, ω_1^2, ω_2^2, and ω_3^2, which are the squares of the natural circular frequencies for vibrations of the rudder with the top of the stock fixed. In terms of these roots, the expression in brackets can be written for any ω as $(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)$. Hence, in general,

$$D = \zeta I_x I_y [\omega^6 + \omega^4 (\ldots) + \omega^2 (\ldots) + \ldots] = 0$$

Solution of Equations [C.1a,b,c] for v, γ, and α by the usual method of determinants gives (the solutions v, γ, and α have been multiplied through by D):

$$\begin{aligned} Dv = {} & (v_b + \tfrac{1}{2}\ell\alpha_b)\left[k_T k\ell^2 - (k_T I_x + k\ell^2 I_z)\omega^2 + (I_x I_z - I_{xz}^2)\omega^4\right] \\ & - \gamma_b k_T \left[hk\ell^2 - (hI_x - b_o I_{xz})\omega^2\right] \\ & + \alpha_b k\ell^2 \left[b_o k_T - (b_o I_z - hI_{xy})\omega^2\right] \end{aligned} \qquad [C.2a]$$

$$\begin{aligned} D\gamma = {} & -(v_b + \tfrac{1}{2}\ell\alpha_b) m_y \omega^2 \left[hk\ell^2 - (hI_x - b_o I_{xz})\omega^2\right] \\ & + \gamma_b k_T \left[k\ell^2 - (b_o^2 m_y + I_x + \zeta k\ell^2)\omega^2 + \zeta I_x \omega^4\right] \\ & - \alpha_b k\ell^2 \omega^2 \left[hb_o m_y + I_{xz} - \zeta I_{xz}\omega^2\right] \end{aligned} \qquad [C.2b]$$

$$\begin{aligned} D\alpha = {} & (v_b + \tfrac{1}{2}\ell\alpha_b) m_y \omega^2 \left[b_o k_T - (b_o I_z - hI_{xz})\omega^2\right] \\ & - \gamma_b k_T \omega^2 (hb_o m_y + I_{xz} - \zeta I_{xz}\omega^2) \\ & + \alpha_b k\ell^2 \left[k_T - (\zeta k_T + h^2 m_y + I_z)\omega^2 + \zeta I_z \omega^4\right] \end{aligned} \qquad [C.2c]$$

The corresponding reactions on the ship at the rudder bearing are (from Equations [15a,b,c] with the left-hand members of Equations [13a,b,c] substituted for Y, T, M):

$$Y_b = m_y \omega^2 v \qquad [C.3a]$$

$$T_b = \omega^2 (-h\, m_y v + I_z \gamma - I_{xz} \alpha) \qquad [C.3b]$$

$$M_b = \omega^2 \left[(\ell + b) m_y v - I_{xz} \gamma + I_x \alpha \right] \qquad [C.3c]$$

From the formulas [C.2a,b,c], v, γ, and α can be calculated provided ω_1^2, ω_2^2, ω_3^2, v_b, γ_b, and α_b are known or assumed[11*] (unless $\omega = \omega_1$, ω_2, or ω_3). If ω lies close to one of the natural frequencies, say ω_1, the latter may be substituted for ω without great error in the right-hand members of Equations [C.2a,b,c] and also in the determinant D, except in the factor $(\omega_1 - \omega)$ that occurs after $\omega_1^2 - \omega^2$ has been replaced by $(\omega_1 - \omega)(\omega_1 + \omega)$. Furthermore, if the ratios of the displacements v, γ, and α during a free ω_1 vibration are known, it may be sufficiently accurate to calculate from the formulas only one displacement, perhaps v, and to find the other two from the known ratios.

The approximate values of v, γ, and α thus obtained are proportional to $\frac{1}{\omega_1 - \omega}$, or, to make the factor of proportionality nondimensional, they are numerically proportional to R, where

$$R = \left| \frac{\omega_1}{\omega_1 - \omega} \right| = \left| \frac{\omega_1}{\omega - \omega_1} \right|$$

When ω is near ω_1, the value of R may serve as a rough measure of the resonance effect. If $\omega < 0.7\,\omega_1$, or $\omega > 1.3\,\omega_1$, $R < 3.4$; if $|\omega - \omega_1| = \frac{1}{5}\omega_1$, $R > 5$; if $|\omega - \omega_1| < \frac{1}{10}\omega_1$, $R > 10$. When $R < 3$ or 4, the approximation

*In Appendix A of Reference 12, rudder-hull calculations were made in which the rudders were treated simply as an equivalent sprung mass tuned to the frequency of the second horizontal mode as computed without the sprung mass effect.

that has been made is likely to be poor, i.e., the resonance effects may probably be ignored, but it becomes better as ω moves toward ω_1. If $R > 10$ there is certainly a decided resonance effect. Of course, either ω_2 or ω_3 may be used instead of ω_1.

It can be shown that the ratio γ/v during an ω_1 free vibration is equal to the ratio of the coefficient of γ_b in Equation [C.2b] to the corresponding coefficient in [C.2a], provided ω is replaced by ω_1 in both; and γ/v is also equal to the similar ratio of the $(v_b + \frac{1}{2}\ell\alpha_b)$ or the α_b coefficients.* Similar statements hold for α/v or α/γ. It follows that as expected near resonance, the vibration pattern approximates the pattern for free vibration independently of the relative magnitudes of v_b, γ_b, and α_b.

It may be that a complete analysis of the rudder-ship system might not predict an infinite rudder amplitude even when $\omega = \omega_1$, because the reactions on the ship would then reduce v_b, γ_b, and α_b either to zero or at least to a combination of values consistent with limited values of v, γ, and α. This feature is well known in the case of a sprung mass whose amplitude at exact resonance becomes only large enough to hold its base at rest.[1,2] Probably the corresponding situation cannot be reached with a rudder except as a result of structural damage.

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*Proof: Set right-hand members of Equations [C.1a,b,c] to zero and let $\omega = \omega_1$. Then D = 0 and nonvanishing values of v, γ, and α can exist. To find γ/v divide [C.1a] and [C.1c] by v and solve the resulting equations for γ/v. We obtain

$$\frac{\gamma}{v} = -\frac{\left[(1 - \zeta\,\omega_1^2)(k\ell^2 - I_x\,\omega_1^2) - b_o^2\, m_y\,\omega_1^2\right]}{\left[h\,(k\ell^2 - I_x\omega_1^2) + b_o\, I_{xz}\,\omega_1^2\right]}$$

$$= \frac{\text{coefficient of } \gamma_b \text{ in [C.2b]}}{\text{coefficient of } \gamma_b \text{ in [C.2a]}} \quad \text{with } \omega \rightarrow \omega_1$$

Other ratios can be checked in the same way. Omitting Equation [C.1a] gives γ/v = ratio of coefficient of $(v_b + \frac{1}{2}\ell\alpha_b)$, etc.

2. Longitudinal or u, w, β Rudder Motion

For longitudinal motion of the rudder, corresponding results are obtained; only the equations need be given. Let

$$k_a = \frac{EA}{\ell} \quad ; \quad \xi = \frac{m_x}{12k_s}$$

By the addition of b_o times Equation [27a] and h times [27b] to [27c], the division of [27a] by $12k_s$, and the assumption of harmonic motion, we convert Equations [27a,b,c] into the following:

$$(1 - \xi\,\omega^2)\,u - b_o\beta = u_b + \frac{1}{2}\,\ell\beta_b \qquad [C.4a]$$

$$(k_a - m_z\,\omega^2)\,w_1 + h\,m_z\,\omega^2\beta = k_a\,w_b \qquad [C.4b]$$

$$- b_o\,m_x\,\omega^2 u + k_a h\,w_1 + (k\ell^2 - I_y\,\omega^2)\,\beta = k_a h\,w_b + k\ell^2\beta_b \qquad [C.4c]$$

The determinant $\overline{D}$ of the coefficients can be written

$$\overline{D} = \xi\,m_z\,I_y\,(\overline{\omega}_1^2 - \omega^2)(\overline{\omega}_2^2 - \omega^2)(\overline{\omega}_3^2 - \omega^2)$$

in terms of the three circular frequencies $\overline{\omega}_1$, $\overline{\omega}_2$, $\overline{\omega}_3$, for free u, w, β vibration of the rudder with the top of the stock fixed. The equations yield

$$\begin{aligned}\overline{D}u = {} & (u_b + \frac{1}{2}\,\ell\beta_b)\left[k_a k\ell^2 - (k_a h^2\,m_z + k\ell^2 m_z + k_a I_y)\,\omega^2 + m_z I_y\,\omega^4\right] \\ & - w_b\,k_a^2\,h\,b_o + (k_a h\,w_b + k\ell^2\beta_b)\;b_o\,(k_a - m_z\,\omega^2)\end{aligned} \qquad [C.5a]$$

$$\begin{aligned}\overline{D}w_1 = {} & - (u_b + \frac{1}{2}\,\ell\,\beta_b)\,h\,b_o\,m_x\,m_z\,\omega^4 \\ & + \omega_b k_a\left[k\ell^2 - (\xi\,k\ell^2 + b_o^2\,m_x + I_y)\,\omega^2 + \xi I_y\,\omega^4\right] \\ & - (k_a h\,w_b + k\,\ell^2\beta_b)\,h\,m_z\,\omega^2\,(1 - \xi\,\omega^2)\end{aligned} \qquad [C.5b]$$

$$\overline{D}\beta = (u_b + \frac{1}{2}\ell\beta_b)b_o m_x \omega^2 (k_a - m_z\omega^2) - w_b k_a^2 h (1 - \xi\omega^2)$$

$$+ (k_a h w_b + k\ell^2\beta_b)[k_a - (\xi k_a + m_z)\omega^2 + \xi m_y \omega^4] \qquad [C.5c]$$

The reactions on the ship are (from Equations [30a,b,c]):

$$X_b = m_x \omega^2 u$$

$$Z_b = \omega^2 (m_z w_1 - h m_z \beta)$$

$$Q_b = \omega^2 [(\ell + b) m_x u - h m_z w_1 + (I_y + h^2 m_z)\beta]$$

Remarks similar to those following Equations [C.3a,b,c] may be made with appropriate changes in the case of longitudinal vibration.

MOBILITY ANALOGS

Apparently there is no simpler way to investigate resonance in a rudder-ship system with use of a mobility analog than to set up the usual network and vary the natural rudder frequency past the ship frequency by suitable variation of one or more of the network elements.

APPENDIX D

SHAFT ROTATION IN S-BENDING

Consider a cantilever shaft with load L at the right end. Imagine the shaft divided at a certain point and moved apart there; see Figure 30a. At the gap, which may be anywhere, there are two shear forces, L acting on Section A and -L acting on Section B. L is here designated as "the shear force" meaning that it represents the shear force acting towards the left. Then in the cantilever the shear force is uniform and equal to L.

In Figure 26, P and P_1 are the shear force acting toward the left or toward 0. Let us now consider the effect of the bearing on the shaft rotations for the cases where $P_1 = P$, $P_1 = -P$, $P_1 \neq P$ or $-P$.

1. $P_1 = P$

In this case the forces on the ends of the short section of the shaft inside the bearing are equal but opposite ($-P_1$ or $-P$ and P), as shown in Figure 30b. These forces tend to bend this section like an S; but this bending may be ignored.

The shear force is P at one end and P_1 at the other end; but $P_1 = P$ in this case, hence the shear force must be uniform throughout the bearing, accompanied by uniform shear warping, which is assumed not to be interfered with by the bearing. Thus there is no shear effect, and $\theta_{b1} = \theta_b = \theta_{b2}$.

2. $P_1 = -P$

This case is illustrated in Figure 30c. Note the reversal of shear warping as the shaft passes through the bearing. The force $-P_1$ or P on the left end of the bearing tips the shaft upward and thereby decreases its slope from θ_b to $\theta_b - 2\sigma P$. Hence, $\theta_{b1} = \theta_b - 2\sigma P$.

3. $P_1 \neq P$ or $-P$

The pair of shear forces P_1 and P can be divided into the following two superposed sets:

(a) $\frac{1}{2}(P + P_1)$ at left and $\frac{1}{2}(P + P_1)$ at right.

(b) $-\frac{1}{2}(P - P_1)$ at left and $\frac{1}{2}(P - P_1)$ at right.

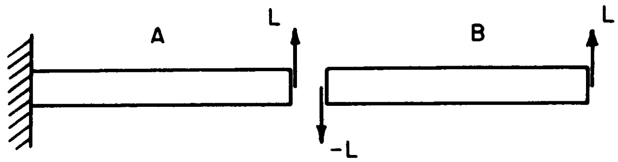

Figure 30a - Shear Forces Acting on Shaft

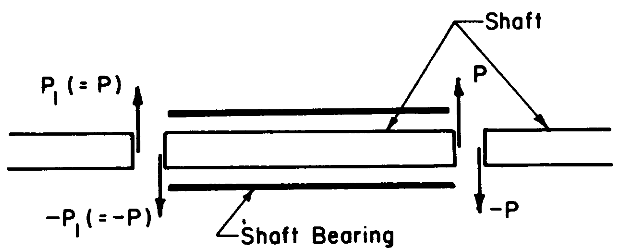

Figure 30b - Shear Forces Acting on Shaft at
Bearing Terminals when $P_1 = P$

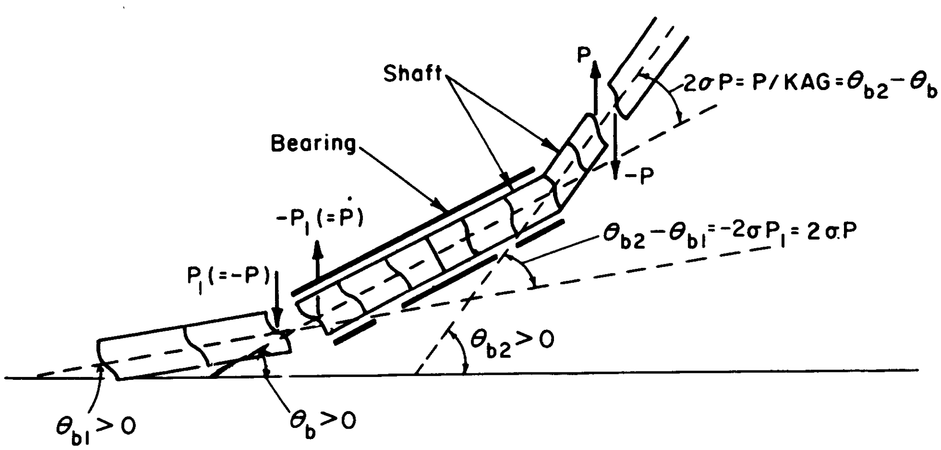

Figure 30c - Shear Forces Acting on Shaft at
Bearing Terminals when $P_1 = -P$

Figure 30 - Shear Forces and Rotations for Shaft Passing through a Bearing

In both cases the force "at left" acts, not on the shaft in the bearing, but toward the left, like P_1 in cases 1 and 2.

In the first set, the two forces $\frac{1}{2}(P + P_1)$ are equal and in the same direction, as were P_1 and P in case 1. Hence, the same reasoning as in case 1 leads to zero shear effect, so that this set introduces no differences between θ_{b1}, θ_b and θ_{b2}.

In the second set, however, the force $-\frac{1}{2}(P - P_1)$ at the left is equal and opposite to the force $\frac{1}{2}(P - P_1)$ at the right, just as in case 2 $P_1 = -P$. Hence, this set introduces differences as in case 2, except that here $\frac{1}{2}(P - P_1)$ replaces P.

Therefore, in general,

$$\theta_{b1} = \theta_b - (P - P_1) \quad , \quad \theta_{b2} = \theta_b + \sigma (P - P_1)$$

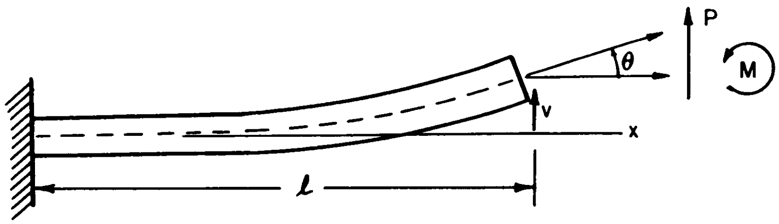

Figure 31a - Forces and Moments Acting on a Uniform Cantilever

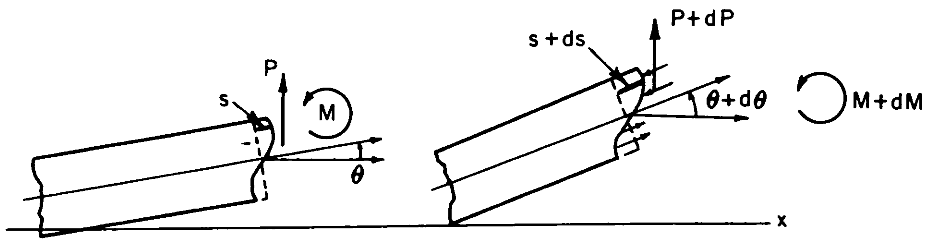

Figure 31b - Shear Warping in Shaft

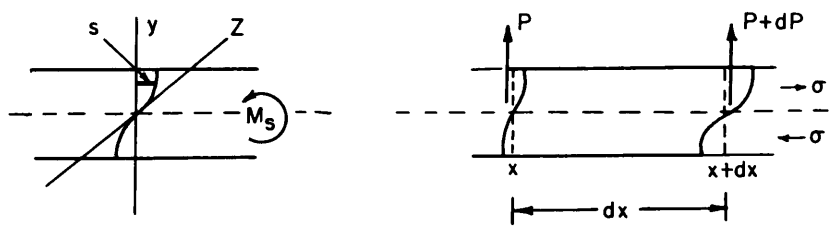

Figure 31c - Variation in Shear Warping for Section of Length dx

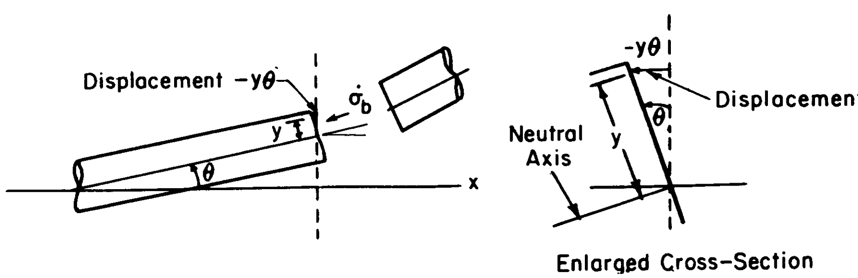

Figure 31d - Displacement and Stress Associated with Pure Bending of a Shaft

Figure 31 - Forces and Moments Acting on Uniform Cantilever and Associated Shear Warping and Bending Displacements

Displacement is positive toward $+x$, hence it is negative here. Actual direction of σ_b, if θ_x is positive, is negative as shown; σ_b is positive toward $+x$.

APPENDIX E

ENERGY RELATIONS FOR A UNIFORM CANTILEVER

Let a uniform cantilever be acted on by a force P and a moment M, resulting in a displacement v and a slope θ at the end; see Figure 31a. The relations of v and θ to P and M can be found from Equations [7] and [4]. These particular equations are not affected by the "fourth boundary condition," which does not hold in the present case because the free end of the cantilever is not connected to anything, i.e., the rudder; this end is, therefore, free to shear-warp to suit itself.* Substitution of $\frac{dy'}{dz} = \theta$, $z = \ell$, and $y = v$ in Equations [7] and [4] yields**

$$EIv = \left(\frac{1}{3}\,\ell^3 + \frac{\ell EI}{KAG}\right)P + \frac{1}{2}\,\ell^2 M$$

$$EI\theta = \left(\frac{1}{2}\,\ell^2 + \frac{EI}{KAG}\right)P + \ell M$$

The coefficients in these equations are not symmetrical; e.g., the coefficients of M in the v equation and P in the θ equation are not equal. How is this consistent with conservation of energy?

If W is the work done by P and M <u>gradually</u> applied, elementary analysis gives

$$dW = Pdv + Md\theta$$

$$= \frac{1}{EI}\left[\left(\frac{1}{3}\,\ell^3 + \frac{\ell EI}{KAG}\right)PdP + \frac{1}{2}\,\ell^2 PdM + \left(\frac{1}{2}\,\ell^2 + \frac{EI}{KAG}\right)MdP + \ell MdM\right]$$

whence, integrating between the limits P = 0, P and M = 0, M, we have

*Of course, the actual shape of the end will depend on how the P stresses are distributed over it; we tacitly assume these stresses to be distributed as they would be if the cantilever extended onward beyond this point.

**Henceforth x replaces z so that $\frac{dy'}{dz}$ becomes $\frac{dv}{dx}$; see Figure 31a.

$$W = \frac{1}{EI}\left[\left(\frac{1}{6}\,\ell^3 + \frac{\ell EI}{2KAG}\right)P^2 + \frac{1}{2}\,\ell^2 PM + \frac{1}{2}\,\ell M^2 + \frac{EI}{KAG}\int_0^P MdP\right]$$

The last term is not unique. For example, if P is applied while M = 0 and the M is added, $\int_0^P MdP = 0\int_0^P dP = 0$, whereas if M is applied first and P is added, $\int_0^P MdP = M\int_0^P dP = MP$. The first three terms of W are the same; however, P and M are applied, and the final state of the cantilever is the same. Hence, it appears as if two different amounts of energy can bring the beam to the same final position, depending upon the manner in which M and P are applied. In other words, energy does not seem to be conserved in a conservative system.

The source of this apparent dilemma is the shear warping effect which has not been taken into account, thus making the expression for W incomplete. As P and M change, the shear warping at the end changes, perhaps as shown in Figure 31b. M does negative work on the end of the cantilever in such a case;* the amount of negative work by M (or by M stresses) when shear distortion of the end cross-section changes, will now be calculated.

Let s (P, y, z) be the x-displacement due to shear warping, or $s = Pf(y, z)$. It may be assumed that $\iint s\,dydz = 0$, hence (see Reference 13) $\iint f(y, z)dydz = 0$ since $P \neq 0$; see Figure 31c. The unknown function of f (y, z) can be connected with the shear-warping constant K by the following analysis:

Consider a case in which P varies along the beam. In such a case $s_{x+dx} > s_x$ by the amount $ds = P_x f(y, z)\,dx$; $P_x = \frac{dP}{dx}$. Thus the strain in the corresponding longitudinal fiber is $\frac{ds}{dx} = P_x f(y, z)$, and the associated normal stress is $\sigma = EP_x f(y, z)$, the stress force acting across the

*When P and M change, the change in shape and position of the cantilever is as shown in Figure 31b. The change in θ gives for work due to M, the value used in the usual "elementary analysis" $Md\theta$. But M is applied by means of stresses, as shown by small arrows. On both the upper and lower part of the end cross section, the additional displacement of elements due to ds caused by dP are opposite to the stresses. Hence, negative work is done by the M stresses when P increases and warps the end cross section more (negative at least, provided M and P have the directions shown).

cross section on the material lying to the left of it being directed toward $+x$ if $P_x f(y, z) > 0$.* Therefore, the shear-warping moment M_s, positive as shown in Figure 31c, is

$$M_s = - \iint y \sigma dy\, dz = - EP_x \iint y f(y, z) dy\, dz$$

The level from which y is measured here does not matter because it has been assumed that $\iint f(y, z) dy\, dz = 0$ so that adding a constant to y does not change this equation.**

From Appendix A2 of Reference 3 (Equations [A25b] and [A26]), when KAG is uniform

$$M_s = - \frac{EI}{KAG} P_x$$

since P here = -V there. Hence, we conclude that

$$\iint y f(y, z) dy\, dz = \frac{I}{KAG}$$

Along the cantilever, however, P is uniform; hence $P_x = 0$ and $\sigma = 0$. Normal stresses on the cross section arise only from bending and have their usual values. In pure bending, on the other hand, the cross sections remain plane, hence the x-displacement is $-y\theta$ at a height y above the neutral axis which is taken as the reference level for bending, the strain is $-y\theta_x$, and the stress is $\sigma_b = -Ey\theta_x$ (σ_b is the force acting on the material lying toward -x, taken positive toward +x as shown in Figure 31d).

Now let P change with time by an amount dP. Then $ds = f(y, z)\, dP$ and

---

*$\sigma = + EP_x f(y, z)$. If $P_x > 0$ and $f(y, z) > 0$, then σ is in tension, so that the stress force acting on the material lying on the side of the cross section toward $x < 0$, taken + in the direction of $x > 0$, is positive. Therefore, $\sigma > 0$.

**$\iint f(y, z) dy\, dz$ was assumed zero to make $\iint s\, dy\, dz = P \iint f(y, z)\, dy\, dz = 0$. Making $\iint s\, dy dz = 0$ makes the total x force on a cross section zero when P varies with x and so stretches or compresses the fibers; the total x-force will then be $E \iint s_x dy\, dz = EP_x \iint f(y, z) dy\, dz = 0$. Thus shear warping will be cleanly separated from longitudinal stretching or compression.

integrating, the work done by σ_b on the end of the cantilever due to dP is

$$\iint \sigma_b \, ds \, dy \, dz = - \iint (E y \theta_x) \, f(y, z) \, dP \, dy \, dz$$

$$= - E\theta_x \, dP \iint y f(y, z) \, dy \, dz$$

or inserting $M = EI\theta_x$ (θ_x being the curvature) and the value found for $\iint y f(y, z) dy dz$, the work is

$$-\left(\frac{M}{I}\right) dP \left(\frac{I}{KAG}\right) = - \frac{MdP}{KAG}$$

Thus the total work done by the applied P and M is not $Pdv + Md\theta$ but

$$dW = Pdv + Md\theta - \frac{MdP}{KAG}$$

$$= \frac{1}{EI}\left[\left(\frac{1}{3} \ell^3 + \frac{\ell EI}{KAG}\right) PdP + \frac{1}{2} \ell^2 PdM + \frac{1}{2} \ell^2 MdP + \ell MdM\right]$$

from which

$$W = \frac{1}{EI}\left[\left(\frac{1}{6} \ell^3 + \frac{\ell EI}{2KAG}\right) P^2 + \frac{1}{2} \ell^2 PM + \frac{1}{2} \ell M^2\right]$$

Thus the law of conservation of energy holds.

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Conclusions

"(1) The tests have established the existence of rudder critical frequencies which, when propeller excited, can produce substantial torsional and flexural modes of rudder vibration and which are predictable by calculation to a practical degree of accuracy. These forces may be of sufficient magnitude to cause appreciable hull vibration, and excessive wear and tear of the rudder and attachments."

"(2) The tests are, however, an introduction to the study of an interesting phenomenon, and nothing more is claimed for them. It is hoped, however, that they will attract the attention of many who have all the facilities for carrying out further tests, with or without the vibration exciter, in order that a more general and comprehensive study may be made of the problem."

"(3) The calculations have omitted on purpose any consideration of the damping and propeller forces which determine the extent of the vibration amplitudes that may be built up, because it is felt that a lot more results are necessary before this aspect of the work is entered upon."

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