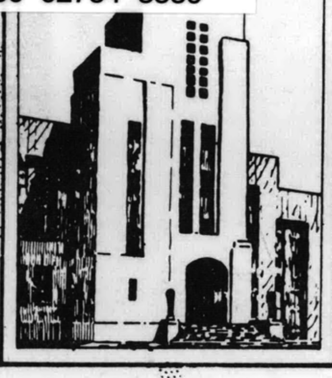


DEPARTMENT OF THE NAVY
DAVID TAYLOR MODEL BASIN

HYDROMECHANICS

AERODYNAMICS

STRUCTURAL MECHANICS

APPLIED MATHEMATICS

LECTURES ON TOPICS IN NONLINEAR DIFFERENTIAL EQUATIONS

by

G. Temple
Oxford University

APPLIED MATHEMATICS LABORATORY

RESEARCH AND DEVELOPMENT REPORT

March 1960

Report 1415

[redacted]

LECTURES ON TOPICS IN NONLINEAR
DIFFERENTIAL EQUATIONS*

by

G. Temple
Oxford University

* Prepared during the author's stay at Applied Mathematics Laboratory, David Taylor Model Basin, January - March 1960.

March 1960

Report 1415

ABSTRACT

These lectures describe some recent researches by the author and one of his graduate students at Oxford University on ordinary nonlinear differential equations. The first lecture is devoted to a search for a "superposition" principle for these nonlinear equations and it determines the class of nonlinear equations for w[redacted]ch a superposition principle exists.

[redacted]e remaining four lectures provide a rigorous, analytical [redacted]ry of the technique invented by Lighthill (1949) for solving nonlinear differential equations with an "irregular" perturbation. Such equations involve a small parameter α and such that the coefficient of the highest derivative vanishes identically, or at the "initial point", when $\alpha = 0$.

The theory is developed from a number of simple examples and given a rigorous form by means of the theory of "dominant functions".

A Superposition Principle for Ordinary, Non Linear Differential Equations

The purpose of this investigation was to advance the theory of the systematic integration of ordinary, non-linear differential equations. It must be admitted at once that, in this attack on the strongly held territory of non-linear equations, the ground occupied and administered is of smaller area than was hoped, and scarcely forms more than a modest bridge head. Nevertheless the tactics of the operation form an elegant, simple and interesting application of the theory of finite, continuous groups.

1. Motivation.

a. It is almost trivial to assert that the comparative simplicity of linear differential equations is due to the existence of a principle of superposition of solutions, and that the comparative difficulty of non-linear differential equations is due to the non-existence of a similar principle. This remark suggests at once that we should seek for a generalization of the principle of superposition which shall be valid for non-linear equations, or, at least, for a substantial class of non-linear equations. Such a generalization should be of considerable value in extending and organizing our methods of solving non linear equations, by providing a process for the construction of general solutions from a number of particular solutions.

b. Modern, high-speed, computing machinery, using methods based on classical existence theories, has provided a means of rapidly calculating particular solutions of non linear differential equations from given initial conditions. But these computational methods can only approach the construction of general solutions by calçulating the particular solutions which correspond to a large number of different initial conditions. A generalization of the principle of superposition should enable us to combine a few particular (numerical) solutions obtained by high-speed computation into a general solution.

c. Finally the engineering problem of testing non linear mechanisms, such as hydraulic servo-mechanisms, might well be simplified if any operation of such a system could be treated as a combination of certain elementary operations, each more easily examined than the general operation.

2. Formulation of the Problem.

In the case of a linear differential equation, or a system of simultaneous linear differential equations, the totality of solutions forms a linear, vector space. Thus, if $y_1(x)$, $y_2(x)$ and $y_3(x)$ are any particular solutions of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(n) \neq 0,$$

the general solution is

$$y = Ay_1 + By_2 + Cy_3$$

where $$A + B + C = 1.$$

For, taking y_3 to be a "particular integral", the functions $y_1 - y_3$ and $y_2 - y_3$ are solutions of the "complementary equation", so that the general solution is

$$y = y_3 + A(y_1 - y_3) + B(y_2 - y_3)$$

$$= Ay_1 + By_2 + (1 - A - B)y_3.$$

The essential features of this particular method of superposition appear to be that

a. The general solution y is expressed as a function F of a certain number p of particular solutions

$$y_1, y_2, --- y_p,$$

and of a certain number n of independent "constants of integration"

$$C_1 = A, \quad C_2 = B.$$

In this example n is fixed by the order (2) of the differential equation, and $p = n + 1$.

b. The form of the function F is the same no matter which particular solutions are employed.

Now there are certain non linear equations for which there is an analogous theory. For example, if y_1, y_2, y_3 are any three independent solutions of the Riccati equation

$$\frac{dy}{dx} = P(x) + Q(x).y + R(x).y^2,$$

then the general ~~solution~~ y is given by the well known cross ratio formula,

$$\frac{y - y_1}{y - y_2} \cdot \frac{y_3 - y_2}{y_3 - y_1} = C, \tag{1}$$

where C is the arbitrary constant of integration. Written more symmetrically this formula is

$$\lambda(y_1 y_2 + y y_3) + \mu(y_1 y_3 + y y_2) + \nu(y_1 y + y_2 y_3) = 0$$

where $\lambda + \mu + \nu = 0.$

Here again where the general solution y expressed as a function $F(y_1, y_2, y_3; C)$ of 3 particular solutions and an arbitrary constant.

This example encourages us to construct a general theory for the construction of general solutions of non linear equations from appropriate numbers of particular solutions.

We note that at present we have no information about the number p of particular solutions which may be required.

3. <u>The Equation</u> $dy/dx = F(x, y)$.

To illustrate the general theory we start with a single equation of the first degree in one independent and one dependent variable,

$$dy/dx = F(x, y), \tag{2}$$

We assume that the general solution of this equation, $y(x)$, can be expressed in the form

$$y = y(y_1, y_2, \ldots, y_x, a), \tag{3}$$

where $y_1 y_2 \ldots y_x$ are n independent particular solutions, and a is an arbitrary constant.

This relation (2 constitutes a law of internal composition between the elements of the set E of solutions of equation (2, and thereby determines the algebraic structure of the set E. In the theory of rings, fields and groups the fundamental law of composition involve only two elements of the corresponding sets E, where here the law of composition involves a number n, which, although unknown a priori, is not necessarily equal to 2.

It seems at first that the composition law (3 will determine a new species of algebraic structure - almost the reverse of the "co-groups" in which two elements are compounded together to yield n elements. The usual law of composition give an "application" F of $E \times E$ in E; the law of composition for a co group gives an application F of E in $E \times E \times E \times \dots \times E$; the law of composition (2 gives an application of $E \times E \times E \times \dots \times E$ in E.

However although the structure given by the composition function F of (3 is not a group, it is easy to see that it can be "embedded" in a group. In fact we have only to form the set of equations

$$y_k' = F(y_1, y_2, \dots, y_n; a_k), \qquad (4$$

$$k = 1, 2, \dots, n,$$

which give n new solutions, $y_1', y_2', \dots y_n'$, in terms of the n old solutions, $y_1, y_2, \dots, y_n$ and n arbitrary constants $a_1, a_2, \dots a_n$. This set of equations define a group, G.

The equations (4 transform any set of n solutions of

$$dy/dx = F(x, y)$$

into another set of n solutions. The transformation is therefore associative.

Also, since by hypothesis any solution is given by (1, we can choose the parameters a_k so that when

$$a_k = \bar{a}_k, \text{ then } y_k' = y_k.$$

i.e. $$y_k = F(y_1, y_2, \dots, y_n; \bar{a}_k)$$

Thus there exists an identical transformation in the set (4. Lastly, for the same reason, we can choose the parameters a_n so that when $a_k = a_k'$ then

$$y_n = g(y_1', y_2', \ldots y_n'; a_k').$$

Hence the equations (4 define a finite continuous group G with n variables y_n and n parameters a_n.

It was E. Vessiot (1893) who first had the happy idea of studying these groups and of classifying the differential equations to which they refer. This drew down upon him a rather severe reprimand from Marius Sophus Lie (1893) whose heavy hand had lain upon group theory for many years.

Lie pointed out quite properly that Vessiot's theory was a special case of Lie's own theory of "Fundamental solutions" of differential equations (Lie, 1893). In Lie's theory it is supposed that there are n <u>special</u> or <u>fundamental</u> solutions $y_1, y_2, — y_n$ such that the general solution can be expressed in the form

$$y = g(y_1, y_2, \ldots, y_n; a),$$

and there is no question of this relation forming a <u>general</u> law of composition. In fact we may not know which particular solutions are fundamental; so that Lie's general theory does not seem so useful as Vessiot's theory, which, moreover, has the advantage of being expressed with great elegance and in French.

4. <u>Vessiot's Lemma on Simply Transitive Groups.</u>

The study of the "embedding" group (4 is considerably simplified by the fact that in the set of equations (4 the roles of variables and parameters can be interchanged to yield a group H which is the direct product of n groups, each of the form

$$b' = h(y_1, y_2, \ldots y_n; b),$$

with one variable b and n parameters $y_1, y_2, \ldots y_n$.

The lemma on the interchange of variables and parameters depends only on the fact that the group G is simply transitive, and is valid for any group with this property.

Consider then any simply transitive group G in n variables x_i and n parameters a_i, with the typical transformation

$$x'_s = g_s(x_1, x_2, \ldots, x_n; a_1, a_2 \ldots a_n) \tag{5}$$

$$\equiv g_s(x; a) \qquad s = 1, 2, \ldots, n$$

The associative law for a group is expressed by the conditions

$$x''_s = g_s(x'; b) = g_s(x; c) \tag{6}$$

then $$c_s = \phi_s(a; b).$$

Take x, a, c as sets of independent variables, and x', b as sets of dependent variables. Differentiate (4 with respect to a_t. Then

$$\frac{\partial g_s(x'; b)}{\partial x'_\alpha} \frac{\partial x'_\alpha}{\partial a_t} + \frac{\partial g_s(x', b)}{\partial b_\alpha} \frac{\partial b_\alpha}{\partial a_t} = 0 \tag{7}$$

(using the Einstein summation convention.)

By a fundamental theorem the necessary and sufficient conditions that equation (5 should define a group are that the functions $g_s(x', b)$ should be n integrals of (7 which reduce to the variables x_s when the parameter $b_1 - b_n$ take certain particular values $b_1^x - b_n^x$. Moreover the determinant $\|\partial g_s / \partial b_\alpha$

must not vanish identically. And if the group is simply transitive the

$\| \partial g_s / \partial x'_\alpha \|$ must not vanish identically either.

It is immediately evident that there is complete symmetry as between the set of variables x'_α and the set of parameters a_t in the condition (5. From (5 we can obtain the usual equations

$$\frac{\partial g_s}{\partial x'_\alpha} = \xi_{s\rho}(x')\, A_{\rho\alpha}(b),$$

and we could also obtain the similar equation

$$\frac{\partial g_s}{\partial b_\alpha} = \eta_{s\rho}(b)\, B_{\rho\alpha}(x')$$

Hence we conclude at once that equation (5 also defines a semi-group in the variables a_s with parameters x'_s. It only remains to establish the existence of the identical transformation. This is easily shown by making a preliminary change of parameters in (4. We introduce the new parameters β_s defined by

$$\beta_s = g_s(\bar{x}\,;\, a),$$

where $\bar{x}_i$, $\bar{x}_2$ —— $\bar{x}_n$ is some set of fixed value of $n_1 - n_n$.

Then the transformation equation becomes

$$x'_s = g_s(x;\, a) \qquad\qquad x'_s = \gamma_s(x;\, \beta),$$

$$\beta_s = g_s(\bar{n};\, a)$$

and it is obvious that, when the <u>parameters</u> x_s take the values $\bar{x}_s$, then $x'_s = \beta_s$, i.e. we have the identical transformation.

Therefore with any transitive group

$$G \quad x'_s = g_s(x;\, b) \qquad \text{(variables x, parameter b)} \qquad (8$$

we can associate another transitive group

$$b'_s = g_s(x;\ b\ ) \qquad \text{(variables } b\text{, parameter } x\text{)} \tag{9}$$

I have not come across this theorem elsewhere in a rapid perusual of the literature. It is possible to give a direct proof without differentiating equations (6.

To elucidate the general theory consider the group associated with the Riccati equation (1

$$\frac{y'_k - y_1}{y'_k - y_2} = a_k \frac{y_3 - y_1}{y_3 - y_2}$$

Introduce new parameters β_k defined by the equations

$$\frac{\beta_k - \omega_1}{\beta_k - \omega_2} = a_k \frac{\omega_3 - \omega_1}{\omega_3 - \omega_2} .$$

Then

$$\frac{y'_k - y_1}{y'_k - y_2} = \frac{y_3 - y_1}{y_3 - y_2} \cdot \frac{\omega_3 - \omega_2}{\omega_3 - \omega_1} \cdot \frac{\beta_n - \omega_1}{\beta_n - \omega_2} \tag{10}$$

$$= c\ \frac{\beta_k - \omega_1}{\beta_k - \omega_2} ,$$

where c is a function only of y_1, y_2, y_3, ω_2, ω_3. It is clear that these equations (10 not only define a transformation from (y_1, y_2, y_3) to (y'_1, y'_2, y'_3) with parameters $(\beta_1, \beta_2, \beta_3)$, but also a transformation from $(\beta_1, \beta_2, \beta_3)$ to (y'_1, y'_2, y'_3) with a parameter c depending on (y_1, y_2, y_3). Also the transformation of the second type clearly form a group. Thus the roles of the variables (y_1, y_2, y_3) and the parameter, $(\beta_1, \beta_2, \beta_3)$ can be interchanged.

5. <u>The Equations of the Type $dy/dx = F(x, y)$ which are Soluble by Comparison</u>

To apply Vessiot's lemma to the group

$$y'_x = g_n(y_1, y_2, \dots ; a_k) \quad n = 1, 2, \dots n \quad (4$$

we introduce the new parameters b_k by means of the relation

$$b_n = g_k(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n; a_k)$$

and then obtain the group

$$b_k = h(y_1, y_2, \dots, y_n; b_k) \qquad n = 1, 2, \dots n,$$

which is clearly the direct product of n groups each of which has the form

$$b' = b(y_1, y_2, \dots, y_2; b),$$

i.e. with one variable and n parameters $y_1, y_2, \dots y_n$.

Now Lie has shown that when the number of variables is unity the number of essential parameters can only be 1, 2, or 3 that by a suitable change of variable and parameters these groups can be put in the forms

1) $\quad b' = yb_1$

2) $\quad b' = yb_1 - y_2$

3) $\quad b' = \dfrac{yb_1 + y_2}{yb_3 + 1}$

An easy calculation then shows that the corresponding differential equation is of the form

$$dy/dx = Qy$$

$$dy/dx = P + Qy$$

$$dy/dx = P + Qy + Ry^2$$

This is a somewhat disappointing result since it implies that the only nonlinear equation in y and dy/dx soluble by rhe present method is the Riccati eqn. We can however considerably extend the range of application, but before doing so we shall consider the generalization of Vessiot's theory to a system of equations of the form

$$\frac{dy_k}{dx} = F_n(y_1, y_2, \ldots, y_n, x) \quad n = 1, 2, \ldots n \tag{11}$$

6. Guldberg's Theory

This generalization was given almost immediately by A. Guldberg (1893). Let

$$y_k = y_k^{(\ell)}(x) \qquad \begin{array}{l} n = 1 - n \\ \ell = 1 - p. \end{array}$$

denote any p sets of particular solutions of (11. It is assumed that any set of solutions, y_k' can be expressed in the form

$$y_k' = g_n(y; a) \tag{12}$$

where y stands for the set of pn functions $y_n^{(\ell)}$ and a for a set of n constants of integration, a_1, a_2 ——— a_n.

The structure defined by the law of composition (12 can be embedded in a group

$$y_k^{(\ell)} = g_k(y; a^{(\ell)}) \tag{13}$$

with pn variables $y_k^{(\ell)}$ and p sets of n parameters

$$a^{(\ell)} = (a_1^{(\ell)}, a_2^{(\ell)} \ldots, a_n^{(\ell)}.$$

Just as before we introduce new parameters $b_n^{(\ell)}$ defined by the equation

$$b_n^{(\ell)} = g_k(\bar{y}; a^{(\ell)})$$

when $\bar{y}$ denotes some fixed set of values of the pn variables y, and eliminate the old parameter a. Thus we obtain a simply transitive group

$$y_k^{(\ell)'} = h_k(y; b) \tag{14}$$

in pn variables and pn parameters.

By Vessiot's lemma the same equations define a group

$$b_n^{(\ell)'} = h_n(y; b)$$

in which the variables are $b_n^{(\ell)}$ and the parameters are $y_n^{(\ell)}$.

This group is the direct product of p groups each of the type

$$b_k = h_k(y; b_1, b_2, \ldots, b_n)$$

with n variables and pn parameters.

This group is p times transitive, thence by a theorem due to Lie

$$p \leqslant n + 2,$$

i.e. the possible values of p are 1, 2, . . . , n + 2. (When p = n + 2 the group is similar to the general projective group.)

It is now possible to drew up a systematic and complete catalogue of the various types of systems of differential equations which can be solved by the method of "composition."

7. Return to Equation $dy/dx = p(x, y)$

The extent of the domain of equations of the 1st degree in x, y which are soluble by composition, can be considerably enlarged by writing the basic differential equation in parametric form as

$$\frac{dx}{dt} = X(x, y) \qquad \frac{dy}{dt} = Y(x, y).$$

Guldberg's theory then establishes the existence of a group H in 2 variables b_1, b_2 and 2p parameters $x^{(1)}$, $y^{(1)}$, $x^{(2)}$, $y^{(2)}$. where $p \leqslant 4$. This result considerably extends the possible field of application, but also leads to the ineluerable conclusion that only a very few types of non linear equations are soluble by composition.

A simple example is furnished by the Clairaut equation

$$y = px + p^2, \qquad p = dy/dx.$$

The general solution is, of course,

$$y = cx + c^2$$

and this can be expressed in terms of any particular solution

$$y_1 = kx_1 + k^2$$

by means of the composition law

$$x = ax_1, \quad y = a^2 y_1.$$

8. Conclusion

The painstaking investigations of Lie have provided an exhaustive classification of all transitive groups in 1, 2 and 3 variables, and this analysis could be extended to more variables. We are therefore in a position to give a complete list of all the types of non linear equations (or systems of such equations) which are soluble by "composition." It would be most satisfactory if we could prove that any non linear equation (or system of equations) is soluble by composition in terms of a finite number of particular solutions. That this conjecture is false can however be shown at once by reference to equations with a general solution of the form

$(y - y_1)(y - y_2) \ldots (y - y_p)$ = constant, where $y_1 y_2 \ldots, y_p$ are given functions of n. Such equations have the form

$$\sum_{n=1}^{p} \frac{p - p_k}{y - y_k} = 0,$$

where $p_n = dy_k/dx$, and $p = dy/dx$. To construct the general solution we need p particular solutions,

$$y = y_k, \qquad k = 1, 2 \ldots, p,$$

and we can make the number p arbitrarily large, and certainly in excess of the maximum, $p = 3$, appropriate to equations soluble by composition.

We conclude therefore that the various types of differential equations soluble by composition certainly does not exhaust all possible types, so that the range of our results is definitely limited. It appears therefore that there is no general analogue of the principle of superposition valid for non linear equations; which implies that the problem of non linear equations is even more difficult than we had anticipated.

9. Acknowledgements.

This work was carried out while the author was at the Institute for Advanced Study at Princeton (under Contract No.
from the Office of Ordnance Research) and described in a lecture at the David Taylor Model Basin at Washington.

References

Darboux, G. Bull. Sc. Math (2), 2, (1878). p. 72

Guldberg, A. Comptes Rendus de l'Academie des Sciences, Paris, 116, 1893, p. 964.

Lie., M. S. Comptes Rendus de l'Academie des Sciences, Paris, 116, 1893, p. 1233

Leipziger, Bexichte, 1893, p. 341

Math. Ann. 25, p. 71.

"Vorlesungen iiber Continuierliche Gruppen," 1893, Chap. 24

Vessiot, E. Annales de l'École Normale Supérieure, (3), 10, 1893, p. 52

Comptes Rendus de l'Academie des Sciences, Paris, 116, 1893, pp. 427, 959, 1112.

The "P-L-K" Method

1. Introduction.

The object of this investigation is to give a rigorous, analytical theory of the technique invented by Lighthill (1949) for solving differential equations with an irregular perturbation - the so-called "PLK" method. The present theory has been developed by the author and by Mr. P. Lampitt, one of his research students at Oxford.

The method adopted is to state the problem just for a single equation of the first order and degree, and to illustrate the difficulties by means of a number of elementary examples. We then describe Lighthill's method of the auxiliary variable and the author's method of regularising the differential equation. The formal proof of the validity of the method of regularisation is then developed by means of the theory of dominant functions, and a number of illustrative examples are given.

The method is then generalized for a system of differential equations of the first order - or for a differential equation of arbitrary order.

2. Regular and Irregular Perturbations.

The standard form for an ordinary differential equation of the first order and degree is

$$\frac{dy}{dx} = F(x, y),$$

but in mathematical physics we are frequently interested in equations which involve a "small" parameter α in the form

$$\frac{dy}{dx} = F(x, y, \alpha).$$

We often think of the terms involving α as introducing a "perturbation" into the equation

$$\frac{dy}{dx} = F(x, y, 0),$$

and this latter is often described as the "unperturbed" or "reduced" equation.

The perturbation is said to be "regular" at a point (x, y) if the function $F(x, y, \alpha)$ is analytic in the complex variable α, near the origin $\alpha = 0$, for the prescribed values of x and y, i.e., if $F(x, y, \alpha)$ possesses a Taylor expansion

$$F(x, y, \alpha) = F(x, y, 0) + \sum_{n=1}^{\infty} \alpha^n F_n(x, y),$$

valid in some disc, $|\alpha| \leqslant A$.

If the perturbation is regular at all points of a domain Δ of the variables x and y then it is said to be regular in Δ.

If, however, the function $F(x, y, \alpha)$ is not analytic in α near $\alpha = 0$, for prescribed values of x and y, then the perturbation is said to be "irregular" at this point.

There are two obvious techniques for searching for solutions of perturbed differential equations.

In the case of equations with a perturbation regular in a domain Δ we naturally look for solutions of the type

$$y = y_0(x) + \sum_{n=1}^{\infty} \alpha^n y_n(x),$$

i.e., as a power series in the parameter α, reducing to a solution of the reduced equation,

$$y = y_0(x),$$

when $\alpha \longrightarrow 0$.

In the case of equations with a perturbation which is irregular at some point of Δ, our only hope seems to be to look for some transformation which will reduce the equation to one with a regular perturbation in Δ. Lighthill's achievement was to discover such a transformation. Our researches are directed to the systematization and validation of Lighthill's discovery.

What makes this problem of particular interest in mathematical physics is that the "reduced" equation is often (although not always) linear and easily soluble, and that the perturbation is regular except at certain points or along certain curves in the x, y — plane.

In order to clarify the situation and to simplify the analytic theory it seems, however, advantageous to begin with the very simplest example, namely the first problem discussed by Lighthill.

3. Typical Equations of the First Order and Degree.

All the characteristic features of the "PLK" method are exemplified in the typical equation

$$(x + \alpha u) \frac{du}{dx} + q(x) u = r(x),$$

studied by Lighthill (Phil. Mag. (7), Vol. 40, 1949, pp. 1179 - 1201) and by Wasow (Journal of Rational Mechanics and Analysis, Vol. 4, 1955, pp. 751 - 767). Here $q(x)$ and $r(x)$ are analytic near $x = 0$. The perturbation is regular except at $x = 0$, and the reduced equation is

$$x \frac{du}{dx} + q(x)u = r(x).$$

In fact it is almost sufficient to examine some trivial specializations of the Lighthull equation, e.g.

$$E_1 \qquad (x + \alpha u)\frac{du}{dx} + u = 0,$$

$$E_2 \qquad (x + \alpha u)\frac{du}{dx} - u = 0,$$

$$E_3 \qquad (x + \alpha u)\frac{du}{dx} - u = x,$$

$$E_4 \qquad (x^2 + \alpha u)\frac{du}{dx} + 2xu = 1,$$

with the initial conditions $u = 1$ at $x = 1$.

First of all we write down the exact solutions of the perturbed and reduced equations in order to gain an appreciation of the nature of the influence of the perturbation. Then we consider the techniques introduced by Carrier (Advances in Applied Mechanics Vol. 3, 1953, pp. 1 - 19, Communications on Pure and Applied Mathematics, Vol. 7, 1954, pp. 11 - 17), Lighthill and Temple (Proc. Int. Congress of Mathematicians, Edinburgh, 1958.)

We list below the exact solutions of these equations and of the reduced equations, together with the leading terms in the expression for u near $\alpha = 0$: -

E_1 $$ux + \frac{1}{2}\alpha u^2 = 1 + \frac{1}{2}\alpha,$$

$$ux = 1, \qquad (\alpha = 0),$$

$$u = \frac{1}{x} + \frac{\alpha}{2x} + 0\left(\frac{\alpha}{2x^3}\right).$$

E_2 $$x = u + \alpha u \log |u|,$$

$$x = u, \qquad (\alpha = 0),$$

$$u = x - \alpha x \log |x| + 0\,(\alpha^2 x\, \overline{\log |x|}\,]^2).$$

E_3 $$\left\{\frac{x + \beta u}{1 + \beta}\right\}^{1-\beta} = \left\{\frac{x - \beta u}{1 - \beta}\right\}^{1+\beta}, \qquad (\alpha = \beta^2),$$

$$u = x + x \log |x|, \qquad (\alpha = 0),$$

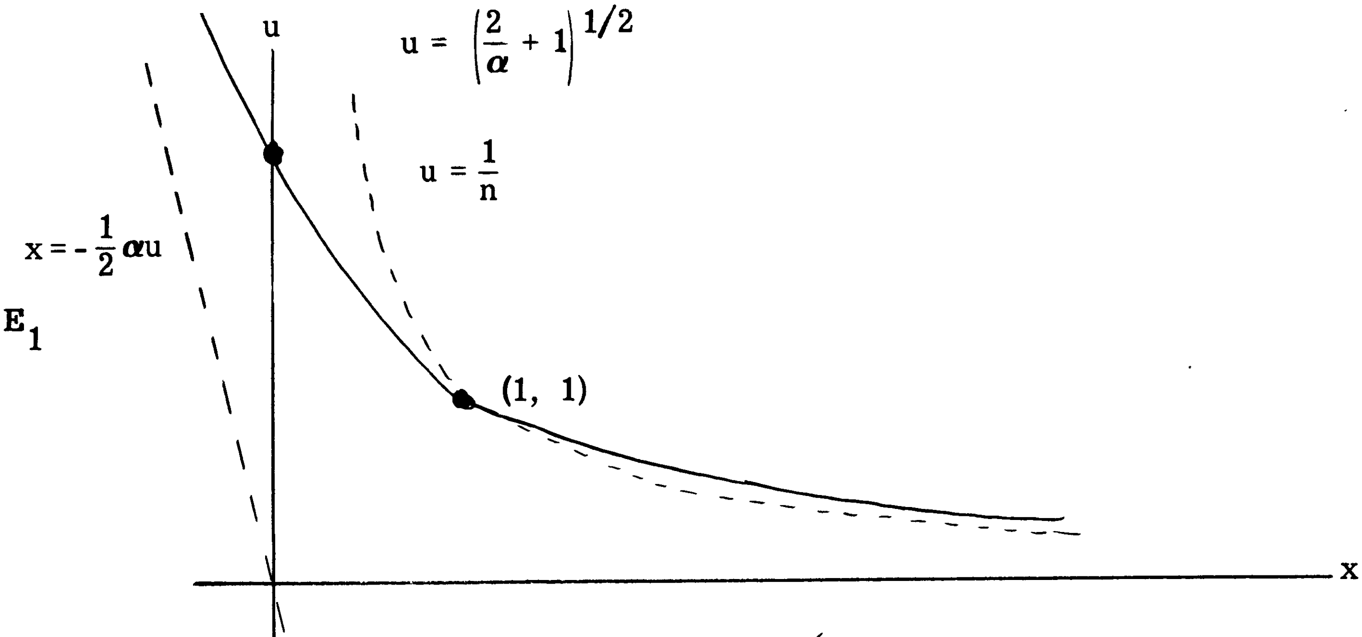
$$u = x + x \log |x| + 0\,(\alpha x\, \overline{\log |x|}\,]^2).$$

E_4 $$x^2 u + \frac{1}{2}\alpha u^2 = x + \frac{1}{2}\alpha,$$

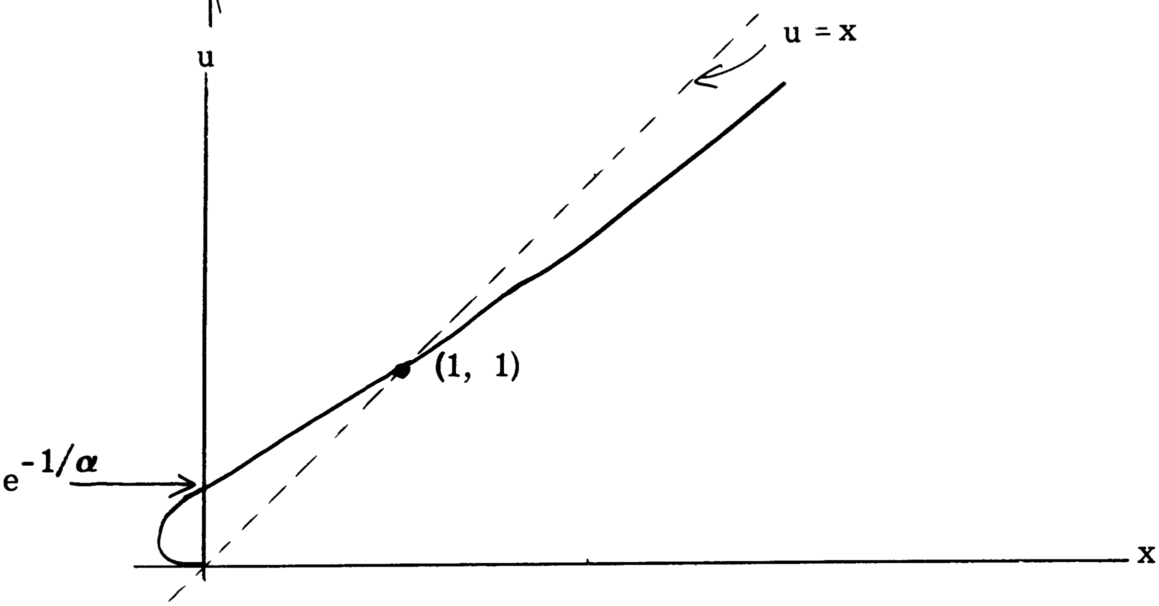
$$u = 1/x, \qquad (\alpha = 0),$$

$$u = \frac{1}{x} + 0\left(\frac{\alpha}{x^2}\right).$$

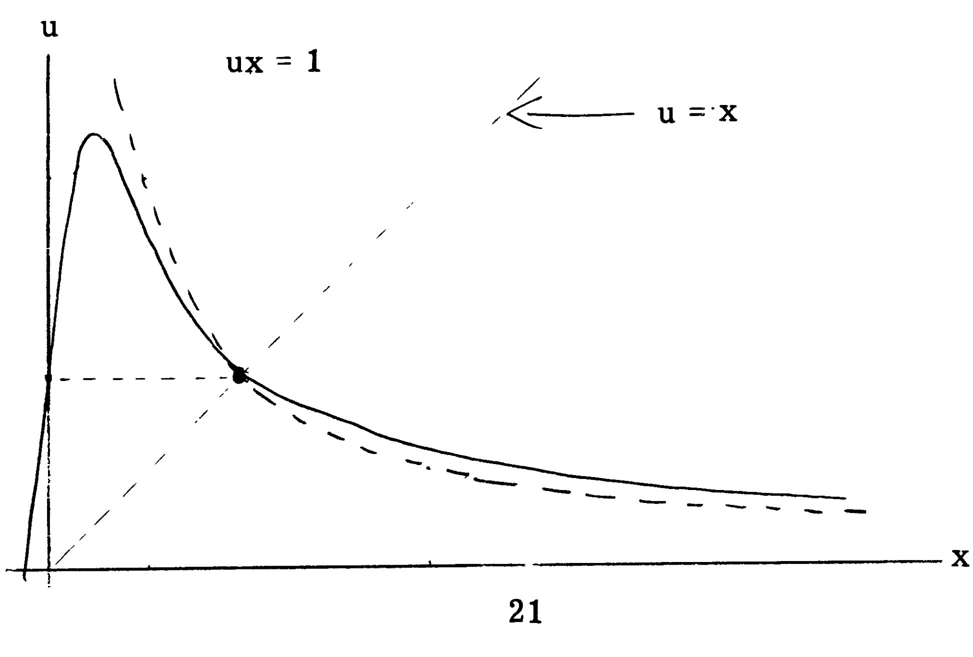
E_1


u
$u = \left(\frac{2}{\alpha} + 1\right)^{1/2}$
$u = \frac{1}{n}$
$x = -\frac{1}{2}\alpha u$
(1, 1)
x

E_2


u
u = x
(1, 1)
$e^{-1/\alpha}$
x

E_4


u
ux = 1
u = x
x

The following general conclusions can be drawn from the eqns. and Figures representing the solutions: -

1) It is impossible to express the solution of the perturbed equation as a power series in the parameter α uniformly convergent near $\alpha = 0$.

2) The solution of the reduced equation is an asymptotic approximation to the solution of the perturbed equation in cases E_1, E_4 but <u>not</u> in case E_2.

3) The importance of the perturbation, as estimated by the ratio $\alpha u/x$, calculated from the reduced solution, is

E_1 $\quad 0(\alpha/x^2)$

E_2 $\quad 0(\alpha)$

E_3 $\quad 0(\alpha \log x)$

E_4 $\quad \alpha u/x^2 = 0(\alpha/x^3)$

Hence the perturbation cannot be neglected near the singular point $x = 0$ - but we note that the rest is misleading in the case E_2.

4) Carrier's method may or may not give information about the perturbed solution.

4. Carrier's "Boundary Layer" Theory

Carrier (loc. cit.) has devised a "boundary layer" technique which can be very effective in improving the reduced solution in the neighborhood of a singular point. This technique however does not seem to apply in cases E_2 or E_4 where all indications given by the reduced solution are misleading.

In case E_1 we can introduce a crude form of boundary layer theory as follows: - The reduced solution

$$u = 1/x$$

may be assumed to approximate to the perturbed solution, except when the perturbation term αu is large, i.e., near the "boundary" $x = 0$. In this region the dominant terms in the equation are, presumably,

$$\alpha u \frac{du}{dx} + u = 0,$$

with the solution $\alpha u + x =$ a constant, c.

We choose the constant c so that this "boundary" solution joins smoothly with the reduced solution $xu = 1$ or some point (x_o, y_o). We easily find that

$$x_o = \alpha^{1/2}, \qquad u_o = \alpha^{-1/2}, \qquad c = 2\alpha^{1/2},$$

so that the boundary layer solution is

$$\alpha u + x = 2\alpha^{1/2},$$

giving $u = 2\alpha^{-1/2}$ at $x = 0$, instead of the accurate value $(2/\alpha + 1)^{1/2}$.

5. Lighthill's Method of the Auxiliary Variable.

The problem attacked by Lighthill was to express the solution of a perturbed equation as a function of the parameter α in a form which is uniformly convergent in the neighborhood of $\alpha = 0$, and which converges as $\alpha \longrightarrow 0$ to the reduced solution. Lighthill's method is to express both the dependent variable u and the independent variable x as power series in α with coefficients which are functions of a new auxiliary variable z. These series are of the form

$$u = u_0(z) + \alpha u_1(z) + \alpha^2 u_2(z) + \ldots$$
$$x = z + \alpha x_1(z) + \alpha^2 x_2(z) + \ldots,$$

and, as $\alpha \longrightarrow 0$, this solution tends to

$$u = u_0(z), \qquad x = z,$$

so that $u = u_0(x)$ must be the solution of the reduced equation.

Thus, in the case of the equations discussed above, the exact solutions of the perturbed equations can be parameterized as follows: -

$$E_1 \qquad u = \frac{1}{z},$$
$$x = (1 + \tfrac{1}{2}\alpha)z - \frac{\alpha}{2z};$$

$$E_2 \qquad u = z,$$
$$x = z + \alpha z \log z;$$

$$E_3 \qquad x = \tfrac{1}{2}(1 + \beta)z^{1+\beta} + \tfrac{1}{2}(1 - \beta)z^{1-\beta},$$
$$(\alpha = \beta^2).$$
$$\beta u = \tfrac{1}{2}(1 + \beta)z^{1+\beta} - \tfrac{1}{2}(1 - \beta)z^{1-\beta},$$

whence $x = z + 0(\alpha)$,

$$u = z + z \log z + 0(\alpha);$$

E_4 Put $u - 1 = m(x - 1)$

$$\text{Then } x = \frac{-\Delta - (1 + \frac{1}{2}\alpha m^2)}{2m}$$

$$u = 1 - m - \frac{1}{2}(1 + \frac{1}{2}\alpha m^2) - \frac{1}{2}\Delta,$$

where $\Delta = -\left\{(1 + \frac{1}{2}\alpha m^2)^2 - 4m\,(\alpha m - \frac{1}{2}\alpha m^2)\right\}^{1/2}$,

and $$m = -\frac{1}{z} + 0(\alpha)$$

whence $$x = z + 0(\alpha),$$

$$u = \frac{1}{z} + 0(\alpha).$$

Of course Lighthill's method is designed to apply precisely when the exact analytical form of the perturbed solution is unknown. In these circumstances the series for u in powers of α are substituted directly into the perturbed differential equation

$$(x + \alpha u)\frac{du}{dx} + q(x)u = r(x),$$

and the coefficients of powers of α are equated to zero.

At each stage this procedure yields one equation for the two unknown coefficients $u_n(z)$ and $x_n(z)$. This equation is then split into two in such a way that the power series for u and z become uniformly convergent in some region $|\alpha| < A$.

It is a tedious matter to carry out the calculation but Lighthill has sketched a proof that it can be successfully carried through to yield the desired result.

A more detailed proof on the same lines has been given by Wasow (loc. cit.), and some notes on the apparent limitations of Lighthill's method have been given by Carrier (loc. cit.).

The importance of Lighthill's technique makes it desirable to provide a rigorous general theory which shall be applicable to as wide a class as possible of both ordinary and partial differential equations. Such a theory is developed in the following sections.

6. Temple's Method of Regularization.

In Lighthill's method the single power series for u,

$$u = \sum_{0}^{\infty} \alpha^n f_n(x),$$

which cannot converge uniformly near $x = 0$, is replaced by two series

$$u = \sum_{0}^{\infty} \alpha^n u_n(z), \qquad x = z + \sum_{1}^{\infty} \alpha^n x_n(z),$$

which are both uniformly convergent near $x = 0$. The coefficients u_n and x_n are determined by splitting the equation deduced from the perturbed differential equation by equating powers of α.

In Temple's method the original differential equation is split into two equations, each of which has a regular perturbation. Thus the equations for u and x are separated once for all at the very beginning of the calculation.

Thus we replace the original equation

$$(x + \alpha u) \frac{du}{dx} + q(x)u = r(x),$$

by the pair of equations,

$$\frac{dx}{dt} = x + \alpha u,$$

and
$$\frac{du}{dt} = r(x) - q(x)u,$$

with the initial condition that $t = 0$ when $x = 0$. It is manifest that the derivatives of x and of u with respect to t are now analytic functions of α near $\alpha = 0$, and physical intuition and mathematical analysis agree in recognizing that such a pair of equations possess a solution in which x and u are power series in α, provided that the functions $r(x)$ and $q(x)$ are analytic in x near $x = 0$. The formal proofs will be given later after we have studied a few examples in detail.

This method of "regularization" provides a systematic means of parameterizing the exact solution of the perturbed equation.

The solution of the "regularized" equations is of the form

$$x = \sum_{n=0}^{\infty} \alpha^n f_n(t) ,$$

$$u = \sum_{n=0}^{\infty} \alpha^n g_n(t).$$

In order to exhibit the relation of these solutions to the reduced solution we make the change or variable

$$t = \log z.$$

Then the regularized equations become

$$z \frac{dx}{dz} = x + \alpha u,$$

and $$z \frac{du}{dz} = r(x) - q(x)u,$$

while their solution is of the form

$$x = \sum_{n=0}^{\infty} \alpha^n x_n(z),$$

$$u = \sum_{n=0}^{\infty} \alpha^n u_n(z).$$

Hence $$z \frac{dx_o}{dz} = x_o,$$

and $$z \frac{du_o}{dz} = r(x_o) - q(x_o)u_o.$$

Thus $x_o = Cz$ and $u_o(Cz)$ is a solution of the reduced equation. There is no loss of generality in taking $C = 1$, and then the solution of the regularised equations is in Lighthill's form -

$$x = z + \sum_{n=1}^{\infty} \alpha^n x_n(z),$$

$$u = u_o(z) + \sum_{n=1}^{\infty} \alpha^n u_n(z).$$

7. Examples of Regularization.

Before developing the general theory it is instructive to consider the four simple examples introduced earlier. We therefore list the regularised equations and their solutions for these four cases: -

$$E_1 \qquad z \frac{dx}{dz} = x + \alpha u,$$

$$z \frac{du}{dz} = - u,$$

$$x = z + \frac{1}{2}\alpha(z - \frac{1}{z}) ,$$

$$u = \frac{1}{z}.$$

E_2 $$z\frac{dx}{dz} = x + \alpha x,$$

$$z\frac{du}{dz} = u ,$$

$$x = z + \alpha z \log|z|,$$

$$x = z .$$

E_3 $$z\frac{dx}{dz} = x + \alpha u,$$

$$z\frac{du}{dz} = u + x,$$

$$x + \beta u = (1 + \beta)z^{1+\beta}, \qquad (\alpha = \beta^2),$$

$$x - \beta u = (1 - \beta)z^{1-\beta}.$$

E_4 $$\frac{dx}{dt} = z^2\frac{dx}{dz} = x^2 + \alpha u$$

$$\frac{du}{dt} = z^2\frac{du}{dz} = 1 - 2xu$$

N. B. In this case we have used the transformation

$$t = - 1/z$$

instead of $t = \log z$ in order to ensure that the series for x begins with the term $x_o = z$. We find that

$$\frac{d^2x}{dt^2} = \alpha + 2x^3,$$

whence $$\left(\frac{dx}{dt}\right)^2 = \alpha^2 + 2\alpha x + x^4$$

But $$\left(\frac{dx}{dt}\right)^2 = (x^2 + \alpha u)^2,$$

therefore $$ux^2 + \frac{1}{2}\alpha u^2 = \frac{1}{2}\alpha + x.$$

8. <u>Systems of Ordinary Differential Equations.</u>

The method of regularization is easily extended to ordinary differential equations of <u>any</u> order, or to systems of ordinary differential equations. The standard form of such a system, involving an independent variable x, n dependent variables $u_1, u_2, \ldots, u_n$, and a parameter α is

$$\frac{du_m}{dx} = F_m(u_1, u_2, \ldots, u_n, x, \alpha)$$

$$(m = 1, 2, \ldots, n)$$

For example an equation of the second order,

$$\frac{d^2u}{dx^2} = f\left(\frac{du}{dx}, u, x, \alpha\right)$$

can be expressed in this form by writing

$$u = u_1, \qquad \frac{du}{dx} = u_2.$$

Then $$\frac{du_1}{dx} = u_2$$

$$\frac{du_2}{dx} = f(u_2, u_1, x, \alpha).$$

If each of the functions F_m is analytic in α, near $\alpha = 0$, for prescribed values of x, u_1, u_2, ..., u_n then the perturbation is said to be regular for these prescribed values. Otherwise it is irregular.

We shall consider the case of an irregular perturbation in which each function F_m is expressible in the form

$$F_m = \frac{N_m}{D_m} \quad ,$$

where N_m and D_n are each analytic in α.

We then replace the perturbed equations by the regularised equations

$$\frac{du_m}{dt} = D_1 D_2 \ldots D_{m-1} N_m D_{m+1} D_{m+2} \ldots D_n ,$$

$$\frac{dx}{dt} = D_1 D_2 \ldots D_n .$$

in which the right hand sides are manifestly analytic in α.

9. Further examples of regularization.

The following examples are discussed by Lighthill (loc. cit.) and it is of interest to see them handled by the method of regularization

(I) Given the equation

$$(x + \alpha u)\, du/dx + (2 + x)\, u = 0$$

and the initial conditions

$$u = e^{-1} \quad \text{at} \quad x = 1,$$

to estimate the value of u at $x = 0$.

The regularized equations are

$$z\, dx/dz = x + \alpha u,$$

$$z\, du/dz = -(2 + x)\, u.$$

Hence

$$x_o = z \text{ and } u_o = e^{-z} \cdot z^{-2}$$

To estimate the value of u at $x = 0$ we need only the first approximation given by the equations

$$x_o(z) + \alpha x_1(z) = 0$$

$$u = u_o(z),$$

and $x_1(z)$ is needed only for values of z near zero. Hence the equation for x_1 can be approximated as

$$z\, dx_1/dz = x_1 + u_o \doteqdot x_1 + z^{-2}.$$

with the solution

$$x_1 = \frac{1}{3}z - \frac{1}{3z^2} \doteqdot -\frac{1}{3z^2} .$$

Therefore at $x = 0$

$$\alpha = -x_o/x_1 = 3z^3,$$

$$z = (\tfrac{1}{3}\alpha)^{1/3},$$

and

$$u = u_o = z^{-2} = (\tfrac{1}{3}\alpha)^{-2/3}.$$

(II) $(x + \alpha u)\, du/dx - \frac{1}{2}u = 1 + x^2,$

with $u = 1$ at $x = -1$.

This equation requires a preliminary transformation before Lighthill's method can be applied. The regularized equations.

$$z\frac{dx}{dz} = x + \alpha u$$

$$z\frac{du}{dz} = \frac{1}{2}u + 1 + z^2$$

Furnish the coefficients

$$x_o = z, \qquad u_o = -2 + \frac{1}{3}z^{1/2} + \frac{2}{3}z^2,$$

$$x_1 = 2 - \frac{2}{3}z^{1/2} + \frac{2}{3}z^2 - 2z,$$

$$u_1 = 8z - \frac{4}{3}z^{3/2} + \frac{14}{15}z^3 - \frac{8}{3}z^2 - \frac{74}{15}z^{1/2}.$$

These differ from those obtained by Lighthill, so that presumably our auxiliary variable z differs from his.

(III) $P\frac{du}{dx} = Q, \quad \frac{dv}{dx} = u,$

$$P = 1 - x^2 + (\gamma + 1)xu - (\gamma - 1)v - \frac{1}{2}(\gamma + 1)u^2,$$

$$Q = -\frac{u}{x}\left\{1 + (\gamma - 1)\left(xu - v - \frac{1}{2}u^2\right)\right\},$$

with the boundary conditions

$$u(\alpha) = \alpha, \quad u(M) = 2(M - M^{-1})/(\gamma + 1),$$

$$v(M) = 0, \qquad M > 1.$$

It is required to estimate the value of M in terms of the small parameter α.

We write

$$M = 1 + \alpha^2 M_1 + \alpha^4 M_2 + \ldots,$$

whence
$$u(M) = \frac{4\alpha^2 M_1 + \alpha^4(4M_2 - 2M_1^2) + \ldots}{\gamma + 1}$$

The regularised equations are

$$\frac{du}{dt} = Qx, \qquad \frac{dx}{dt} = Px,$$

and, in order to facilitate comparison with Lighthill's solution, we write

$$u = u_{-1} + \alpha^2 u_o + \alpha^4 u_1 + \ldots,$$

$$v = v_{-1} + \alpha^2 v_o + \alpha^4 v_1 + \ldots,$$

$$x = x_o + \alpha^2 x_1 + \alpha^4 x_2 + \ldots,$$

Then the initial conditions are

$$x = M, \quad x_o = 1, \quad x_1 = M_1, \quad x_2 = M_2, \ldots$$

$$v_{-1} = 0, \quad v_o = 0, \quad v_1 = 0, \ \ldots$$

$$u_{-1} = 0, \quad u_o = \frac{4M_1}{\gamma+1}, \quad u_1 = \frac{4M_2 - 2M_1^2}{\gamma + 1}$$

We easily find that $u_{-1} = 0$ and $v_{-1} = 0$,

$$\frac{dx_o}{dt} = x_o(1 - x_o^2),$$

whence
$$e^{-t} = \frac{(1 - x_o^2)^{1/2}}{x_o}$$

It is convenient to write $t = - \log \tau$, and then

$$x_o = (1 + \tau^2)^{-1/2}.$$

Also
$$\frac{du_o}{dt} = - u_o,$$

whence
$$u_o = e^{-t} = \tau, \text{ and } M_1 = 0.$$

Now
$$\frac{dv_o}{dx} = u_o,$$

so that
$$v_o = u_o x_o - \int x_o du_o$$

$$= \tau (1 + \tau^2)^{-1/2} - \log \left\{ \tau + (1 + \tau^2)^{1/2} \right\}$$

The equation for x_1 is

$$dx_1/dt = x_1 - 3x_o^2 x_1 + (\gamma + 1) u_o x_o^2 - (\gamma - 1) x_o v_o,$$

whence
$$\frac{d}{d\tau}\left\{ \frac{(1 + \tau^2)^{3/2} x_1}{\tau^2} \right\} = - 2 \frac{(1 + \tau^2)^{1/2}}{\tau^2} - (\gamma - 1) \frac{1 + \tau^2}{\tau^3} \log\left\{ \tau + (1 + \tau^2)^{1/2} \right\}$$

When τ is small, we find that

$$\frac{d}{d\tau}\left(\frac{x_1}{\tau^2}\right) \doteq -\frac{2}{\tau^2} - (\gamma - 1)\frac{1}{\tau^2},$$

and $$\frac{x_1}{\tau^2} \doteq \frac{(\gamma + 1)}{\tau} + \text{a constant.}$$

Therefore $$x_1 \longrightarrow 0 \text{ as } \tau \longrightarrow 0, \quad \text{and } M_1 = 0,$$

in agreement with the result found above.

Now

$$\frac{du_1}{dt} = - u_1 - (\gamma - 1)\,(x_o u_o^2 - u_o v_o),$$

whence

$$- \tau^2 \frac{d(u_1/\tau)}{d\tau} = 0(\tau^2)$$

and $$u_1 = 0(\tau) \text{ for small } \tau .$$

Therefore $$u_1 \longrightarrow 0 \text{ as } \tau \longrightarrow 0 \quad \text{and} \quad M_2 = 0.$$

This, unfortunately, disagrees with Lighthill's result, which seems to proceed from a slip in his general theory. If the preceding analysis is correct it is necessary to evaluate x_2 and possibly u_2 in order to estimate M_3 —— and we leave this task as an exercise for the reader.

(IV) One final example, cited by Carrier, as an instance of the failure of Lighthill's method: -

$$(x^2 + \alpha u)\frac{du}{dx} + u = 2x^3 + x^2,$$

with $$u = 1 + e \text{ at } x = 1.$$

It is required to estimate the value of u at x = 0.

The appropriate regularized equations are

$$z^2 \frac{dx}{dz} = x^2 + \alpha u$$

$$z^2 \frac{du}{dz} = 2x^3 + x^2 - u,$$

whence $$x_o = z, \qquad u_o = z^2 + e^{1/z}.$$

$$x_1 = e^{1/z} \left\{ - 1 + 2z - 2z^2 \right\} - z + z^2 + ez^2 .$$

Near z = 0 the dominant terms are

$$x \doteq z - \alpha e^{1/z},$$

Thence, at x = 0, $e^{1/z} = \frac{z}{\alpha}$

or $$\frac{1}{z} = \log z + \log \frac{1}{\alpha}.$$

The first approximate solution of this equation is

$$\frac{1}{z} = \log \frac{1}{\alpha},$$

and the second approximation is

$$\frac{1}{z} = \log \frac{1}{\alpha} - \log \log \frac{1}{\alpha}.$$

Therefore the required estimate of u at x = 0 is

$$u_o \doteq e^{1/z} = \exp \left\{ \log \frac{1}{\alpha} - \log \log \frac{1}{\alpha} \right\}$$

$$= \frac{1}{\alpha} (\log \frac{1}{\alpha})^{-1}.$$

10. The Existence Theorem.

In most physical problems we do not need any precise information about the radius of convergence of the series for x and u in terms of α. It is sufficient if we can be assured that there is some radius ρ, such that the series converges if $|\alpha| < \rho$. This rather limited information can be easily and rigorously deduced by the method of "dominant functions". In some cases we can also obtain estimates of the error introduced by truncating the series for x and u after a few terms.

The method of dominant functions was invented by Cauchy, rediscovered by Weierstrass, improved by Goursat and Sophie Kowalevsky, and is conveniently accessible in Forsyth's 6-volume Treatise on Differential Equations.

The relevant definitions are as follows: -

(1) If $f(x_1, x_2, \ldots, x_n)$ and $g(x_1, x_2, \ldots, x_n)$ are analytic functions of the variables $(x_1, x_2, \ldots, x_n)$ in the region $|x_i| < R$, $(i = 1, 2, \ldots, n)$, so that f and g are expressible as power series

$$f = \Sigma\, a_{mn\ldots}\, x_1^m\, x_2^n \ldots,$$

$$g = \Sigma\, b_{mn\ldots}\, x_1^m\, x_2^n \ldots,$$

and, if $\quad |a_{mn\ldots}| \leqslant b_{mn\ldots} \quad$ for all $m_1 n_1 \ldots$,

then g is said to be a dominant function for f.

(2) If, in the two systems of simultaneous differential equations,

$$F: \; dx_i/dt = f_i(x_1, x_2, \ldots, x_n t)$$

$$G: \; dx_i/dt = g_i(x_1, x_2, \ldots, x_n t),$$

g_i is a dominant function for f_i, $(i = 1, 2, \ldots, n)$, then the system G is said to dominate the system F.

The main theorem is the following: -

The solution of the dominant system G dominates the solution of the system F, the same initial conditions being imposed in each case;

i.e. If $x_i = \Sigma\, a_{in} t^n$ $\qquad (i = 1, 2, \ldots n)$

are formal power series satisfying F, and

$$x_i = a_{io} \quad \text{at } t = 0,$$

and if $\qquad x_i = \Sigma\, b_{in} t^n \qquad (i = 1, 2, \ldots, n)$

is the solution of G, analytic in

$$|t| < T$$

and satisfying $\qquad x_i = a_{io}$ at $t = 0$,

then $\qquad |a_{in}| \leqslant b_{in}.$

Hence the formal power series solution of F is analytic in

$$|t| < T.$$

The value of this theorem is largely due to the fact that there is considerable flexibility in the choice of dominant functions. Thus two useful types of dominant functions are the following: -

If $f(x_1, x_2, \ldots, x_n)$ is analytic in $|x_i| < R$,

and if $|f(x_1, x_2, \ldots, x_n)| < M$ in $|x_i| < R$,

then f is dominated by

$$M(1 - x_1/R)^{-1} (1 - x_2/R)^{-1} \ldots (1 - x_n/R)^{-1},$$

or by

$$M \left\{ 1 - (x_1 + x_2 + \ldots + x_n)/R \right\}^{-1}.$$

The choice of a dominant function is guided by two conditions: -

(1) It should be as close as possible to the original function, so that the two systems of differential equations differ as little as possible, and

(2) It should be sufficiently simple to enable the system G to be integrated and the radius of convergence of its power series solution to be determined. In practice we are usually content with a compromise.

As an elementary illustration consider the equation for the Jacobian elliptic function $y = \operatorname{sn} x$ with modulus k, viz.

$$\frac{dy}{dx} = (1 - y^2)^{1/2} (1 - k^2 y^2)^{1/2} , \quad y = 0 \text{ at } n = 0.$$

The right hand side of this equation is analytic if $|y| < 1$, and its upper bound in this region is

$$M = \left[2(1 + k^2) \right]^{1/2}.$$

Hence a dominant function is

$$M (1 - y^2)^{-1},$$

and the solution of the dominant equation is

$$Mx = y - \tfrac{1}{3}y^3.$$

This makes y an analytic function of x if

$$|x| < \tfrac{2}{3}M^{-1}.$$

Hence we can be sure that the original equation also possesses a solution analytic in x in the region $|x| < \frac{2}{3}\left[2(1+k^2)\right]^{-1/2}$.

In fact snx is analytic in a larger region $|x| < K'$, in the usual notation. This emphasizes the fact that the method of dominant functions yields only a lower bound to the radius of convergence of the power series of a differential equation.

11. The Proof of the Existence Theorem.

The regularized equations to which we intend to apply the existence theorem are always "autonomous" systems, i.e. the independent variable does not appear on the right hand side of the equations, which have the form

$$F: dx_i/dt = f_i(x_1, x_2, \ldots, x_n), \qquad i = 1, 2, \ldots, n,$$

with initial conditions

$$x_i = a_i \quad \text{at} \quad t = 0.$$

The functions f_i are analytic functions of $x_1, x_2, \ldots, x_n$ in the region

$$|x_i| < R,$$

which includes the point $x_i = a_i$. $(i = 1, 2, \ldots, n)$

We suppose that the system F possesses a dominant system

$$\left.\begin{aligned} G: dy_i/dt &= g_i(y_1, y_2, \ldots, y_n), \\ y_i &= b_i > 0, \end{aligned}\right\} \quad i = 1, 2, \ldots, n,$$

with a solution

$$y_i = \psi_i(t; b_1, b_2, \ldots, b_n)$$

which is an analytic function of t, $b_1, \ldots, b_n$ in a region

$$|t| < T, \qquad |b_i| < B.$$

Since the functions f_i are analytic in $|x_i| < R$, they can be expanded in power series

$$f_i = \Sigma\, a_i(\alpha_1, \alpha_2, \ldots, \alpha_n) x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} ,$$

uniformly convergent in this region, and we can obtain a formal solution in the form

$$x_i = \sum_{p=0}^{\infty} c_i(p)\, t^p/p! ,$$

by calculating the coefficients $c_i(p)$ from the equations

$$c_i(p) = \left\{\frac{d^p x_i}{dt^p}\right\}_{t=0} = \left\{\frac{d^{p-1} f_i}{dt^{p-1}}\right\}_{t=0} .$$

The initial conditions determine the coefficients

$$c_i(o) = a_i.$$

The expression for $c_i(p + 1)$ has the form

$$c_i(p + 1) = \Sigma\, a_i P_i,$$

where $$a_i = a_i(\alpha_1, \alpha_2, \ldots, \alpha_n) ,$$

and $$P_i = P_i(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$= \frac{d^p}{dt^p} \left\{ x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \right\}_{t=0} .$$

Now we can prove by induction that P_i is a polynomial in $c_i(q)$ for $i = 1, 2, \ldots, n$ and $q = 0, 1, 2, \ldots, p$, <u>with coefficients which are positive integers.</u>

Similar considerations apply to the dominant system G, in which

$$g_i = \Sigma\, b_i(\alpha_1, \alpha_2, \ldots, \alpha_n)\, y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n} ,$$

and

$$d_i(p) = \left\{ d^p x_i / \ t^p \right\}_{t=0} .$$

We find that

$$d_i(p + 1) = \Sigma\, b_i Q_i ,$$

where Q_i is the <u>same</u> polynomial as P_i, except that the $c_i(q)$ are replaced by $d_i(q)$.

Now $c_i(0) = a_i$, $d_i(0) = b_i$,

and $$|a_i| \leqslant b_i ,$$

since the system G dominates the system F.

We can now prove by induction that $c_i(p)$ and $d_i(p)$ are polynomials in a_i and in b_i respectively, and that

$$|c_i(p)| \leqslant d_i(p).$$

Now by hypothesis the dominant system possesses a solution which must have the form of power series,

$$y_i = \Sigma\, d_i(p) t^p/p!\,,$$

convergent in $|t| < T$. Hence the formal power series

$$x_i = \Sigma\, c_i(p)\, t^p/p!$$

must also converge in $|t| < T$. The formal power series is therefore a solution of the system F and it defines functions x_i $(i = 1, 2, \ldots, n)$ which are analytic in t and in $a_1, a_2, \ldots, a_n$ in the region

$$|t| < T, \qquad |a_i| < B.$$

Corollary 1: - We can immediately extend this theorem to an autonomous system with a parameter α,

$$F_\alpha:- \quad dx_i/dt = f_i(x_1, x_2, \ldots, x_n, \alpha) \qquad (i = 1, 2, \ldots, n),$$

$$x_i = a_i \quad \text{at} \quad t = 0.$$

It is sufficient to consider the augmented system

$$F_\alpha^*:- \quad dx_i/dt = f_i(x_1, x_2, \ldots, x_n, x_{n+1}) \qquad (i = 1, 2, \ldots, n)$$

$$dx_{n+1}/dt = 0,$$

$$\left.\begin{aligned} x_i &= a_i, \\ x_{n+1} &= \alpha, \end{aligned}\right\} \text{at } t = 0.$$

These two systems possess the same solution which is analytic in t, $a_1, a_2, \ldots, a_n$ and α in a region, $|t| \leqslant T$, $|a_i| < B$, $\alpha < B$.

Corollary 2: - A further extension is to the system

$$F_\alpha^\dagger : - \quad dx_i/dt = f_i(x_1, x_2, \ldots, x_n; \alpha)$$

$$(i = 1, 2, \ldots, n)$$

with the initial conditions

$$x_i = \xi_i(\alpha) \quad \text{at} \quad t = 0,$$

where f_i and ξ_i are analytic functions of $x_1, x_2, \ldots, x_n$ and α in a region

$$|x_i| < R, \quad |\alpha| < A.$$

The preceding argument shows that the system $F_\alpha^\dagger$ possesses a solution which is analytic in t and in $\xi_1, \xi_2, \ldots, \xi_n$ and α in a region $|t| < T$, $|\xi_i| < B$, $|\alpha| < A$. But the functions ξ_i are analytic in α in $|\alpha| < A$. Hence the solution is analytic in t and α in a region $|t| < T$, $|\alpha| < A$.

12. Application to Lighthill's Equation.

To construct a simple dominant system for the regularised equations

$$\frac{dx}{dt} = x + \alpha u,$$

$$\frac{du}{dt} = r(x) - q(x)u,$$

let $\quad |r(x) - q(x)u| < M$

when $\quad |x|, |u| > R,$

and let $\quad |\alpha| < A.$

Then $r(x) - q(x)u$ is dominated by

$$M\left\{1 - (x+u)/R\right\}^{-1}$$

In order to obtain an easily soluble set of dominant equations we dominate $x + \alpha u$ by

$$N\left\{1 - (x+u)/R\right\}^{-1},$$

where $\qquad N = R + AR.$

In fact we go further and take the dominant system to be

$$G: \quad \begin{cases} \dfrac{dx}{dt} = C\left\{1 - (x+u)/R\right\}^{-1} \\[2ex] \dfrac{du}{dt} = C\left\{1 - (x+u)/R\right\}^{-1} \end{cases}$$

where $\qquad C = \max\,(M,\ N).$

The solution of this system, subject to the initial conditions

$$x = 0,\ u = a \ \text{ at } \ t = 0,$$

is given by

$$u - x = a,$$

$$u + x = v,$$

$$\frac{v}{R} - 1 = \pm\left\{\left(\frac{a}{R} - 1\right)^2 - \frac{4ct}{R}\right\}^{1/2},$$

and this solution is analytic in

$$4C\,|t| < R\left(\frac{a}{R} - 1\right)^2.$$

Now in the region which we are considering

$$|u| < R,$$

whence $\qquad |a| < R,$

so that $(\frac{a}{R} - 1)^2$ cannot vanish.

Hence the dominant equations have a solution which is analytic in t and α in the region

$$|t| < \frac{1}{4}K\left(\frac{a^2}{R} - 2a + R\right),$$

where

$$K = \min\left(\frac{1}{M}, \frac{1}{R + RA}\right)$$

Initial Distribution

9 CHBUSHIPS
  3 Tech Info Br (Code 335)
  1 Tech Asst to Chief (Code 106)
  1 Asst Chief for Field Activities (Code 700)
  3 Computer Systems and Applications Branch (732)
  1 Asst Chief for Nuclear Propulsion (Code 1500)

1 CHBUWEPS

1 CHBUSANDA

1 CHBUCENSUS

1 CHONR

1 NAVSHIPYD BSN

1 NAVSHIPYD CHASN

1 NAVSHIPYD LBEACH

2 NAVSHIPYD NYK
  1 MATL LAB

1 NAVSHIPYD MARE

1 NAVSHIPYD NORVA

1 NAVSHIPYD SFRAN

1 NAVSHIPYD PHILA

1 NAVSHIPYD PTSMH

1 NAVSHIPYD PUG

1 NAVSHIPYD PEARL

1 CO&DIR, USNEL

1 CO&DIR, USNRDL

1 CO&DIR, USNAVTRADEVCEN

1 CO&DIR, USNMDL

1 CO&DIR, USNUSL

1 CDR, US Naval Weapons Lab.

3 CDR, USNOTS, China Lake
  1 Michelson Lab (Code 5038)
  1 Library

1 CDR, USNOL, White Oak, Md.

1 CO&DIR, USNBTL

1 DIR, USNEES

1 DIR, USNRL

1 SUPT, USNAVPGSCOL
  Attn: Library, Technical Reports Section

1 SUPT, USNA
  Dept. of Math.

1 CG, Aberdeen PG

1 DIR, Natl BuStand

1 Chief, AFSWP

1 DIR, Langley Res Ctr, NASA

1 DIR, Lewis Res Ctr, NASA

1 CG, W-PAFB

2 New York University
  1 Inst. of Math. Sci.
  1 AEC Computing Facility

1 Ohio State University Res Ctr

1 Pennsylvania State University
Dept. of Math.

1 Princeton University, Library

1 University of California

1 University of Illinois
Dig Comp Lab

2 Harvard University
  1 Dept. of Math.
    Attn: Prof. J. L. Walsh
  1 Computation Lab

1 Inst for Advanced Study

2 MIT Computation Center

1 State College of Washington
Dept. of Math.

1 University of California
Attn: Dr. G. E. Forsythe

1 Johns Hopkins University
Baltimore, Md.

1 Rutgers University
Attn: Prof. E. P. Starke

1 Midwest Research Inst.
Attn: Mr. Yudell L. Luke

1 Brown University
Attn: Dr. R. D. Kodis

1 Princeton University
Attn: Prof. H. J. Maehly

2 University of Maryland
  1 Dept. of Math.
  1 Inst. for Fluid Dyn. & Applied Math.

1 University of Michigan

1 University of Washington
Dept. of Math.

1 Yale University

1 Combustion Engineering, Inc.
Head of Nuclear Physics

1 DIR, Westinghouse Elect. Corp.
Bettis Atomic Power Div.

2 Argonne National Lab

1 Illinois Inst. of Tech.
Armour Res Foundation

1 Battelle Memorial Inst.

1 Brookhaven National Lab

1 Cornell Aero Lab, Inc.

1 Curtiss-Wright Corp.
Res Div

1 Douglas Aircraft Co., Inc.

1 University of Rochester
Dept. of Math.

1 Scripps Inst. of Oceanography
Attn: Dr. Walter Munk

1 Syracuse Univ. Res Inst.
Attn: Dr. Bruce Gilchrist

1 Lockheed Aircraft Corp

1 Los Alamos Scientific Lab

1 Remington Rand Univac
Electronic Computer Dept.

1 Ramo Wooldridge Corp.

1 Rand Corporation

1 Sandia Corp., Library

1 United Aircraft Corp.

1 Vitro Corp. of America

1 Remington Rand Univac
Engineering Research Associates
St. Paul 4, Minnesota

1 IBM Corporation

1 General Electric Co.
Knolls Atomic Power Lab.
Math Analysis Unit

1 Lincoln Lab, B-125

1 Lockheed Aircraft Corp.

1 Prof. R. E. Langer
Math Res Ctr
University of Wisconsin

1 Prof. Wallace Givens
Dept of Math
Wayne State University

1 Dr. A. S. Householder
Oak Ridge National Lab

1 Prof. John Todd
Dept of Math
California Inst. of Technology

1 Dr. Franz Alt
Math Comp Lab
National Bureau of Standards

1 Prof. Dr. E. G. Kogbetliantz
IBM Corp., New York 22, N.Y.

1 Prof. D. H. Lehmer
Dept of Math
University of California

1 Prof. E. J. McShane
University of Virginia

1 Prof. J. B. Rosser
Dept of Math
Cornell University

1 CG, Frankford Arsenal
Math. Sec., Pitman Dunn Lab
Philadelphia 27, Pa.

1 CG, White Sands Proving Ground
Flight Determination Lab

1 USAEC Tech Info Serv
Oak Ridge, Tenn.

1 CO, Diamond Ordnance Fuze Lab
Wash., D.C.
Attn: Library

2 USAEC, Wash 25, D.C.
Tech Lib.

1 George Washington University
Logistics Research Project

1 Johns Hopkins University
Applied Physics Lab
Silver Spring, Md.

2 University of California
1 Librarian, Numerical Analysis
1 Dr. C. B. Tompkins, Dept of Math

1 Hudson Laboratories
Columbia University

1 M. Paul Germain
Collège de France
Paris, France

5 Prof. G. Temple
Oxford University
Oxford, England

David Taylor Model Basin. Report 1415.
LECTURES ON TOPICS IN NONLINEAR DIFFERENTIAL EQUATIONS, by G. Temple. March 1960. ii, 51p.
UNCLASSIFIED

1. Nonlinear differential equations
I. Temple, George

These lectures describe some recent researches by the author and one of his graduate students at Oxford University on ordinary nonlinear differential equations. The first lecture is devoted to a search for a "superposition" principle for these nonlinear equations and it determines the class of nonlinear equations for which such a superposition principle exists.

The remaining four lectures provide a rigorous, analytical theory of the technique invented by Lighthill (1949) for solving nonlinear differential equations with an "irregular" perturbation. Such equations involve a small parameter α and such that the coefficient of the highest derivative vanishes identially, or at the "initial point," when $\alpha = 0$.

The theory is developed from a number of simple examples and given a rigorous form by means of the theory of "dominant functions."

David Taylor Model Basin. Report 1415.
LECTURES ON TOPICS IN NONLINEAR DIFFERENTIAL EQUATIONS, by G. Temple. March 1960. ii, 51p.
UNCLASSIFIED

1. Nonlinear differential equations
I. Temple, George

These lectures describe some recent researches by the author and one of his graduate students at Oxford University on ordinary nonlinear differential equations. The first lecture is devoted to a search for a "superposition" principle for these nonlinear equations and it determines the class of nonlinear equations for which such a superposition principle exists.

The remaining four lectures provide a rigorous, analytical theory of the technique invented by Lighthill (1949) for solving nonlinear differential equations with an "irregular" perturbation. Such equations involve a small parameter α and such that the coefficient of the highest derivative vanishes identially, or at the "initial point," when $\alpha = 0$.

The theory is developed from a number of simple examples and given a rigorous form by means of the theory of "dominant functions."

David Taylor Model Basin. Report 1415.
LECTURES ON TOPICS IN NONLINEAR DIFFERENTIAL EQUATIONS, by G. Temple. March 1960. ii, 51p.
UNCLASSIFIED

1. Nonlinear differential equations
I. Temple, George

These lectures describe some recent researches by the author and one of his graduate students at Oxford University on ordinary nonlinear differential equations. The first lecture is devoted to a search for a "superposition" principle for these nonlinear equations and it determines the class of nonlinear equations for which such a superposition principle exists.

The remaining four lectures provide a rigorous, analytical theory of the technique invented by Lighthill (1949) for solving nonlinear differential equations with an "irregular" perturbation. Such equations involve a small parameter α and such that the coefficient of the highest derivative vanishes identially, or at the "initial point," when $\alpha = 0$.

The theory is developed from a number of simple examples and given a rigorous form by means of the theory of "dominant functions."

David Taylor Model Basin. Report 1415.
LECTURES ON TOPICS IN NONLINEAR DIFFERENTIAL EQUATIONS, by G. Temple. March 1960. ii, 51p.
UNCLASSIFIED

1. Nonlinear differential equations
I. Temple, George

These lectures describe some recent researches by the author and one of his graduate students at Oxford University on ordinary nonlinear differential equations. The first lecture is devoted to a search for a "superposition" principle for these nonlinear equations and it determines the class of nonlinear equations for which such a superposition principle exists.

The remaining four lectures provide a rigorous, analytical theory of the technique invented by Lighthill (1949) for solving nonlinear differential equations with an "irregular" perturbation. Such equations involve a small parameter α and such that the coefficient of the highest derivative vanishes identially, or at the "initial point," when $\alpha = 0$.

The theory is developed from a number of simple examples and given a rigorous form by means of the theory of "dominant functions."

David Taylor Model Basin. Report 1415.
LECTURES ON TOPICS IN NONLINEAR DIFFERENTIAL EQUATIONS, by G. Temple. March 1960. ii, 51p.
UNCLASSIFIED

These lectures describe some recent researches by the author and one of his graduate students at Oxford University on ordinary nonlinear differential equations. The first lecture is devoted to a search for a "superposition" principle for these nonlinear equations and it determines the class of nonlinear equations for which such a superposition principle exists.

The remaining four lectures provide a rigorous, analytical theory of the technique invented by Lighthill (1949) for solving nonlinear differential equations with an "irregular" perturbation. Such equations involve a small parameter α and such that the coefficient of the highest derivative vanishes identially, or at the "initial point," when $\alpha = 0$.

The theory is developed from a number of simple examples and given a rigorous form by means of the theory of "dominant functions."

1. Nonlinear differential equations
I. Temple, George

David Taylor Model Basin. Report 1415.
LECTURES ON TOPICS IN NONLINEAR DIFFERENTIAL EQUATIONS, by G. Temple. March 1960. ii, 51p.
UNCLASSIFIED

These lectures describe some recent researches by the author and one of his graduate students at Oxford University on ordinary nonlinear differential equations. The first lecture is devoted to a search for a "superposition" principle for these nonlinear equations and it determines the class of nonlinear equations for which such a superposition principle exists.

The remaining four lectures provide a rigorous, analytical theory of the technique invented by Lighthill (1949) for solving nonlinear differential equations with an "irregular" perturbation. Such equations involve a small parameter α and such that the coefficient of the highest derivative vanishes identially, or at the "initial point," when $\alpha = 0$.

The theory is developed from a number of simple examples and given a rigorous form by means of the theory of "dominant functions."

1. Nonlinear differential equations
I. Temple, George