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THE POTENTIAL PROBLEM OF THE OPTIMUM PROPELLER WITH FINITE NUMBER OF BLADES OPERATING IN A CYLINDRICAL DUCT

by

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HYDROMECHANICS LABORATORY

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NOTATION

A_m	Coefficient given by Equation [27]
a_m	Coefficients in the internal field
B_m	Coefficient given by Equation [28]
b_m	Coefficients in the internal field
c_n, c	Coefficients in the external field
D	Coefficients in the internal field
d_n	Coefficient in the external field
F_{pn}	Function defined by Equation [23]
$I_{pn}(pn\mu)$	Modified Bessel function of the first kind
$K_{pn}(pn\mu)$	Modified Bessel function of the second kind
m, n	Indices of summation
p	Number of blades
R	Propeller radius
r	Radial coordinate
$T_{1,\nu}(\nu\mu)$	Lommel function defined by Equation [8]
T	Propeller thrust in open water
V	Velocity of advance of propeller
w	Velocity of advance of helical sheets
x	Nondimensional radius (r/R)
z	Axial distance from propeller
Γ	Bound circulation for a propeller operating in open water
Γ_t	Bound circulation for a propeller operating in a cylindrical duct
$\Delta\Gamma$	Increase in circulation due to the presence of the duct
ζ	$\theta - \frac{\omega z}{V+w}$
θ	Angular coordinate
κ_0	Goldstein factor for zero hub
λ_i	Advance ratio
μ	$\frac{\omega r}{V+w}$

μ_0	$\dfrac{1}{\lambda_i} = \dfrac{\omega R}{V + w}$
μ_2	$\dfrac{\omega r_2}{V + w}$
ν	$\dfrac{p}{2}(2m + 1)$
τ	$\dfrac{\Gamma_t - \Gamma}{\Gamma}$
ϕ	Velocity potential
ω	Propeller angular velocity

Subscripts

0	Pertaining to quantities at $r = R$
2	Pertaining to quantities at the duct

Primes denote differentiation with respect to the argument

ABSTRACT

The potential problem is solved for the circulation distribution of an optimum propeller with finite number of blades when operating in a cylindrical duct. The effects of the duct on both the circulation and the thrust increase have been calculated for specific cases, showing that they become important for propellers operating in a duct having a diameter which approaches the propeller diameter. The effect of the duct increases with decreasing number of blades and with increasing pitch. The percentage increase in thrust which has been computed is additive to the Wood and Harris correction resulting from the contraction of the slipstream.

INTRODUCTION

The increasing use of propellers operating within circular ducts has indicated the need for a better understanding of the effect that fixed boundaries have on propeller operation, particularly regarding their circulation distribution and total thrust. Furthermore, the investigation of propeller characteristics and cavitation phenomena in "closed throat" propeller tunnels requires also a knowledge of the magnitude of such effects.

A large amount of effort has been devoted to the investigation of "slotted wall" tunnels for axial bodies of revolution in order to decrease the "blockage" interference, but little work has been done in this field regarding the operation of a propeller in a circular duct. The duct walls, whether "slotted" or continuous, will affect the local flow near the tips, thereby altering the radial distribution of circulation and the cavitation characteristics. The present investigation has been limited to the effect of a continuous wall.

Correction factors pertaining to the increase in thrust for a propeller operating in a duct were given in 1920 by Wood and Harris.[1] They considered the propeller as an actuator disc and determined the "blockage" interference due to the contraction of the slipstream. This correction is, therefore, applicable to moderately heavily loaded propellers when the contraction of the slipstream is appreciable. The presence of the walls will also affect lightly loaded propellers with finite number of blades, and it is for such propellers that the following theory has been developed. A similar investigation has been made by Goodman,[2] who, however, assumed the propeller slipstream to be identical with the Prandtl model; i.e., the helical vortex sheets were replaced by parallel lines.

The analysis which follows is conducted for a propeller having a minimum energy loss and operating in a circular duct of constant diameter and of infinite or semi-infinite length. The theory is applicable to the case of ducted propellers even where the clearance from the

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[1] References are listed on page 23.

propeller tips to the walls is small, giving the optimum distribution of circulation for such propellers when they have a finite number of blades. A circulation distribution factor relating the circulation for a propeller operating within a duct to the circulation for the same propeller operating in open water, has also been derived and can be used in the design of ducted systems. This theory might possibly be extended also for propellers operating in relatively short ducts.

The potential solution given is not dissimilar to the solution already published[3] for a propeller having a finite hub. The boundary conditions in this case are changed.

The mathematical analysis is followed by curves showing the effect of the tunnel walls on the circulation distribution, as well as the percentage increase in thrust for 3-, 4-, and 5-bladed propellers over a range of propeller-tunnel diameter ratios. Tables are also given for the change in circulation distribution for 3-, 4-, 5-, and 6-bladed ducted propellers when operating in a tunnel whose propeller-tunnel diameter ratio is unity.

STATEMENT OF PROBLEM

The potential problem is analogous to that of a propeller with finite hub[3] but with modified boundary conditions. Laplace's equation for this type of flow can be reduced to

$$\left(\mu \frac{\partial}{\partial \mu}\right)^2 \phi + (1 + \mu^2) \frac{\partial^2 \phi}{\partial \zeta^2} = 0 \qquad [1]$$

where ϕ is the velocity potential,

$\mu = \dfrac{\omega}{V + w} r,$

V is the velocity of advance of the propeller,

w is the velocity of advance of the vortex sheets,

ω is the propeller angular velocity,

r is the distance from axis of rotation,

$\zeta = \theta - \dfrac{\omega}{V + w} r$ = angle measured on helical surface, and

r, θ, z are the cylindrical coordinates.

The boundary conditions which have to be satisfied on the vortex sheets are derived from the velocity of the sheets, and can be written as

$$\frac{\partial \phi}{\partial \zeta} = - \frac{\mu^2}{1 + \mu^2} \quad \text{on} \quad \zeta = 0 \text{ or } \frac{2\pi}{p} \text{ for } 0 < r < R \text{ or } 0 < \mu < \mu_0 \qquad [2]$$

where p is the number of blades,

R is the propeller radius, and

$$\mu_0 = \frac{\omega}{V + w} R.$$

Additional boundary conditions are necessary at the slipstream radius and at the propeller axis, thus

$$\phi \text{ and } \frac{\partial \phi}{\partial \mu} \text{ are continuous at } r = R \text{ or } \mu = \mu_0 \qquad [3]$$

and

$$\phi \quad \text{is finite at} \quad r = 0 \text{ or } \mu = 0. \qquad [4]$$

The presence of the duct also requires that the radial component of the velocity be zero on the duct. By this condition

$$\frac{\partial \phi}{\partial \mu} = 0 \quad \text{at} \quad r = r_2 \text{ or } \mu = \mu_2 \quad \text{for } r_2 > R \qquad [5]$$

where $\mu_2 = \frac{\omega}{V + w} r_2$ and r_2 is the duct radius.

SOLUTION OF POTENTIAL PROBLEM

The generalized solution which satisfies Equation [1] and the boundary condition given by Equation [2] has already been derived in Reference 3 and is given by:

for the external field, $r > R$

$$\phi = \sum_{n=1}^{\infty} \left\{ \frac{2 c_n}{p} \frac{K_{pn}(pn\mu)}{K_{pn}(pn\mu_0)} + \frac{2 d_n}{p} \frac{I_{pn}(pn\mu)}{I_{pn}(pn\mu_0)} \right\} \sin pn\zeta + c \log \frac{\mu}{\mu_0} \qquad [6]$$

and for the internal field, $0 < r < R$

$$\phi = \frac{2}{p} \sum_{m=0}^{\infty} \left[\frac{4}{\pi} \frac{T_{1,\nu}(\nu\mu)}{(2m+1)^2} + a_m \frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)} + b_m \frac{K_\nu(\nu\mu)}{K_\nu(\nu\mu_0)} \right] \cos\nu\zeta + D \log \frac{\mu}{\mu_0} \qquad [7]$$

where $\nu = \frac{p}{2}(2m+1)$,

$I_{pn}(pn\mu)$, $K_{pn}(pn\mu)$ are the modified Bessel functions of the first and second kind, and

a_m, b_m, c_n, d_n, c, d are arbitrary constants.

$$T_{1,\nu}(\nu\mu) = \text{Lommel function}$$

$$= \frac{\mu^2}{1+\mu^2} + \frac{4\mu^2(1-\mu^2)}{\nu^2(1+\mu^2)^4} + \ldots\ldots \qquad [8]$$

As ϕ has to be finite at $\mu = 0$, the coefficients of $K_\nu(\nu\mu)$ and of the log term in Equation [7] have to be zero; then $b_m = D = 0$. In order to evaluate the additional arbitrary constants, it is first necessary to express ϕ for the internal field in terms of $\sin pn\zeta$ in the interval $0 < \zeta < 2\pi/p$. In this range we write

$$\cos\nu\zeta = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-(2m+1)^2} \sin pn\zeta \qquad [9]$$

Substituting Equation [9] in Equation [7] and equating to Equation [6] for $\mu = \mu_0$ and all ζ, we satisfy the boundary condition [3] when

$$c_n + d_n = \frac{8}{\pi} \sum_{m=0}^{\infty} \frac{n}{4n^2-(2m+1)^2} \left[\frac{4}{\pi} \frac{T_{1,\nu}(\nu\mu)}{(2m+1)^2} + a_m \right] \qquad [10]$$

and $c = 0$. Also,

$$c_n \frac{K'_{pn}(pn\mu_0)}{K_{pn}(pn\mu_0)} + d_n \frac{I'_{pn}(pn\mu_0)}{I_{pn}(pn\mu)}$$

$$= \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(2m+1)}{4n^2-(2m+1)^2} \left[\frac{4}{\pi} \frac{T'_{1,\nu}(\nu\mu_0)}{(2m+1)^2} + a_m \frac{I'_\nu(\nu\mu_0)}{I_\nu(\nu\mu_0)} \right] \qquad [11]$$

where primes denote differentiation with respect to the argument.

For zero radial velocity on the duct, from Equation [5], we get

$$\sum_{n=1}^{\infty} n \left[c_n \frac{K'_{pn}(pn\mu_2)}{K_{pn}(pn\mu_0)} + d_n \frac{I'_{pn}(pn\mu_2)}{I_{pn}(pn\mu_0)} \right] = 0 \qquad [12]$$

Equations [10], [11], and [12] can be solved for the coefficients a_m, c_n, and d_n. We obtain

$$\left\{ \frac{I'_{pn}(pn\mu_0)}{I_{pn}(pn\mu_0)} - \frac{I'_{pn}(pn\mu_2)}{I_{pn}(pn\mu_0)} \frac{K'_{pn}(pn\mu_0)}{K_{pn}(pn\mu_2)} \right\} \sum_{m=0}^{\infty} \frac{2n}{4n^2-(2m+1)^2} \left\{ \frac{4}{\pi} \frac{T_{1,\nu}(\nu\mu_0)}{(2m+1)^2} + a_m \right\}$$

$$= \left\{ 1 - \frac{I'_{pn}(pn\mu_2)}{I_{pn}(pn\mu_0)} \frac{K_{pn}(pn\mu_0)}{K'_{pn}(pn\mu_2)} \right\} \sum_{m=0}^{\infty} \frac{(2m+1)}{4n^2-(2m+1)^2} \left\{ \frac{4}{\pi} \frac{T'_{1,\nu}(\nu\mu_0)}{(2m+1)^2} + a_m \frac{I'_{\nu}(\nu\mu_0)}{I_{\nu}(\nu\mu_0)} \right\} \qquad [13]$$

The coefficients a_m can now be determined from Equation [13] by a method of successive approximations. It is expected, however, that such a method would result in slow convergence particularly near $\zeta = 0$ and $\frac{2\pi}{p}$ where singularities exists. If the solution is restricted to values of μ_0 which are not too small, we can write

$$T_{1,\nu}(\nu\mu_0) \simeq \frac{\mu_0^2}{1+\mu_0^2} \qquad [14]$$

and

$$\nu\, T'_{1,\nu}(\nu\mu_0) \simeq \frac{2\mu_0}{(1+\mu_0^2)^2} \qquad [15]$$

From Nicholson's expansions,[4] we know that

$$\frac{K'_{pn}(pn\mu_0)}{K_{pn}(pn\mu_0)} \simeq -\frac{(1+\mu_0^2)^{1/2}}{\mu_0} \qquad [16]$$

and

$$\frac{I'_{pn}(pn\mu_0)}{I_{pn}(pn\mu_0)} \simeq \frac{(1+\mu_0^2)^{1/2}}{\mu_0} \qquad [17]$$

and we can, therefore, write

$$\left[\frac{I'_{pn}(pn\mu_0)}{I_{pn}(pn\mu_0)} - \frac{I'_{pn}(pn\mu_2)}{I_{pn}(pn\mu_0)}\frac{K'_{pn}(pn\mu_0)}{K_{pn}(pn\mu_2)}\right] = \frac{(1+\mu_0^2)^{1/2}}{\mu_0}\left\{1 - \frac{I_{pn}(pn\mu_2)}{I_{pn}(pn\mu_0)}\frac{K_{pn}(pn\mu_0)}{K_{pn}(pn\mu_2)}\right\} \qquad [18]$$

and also

$$\left\{1 - \frac{I'_{pn}(pn\mu_2)}{I_{pn}(pn\mu_0)}\frac{K_{pn}(pn\mu_0)}{K'_{pn}(pn\mu_2)}\right\} = \left\{1 + \frac{I_{pn}(pn\mu_2)}{I_{pn}(pn\mu_0)}\frac{K_{pn}(pn\mu_0)}{K_{pn}(pn\mu_2)}\right\} \qquad [19]$$

Substituting Equations [14], [15], [16], [17], [18], and [19] into Equation [13], we get

$$\sum_{m=0}^{\infty}\frac{a_m}{4n^2-(2m+1)^2}\left\{2n-(2m+1) - \frac{I_{pn}(pn\mu_2)}{I_{pn}(pn\mu_0)}\frac{K_{pn}(pn\mu_0)}{K_{pn}(pn\mu_2)}[2n+2m+1]\right\}$$

$$= -\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{[4n^2-(2m+1)^2](2m+1)^2}\frac{\mu_0^2}{1+\mu_0^2}\left\{2n - \frac{4}{p(1+\mu_0^2)^{3/2}}\right. \qquad [20]$$

$$\left. - \frac{I_{pn}(pn\mu_2)}{I_{pn}(pn\mu_0)}\frac{K_{pn}(pn\mu_0)}{K_{pn}(pn\mu_2)}\left[2n + \frac{4}{p(1+\mu_0^2)^{3/2}}\right]\right\}$$

For the usual values of μ_0 it is possible to neglect the second term $\frac{4}{p(1+\mu_0^2)^{3/2}}$ in comparison with $2n$ and also noting that

$$\sum_{m=0}^{\infty}\frac{2n}{[4n^2-(2m+1)^2](2m+1)^2} = \frac{\pi^2}{16n} \qquad [21]$$

Equation [20] finally becomes

$$\sum_{m=0}^{\infty} a_m\left\{\frac{1}{2n-(2m+1)} - \frac{F_{pn}}{2n+(2m+1)}\right\} = \frac{\pi}{4n}\frac{\mu_0^2}{1+\mu_0^2}(F_{pn}-1) \qquad [22]$$

where

$$F_{pn} = \frac{I_{pn}(pn\mu_0)\, K_{pn}(pn\mu_2)}{I_{pn}(pn\mu_2)\, K_{pn}(pn\mu_0)} \qquad [23]$$

for $n = 1, 2, 3 \ldots\ldots$

The solution of the infinite system given by Equation [22] can be found by the standard method of successive approximations.

It will be noted that as the duct diameter tends to infinity, i.e., μ_2 tends to infinity, then F_{pn} tends to zero and Equation [22] degenerates into a system of equations given by Goldstein[5] for a propeller operating in open water.

If we now examine the case when the duct diameter equals the propeller diameter; i.e., when μ_2 equals μ_0, then $F_{pn} = 1$ and Equation [22] becomes

$$\sum_{m=0}^{\infty} \frac{a_m}{2n-(2m+1)} = 0 \qquad [24]$$

for $n = 1, 2, 3 \ldots\ldots$ and $\mu_2 = \mu_0$ which is satisfied only when $a_m = 0$.

In order to simplify the expressions for the circulation it is convenient to define now arbitrary coefficients A_m and B_m, as

$$a_m = -\frac{\mu_0^{\,2}}{1+\mu_0^{\,2}}(A_m - B_m) \qquad [25]$$

and such that A_m satisfies the equation

$$\sum_{m=0}^{\infty} \frac{A_m}{2n-(2m+1)} = \frac{\pi}{4n} \qquad [26]$$

We know[5] that the solution to Equation [26] is given by

$$A_m = \frac{(2m)!}{2^{2m}(m!)^2(2m+1)} \qquad [27]$$

Then Equation [22] becomes

$$\sum_{m=0}^{\infty} B_m \left[\frac{1}{2n-(2m+1)} - \frac{F_{pn}}{2n+(2m+1)} \right] = F_{pn} \left[\frac{\pi}{4n} - \sum_{m=0}^{\infty} \frac{A_m}{2n+(2m+1)} \right] \qquad [28]$$

for $n = 1, 2, 3 \ldots .$

An approximate solution for the coefficients B_m as determined from the above equation is given in Appendix A.

DETERMINATION OF BOUND CIRCULATION

The circulation distribution is proportional to the change in potential $[\phi]$ on the vortex sheets at $\zeta = 0$ and $\frac{2\pi}{p}$, then from Equation [7]

$$[\phi] = \frac{4}{p} \sum_{m=0}^{\infty} \left[\frac{4}{\pi} \frac{T_{1,\nu}(\nu\mu)}{(2m+1)^2} - \frac{\mu_0^{\,2}}{1+\mu_0^{\,2}} (A_m - B_m) \frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)} \right]$$

where A_m and B_m are given by Equations [27] and [28], respectively. If Γ_t denotes the circulation of a propeller operating within a tunnel, then

$$\frac{p\omega\Gamma_t}{2\pi w(V+w)} = \frac{p}{2\pi}[\phi] = \frac{2}{\pi} \sum_{m=0}^{\infty} \left[\frac{4}{\pi} \frac{T_{1,\nu}(\nu\mu)}{(2m+1)^2} - \frac{\mu_0^{\,2}}{1+\mu_0^{\,2}} (A_m - B_m) \frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)} \right] \qquad [29]$$

If Γ is the circulation for the same propeller operating in open water, we know that

$$\frac{p\omega\Gamma}{2\pi w\,(V+w)} = \kappa_0 \frac{\mu^2}{1+\mu^2} = \frac{2}{\pi} \sum_{m=0}^{\infty} \left[\frac{4}{\pi} \frac{T_{1,\nu}(\nu\mu)}{(2m+1)^2} - \frac{\mu_0^{\,2}}{1+\mu_0^{\,2}} A_m \frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)} \right] \qquad [30]$$

where κ_0 is the circulation distribution factor for a propeller having zero hub diameter (Goldstein Factor). Denoting by $\Delta\Gamma$ the increase in circulation at any radius due to the presence of the duct, such that

$$\Delta\Gamma = \Gamma_t - \Gamma \qquad [31]$$

and defining a new factor $\tau = \frac{\Delta\Gamma}{\Gamma}$, then

$$\tau = \frac{\Gamma_t - \Gamma}{\Gamma} = \frac{2}{\pi} \frac{(1+x^2\mu_0^2)}{\kappa_0 x^2 (1+\mu_0^2)} \sum_{m=0}^{\infty} B_m \frac{I_\nu(\nu\mu_0 x)}{I_\nu(\nu\mu_0)} \qquad [32]$$

Calculation of the factor τ will show the effect that the number of blades, pitch ratio, and propeller-tunnel diameter ratio have on the circulation distribution.

The increase in thrust for a propeller operating in a duct can now be determined in comparison with the same propeller operating in open water

$$\frac{\Delta T}{T} = \frac{T_t - T}{T} = \frac{2}{\pi} \frac{\mu_0^2}{1+\mu_0^2} \cdot \frac{\int_0^1 x \sum_{m=0}^{\infty} B_m \frac{I_\nu(\nu\mu_0 x)}{I_\nu(\nu\mu_0)} dx}{\int_0^1 \kappa_0 \frac{\mu_0^2 x^3}{1+x^2\mu_0^2} dx} \qquad [33]$$

The numerator of Equation [33] has been integrated and the expression is given in Appendix B.

NUMERICAL RESULTS

Through the use of Equations [28], [29], [32], and [33] and the coefficients computed in Appendix A, calculations were performed showing the effect of propeller-duct diameter ratio over a range of values from 0.85 to 1.0. The circulation distribution is shown in Figure 1 for a four-bladed propeller operating at $1/\lambda_i = \mu_0 = 3.0$ for open water and various propeller-duct diameter ratios. The factor τ has been computed and is shown in Table 1 for 3-, 4-, 5-, and 6-bladed propellers over a range of $1/\lambda_i$ from 1.5 to 6.0 for a propeller-duct diameter ratio of unity. In practice, this is equivalent to an infinitely small tip clearance (less than 10^{-4} of the propeller diameter). The variation of τ along the radius is shown in Figure 2.

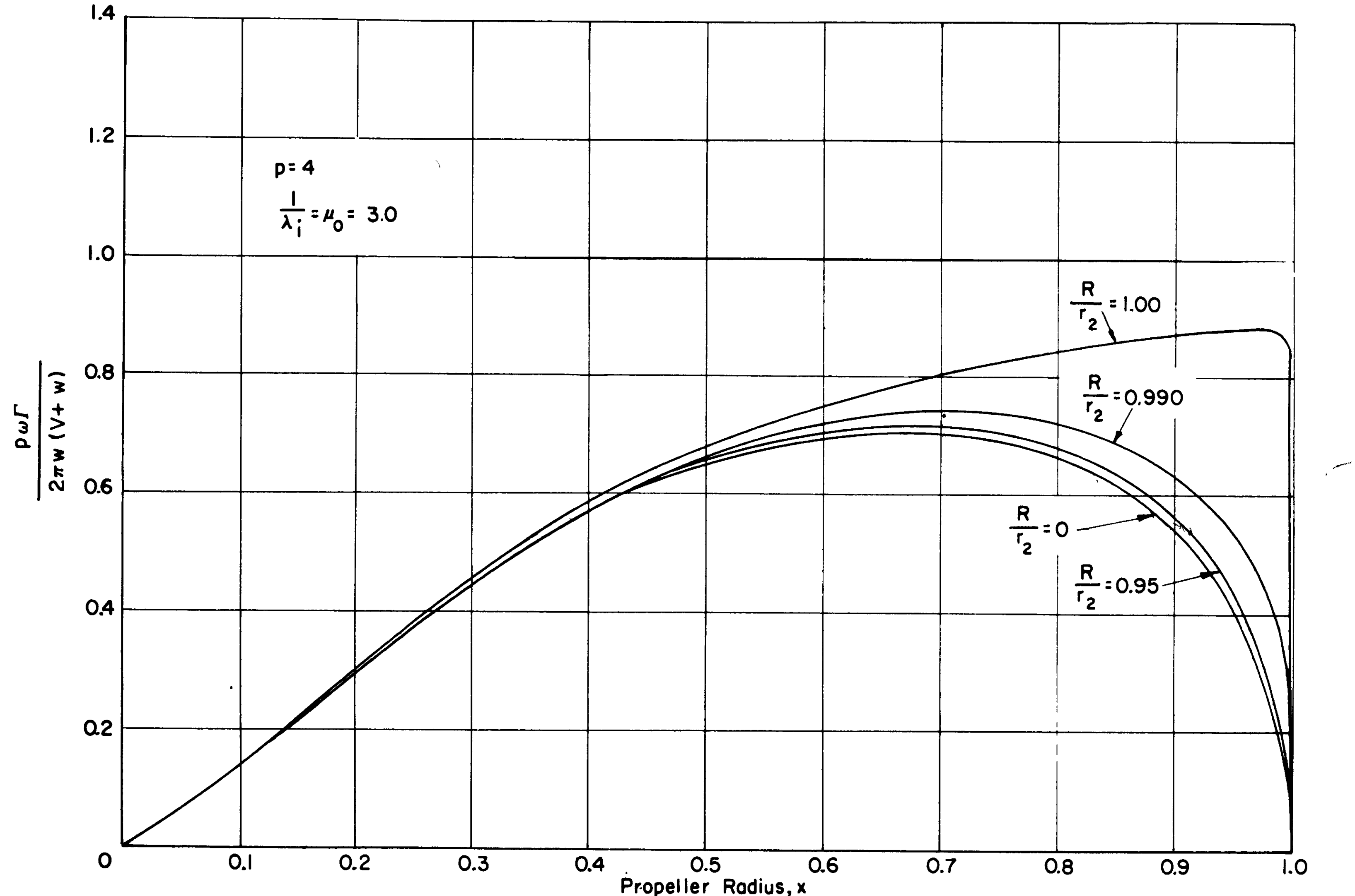
It is noticed from Figure 1 that the optimum load distribution shifts towards the propeller tip as the tip clearance to the duct decreases. This condition is undesirable from the point of view of tip vortex cavitation and will be aggravated by the boundary layer of the duct when viscous flows are considered.

Figure 3 shows the effect of the propeller-duct diameter ratio on the percentage increase of thrust for 3-, 4-, and 5-bladed propellers. It is noticed that the increase in thrust increases with decreasing number of blades, and agrees with the fact that the equalization of pressure at the propeller tip is inversely proportional to the number of blades.

TABLE 1

Computed Values of Circulation Increase Factor τ for $R/r_2 = 1.0$

$\frac{1}{\lambda_i}$ \ x	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.85	0.9	0.95	0.975
Three-Bladed Propeller											
1.5	0.2161	0.2607	0.3186	0.3977	0.5117	0.6890	1.0043	1.2822	1.7657	2.9190	4.6878
2.0	0.1165	0.1464	0.1874	0.2446	0.3304	0.4663	0.7146	0.9318	1.3201	2.2029	3.5657
2.5	0.0635	0.0837	0.1129	0.1564	0.2236	0.3343	0.5397	0.7236	1.0483	1.8080	2.9250
3.0	0.0348	0.0485	0.0694	0.1022	0.1554	0.2468	0.4228	0.5839	0.8681	1.5414	2.5317
3.5	0.0192	0.0283	0.0432	0.0678	0.1098	0.1858	0.3383	0.4811	0.7364	1.3458	2.2425
4.0	0.0106	0.0166	0.0270	0.0454	0.0787	0.1420	0.2750	0.4029	0.6349	1.1949	2.0200
4.5	0.0058	0.0098	0.0170	0.0306	0.0569	0.1097	0.2262	0.3416	0.5542	1.0739	1.8434
5.0	0.0032	0.0057	0.0108	0.0208	0.0414	0.0854	0.1878	0.2923	0.4884	0.9743	1.6982
5.5	0.0018	0.0034	0.0068	0.0142	0.0303	0.0669	0.1570	0.2520	0.4335	0.8903	1.5757
6.0	0.0010	0.0020	0.0043	0.0097	0.0222	0.0527	0.1319	0.2184	0.3871	0.8182	1.4701
Four-Bladed Propeller											
1.5	0.0816	0.1130	0.1544	0.2123	0.2978	0.4343	0.6828	0.9047	1.2938	2.2296	3.6764
2.0	0.0384	0.0567	0.0825	0.1212	0.1813	0.2843	0.4757	0.6521	0.9619	1.7045	2.8180
2.5	0.0180	0.0283	0.0446	0.0710	0.1155	0.1950	0.3526	0.4995	0.7627	1.3920	2.3241
3.0	0.0083	0.0142	0.0243	0.0421	0.0746	0.1369	0.2684	0.3951	0.6254	1.1821	2.0046
3.5	0.0038	0.0071	0.0133	0.0252	0.0488	0.0977	0.2080	0.3187	0.5243	1.0297	1.7806
4.0	0.0018	0.0036	0.0073	0.0152	0.0322	0.0705	0.1634	0.2607	0.4459	0.9099	1.6054
4.5	0.0008	0.0018	0.0040	0.0092	0.0214	0.0513	0.1297	0.2155	0.3834	0.8128	1.4628
5.0	0.0004	0.0009	0.0022	0.0055	0.0142	0.0375	0.1036	0.1796	0.3324	0.7322	1.3437
5.5	0.0002	0.0004	0.0012	0.0034	0.0095	0.0276	0.0833	0.1506	0.2900	0.6638	1.2425
6.0	0.0001	0.0002	0.0007	0.0020	0.0064	0.0203	0.0672	0.1268	0.2544	0.6052	1.1547
Five-Bladed Propeller											
1.5	0.0311	0.0506	0.0782	0.1194	0.1836	0.2908	0.4932	0.6776	1.0042	1.7946	3.0116
2.0	0.0127	0.0225	0.0377	0.0629	0.1059	0.1832	0.3382	0.4829	0.7465	1.3683	2.3206
2.5	0.0051	0.0098	0.0181	0.0334	0.0625	0.1194	0.2434	0.3646	0.5859	1.1258	1.9300
3.0	0.0020	0.0042	0.0087	0.0179	0.0373	0.0796	0.1793	0.2818	0.4751	0.9555	1.6730
3.5	0.0008	0.0018	0.0042	0.0096	0.0224	0.0535	0.1342	0.2133	0.3925	0.8275	1.4850
4.0	0.0003	0.0008	0.0020	0.0052	0.0136	0.0363	0.1015	0.1768	0.3285	0.7264	1.3357
4.5	0.0001	0.0003	0.0010	0.0028	0.0082	0.0248	0.0774	0.1421	0.2777	0.6439	1.2130
5.0	0.0001	0.0001	0.0005	0.0015	0.0050	0.0170	0.0594	0.1150	0.2365	0.5753	1.1101
5.5	0.0000	0.0001	0.0002	0.0008	0.0030	0.0116	0.0458	0.0936	0.2025	0.5172	1.0222
6.0	0.0000	0.0000	0.0001	0.0004	0.0019	0.0080	0.0354	0.0764	0.1743	0.4674	0.9459
Six-Bladed Propeller											
1.5	0.0118	0.0229	0.0405	0.0691	0.1170	0.2021	0.3703	0.5275	0.8096	1.4983	2.5623
2.0	0.0042	0.0089	0.0175	0.0334	0.0633	0.1223	0.2492	0.3722	0.6004	1.1619	2.0131
2.5	0.0014	0.0034	0.0074	0.0161	0.0346	0.0756	0.1732	0.2744	0.4661	0.9605	1.6658
3.0	0.0005	0.0012	0.0031	0.0077	0.0190	0.0474	0.1233	0.2072	0.3723	0.7963	1.4397
3.5	0.0002	0.0005	0.0013	0.0037	0.0105	0.0300	0.0888	0.1588	0.3027	0.6851	1.2747
4.0	0.0001	0.0002	0.0006	0.0018	0.0058	0.0191	0.0646	0.1230	0.2490	0.5966	1.1423
4.5	0.0000	0.0001	0.0002	0.0009	0.0032	0.0122	0.0473	0.0960	0.2066	0.5245	1.0334
5.0	0.0000	0.0000	0.0001	0.0004	0.0018	0.0078	0.0347	0.0754	0.1727	0.4646	0.9418
5.5	0.0000	0.0000	0.0000	0.0002	0.0010	0.0050	0.0256	0.0594	0.1450	0.4140	0.8635
6.0	0.0000	0.0000	0.0000	0.0001	0.0006	0.0032	0.0189	0.0470	0.1224	0.3708	0.7955



R = prop radius

Figure 1 – Effect of Duct Diameter on the Circulation Distribution for $\mu_0 = 3.0$, $p = 4$ r_2 = DUCT RADIUS

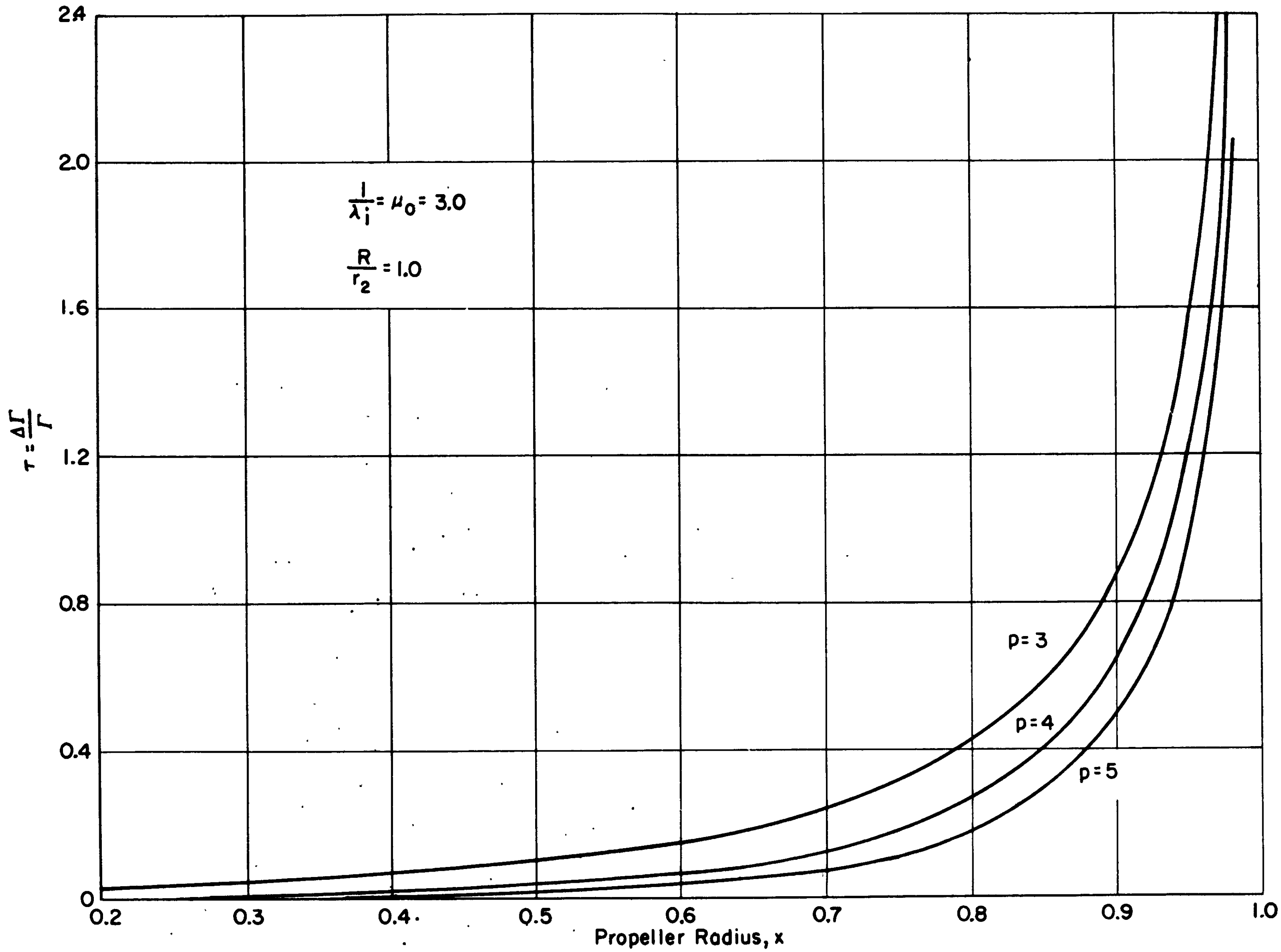


Figure 2 – Variation of Function τ for $\mu_0 = 3.0$, $R/r_2 = 1.0$

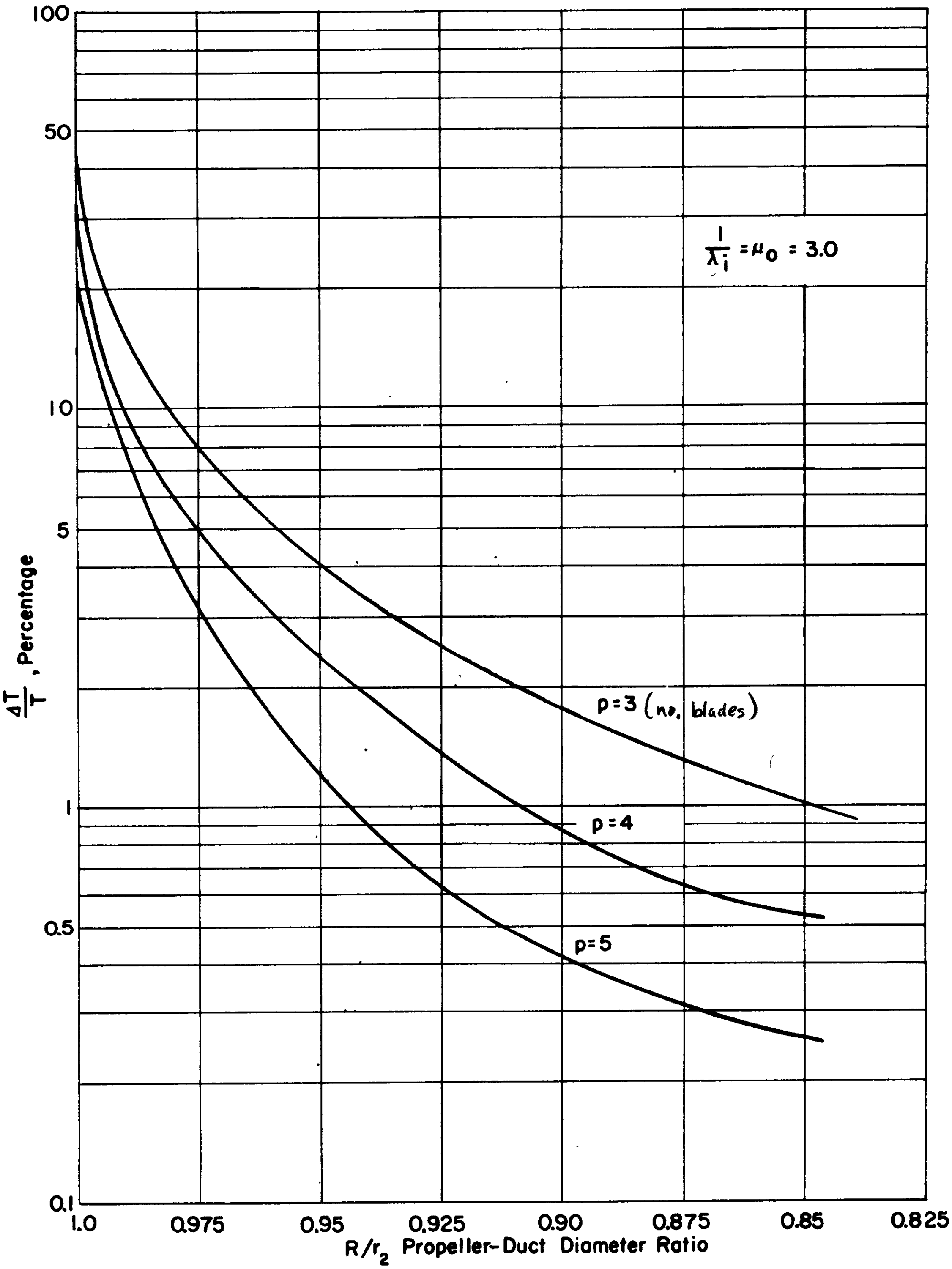


Figure 3 – Effect of Duct Diameter on Thrust Increase for μ_0 = 3.0

CONCLUSION

The solution of the potential problem for the optimum propeller operating within a duct indicates that the presence of the duct results in an increased circulation near the propeller tips and in an increase in thrust of the propeller. This increase in thrust should be added to the correction derived by Wood and Harris and resulting from the slipstream contraction. The percentage increase in thrust is a function of the number of blades and pitch ratio of the propeller and becomes important for small clearances between the propeller tips and the duct.

APPENDIX A

CALCULATION OF COEFFICIENTS

The coefficients B_m are calculated by solving the infinite set of simultaneous equations given by Equation [28].

$$\sum_{m=0}^{\infty} B_m \left[\frac{1}{2n-(2m+1)} - \frac{F_{pn}}{2n+(2m+1)} \right] = F_{pn} \left[\frac{\pi}{4n} - \sum_{m=0}^{\infty} \frac{A_m}{2n+(2m+1)} \right] \qquad [28]$$

where

$$F_{pn} = \frac{I_{pn}(pn\mu_0)}{I_{pn}(pn\mu_2)} \frac{K_{pn}(pn\mu_2)}{K_{pn}(pn\mu_0)}$$

for $n = 1, 2, 3 \ldots$ and where A_m is given by Equation [27]. We can write[4] approximately for the I and K functions

$$I_{pn}(pn\mu) = \left(\frac{1}{2\pi pn} \right)^{1/2} \frac{1}{(1+\mu^2)^{1/4}} e^{pny}$$

and

$$K_{pn}(pn\mu) = \left(\frac{\pi}{2pn} \right)^{1/2} \frac{1}{(1+\mu^2)^{1/4}} e^{-pny}$$

where

$$y = \sqrt{1+\mu^2} - \frac{1}{2} \log \frac{\sqrt{1+\mu^2}+1}{\sqrt{1+\mu^2}-1}$$

then

$$F_{pn} = e^{-2pn(y_2 - y_0)}$$

where y_2 and y_0 refer to μ_2 and μ_0.

The system of Equations [28] was solved by considering the exponential form for F_{pn} as twenty simultaneous equations with twenty unknowns, and the values of B_m determined. The first four values of B_m, i.e., $B_0 \ldots B_3$, are given in Figures 4, 5, 6, and 7 as a function of $2p(y_2 - y_0)$, which can be determined for any value of advance ratio, propeller-tunnel diameter and number of blades.

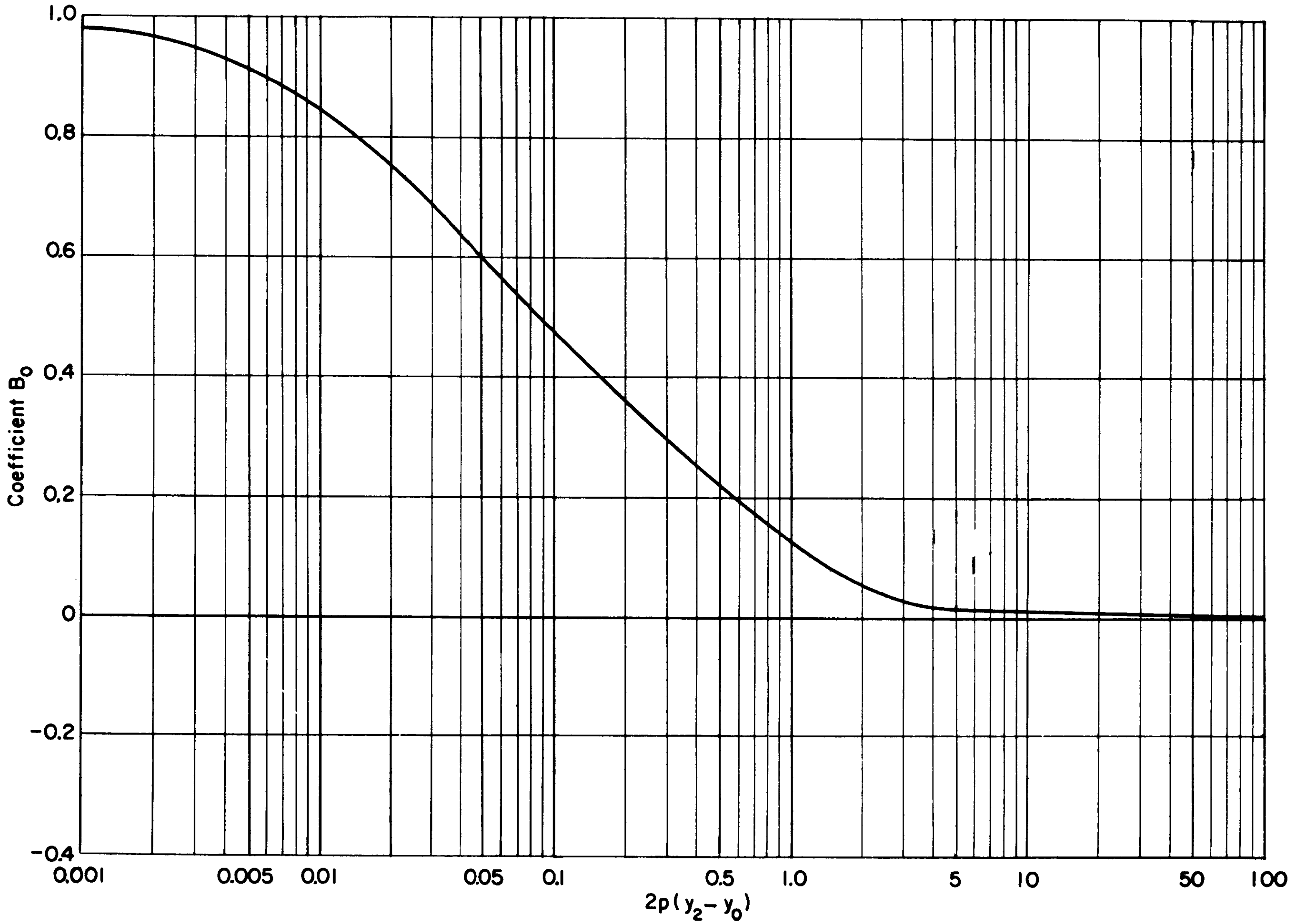


Figure 4 – Coefficient B_0

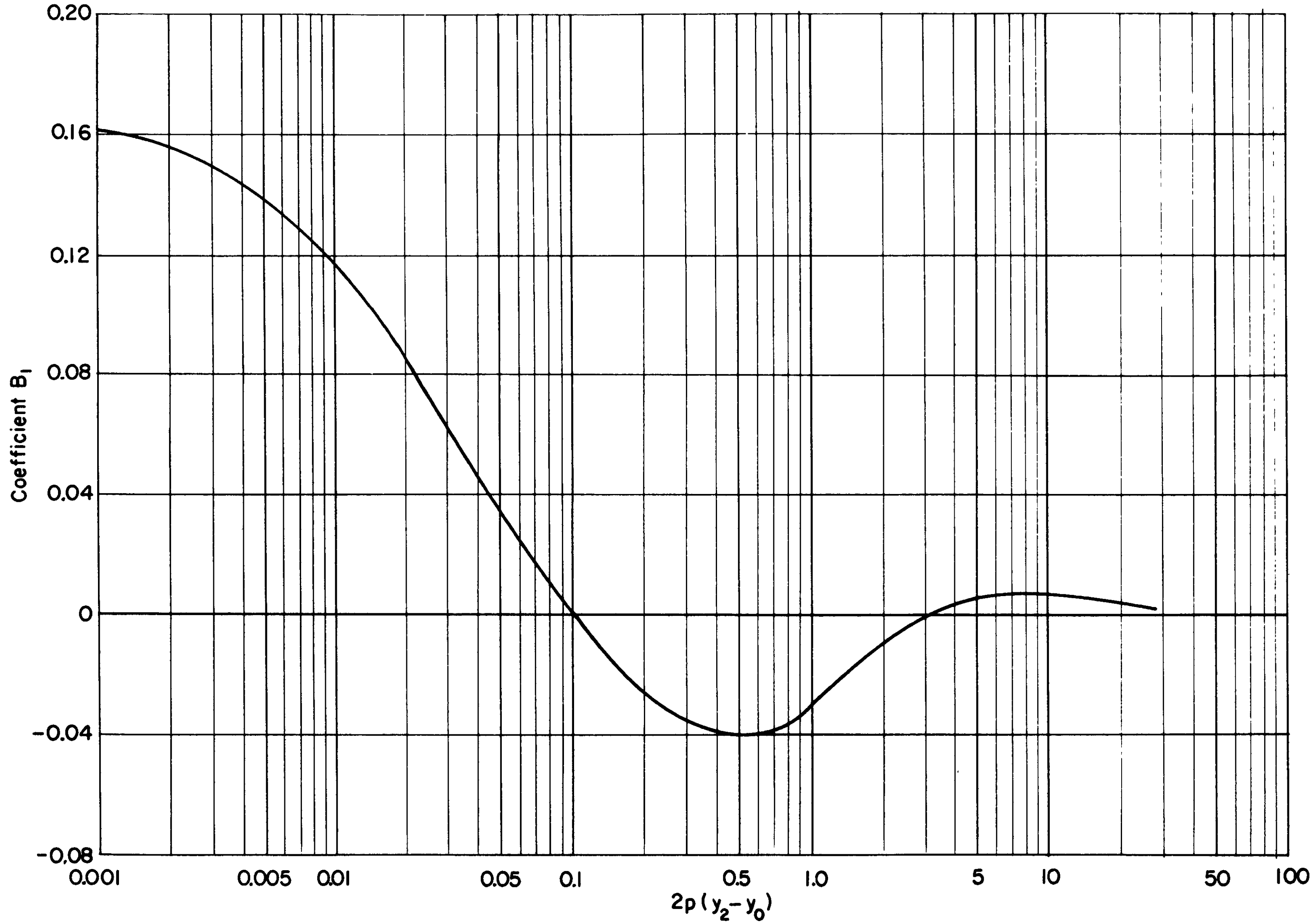


Figure 5 – Coefficient B_1

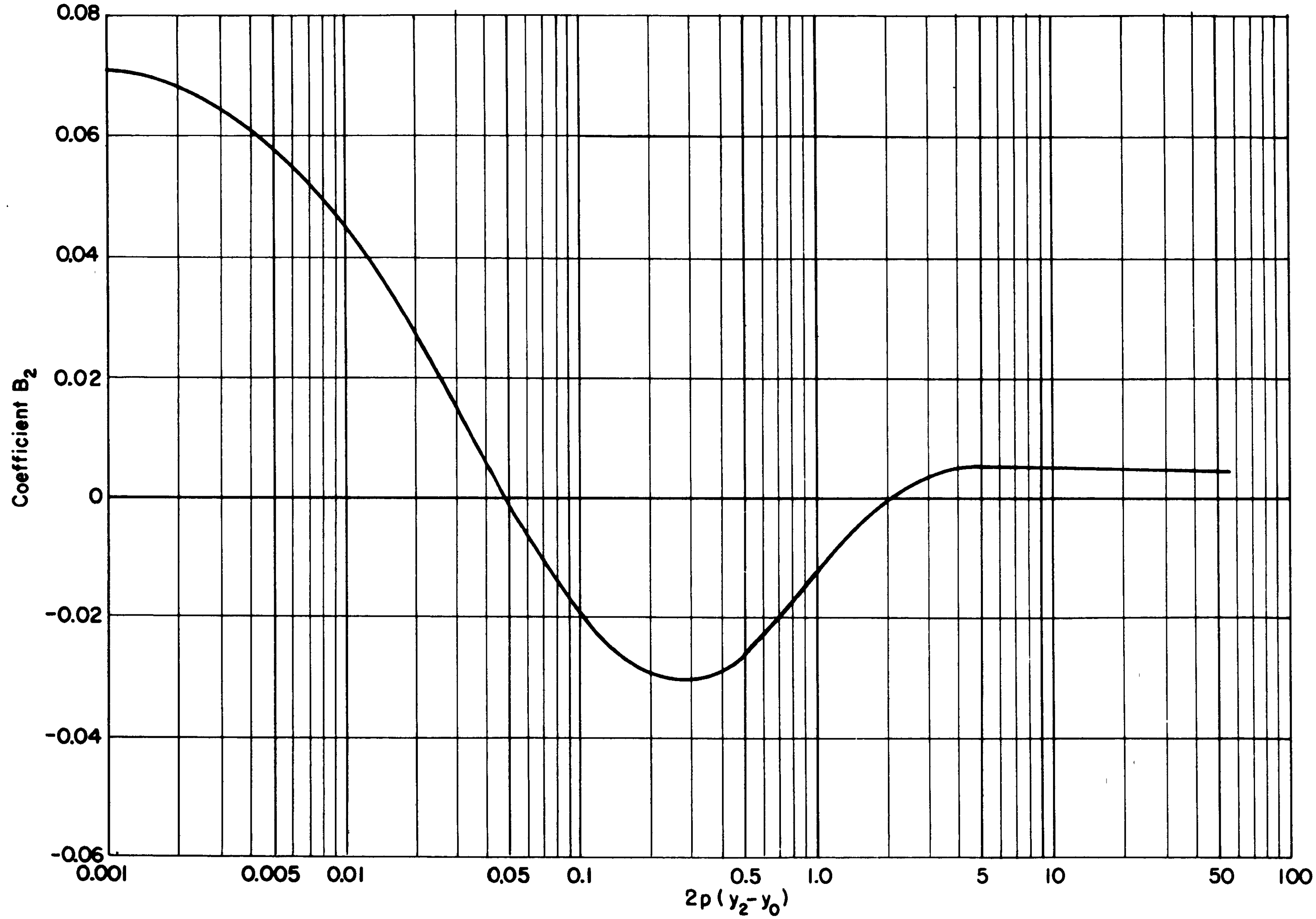


Figure 6 – Coefficient B_2

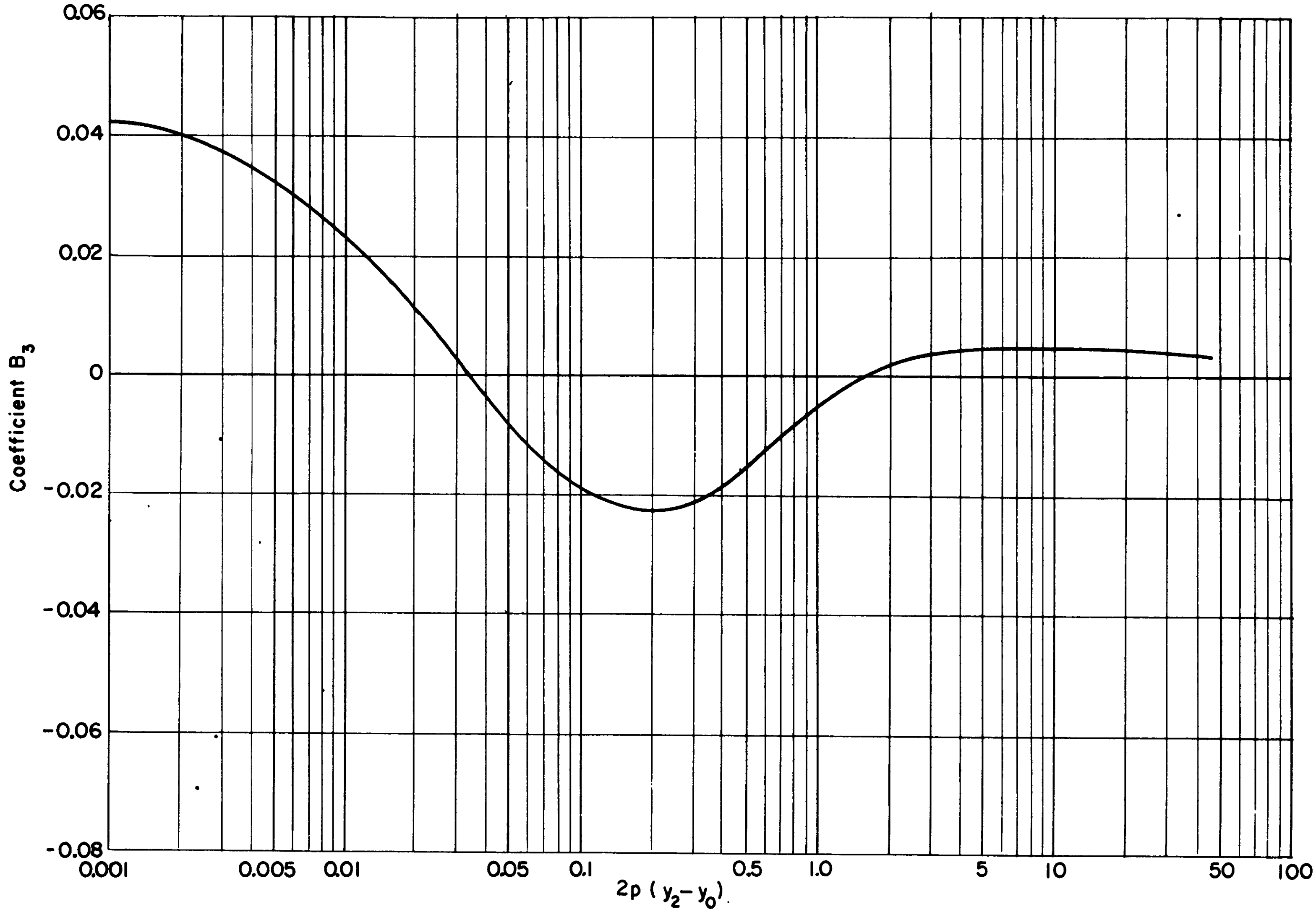


Figure 7 – Coefficient B_3

APPENDIX B

DETERMINATION OF THRUST INCREASE

The percentage increase in thrust due to the presence of the duct is given by Equation [33] as

$$\frac{\Delta T}{T} = \frac{2}{\pi}\,\frac{\mu_0^2}{1+\mu_0^2}\;\frac{\displaystyle\int_0^1 x \sum_{m=0}^{\infty} B_m \frac{I_\nu(\nu\mu_0 x)}{I_\nu(\nu\mu_0)}\,dx}{\displaystyle\int_0^1 \kappa_0 \frac{\mu_0^2 x^3}{1+x^2\mu_0^2}\,dx} \qquad [33]$$

In order to integrate the numerator we use[6] the relationship

$$\int z J_\nu(z)\,dz = \nu z J_\nu(z) S_{0,\nu-1}(z) - z J_{\nu-1}(z) S_{1,\nu}(z)$$

If we let $z = xe^{\frac{\pi i}{2}}$ and noting that $J_\nu\left(xe^{\frac{\pi i}{2}}\right) = e^{\frac{\nu\pi i}{2}} I_\nu(x)$

then

$$i\int x I_\nu(x)\,dx = \nu x I_\nu(x) S_{0,\nu-1}(ix) + ix I_{\nu-1}(x) S_{1,\nu}(ix)$$

where

$$S_{1,\nu}(ix) = s_{1,\nu}(ix) + \Gamma\left(1-\frac{\nu}{2}\right)\Gamma\left(1+\frac{\nu}{2}\right)\left\{\cos\frac{\nu\pi}{2}\,J_\nu(ix) - \sin\frac{\nu\pi}{2}\,Y_\nu(ix)\right\}$$

$$= s_{1,\nu}(ix) + \frac{\nu\pi}{2\sin\frac{\nu\pi}{2}}\,I_\nu(x) + \nu e^{-\frac{\pi i}{2}}\,K_\nu(x) \qquad [34]$$

for $p \neq 4, 8, 12$, and where

$$s_{1,\nu}(ix) = -\left\{\frac{x^2}{2^2-\nu^2} + \frac{x^4}{(2^2-\nu^2)(4^2-\nu^2)} + \frac{x^6}{(2^2-\nu^2)(4^2-\nu^2)(6^2-\nu^2)} + \cdots\right\}$$

$$= -\,t_{1,\nu}(x) \qquad [35]$$

similarly we obtain

$$S_{0,\nu-1}(ix) = s_{0,\nu-1}(ix) - \frac{i\pi}{2\sin\frac{\nu\pi}{2}} I_{\nu-1}(x) - ie^{-\frac{\nu\pi i}{2}} K_{\nu-1}(x) \qquad [36]$$

where

$$s_{0,\nu-1}(ix) = i\left\{\frac{x}{[1-(\nu-1)^2]} + \frac{x^3}{[1-(\nu-1)^2][3^2-(\nu-1)^2]} + \cdots\right\}$$
$$= i\, t_{0,\nu-1}(x) \qquad [37]$$

Therefore

$$\int x\, I(x)\, dx = \nu x I_\nu(x)\, t_{0,\nu-1}(x) - x I_{\nu-1}(x)\, t_{1,\nu}(x)$$

Performing the integration for the known equation

$$\int_0^1 x J_\nu(\nu\mu_0 x)\, dx = \frac{1}{\nu\mu_0}\left\{\nu I_\nu(\nu\mu_0)\, t_{0,\nu-1}(\nu\mu_0) - I_{\nu-1}(\nu\mu_0)\, t_{1,\nu}(\nu\mu_0)\right\}$$

Equation [33] finally becomes

$$\frac{\Delta T}{T} = \frac{\mu_0}{1+\mu_0^2}\, \frac{\sum_{m=0}^{\infty} B_m\left[t_{0,\nu-1}(\nu\mu_0) - \frac{I_{\nu-1}(\nu\mu_0)}{\nu I_\nu(\nu\mu_0)}\, t_{1,\nu}(\nu\mu_0)\right]}{\int_0^1 \kappa_0\, \frac{x^3\mu_0^2}{1+x^2\mu_0^2}\, dx} \qquad [38]$$

for $p \neq 4, 8, 12$ where $t_{0,\nu-1}(\nu\mu_0)$ and $t_{1,\nu}(\nu\mu_0)$ are given by Equations [35] and [37]. For propellers having four blades or a multiple of four, Equation [38] does not hold and the following should be used

$$\frac{\Delta T}{T} = \frac{\mu_0}{1+\mu_0^2} \cdot \frac{\sum_{m=0}^{\infty} B_m \left[-i S_{0,\nu-1}(i\nu\mu_0) + \frac{I_{\nu-1}(\nu\mu_0)}{\nu I_\nu(\nu\mu_0)} \; S_{1,\nu}(i\nu\mu_0) \right]}{\int_0^1 \kappa_0 \; \frac{\mu_0^2 x^3}{1 + x^2 \mu_0^2} \; dx} \qquad [39]$$

for $p = 4, 8, 12 \ldots .$, where

$$S_{0,\nu-1}(i\nu\mu_0) = \frac{1}{(i\nu\mu_0)} \left\{ 1 + \frac{[1-(\nu-1)^2]}{(\nu\mu_0)^2} + \frac{[1-(\nu-1)^2]\,[3^2-(\nu-1)^2]}{\nu\mu_0^4} + \cdots \right\} \qquad [40]$$

and

$$S_{1,\nu}(i\nu\mu_0) = 1 + \frac{2^2-\nu^2}{(\nu\mu_0)^2} + \frac{(2^2-\nu^2)(4^2-\nu^2)}{(\nu\mu_0)^4} + \cdots . \qquad [41]$$

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