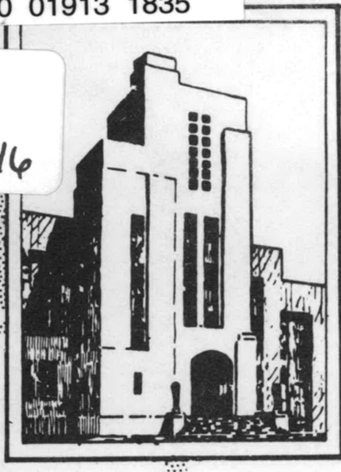

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APPLIED MATHEMATICS

LECTURES ON APPLIED MATHEMATICS
THE LAPLACE TRANSFORMATION

by

Francis D. Murnaghan

APPLIED MATHEMATICS LABORATORY

RESEARCH AND DEVELOPMENT REPORT

June 1959

Report 1346

PRNC-TMB-648 (Rev. 3-58)

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FOREWORD

These lecture notes were prepared by Professor Francis D. Murnaghan for use in a series of nineteen lectures on the Laplace Transformation given by him during the spring of 1959 at the Applied Mathematics Laboratory, David Taylor Model Basin. The lectures were well attended by the technical staff of the David Taylor Model Basin laboratories as well as by scientists throughout the Washington community. It was an inspiring experience to all who had the opportunity to be present.

The Applied Mathematics Laboratory is proud to present these lecture notes in report form. Although the warm humor and the pedagogical skill of the delivery are missing from the notes, these lectures constitute an unusually clear and complete presentation of the theory and application of the Laplace Transformation which will remain of permanent value in the instruction of applied mathematics.

Harry J Polachek
Harry Polachek
Technical Director
Applied Mathematics Laboratory

ABSTRACT

These lectures on applied mathematics are devoted to the Laplace Transformation and its application to linear ordinary differential equations with variable coefficients, to linear partial differential equations, with two independent variables and constant coefficients, and to the determination of asymptotic series. The treatment of the Laplace Transformation is based on the Fourier Integral Theorem and the ordinary differential equations selected for detailed treatment are those of Laguerre and Bessel. The partial differential equation governing the motion of a tightly stretched vibrating string and a generalization of this equation are fully treated. Asymptotic series for the integral $\int_p^\infty \exp(-z^2)dz$ and for modified Bessel function $I_n(p)$, $\left|\arg p\right| < \frac{\pi}{2}$, are obtained by means of the Laplace Transformation and, finally, asymptotic series useful in the calculation of the ordinary Bessel functions $J_n(t)$ are treated.

Care has been taken to make the treatment self-contained and details of the proofs of the basic mathematical theorems are given.

Lectures on Applied Mathematics

Lecture 1

Absolutely Integrable Piecewise Continuous Functions

Let f(t) be a complex-valued function of the unrestricted real variable t, $-\infty < t < \infty$, it being understood that real-valued functions of t are included in the class of complex-valued functions of t, a real-valued function being a complex-valued function whose imaginary part is identically zero. The class of continuous functions is too restricted for our purpose and we shall merely suppose that the number of points of discontinuity of f(t), if any such exist, in any finite interval is finite. This will be the case if the number of points of discontinuity of f(t) is finite but this sufficient condition is not necessary; for example, f(t) may be discontinuous for all integral values of t or it may be a periodic function, of period T, with a finite number of points of discontinuity in the interval $0 \leq t < T$. When f(t) possesses not more than a finite number of points of discontinuity in any finite interval we shall say that is possesses Property 1 and we shall term any function f(t) which possesses Property 1 a piecewise continuous function.

In enlarging the class of functions which we propose to consider from continuous to piecewise continuous functions we lose some of the most convenient properties of the class of continuous functions. For example, every continuous function is bounded over every finite interval but this is not true for piecewise continuous functions. For example, the function f(t) which is equal to t^{-1} if $t \neq 0$ and which is assigned any value at $t=0$ (the particular value assigned to it at $t=0$ being immaterial) is piecewise continuous, since it has only one point of discontinuity, but it is not bounded over any interval which contains the point $t=0$. Furthermore, every continuous function is integrable, in the sense of Riemann, over any closed interval $a \leq t \leq b$ but this is not necessarily true for a piecewise continuous function if the interval contains a point of discontinuity of the function at which the function is unbounded. If c is such a point of discontinuity of f(t) in the interval $a \leq t \leq b$ and if f(t) is continuous at all other points of this interval we say that f(t) is improperly integrable, in the sense of Riemann, over the interval $a \leq t \leq b$ if the function $F(\delta_1, \delta_2)$ of the two non-negative variables δ_1 and δ_2 which is furnished by the sum of the integrals of f(t) over the intervals $a \leq t \leq c - \delta_1$ and $c + \delta_2 \leq t \leq b$, where $\delta_1 \leq c-a$ and $\delta_2 \leq b-c$, possesses a limit as δ_1 and δ_2 tend, independently of each other, to zero and we term this limit the improper integral of f(t) over the interval $a \leq t \leq b$. <u>Note.</u> We assume that the non-negative variables δ_1 and δ_2 are actually positive unless $c=a$, in which case $\delta_1=0$ and δ_2 is positive, or $c=b$, in which case $\delta_2=0$ and δ_1 is positive. There is no lack of generality in assuming that f(t) has only one point of discontinuity, at which it is unbounded, in the interval $a \leq t \leq b$ since the number of its points of discontinuity in this interval is, by hypothesis, finite and if it has more than one point of discontinuity at which it is unbounded in the interval $a \leq t \leq b$ we can break this interval down into a number of sub-intervals each containing only one point of discontinuity of f(t), at which it is unbounded, and define the improper integral of f(t) over the interval $a \leq t \leq b$ as the sum of the improper integrals of f(t) over the sub-intervals, assuming that each of these improper integrals exists. If any single one of these improper integrals fails to exist the improper integral of f(t) over the interval $a \leq t \leq b$ fails to exist.

The function f(t) of the unrestricted real variable t which is 0 if $t < 0$ and is $t^{-1/2}$ if $t > 0$, the value assigned to f(t) at t=0 being immaterial, is improperly integrable over any interval $a \leq t \leq b$ which contains the point t=0. For example, the improper integral of f(t) over the interval $0 \leq t \leq b$ is $2b^{1/2}$. On the other hand, the function f(t) which is 0 if $t < 0$ and is t^{-1} if $t > 0$, the value assigned to f(t) at t=0 being, again, immaterial, is not improperly integrable over any interval which contains the point t=0. In general the point t=0 is a point of discontinuity of $f(t) = t^{\alpha}$, $t > 0$, α real, $f(t) = 0$, $t < 0$, at which f(t) is unbounded if $\alpha < 0$. If $\alpha > -1$ this piecewise continuous function is improperly integrable over any interval $a \leq t \leq b$ which contains the point t = 0 and, if $\alpha \leq -1$, f(t) is not improperly integrable over any such interval.

If f(t) is a piecewise continuous function so also is $|f(t)|$ and it is easy to see that if $|f(t)|$ is improperly integrable over an an interval $a \leq t \leq b$ which contains a single point c of discontinuity of f(t) at which f(t) is unbounded then f(t) is also improperly integrable over the interval $a \leq t \leq b$. Indeed to prove this we must show that each of the two integrals $\int_{c-\delta_1}^{c-\delta_1'} f(t)\,dt$ and $\int_{c+\delta_2'}^{c+\delta_2} f(t)\,dt$, where $0 < \delta_1' < \delta_1$ and $0 < \delta_2' < \delta_2$, may be made arbitrarily small by making δ_1 and δ_2, respectively, sufficiently small. The moduli of these integrals are dominated by, i. e., are not greater than, $\int_{c-\delta_1}^{c-\delta_1'} |f(t)|\,dt$ and $\int_{c+\delta_2'}^{c+\delta_2} |f(t)|\,dt$, respectively, and our hypothesis that f(t) is absolutely integrable (i. e., that $|f(t)|$ is integrable) over $a \leq t \leq b$ assures us that each of these two dominating numbers may be made arbitrarily small by merely making δ_1 and δ_2, respectively, sufficiently small. We shall assume that our piecewise continuous functions f(t) are such that $|f(t)|$ is improperly integrable over any finite interval $a \leq t \leq b$ which implies, as we have just seen, that f(t) is improperly integrable over any finite interval $a \leq t \leq b$. Note. If the interval $a \leq t \leq b$ does not contain a point of discontinuity of f(t) at which f(t) is unbounded both f(t) and $|f(t)|$ are properly integrable over the interval $a \leq t \leq b$ since they are bounded over this interval and continuous save, possibly, for a finite number of points.

We now make a final assumption concerning the class of complex-valued functions of the unrestricted real variable t which we propose to consider. We assume that not only is $|f(t)|$ integrable, properly or improperly, over every finite interval $a \leq t \leq b$ but that the function $F(a, b) = \int_a^b |f(t)|\,dt$ of the two real variables a and b possesses a finite limit as a and b tend, independently, to $-\infty$ and $+\infty$, respectively. When this is the case we say that f (t) is absolutely integrable over $-\infty < t < \infty$ and we term the limit of F(a, b), as $a \longrightarrow -\infty$ and $b \longrightarrow \infty$, the integral of $|f(t)|$ from $-\infty$ to ∞, this integral being denoted by the symbol $\int_{-\infty}^{\infty} |f(t)|\, dt$. When a complex-valued function f(t) of the unrestricted real variable t is such that it is absolutely integrable over $-\infty < t < \infty$ we say that it possesses Property 2. The functions f(t) which we propose to consider are those which possess both Property 1 and Property 2; in other words, they are piecewise continuous functions which are absolutely integrable over $-\infty < t < \infty$.

It is clear that if f(t) possesses Properties 1 and 2 then the function $\int_a^b f(t)\, dt$ of the two real variables a and b possesses a finite limit as a and b tend, independently, to $-\infty$ and $+\infty$, respectively. Indeed, in order to prove this we have to show that $\int_b^{b'} f(t)\, dt$ and $\int_{a'}^{a} f(t)\, dt$, where $b' > b$ and $a' < a$, may be made arbitrarily small by making b and -a sufficiently large. However $\left|\int_b^{b'} f(t)\, dt\right|$ and $\left|\int_{a'}^{a} f(t)\, dt\right|$ are dominated by $\int_b^{b'} |f(t)|\, dt$ and $\int_{a'}^{a} |f(t)|\, dt$, respectively, and each of these dominating numbers may be made arbitrarily small by making b and -a, respectively, sufficiently large (since f(t) is, by hypothesis, absolutely integrable over $-\infty < t < \infty$). Thus $\int_{-\infty}^{\infty} f(t)\, dt$ exists. The converse of this result is not true; $\int_{-\infty}^{\infty} f(t)\, dt$ may well exist without $\int_{-\infty}^{\infty} |f(t)|\, dt$ existing.

An example is furnished by the everywhere continuous function $f(t) = \frac{\sin t}{t}$, $t \neq 0$, $f(0) = 1$. $f(t)$ is an even function and so it suffices to consider $\int_0^b \frac{\sin t}{t} dt$. If $n\pi < b \leq (n+1)\pi$, $n = 0, 1, 2, \ldots$, we have

$$\int_0^b \frac{\sin t}{t} dt = \int_0^\pi \frac{\sin t}{t} dt + \int_\pi^{2\pi} \frac{\sin t}{t} dt + \ldots + \int_{(n-1)\pi}^{n\pi} \frac{\sin t}{t} dt + \int_{n\pi}^b \frac{\sin t}{t} dt$$

$$= I_1 - I_2 + \ldots + (-1)^{n-1} I_n + \int_{n\pi}^b \frac{\sin t}{t} dt$$

where $I_1 = \int_0^\pi \frac{\sin t}{t} dt > 0$, $I_2 = -\int_\pi^{2\pi} \frac{\sin t}{t} dt > 0$ and so on. On writing $t = u+\pi$ in the formula for I_2 we have $I_2 = \int_0^\pi \frac{\sin u}{u+\pi} du$ so that $I_2 < I_1$. Similarly $I_3 < I_2$, $I_4 < I_3$ and, generally, $I_{n+1} < I_n$, $n = 1, 2, 3, \ldots$. In order to appraise the integral $\int_{n\pi}^b \frac{\sin t}{t} dt$ we use the second theorem of the mean of integral calculus which tells us that, since $\frac{1}{t}$ is monotone decreasing over $n\pi \leq t \leq b$ and continuous at $t = n\pi$, $\int_{n\pi}^b \frac{\sin t}{t} dt = \frac{1}{n\pi} \int_{n\pi}^c \sin t\, dt$ $= \frac{\cos n\pi - \cos c}{n\pi}$, $n\pi \leq c \leq b$. Thus $\left|\int_{n\pi}^b \frac{\sin t}{t} dt\right| \leq \frac{2}{n\pi}$ and this may be made arbitrarily small, no matter what is the value of $b > n\pi$, by merely making n sufficiently large. In particular I_{n+1} may be made arbitrarily small by making n sufficiently large so that the alternating infinite series $I_1 - I_2 + I_3 - \ldots$ is convergent, its sum being the infinite integral $\int_0^\infty \frac{\sin t}{t} dt$. It follows that this infinite integral, whose existence we have just proven, lies between $I_1 - I_2$ and I_1. A simple application of Simpson's Rule shows that $I_1 = 1.86$, $I_2 = 0.44$, approximately, so that $\int_0^\infty \frac{\sin t}{t} dt$ lies between 1.42 (approximately) and 1.86 (approximately) and this implies that the integral of $\frac{\sin t}{t}$ over $-\infty < t < \infty$ exists and lies between 2.84 (approximately) and 3.72 (approximately). We shall shortly see that $\int_{-\infty}^\infty \frac{\sin t}{t} dt = \pi$. That $\frac{\sin t}{t}$ is not absolutely integrable over $-\infty < t < \infty$ is clear since $\int_0^b \frac{|\sin t|}{t} dt = I_1 + I_2 + \ldots + I_n + \int_{n\pi}^b \frac{|\sin t|}{t} dt$ and

$$I_1 > \frac{1}{\pi}\int_0^\pi \sin t \, dt = \frac{2}{\pi}, \; I_2 > \frac{1}{2\pi}\int_0^\pi \sin u \, du = \frac{2}{2\pi} \text{ and, generally,}$$

$$I_n > \frac{2}{n\pi} \text{ so that } \int_0^b \frac{|\sin t|}{t} dt > \frac{2}{\pi}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right), \; n = 1, 2, \ldots .$$

Thus, since the partial sums of the infinite series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ are unbounded, $\int_0^b \frac{|\sin t|}{t} dt$ may be made arbitrarily large by making b sufficiently large and the infinite integral $\int_0^\infty \frac{|\sin t|}{t} dt$ does not exist. This implies that $\left|\frac{\sin t}{t}\right|$ is not integrable over $-\infty < t < \infty$.

<u>Exercise</u>. Show that if ω is any real number the infinite integral $\int_{-\infty}^\infty \frac{\sin \omega t}{t} dt$ exists, its value being C, 0, $-C$ according as $\omega > 0$, $\omega = 0$, $\omega < 0$, respectively, where C is the value of the infinite integral $\int_{-\infty}^\infty \frac{\sin t}{t} dt$.

Lecture 2

The Fourier Transform of an Absolutely Integrable Piecewise Continuous Function

If $f(t)$ is a complex-valued function of the unrestricted real variable t which possesses Properties 1 and 2, so that it is piecewise continuous and absolutely integrable over $-\infty < t < \infty$, it does not lose these properties on multiplication by $\exp(-i\omega t)$, ω any real number. Indeed $\exp(-i\omega t)$ is everywhere continuous so that $f(t)\exp(-i\omega t)$ is piecewise continuous and $|\exp(-i\omega t)| = 1$ so that $|f(t)\exp(-i\omega t)| = |f(t)|$. Since $f(t)\exp(-i\omega t)$ is absolutely integrable over $-\infty < t < \infty$, the infinite integral $\int_{-\infty}^{\infty} f(t)\exp(-i\omega t)dt$ exists for each value of ω. Introducing, for our later convenience, the numerical factor $(2\pi)^{-1/2}$ we set

$$g(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt$$

and we term $g(\omega)$ the Fourier Transform of $f(t)$. Since

$$\left|\int_a^b f(t)\exp(-i\omega t)dt\right| \le \int_a^b \left|f(t)\exp(-i\omega t)\right| dt = \int_a^b |f(t)|dt \le \int_{-\infty}^{\infty} |f(t)|dt$$

we have $\left|\int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt\right| \le \int_{-\infty}^{\infty} |f(t)|\,dt$ so that $|g(\omega)| \le$ $(2\pi)^{-1/2}\int_{-\infty}^{\infty} |f(t)|\,dt$.

Thus $g(\omega)$ is bounded over $-\infty < \omega < \infty$. It is easy to see that $\int_a^b f(t)\exp(-i\omega t)\,dt$ is an everywhere continuous function of ω, no matter what is the interval $a \le t \le b$. To prove this we first consider the case where $|f(t)|$ is bounded, say $\leq M$, over $a \leq t \leq b$ (so that the integral $\int_a^b f(t) \exp(-i\omega t)\, dt$ is a proper Riemann integral). If ω and $\omega + \Delta\omega$ are any two real numbers we have

$$\left| \Delta \int_a^b f(t) \exp(-i\omega t)\, dt \right| = \left| \int_a^b f(t) \exp(-i\omega t) \left\{ \exp(-i\Delta\omega t) - 1 \right\} dt \right|$$

$$\leq M \int_a^b \left| \left\{ \exp(-i\Delta\omega t) - 1 \right\} \right| dt.$$

Since the function $\exp z$ of the complex variable z is continuous at $z = 0$, where it has the value 1, we can make $|\exp(-i\Delta\omega t) - 1|$ arbitrarily small, say $< \epsilon/M(b-a)$, by making $|-i\Delta\omega . t|$ sufficiently small, say $< \delta$, ϵ being an arbitrarily assigned positive number. Denoting, for a moment, by α the greater of the two numbers $|a|$ and $|b|$, $|-i\Delta\omega t| < \delta$ over the interval $a \leq t \leq b$ if $|-i\Delta\omega\alpha| < \delta$, i.e. if $|\Delta\omega| < \frac{\delta}{\alpha}$; hence, $\left| \Delta \int_a^b f(t) \exp(-i\omega t)\, dt \right| < \epsilon$ if $|\Delta\omega| < \frac{\delta}{\alpha}$ so that $\int_a^b f(t) \exp(-i\omega t) dt$ is an everywhere continuous function of ω. If f(t) is not bounded over $a \leq t \leq b$ it suffices to consider the case where f(t) has a single point of discontinuity c, at which it is unbounded, in the interval $a \leq t \leq b$. We write $\int_a^b f(t) \exp(-i\omega t)\, dt$ in the form $\int_a^{c-\delta_1} f(t) \exp(-i\omega t)\, dt$ $+ \int_{c-\delta_1}^{c+\delta_2} f(t) \exp(-i\omega t)\, dt + \int_{c+\delta_2}^{b} f(t) \exp(-i\omega t)\, dt$ where δ_1 and δ_2 are any two positive numbers which are less than c-a and b-c, respectively, save when c = a, in which case $\delta_1 = 0$ and δ_2 is any positive number $< b-a$, or when c = b, in which case $\delta_2 = 0$, and δ_1 is any positive number $< b-a$. It suffices to consider the first case, where $a < c < b$, the argument in the other two cases being precisely the same. The integrals

$\int_a^{c-\delta_1} f(t)\exp(-i\omega t)\,dt$ and $\int_{c+\delta_2}^{b} f(t)\exp(-i\omega t)\,dt$ are everywhere continuous functions of ω, since f(t) is continuous, by hypothesis, over the intervals $a \leq t \leq c-\delta_1$ and $c+\delta_2 \leq t \leq b$ and so we have merely to consider the integral $\int_{c-\delta_1}^{c+\delta_2} f(t)\exp(-i\omega t)\,dt$. The modulus of this integral, being dominated by $\int_{c-\delta_1}^{c+\delta_2} |f(t)|\,dt$, may be made arbitrarily small, say $< \epsilon$, since f(t) possesses Property 2, by making δ_1 and δ_2 sufficiently small, the choice of δ_1 and δ_2 being independent of ω. Once this choice of δ_1 and δ_2 has been made it follows that the modulus of $\Delta\int_{c-\delta_1}^{c+\delta_2} f(t)\exp(-i\omega t)\,dt$ is less than 2ϵ, no matter what are the values assigned to ω and $\Delta\omega$. Hence $\left|\Delta\int_a^b f(t)\exp(-i\omega t)dt\right|$ may be made arbitrarily small, no matter what is the value of ω, by making $|\Delta\omega|$ sufficiently small so that $\int_a^b f(t)\exp(-i\omega t)\,dt$ is an everywhere continuous function of ω. This implies, in view of the fact that f(t) is, by hypothesis, absolutely integrable over $-\infty < t < \infty$, that the infinite integral $\int_{-\infty}^{\infty} f(t)\exp(-i\omega t)dt$ is an everywhere continuous function of ω; indeed, the modulus of the difference between $\int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt$ and $\int_a^b f(t)\exp(-i\omega t)dt$ is dominated by the sum of the two infinite integrals $\int_{-\infty}^{a} |f(t)|dt$, $\int_b^{\infty} |f(t)|\,dt$ and this sum may be made arbitrarily small, say $< \epsilon$, by making a negative and b positive and choosing -a and b sufficiently large, the choice of a and b being independent of ω. Once this choice of a and b has been made, it follows that the modulus of $\Delta\left\{\int_{-\infty}^{a} f(t)\exp(-i\omega t)dt + \int_b^{\infty} f(t)\exp(-i\omega t)dt\right\}$ is less that 2ϵ no matter what are the values assigned to ω and $\omega + \Delta\omega$.

Hence $\left| \Delta \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)dt \right|$ may be made arbitrarily small, no matter what is the value of ω, by making $|\Delta\omega|$ sufficiently small, so that $\int_{-\infty}^{\infty} f(t) \exp(-i\omega t)dt$ is an everywhere continuous function of ω. This, combined with the boundedness of $g(\omega)$ over $-\infty < \omega < \infty$, is our first result which may be stated as follows:

The Fourier Transform $g(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)dt$ of any piecewise continuous complex-valued function $f(t)$ of the unrestricted real variable t which is absolutely integrable over $-\infty < t < \infty$ is an everywhere continuous function of the unrestricted real variable ω; moreover, $g(\omega)$ is bounded over $-\infty < \omega < \infty$.

Example 1. Let $f(t) = 0$ if $t < -b$ and if $t > b$, where b is any positive real number, and let $f(t) = 1$ if $-b < t < b$, the values assigned to $f(t)$ when $t = -b$ and when $t = b$ being immaterial. Then

$$g(\omega) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin b\omega}{\omega}, \text{ if } \omega \neq 0, \text{ while } g(0) = \left(\frac{2}{\pi}\right)^{1/2} b.$$

Note. This Example shows that, while the Fourier Transform operation is a smoothing, or strengthening, one as far as Property 1 is concerned ($g(\omega)$ being everywhere continuous and bounded over $-\infty < \omega < \infty$ while $f(t)$ may be only piecewise continuous and may not be bounded over $-\infty < t < \infty$) it is a roughening, or weakening, one as far as Property 2 is concerned: $g(\omega)$ may not be absolutely integrable over $-\infty < \omega < \infty$. In the present example $g(\omega)$ is integrable, in the Riemann sense, over $-\infty < \omega < \infty$ but the following Example shows that $g(\omega)$ may not be integrable over $-\infty < \omega < \infty$.

Example 2. Let $f(t) = 0$ if $t < 0$ and $= \exp(izt)$, where $z = x+iy$ is any complex number whose imaginary part y is positive, if $t > 0$, the value assigned to $f(t)$ at $t = 0$ being immaterial. $f(t)$ possesses

Property 1, since its only point of discontinuity is $t = 0$, and it possesses Property 2, i.e., it is absolutely integrable over $-\infty < t < \infty$, since $|f(t)| = 0$ if $t < 0$ and $|f(t)| = \exp(-yt)$ if $t > 0$ and $t^2 \exp(-yt)$ is arbitrarily small, say < 1, if t is sufficiently large since y is, by hypothesis, positive. The Fourier Transform of f(t) is

$$g(\omega) = (2\pi)^{-1/2} \int_0^\infty \exp\left[-i(\omega - z)t\right] dt = \frac{1}{(2\pi)^{1/2} i(\omega - z)}$$

and, since $|\omega - z| \leq |\omega| + |z| \leq 2|\omega|$, if $|\omega| > |z|$, we have $|g(\omega)| > \frac{1}{(2\pi)^{1/2} \; 2|\omega|}$, if $|\omega| > |z|$, so that $|g(\omega)|$ is not integrable over $-\infty < \omega < \infty$. Also, since $\log(\omega - z)$ is unbounded at $\omega = -\infty$ and at $\omega = \infty$, $g(\omega)$ is not integrable over $-\infty < \omega < \infty$.

Note. We shall see shortly that if f(t), in addition to possessing Properties 1 and 2, is such that its real and imaginary parts are monotone over sufficiently small intervals to the right and to the left of $t = 0$, or if f(t) possesses a right-hand and a left-hand derivative at $t = 0$, then the Cauchy principal value, $\lim_{\alpha \to \infty} \int_{-\alpha}^{\alpha} g(\omega)\, d\omega$ of the integral of $g(\omega)$ over $-\infty < \omega < \infty$ exists, despite the fact that the integral of $g(\omega)$ over $-\infty < \omega < \infty$ may not exist. We shall denote this Cauchy principal value of the integral of $g(\omega)$ over $-\infty < \omega < \infty$ by the symbol $\int_{(-\infty)}^{(\infty)} g(\omega) d\omega$. Thus in the present Example we have $\int_{(-\infty)}^{(\infty)} g(\omega) d\omega = \frac{1}{(2\pi)^{1/2} i} \lim_{\alpha \to \infty} \int_{-\alpha}^{\alpha} \frac{d\omega}{\omega - z} =$

$$\frac{1}{(2\pi)^{1/2} i} \lim_{\alpha \to \infty} \log(\omega - z) \Big|_{-\alpha}^{\alpha} = \left(\frac{\pi}{2}\right)^{1/2}$$

since the argument of $\frac{\alpha - z}{-\alpha - z}$ tends to π as $\alpha \to \infty$ while the modulus of $\frac{\alpha - z}{-\alpha - z}$ tends to 1 as $\alpha \to \infty$.

Lecture 3

The Fourier Integral Theorem

Let $f(t) = f_1(t) + if_2(t)$ be a complex-valued function, whose real and imaginary parts are $f_1(t)$ and $f_2(t)$, respectively, of the unrestricted real variable t and let f(t) possess Properties 1 and 2. At any point t at which f(t) is continuous the two limits

$$f(t+0) = \lim_{\delta\to 0} f(t+\delta), \quad \delta > 0$$

$$f(t-0) = \lim_{\delta\to 0} f(t-\delta), \quad \delta > 0$$

exist and are equal, their common value being f(t) (this being the definition of the concept of continuity). At a point t where f(t) fails to be continuous the limits f(t+0), f(t-0) need not exist, and, if they do exist, they need not be f(t). However, these limits will exist, by hypothesis, if f(t) possesses a right-hand derivative and a left-hand derivative at t, the definition of the right-hand derivative, for example, being $\lim_{\delta\to 0} \dfrac{f(t+\delta) - f(t+0)}{\delta}$, $\delta > 0$; they will exist also if the real and imaginary parts, $f_1(t)$ and $f_2(t)$, respectively, of f(t) are monotone and bounded over sufficiently small intervals to the right and to the left of t, f(t+0), for example, being $f_1(t+0) + if_2(t+0)$ where $f_1(t+0)$, for example, is the greatest lower bound, or least upper bound, of $f_1(t+\delta)$, $\delta > 0$ and sufficiently small, according as $f_1(t)$ is monotone non-decreasing, or monotone non-increasing, over a sufficiently small interval to the right of t. The values assigned to f(t) at its points of discontinuity are immaterial, as far as the definition of the Fourier Transform $g(\omega)$ of $f(t)$ is concerned, but for the purposes of the Fourier Integral Theorem, which we now propose to study, it is convenient to assign to $f(t)$ at any of its points of discontinuity at which both the limits $f(t+0)$ and $f(t-0)$ exist the mean of these two limits; i.e., we set

$$f(t) = \frac{1}{2}\left\{f(t+0) + f(t-0)\right\}$$

For example, if $f(t) = 0$, $t < 0$, and $f(t) = \exp(izt)$, $t > 0$, where the imaginary part y of $z = x + yi$ is positive, we have $f(0-0) = 0$, $f(0+0) = 1$ and so we set $f(0) = \frac{1}{2}$. We have seen that, for this particular function, $\int_{(-\infty)}^{(\infty)} g(\omega)\, d\omega = \left(\frac{\pi}{2}\right)^{1/2}$ so that $(2\pi)^{-1/2}\int_{(-\infty)}^{(\infty)} g(\omega) d\omega = \frac{1}{2}$ $= f(0)$. This result is not an accident, peculiar to this particular function; if $f(t)$ is any complex-valued function of the unrestricted real variable t, which possesses Properties 1 and 2, and which, in addition, is such that its real and imaginary parts are monotone and bounded over sufficiently small intervals to the right and to the left of $t = 0$, or such that it possesses a right-hand and a left-hand derivative at $t = 0$, then $(2\pi)^{-1/2}\int_{(-\infty)}^{(\infty)} g(\omega) d\omega$ exists, its value being $f(0)$, on the understanding that $f(0)$ is defined as the mean of the two limits $f(0+0)$, $f(0-0)$. The Fourier Integral Theorem is merely the extension of this result from $t = 0$ to an arbitrary value $t = \tau$ of the unrestricted real variable t. To make this extension we must multiply $g(\omega)$ by $\exp(i\omega t)$ before taking the Cauchy principal value of the integral over $-\infty < \omega < \infty$. Thus the Fourier Integral Theorem (in the form in which we propose to prove it and which is satisfactory for our purposes) may be stated as follows:

Let f(t) be any complex-valued piecewise continuous function of the unrestricted real variable t which is absolutely integrable over

$-\infty < t < \infty$ and let τ be any value of t at which f(t) possesses either of the two properties

a) The real and imaginary parts of f(t) are monotone and bounded over sufficiently small intervals to the right and to the left of τ

b) f(t) possesses a right-hand and a left-hand derivative at τ. Then

$$(2\pi)^{-1/2} \int_{(-\infty)}^{(\infty)} g(\omega) \exp(i\,\omega\tau)\, d\omega \text{ exists with the value } f(\tau)$$

(on the understanding that $f(\tau)$ is defined as the mean of the two limits $f(\tau+0)$, $f(\tau-0)$).

<u>Note</u>. Pay attention to the fact that $g(\omega)$ is multiplied by $\exp(i\omega\tau)$ and not by $\exp(-i\omega\tau)$ while, in the definition of $g(\omega)$, f(t) was multiplied by $\exp(-i\omega t)$ and not by $\exp(i\omega t)$. Furthermore, note that mere continuity of f(t) at $t = \tau$ does not suffice for the validity of the Fourier Integral Theorem. When f(t) is continuous at $t = \tau$ both the limits $f(\tau+0)$ and $f(\tau-0)$ exist and are equal, their common value being $f(\tau)$, but our proof of the Fourier Integral Theorem requires either Property a) or Property b) above and these are not guaranteed by mere continuity of f(t) at $t = \tau$. If f(t), in addition to being continuous at $t = \tau$, is differentiable at $t = \tau$ it possesses Property b), with the added equality of the right-hand and left-hand derivatives, and the

Fourier Integral Theorem is valid at $t = \tau$.

We begin the proof of the Fourier Integral Theorem by writing $(2\pi)^{-1/2} \int_a^b f(t) \exp(-i\omega t)\, dt = g_a^b(\omega)$ so that $\left| g(\omega) - g_a^b(\omega) \right|$

may be made arbitrarily small, say $\leq \epsilon$, by choosing the positive numbers -a and b to be sufficiently large, the choice of a and b being independent of ω. If τ is any real number and α any positive real number we denote $(2\pi)^{-1/2} \int_{-\alpha}^{\alpha} g(\omega) \exp(i\tau\omega) d\omega$, which exists since $g(\omega)$ is

everywhere continuous, by $F_\alpha(\tau)$ so that

$$\left| F_\alpha(\tau) - (2\pi)^{-1/2} \int_{-\alpha}^{\alpha} g_a^b(\omega) \exp(i\tau\omega) d\omega \right| =$$

$$(2\pi)^{-1/2} \left| \int_{-\alpha}^{\alpha} \left[g(\omega) - g_a^b(\omega) \right] \exp(i\tau\omega)\, d\omega \right|$$

$$\leq (2\pi)^{-1/2} \int_{-\alpha}^{\alpha} \left| g(\omega) - g_a^b(\omega) \right| d\omega \leq (2\pi)^{-1/2}\, 2\alpha\epsilon$$

on the understanding that -a and b have been chosen sufficiently large to ensure that $\left| g(\omega) - g_a^b(\omega) \right| \leq \epsilon$, ω arbitrary. Thus the difference between $F_\alpha(\tau)$ and $(2\pi)^{-1/2} \int_{-\alpha}^{\alpha} g_a^b(\omega) \exp(i\tau\omega)\, d\omega$ may be made arbitrarily small, once the positive number α is given, by making -a and b sufficiently large. The integral $(2\pi)^{-1/2} \int_{-\alpha}^{\alpha} g_a^b(\omega) \exp(i\tau\omega) d\omega$ is the repeated integral $(2\pi)^{-1} \int_{-\alpha}^{\alpha} \left\{ \int_a^b f(t) \exp(-i\omega t)\, dt \right\} \exp(i\tau\omega) d\omega$ and we consider the associated double integral of $(2\pi)^{-1} f(t) \exp\left[-i\omega(t-\tau)\right]$ over the rectangle $a \leq t \leq b$, $-\alpha \leq \omega \leq \alpha$. If f(t) is continuous over the interval $a \leq t \leq b$, the integrand of this double integral is a continuous function of the variables (t, ω) over the rectangle of integration and the order of integration in the repeated integral may be changed. This change of order is also valid when f(t) is not continuous over $a \leq t \leq b$ by virtue of the fact that f(t) is, by hypothesis, piecewise continuous and absolutely integrable over $a \leq t \leq b$. To see this it is sufficient to consider the case where f(t) has a single point c of discontinuity in the interval $a \leq t \leq b$, c being an interior point of this interval. Writing $\int_a^b f(t) \exp(-i\omega t)\,dt$ in the form

$$\int_a^{c-\delta_1} f(t) \exp(-i\omega t)dt + \int_{c-\delta_1}^{c+\delta_2} f(t) \exp(-i\omega t)dt + \int_{c+\delta_2}^{b} f(t) \exp(-i\omega t)dt,$$

where δ_1 and δ_2 are any positive numbers which are less than c-a and b-c, respectively, we have to consider three repeated integrals whose associated double integrals are extended over the rectangles $a \leq t \leq c-\delta_1$, $-\alpha \leq \omega \leq \alpha$; $c-\delta_1 \leq t \leq c+\delta_2$, $-\alpha \leq \omega \leq \alpha$; $c+\delta_2 \leq t \leq b$, $-\alpha < \omega < \alpha$, respectively, and, since $(2\pi)^{-1} f(t) \exp\left[-i\omega(t-\tau)\right]$ is a continuous function of the two variables (t, ω) over the first and third of these rectangles, the order of integration in the first and third of these repeated integrals may be changed. The modulus of the second of our repeated integrals may, since f(t) is absolutely integrable over $a \leq t \leq b$, be made arbitrarily small, say $< \epsilon$, by taking δ_1 and δ_2 sufficiently small. Thus the difference between $(2\pi)^{-1}\int_{-\alpha}^{\alpha} \exp(i\tau\omega)\left\{\int_a^b f(t) \exp(-i\omega t)dt\right\} d\omega$ and the sum of the two repeated integrals

$$(2\pi)^{-1}\int_a^{c-\delta_1} f(t)\left\{\int_{-\alpha}^{\alpha} \exp\left[-i\omega(t-\tau)\right] d\omega\right\}dt \text{ and}$$

$$(2\pi)^{-1}\int_{c+\delta_2}^{b} f(t)\left\{\int_{-\alpha}^{\alpha} \exp\left[-i\omega(t-\tau)\right] d\omega\right\}dt$$

may be made arbitrarily small by choosing δ_1 and δ_2 to be sufficiently small. Furthermore, since $\left|\exp\left[-i\omega(t-\tau)\right]\right| = 1$, so that

$$\left|\int_{-\alpha}^{\alpha} \exp\left[-i\omega(t-\tau)\right] d\omega\right| \leq 2\alpha,$$ the product of

$$\int_{-\alpha}^{\alpha} \exp\left[-i\omega(t-\tau)\right] d\omega$$ by f(t) is integrable over the intervals $a \leq t \leq c$ and $c \leq t \leq b$ and the differences between

$$(2\pi)^{-1}\int_a^c f(t)\left\{\int_{-\alpha}^{\alpha} \exp\left[-i\omega(t-\tau)\right] d\omega\right\}dt$$ and

$$(2\pi)^{-1}\int_a^{c-\delta_1} f(t)\left\{\int_{-\alpha}^{\alpha} \exp\left[-i\omega(t-\tau)\right] d\omega\right\}dt$$ and between

$$(2\pi)^{-1}\int_c^b f(t)\left\{\int_{-\alpha}^{\alpha} \exp\left[-i\omega(t-\tau)\right] d\omega\right\}dt$$ and

$$(2\pi)^{-1}\int_{c+\delta_2}^b f(t)\left\{\int_{-\alpha}^{\alpha} \exp-i\omega(t-\tau)\, d\omega\right\}dt$$ may be made

arbitrarily small by choosing δ_1 and δ_2, respectively, to be sufficiently small. Thus the difference between

$$(2\pi)^{-1}\int_{-\alpha}^{\alpha} \exp(i\tau\omega)\left\{\int_a^b f(t)\exp(-i\omega t)\, dt\right\} d\omega$$ and the sum of the two repeated integrals $(2\pi)^{-1}\int_a^c f(t)\left\{\int_{-\alpha}^{\alpha} \exp\left[-i\omega(t-\tau)\right] d\omega\right\}dt$

and $(2\pi)^{-1}\int_c^b f(t)\left\{\int_{-\alpha}^{\alpha} \exp-i\omega(t-\tau)\, d\omega\right\}dt$ may be made arbitrarily small by choosing δ_1 and δ_2 to be sufficiently small. Since this difference is independent of δ_1 and δ_2 it must be zero and so

$$(2\pi)^{-1} \int_{-\alpha}^{\alpha} \exp(i\tau\omega) \left\{ \int_a^b f(t) \quad \exp(-i\,\omega\, t)\, dt \right\} d\omega$$

$$= (2\pi)^{-1} \int_a^c f(t) \left\{ \int_{-\alpha}^{\alpha} \exp\left[-i\,\omega\,(t - \tau) \right] d\omega \right\} dt$$

$$+ (2\pi)^{-1} \int_c^b f(t) \left\{ \int_{-\alpha}^{\alpha} \exp\left[-i\,\omega\,(t - \tau) \right] d\omega \right\} dt$$

$$= (2\pi)^{-1} \int_a^b f(t) \left\{ \int_{-\alpha}^{\alpha} \exp\left[-i\,\omega(t - \tau) \right] d\omega \right\} dt$$

which proves the legitimacy of the interchange of the order of integration in the repeated integral

$$(2\pi)^{-1} \int_{-\alpha}^{\alpha} \exp(i\tau\omega) \left\{ \int_a^b f(t) \exp(-i\omega t)\, dt \right\} d\omega$$

even when f(t) is not continuous over the interval $a \leq t \leq b$. We have, then, proved that the difference between $F_\alpha(\tau)$ and $(2\pi)^{-1} \int_a^b f(t) \left\{ \int_{-\alpha}^{\alpha} \exp\left[-i\omega\,(t - \tau) \right] d\omega \right\} dt$ may be made arbitrarily small, once the positive real number α is given, by choosing the numbers -a and b to be positive and sufficiently large. In other words, the infinite integral $(2\pi)^{-1} \int_{-\infty}^{\infty} f(t) \left\{ \int_{-\alpha}^{\alpha} \exp -i\omega(t-\tau) d\omega \right\} dt$ exists, no matter what is the positive number α, its value being $F_\alpha(\tau)$. Since $\int_{-\alpha}^{\alpha} \exp\left[-i\omega(t - \tau) \right] d\omega = 2\dfrac{\sin \alpha\,(t-\tau)}{t - \tau}$ this result may be stated as follows:

The infinite integral $\dfrac{1}{\pi} \int_{-\infty}^{\infty} f(t) \; \dfrac{\sin \alpha\,(t-\tau)}{t - \tau} dt$ exists, no matter what is the positive number α, its value being

$$F_\alpha(\tau) = (2\pi)^{-1/2} \int_{-\alpha}^{\alpha} g(\omega) \exp(i\,\tau\,\omega)\, d\omega$$

This is the first, and most crucial, step in the proof of the Fourier Integral Theorem. In our next lecture we shall complete the proof of this theorem by showing that, if f(t) is either a) such that its real and imaginary parts are monotone and bounded over sufficiently small intervals to the right and to the left of τ or b) such that is possesses a right-hand and a left-hand derivative at τ, then $\lim_{\alpha\to\infty} F_\alpha(\tau)$ exists with the value $f(\tau)$, it being understood that, if τ is a point of discontinuity of f(t), the value assigned to f(t) at $t = \tau$ is the mean of the two limits $f(\tau + 0)$ and $f(\tau - 0)$.

Lecture 4

Completion of The Proof of The Fourier Integral Theorem. The Laplace Version of The Fourier Integral Theorem.

We now examine the behavior of the infinite integral $\frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin \alpha (t-\tau)}{t-\tau} dt$ as $\alpha \longrightarrow \infty$. f(t) is a complex-valued function $f_1(t) + if_2(t)$ of the unrestricted real variable t and so this infinite integral is the sum of the two infinite integrals $\frac{1}{\pi} \int_{-\infty}^{\infty} f_1(t) \frac{\sin \alpha (t-\tau)}{t-\tau} dt$ and $\frac{i}{\pi} \int_{-\infty}^{\infty} f_2(t) \frac{\sin \alpha (t-\tau)}{t-\tau} dt$; it suffices to treat the first of these two infinite integrals, the treatment of the second being precisely the same. We write $\frac{1}{\pi} \int_{-\infty}^{\infty} f_1(t) \frac{\sin \alpha(t-\tau)}{t-\tau} dt$ as the sum of the two infinite integrals

$$I_1 = \frac{1}{\pi} \int_{-\infty}^{\tau} f_1(t) \frac{\sin \alpha (t-\tau)}{t-\tau} dt =$$

$$\frac{1}{\pi} \int_{0}^{\infty} f_1(\tau - u) \frac{\sin \alpha u}{u} du; \quad u = \tau - t$$

$$I_2 = \frac{1}{\pi} \int_{\tau}^{\infty} f_1(t) \frac{\sin \alpha (t-\tau)}{t-\tau} dt =$$

$$\frac{1}{\pi} \int_{0}^{\infty} f_1(\tau + v) \frac{\sin \alpha v}{v} dv; \quad v = t - \tau$$

and it again suffices to treat the first of these two infinite integrals, the treatment of the second being precisely the same. We write I_1 as the sum of the three integrals

$$J_1 = \frac{1}{\pi} \int_{0}^{a} f_1(\tau - u) \frac{\sin \alpha u}{u} du;$$

$$J_2 = \frac{1}{\pi} \int_{a}^{b} f_1(\tau - u) \frac{\sin \alpha u}{u} du;$$

$$J_3 = \frac{1}{\pi} \int_{b}^{\infty} f_1(\tau - u) \frac{\sin \alpha u}{u} du$$

where a and b are any two positive numbers which are such that $a < b$. It is clear that $\left| J_3 \right| \leq \frac{1}{\pi b} \int_b^{\infty} \left| f_1(\mathcal{T} - u) \right| du =$

$\frac{1}{\pi b} \int_{-\infty}^{\mathcal{T}-b} \left| f_1(t) \right| dt \leq \frac{1}{\pi b} \int_{-\infty}^{\infty} \left| f_1(t) \right| dt$ so that $\left| J_3 \right|$ may be made arbitrarily small, say $< \epsilon$, by choosing b sufficiently large, the choice of b being independent of α. If $f_1(t)$ is monotone and bounded over a sufficiently small interval to the left of $t = \mathcal{T}$, $f_1(\mathcal{T}-0)$ exists and

$$J_1 - \frac{f_1(\mathcal{T}-0)}{\pi} \int_0^a \frac{\sin \alpha u}{u} du =$$

$$\frac{1}{\pi} \int_0^a \left\{ f_1(\mathcal{T} - u) - f_1(\mathcal{T} - 0) \right\} \frac{\sin \alpha u}{u} du$$

If a is sufficiently small the function $f_1(\mathcal{T} - u) - f_1(\mathcal{T} - 0)$ of u is monotone over the interval $0 \leq u \leq a$, being either positive and non-decreasing (when $f_1(t)$ is monotone non-increasing to the left of $t = \mathcal{T}$) or negative and non-increasing (when $f_1(t)$ is monotone non-decreasing to the left of $t = \mathcal{T}$). Hence we may apply the second Theorem of the Mean of integral calculus to obtain

$$J_1 - \frac{f_1(\mathcal{T}-0)}{\pi} \int_0^a \frac{\sin \alpha u}{u} du =$$

$$\frac{1}{\pi} \left\{ f_1(\mathcal{T} - a + 0) - f_1(\mathcal{T} - 0) \right\} \int_{a'}^a \frac{\sin \alpha u}{u} du$$

where a' is some positive number $< a$. The integral $\int_{a'}^a \frac{\sin \alpha u}{u} du$ is the difference of the two integrals $\int_0^a \frac{\sin \alpha u}{u} du = \int_0^{\alpha a} \frac{\sin t}{t} dt$, $t = \alpha u$, and $\int_0^{a'} \frac{\sin \alpha u}{u} du = \int_0^{\alpha a'} \frac{\sin t}{t} dt$ and each of these integrals is dominated, no matter what are the values of the positive numbers a', a and α by the number $\int_0^{\pi} \frac{\sin t}{t}\, dt$. Hence

$\left| \int_{a'}^{a} \frac{\sin \alpha u}{u}\, du \right| \leq 2 \int_0^{\pi} \frac{\sin t}{t}\, dt$ and this assures us, in view of the existence of the limit $f_1(\mathcal{T} - 0)$, that $\left| J_1 - \frac{f_1(\mathcal{T} - 0)}{\pi} \int_0^a \frac{\sin \alpha u}{u}\, du \right|$ may be made arbitrarily small, say $< \epsilon$, by choosing a sufficiently small, the choice of a being independent of α. If $f_1(t)$ possesses a left-hand derivative, d, say, at $t = \mathcal{T}$, $\left| \frac{f_1(\mathcal{T} - u) - f_1(\mathcal{T} - 0)}{-u} - d \right|$ is arbitrarily small, say ≤ 1, over the interval $0 < u \leq a$, if a is sufficiently small and so $\left| \frac{f_1(\mathcal{T} - u) - f_1(\mathcal{T} - 0)}{u} \right| \leq |d| + 1$ over $0 < u \leq a$ if a is sufficiently small, the choice of a being independent of α. Hence

$$\left| \int_0^a \left\{ f_1(\mathcal{T} - u) - f_1(\mathcal{T} - 0) \right\} \frac{\sin \alpha u}{u}\, du \right| \leq (|d|+1)\, a,$$

if a is sufficiently small, so that $\left| J_1 - \frac{f_1(\mathcal{T} - 0)}{\pi} \int_0^a \frac{\sin \alpha u}{u}\, du \right|$ may be made arbitrarily small, say $< \epsilon$, by choosing a sufficiently small, the choice of a being, again, independent of α. Supposing, then, that a and b are so chosen that, for all values of the positive real number α,

1) $|J_3| < \epsilon$

2) $\left| J_1 - \frac{f_1(\mathcal{T} - 0)}{\pi} \int_0^a \frac{\sin \alpha u}{u}\, du \right| < \epsilon$

we have, for every $\alpha > 0$, $\left| J_1 + J_3 - \frac{f_1(\mathcal{T} - 0)}{\pi} \int_0^a \frac{\sin \alpha u}{u}\, du \right| < 2\epsilon$.

The integral $\int_0^a \frac{\sin \alpha u}{u}\, du = \int_0^{\alpha a} \frac{\sin v}{v}\, dv$, $v = \alpha u$, and as $\alpha \to \infty$, a remaining fixed, this tends to $\int_0^\infty \frac{\sin v}{v}\, dv$ whose value we denote for the moment, by C. Thus $\left| \frac{f_1(\mathcal{T} - 0)}{\pi} \int_0^a \frac{\sin \alpha u}{u} du - \frac{f_1(\mathcal{T} - 0)}{\pi} C \right|$ may be made arbitrarily small, say $< \epsilon$, by choosing α sufficiently large so that $\left| J_1 + J_3 - \frac{f_1(\mathcal{T} - 0)}{\pi} C \right| < 3\epsilon$ if α is sufficiently large.

It remains to investigate the behavior of $J_2 = \frac{1}{\pi} \int_a^b f_1(\mathcal{T} - u) \frac{\sin \alpha u}{u} du$ as $\alpha \longrightarrow \infty$. If $f_1(\mathcal{T} - u)$ is bounded over $a \leq u \leq b$ so also is $f_1(\mathcal{T} - u)/u$ which we denote, for a moment, by h(u) so that h(u) is bounded and, hence, since it is piecewise continuous, properly integrable over the interval $a \leq u \leq b$. If, then, ϵ' is an arbitrarily given positive number we may construct a net of points $a = u_0 < u_1 < \ldots < u_n = b$ on the interval $a \leq u \leq b$ with the following property: Let $h^*(u)$ be the function defined by setting, over any open cell $u_j < u < u_{j+1}$, $j = 0, \ldots, n - 1$, of the net, $h^*(u)$ equal to the greatest lower bound m_j of h(u) over the corresponding closed cell $u_j \leq u \leq u_{j+1}$ and setting, at the points a, $u_1, \ldots, u_{n-1}$, b of the net, $h^*(u)$ equal to the greatest lower bound of h(u) over $a \leq u \leq b$. Then $h(u) - h^*(u) \geq 0$ over $a \leq u \leq b$ and the net a, $u_1, \ldots, u_{n-1}$, b can be so chosen that $0 \leq \int_a^b \left\{ h(u) - h^*(u) \right\} du \leq \epsilon'$. Since $\left| \sin \alpha u \right| \leq 1$, it follows that $\left| \int_a^b \left\{ h(u) - h^*(u) \right\} \sin (\alpha u)\, du \right| \leq \epsilon'$ and it is easy to see that $\left| \int_a^b h^*(u) \sin (\alpha u)\, du \right|$ may be made arbitrarily small, say $\leq \epsilon'$, by choosing the positive number α sufficiently large.

Indeed $\int_a^b h^*(u)\sin(\alpha u)\,du$ is the sum of n terms of the form

$$m_j \int_{u_j}^{u_{j+1}} \sin(\alpha u)\,du = m_j \frac{\cos(\alpha u_j) - \cos(\alpha u_{j+1})}{\alpha} \quad \text{so that}$$

$$\left| \int_a^b h^*(u)\sin(\alpha u)\,du \right| \leq \frac{2}{\alpha} \sum_{j=0}^{n-1} \left| m_j \right| . \text{ Hence}$$

$\left| \int_a^b h(u)\sin(\alpha u)\,du \right| \leq 2\epsilon'$ if α is sufficiently large. This result remains valid even when $f_1(\mathcal{T} - u)$ fails to be bounded over $a \leq u \leq b$; to show this, it suffices to consider the case where $f_1(\mathcal{T} - u)$ is unbounded at a single interior point c of the interval $a \leq u \leq b$. Writing $\int_a^b h(u)\sin(\alpha u)\,du = \int_a^{c-\delta_1} h(u)\sin(\alpha u)du + \int_{c-\delta_1}^{c+\delta_2} h(u)\sin(\alpha u)\,du + \int_{c+\delta_2}^{b} h(u)\sin(\alpha u)\,du$ the first and third of the integrals on the right may be made arbitrarily small by choosing α sufficiently large, since h(u) is properly integrable over the intervals $a \leq u \leq c - \delta_1$, $c+\delta_2 \leq u \leq b$. The modulus of the second integral on the right is dominated by $\int_{c-\delta_1}^{c+\delta_2} \left| h(u) \right| du$ and this is again dominated by $\frac{1}{c-\delta_1}\int_{c-\delta_1}^{c+\delta_2} \left| f_1(\mathcal{T} - u) \right| du$ which may be made arbitrarily small by choosing δ_1 and δ_2 sufficiently small, the choice of δ_1 and δ_2 being independent of α. Thus $\left| J_2 \right|$ may be made arbitrarily small, say $< \epsilon$, by choosing α sufficiently large and, since $I_1 = J_1 + J_2 + J_3$, it follows that $\left| I_1 - \frac{f_1(\tau - 0)}{\pi} C \right| < 4\epsilon$ if α is sufficiently large. Similarly, $\left| I_2 - \frac{f_1(\tau + 0)}{\pi} C \right| < 4\epsilon$ if α is sufficiently large and, since

$$\frac{1}{\pi} \int_{-\infty}^{\infty} f_1(t) \frac{\sin\alpha(t-\tau)}{t-\tau} dt = I_1 + I_2, \text{ this implies that}$$

$$\frac{1}{\pi} \left| \int_{-\infty}^{\infty} f_1(t) \frac{\sin\alpha(t-\tau)}{t-\tau} dt - \frac{2f_1(\tau)}{\pi} C \right| < 8\epsilon$$

if α is sufficiently large (it being understood that, at any point τ of discontinuity of $f_1(t)$ at which the limits $f_1(\tau + 0)$ and $f_1(\tau - 0)$ exist, $f_1(\tau)$ is defined as the mean of these two limits). Thus

$$\lim_{\alpha\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f_1(t) \frac{\sin\alpha(t-\tau)}{t-\tau} dt = \frac{2C}{\pi} f_1(\tau)$$

and similarly for $f_2(t)$ so that

$$\lim_{\alpha\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin\alpha(t-\tau)}{t-\tau} dt = \frac{2C}{\pi} f(\tau)$$

which implies that $(2\pi)^{-1/2} \int_{(-\infty)}^{(\infty)} g(\omega) \exp(i\tau\omega)\, d\omega = \frac{2C}{\pi} f(\tau)$. The constant C is independent of the function f(t) and, to determine it, we choose the function f(t) which is 0 if $t < 0$ and $= \exp(izt)$, where the imaginary part y of z is positive, if $t > 0$. Setting $\tau = 0$ and using the already proved fact that $(2\pi)^{-1/2} \int_{(-\infty)}^{(\infty)} g(\omega)\, d\omega = f(0)$ we see that $C = \frac{\pi}{2}$. Thus

$$(2\pi)^{-1/2} \int_{(-\infty)}^{(\infty)} g(\omega) \exp(i\tau\omega)\, d\omega = f(\tau)$$

which completes the proof of the Fourier Integral Theorem.

The Fourier Integral Theorem is one of the most useful theorems of applied mathematics but, in the form in which we have stated it, it suffers from a serious disadvantage. The class of complex-valued functions of the unrestricted real variable t which possess Property 2 is too restricted. For example, the Heaviside unit-function u(t) which is defined as follows:

$$u(t) = 0 \text{ if } t < 0; \quad u(t) = 1 \text{ if } t > 0; \quad u(0) = \frac{1}{2}$$

while possessing Property 1 (since it is continuous save at $t = 0$) does not possess Property 2. To remove this disadvantage we introduce a complex variable $p = c + i\omega$ whose real part c is not, necessarily, zero. Then $\omega = i(c-p)$ and the Fourier Integral Theorem may be written in the form

$$f(\tau) = \frac{(2\pi)^{-1/2}}{i} \int_{(c-i\infty)}^{(c+i\infty)} g\left[i(c-p)\right] \exp\left[\tau(p-c)\right] dp$$

the integration in the complex p-plane being along the line $p = c$ which is parallel to the ω - axis and the Cauchy principal value of the integral being taken. On multiplying by $(2\pi)^{-1/2} \exp(c\tau)$ and setting $(2\pi)^{-1/2} \exp(c\tau) f(\tau) = h(\tau)$, $g\left[i(c-p)\right] = k(p)$ we obtain

$$h(t) = \frac{1}{2\pi i} \int_{(c-i\infty)}^{(c+i\infty)} k(p) \exp(\tau p)\, dp$$

where $k(p) = g(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt =$

$$\int_{-\infty}^{\infty} h(t) \exp(-pt)\, dt.$$

k(p) is termed the Laplace Transform of the complex-valued function h(t) of the unrestricted real variable t and the relation

$h(\tau) = \frac{1}{2\pi i} \int_{(c-i\infty)}^{(c+i\infty)} k(p) \exp(\tau p)\, dp$ is the Laplace version of the

Fourier Integral Theorem. The great advantage of this version is that $h(t)$ is not required, as is $f(t)$, to be absolutely integrable over $-\infty < t < \infty$. It suffices that there exist a real number c such that $\exp(-ct)\, h(t)$ is absolutely integrable over $-\infty < t < \infty$. Thus, for example, if $h(t)$ is zero when $t < 0$, $h(t)$ may be furnished, if $c > 0$, for positive values of t, by any polynomial function of t. In particular, the Heaviside unit function $u(t)$ possesses, if $c > 0$, the Laplace Transform $\int_0^\infty \exp(-pt)\, dt = \frac{1}{p}$ and the Laplace version of the Fourier Integral Theorem tells us that

$$\frac{1}{2\pi i} \int_{(c-i\infty)}^{(c+i\infty)} \frac{\exp(\tau p)}{p}\, dp,\ c > 0, \text{ is } 1 \text{ if } \tau > 0,\ \frac{1}{2} \text{ if } \tau = 0$$

and 0 if $\tau < 0$.

Lecture 5

The Laplace Transform of a Right-sided Function

The function h(t) which appears in the Laplace version of the Fourier Integral Theorem is connected with the function f(t) which appeared in the original version by the relation

$$h(t) = (2\pi)^{-1/2} \exp(ct)\, f(t)$$

Since exp(ct) is everywhere continuous, h(t) possesses, like f(t), Property 1, i.e., it is piecewise continuous. Since f(t) possesses, by hypothesis, Property 2, h(t), which need not possess this property, must be such that there exists a real number c such that exp(-ct)h(t) possesses Property 2, i.e., is absolutely integrable over $-\infty < t < \infty$. For example, h(t) may be the Heaviside unit-function u(t) which is defined as follows:

$$u(t) = 0,\ t < 0;\ u(t) = 1,\ t > 0;\ u(0) = \frac{1}{2}$$

since, if c is any positive real number, exp(-ct) u(t) is absolutely integrable over $-\infty < t < \infty$, the value of the infinite integral $\int_{-\infty}^{\infty} |\exp(-ct)\, u(t)|\, dt = \int_0^{\infty} \exp(-ct)dt$ being $\frac{1}{c}$. The product of any complex-valued function of the unrestricted real variable t by u(t) is zero if $t < 0$ and we term any piecewise continuous complex-valued function of t which is zero if $t < 0$ a right-sided function. Similarly, we term any piecewise continuous complex-valued function of t which is zero if $t > 0$ a left-sided function; for example, the product of any piecewise continuous complex-valued function of t by u(-t) is a left-sided function. The Laplace Transform, Lh, of a right-sided function h(t) is defined by the formula

$$Lh = \int_0^\infty h(t)\exp(-pt)dt$$

where p is any complex number for which the infinite integral on the right exists (it being not required that this infinite integral converge absolutely, i.e., that the infinite integral $\int_0^\infty |h(t)|\exp(-ct)dt$, where c is the real part of p, exist). Let us now suppose that Lh exists at some point c_1 of the real axis of the complex p-plane. We propose to prove that this implies the existence of Lh at any point p of the complex p-plane whose real part c is $> c_1$; not only this, but also that Lh is an analytic function of the complex variable p over the half-plane $c > c_1$.

Since h(t) exp(-ct) is integrable, by hypothesis, over $-\infty < t < \infty$ it is integrable over the interval $0 \leq t \leq T$, where T is any positive real number, and we denote by $H_{c_1}(T)$ the integral $\int_0^T h(t)\exp(-c_1 t)\,dt$ so that $H_{c_1}(T)$ is everywhere continuous and, at every point of continuity of h(t), differentiable with the derivative $h(T)\exp(-c_1 T)$. In view of the continuity of $H_{c_1}(T)$, $H_{c_1}(T)$ is bounded over any interval $0 \leq T \leq b$, where b is any positive number, and this implies, since the infinite integral $\int_0^\infty h(t)\exp(-c_1 t)dt$ exists, by hypothesis, that $H_{c_1}(T)$ is bounded over $0 \leq T < \infty$; in other words, there exists a positive number M which dominates $\left| H_{c_1}(T) \right|$, T any non-negative real number. On writing $Lh = \int_0^\infty h(t)\exp(-pt)dt$ in the form $\int_0^\infty h(t)\exp(-c_1 t)\exp\left[-(p-c_1)t\right]dt$ we obtain, on integration by parts,

$$Lh = H_{c_1}(t) \exp\left[-(p-c_1)t\right]\Big|_0^{\infty} + (p-c_1)\int_0^{\infty} H_{c_1}(t)\exp\left[-(p-c_1)t\right] dt$$

$$= (p-c_1) \int_0^{\infty} H_{c_1}(t) \exp\left[-(p-c_1)t\right] dt$$

provided that the real part c of $p > c_1$. Since $\left|H_{c_1}(t) \exp\left[-(p-c_1)t\right]\right| \leq M \exp\left[-(c-c_1)t\right]$ the infinite integral $\int_0^{\infty} H_{c_1}(t) \exp\left[-(p-c_1)\, t\right] dt$ exists over the half-plane $c > c_1$, its convergence being absolute. Thus, although the convergence of the infinite integral $\int_0^{\infty} h(t) \exp(-pt) dt$ which defines Lh need not be absolute at $p = c_1$, nor at points of the half-plane $c > c_1$, L h exists over this half-plane and may be expressed, over this half-plane, as the product of $p-c_1$ by an infinite integral $\int_0^{\infty} H_{c_1}(t) \exp\left[-(p-c_1)t\right] dt$ which converges absolutely over the half-plane. Let, now, T be any positive real number and let us consider the integral $\phi_T(p) = \int_0^{T} H_{c_1}(t) \exp\left[-(p-c_1)\, t\right] dt$. Since $H_{c_1}(t)$ is bounded over the interval $0 \leq t \leq T$, $\phi_T(p)$ is a differentiable function, i. e, an analytic function, of the complex variable p, its derivative being $-\int_0^{T} t\, H_{c_1}(t) \exp\left[-(p-c_1)t\right] dt$ no matter what is the value of p. Assigning to T, in turn, the values 1, 2, 3, ..., we obtain a sequence of functions $\phi_1(p)$, $\phi_2(p)$, ..., of the complex variable p which are analytic over the entire finite complex p-plane. At any point of the half-plane $c > c_1$ this sequence converges to $\int_0^{\infty} H_{c_1}(t) \exp\left[-(p-c_1)t\right] dt = \phi(p)$, say, and it is easy to see that the convergence is uniform over the half-plane $c_1 + \delta \leq c$, where δ is any positive number. Indeed, the modulus of

$\phi(p) - \phi_n(p) = \int_0^\infty H_{c_1}(t) \exp\left[-(p-c_1)t\right]dt - \phi_n(p) =$

$\int_n^\infty H_{c_1}(t) \exp\left[-(p-c_1)t\right]dt$ is dominated, over the half-plane $c_1 + \delta \leq c$, by $M\int_n^\infty \exp\left[-(c-c_1)t\right]dt$ which is, in turn, dominated by $M\int_n^\infty \exp(-\delta t)dt = \frac{M}{\delta}\exp(-n\delta)$, which is arbitrarily small if n is sufficiently large, the choice of n being independent of c. Since $\phi_n(p)$ is analytic over the half-plane $c > c_1$ the integral $\int_C \phi_n(p)dp$, where C is any simple closed curve of finite length which is covered by this half-plane, is zero. Since the points of C constitute a closed set their distances from the line $c = c_1$ possess a positive lower bound so that C is covered by a half-plane $c_1 + \delta \leq c$, if δ is sufficiently small, and so $\int_C \phi(p)\,dp$, which is the same as $\int_C \left\{\phi(p) - \phi_n(p)\right\}dp$ is dominated by $\frac{M}{\delta}\exp(-n\delta)\,l$, where l is the length of C. Since $\int_C \phi(p)dp$ is independent of n it follows that $\int_C \phi(p)\,dp = 0$ and this implies that $\phi(p)$ is an analytic function of the complex variable p over the half-plane $c > c_1$. Hence $Lh = (p-c_1)\,\phi(p)$ is an analytic function of the complex variable p over the half-plane $c > c_1$.

Example 1. $h(t) = u(t)$

Here $Lh = \int_0^\infty \exp(-pt)dt = \frac{1}{p}$, $c > 0$

<u>Note</u> Lh does not exist at $p = 0$, but is exists at $c = c_1$ where c_1 is any positive number and, if $c > 0$, there exists a positive number c_1, $\frac{1}{2}c$ for example, such that $c > c_1$.

Example 2. $h(t) = \exp(\alpha t)\,u(t)$, α an arbitrary complex number.

Here $Lh = \int_0^\infty \exp\left[-(p-\alpha)t\right]dt = \frac{1}{p-\alpha}$, $c >$ real part α_r of α

Note Similarly, if, for any right-sided function $h(t)$, $Lh = \phi(p)$, $c > c_1$, and $h'(t) = \exp(\alpha t)\, h(t)$, then $Lh' = \phi(p-\alpha)$, $c > c_1 + \alpha_r$. This useful property of the Laplace Transform of a right-sided function is known as the Translation Theorem.

Example 3. $h(t) = t^{\alpha}\, u(t)$, α a complex number $\alpha_r + i\alpha_i$.

The complex power, t^{α}, of a positive real number t is defined by the relation $t^{\alpha} = \exp(\alpha \log t)$ so that $\left| t^{\alpha} \right| = \exp(\alpha_r \log t) = t^{\alpha_r}$, where α_r is the real part of α. In order that $Lh = \int_0^{\infty} t^{\alpha} \exp(-pt)\, dt$ exist at the point $p = c_1$ of the real axis in the complex p-plane we must have $\alpha_r > -1$ (to take care of the small values of t) and $c_1 > 0$ (to take care of the large values of t). Thus the Laplace Transform of $t^{\alpha}\, u(t)$, where the real part α_r of α is > -1, exists, and is an analytic function of the complex variable p, over the half-plane $c > 0$, c being the real part of p. On setting $pt = s$ in the infinite integral $\int_0^{\infty} t^{\alpha} \exp(-pt)dt$ which furnishes this Laplace Transform this infinite integral appears as $\frac{1}{p^{\alpha+1}} \int_0^{\infty} s^{\alpha} \exp(-s)ds$, the integration being along the ray from o to ∞ in the complex p-plane which passes through the point p. If R and θ are the modulus and argument, respectively, of any point s in the complex p-plane,

$$s^{\alpha} = \exp(\alpha \log s) = \exp\left\{ (\alpha_r \log R - \alpha_i\, \theta) + i\,(\alpha_r\, \theta + \alpha_i \log R) \right\}$$

so that $\left| s^{\alpha} \right| = \exp(\alpha_r \log R - \alpha_i\, \theta) = R^{\alpha_r} \exp(-\alpha_i \theta)$ and, since $\left| \exp(-s) \right| = \exp(-R \cos\theta)$, we have $\left| s^{\alpha} \exp(-s) \right| = R^{\alpha_r} \exp(-\alpha_i\, \theta) \exp(-R \cos\theta)$. Denoting, for a moment, by β the argument of p, so that $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, it follows that along the arc of the circle $s = R \exp(\theta i)$ in the complex p-plane from $\theta = 0$ to

$\theta = \beta$, this arc lying in the first quadrant if $\beta > 0$ and in the fourth quadrant if $\beta < 0$, $\left| s^{\alpha} \exp(-s) \right| \leq R^{\alpha_r} \exp(|\alpha_i \beta|)\exp(-R\cos\beta)$ and this implies, since $R^{\alpha_r+1} \exp(-R\cos\beta)$ tends to zero as $R \longrightarrow \infty$, that the integral of $s^{\alpha} \exp(-s)$ along this arc of the circle $s = R\exp(\theta i)$ tends to zero as $R \longrightarrow \infty$, or, equivalently, that the integral of $s^{\alpha} \exp(-s)$ along the ray of argument β from 0 to ∞ in the complex p-plane is the same as the integral of $s^{\alpha}\exp(-s)$ along the ray of argument zero from 0 to ∞ in the complex p-plane. This integral, $\int_0^{\infty} t^{\alpha} \exp(-t)dt$, is the Gamma Function, $\Gamma(\alpha+1)$, of argument $\alpha + 1$ and so:

The Laplace Transform of $t^{\alpha} u(t)$, where the real part of α is > -1, is $\dfrac{\Gamma(\alpha+1)}{p^{\alpha+1}}$, over the half-plane $c > 0$.

A simple integration by parts shows that if the real part of α is not only > -1 but also > 0, then $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and, since $\Gamma(1) = \int_0^{\infty} \exp(-t)dt = 1$, it follows that, if α is a positive integer, $\Gamma(\alpha+1) = \alpha!$. The Laplace version of the Fourier Integral Theorem tells us that

$$t^{\alpha} u(t) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{(c-i\infty)}^{(c+i\infty)} \frac{\exp(pt)}{p^{\alpha+1}} dp;$$

real part of $\alpha > -1$; $c > 0$ and in particular, on setting $t = 1$, that

$$1 = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{(c-i\infty)}^{(c+i\infty)} \frac{\exp(p)}{p^{\alpha+1}} dp; \text{ real part of } \alpha > -1;\ c > 0$$

Thus $\Gamma(\alpha+1)$ is never zero over the half-plane $\alpha_r > -1$, where α_r is the real part of α, for which the Laplace Transform of $t^{\alpha} u(t)$ is defined. If α is real and > -1, $\Gamma(\alpha+1)$ is real and, since it is continuous and never zero, one-signed. Since $\Gamma(1) = 1$ is positive it follows that $\Gamma(\alpha+1)$ is positive for every real value of $\alpha > -1$.

Exercise 1. Show that the Laplace Transform operator $Lh = \int_{-\infty}^{\infty} h(t)\exp(-pt)dt$ is linear, i.e., $L(h_1 + h_2) = Lh_1 + Lh_2$, $L(\alpha h) = \alpha Lh$, α any complex number. and use this property to determine the Laplace Transforms of the right-sided functions $\sin(\beta t)u(t)$, $\cos(\beta t)u(t)$, β any complex number, indicating in each case the half-planes over which the Laplace Transforms are analytic functions of the complex variable p.

Exercise 2. Show that if the Laplace Transform, Lh, of a right-sided function h(t) exists at a point $p_1 = c_1 + i\omega_1$ of the complex p-plane then Lh exists, and is an analytic function of the complex variable p, over the half-plane $c > c_1$.

Hint. The Laplace Transform of h(t) at p_1 is the same as the Laplace Transform of $h(t)\exp(-i\omega_1 t)$ at c_1 and the Laplace Transform of $h(t)\exp(-i\omega_1 t)$ at $p - i\omega_1$ is the same as the Laplace Transform of h(t) at p.

An important consequence of the Laplace version of the Fourier Integral Theorem is the following uniqueness theorem: If two piecewise continuous right-sided functions $h_1(t)$, $h_2(t)$, possess Laplace Transforms at a point c_1 of the real axis in the complex p-plane and if their Laplace Transforms coincide over the half-plane $c > c_1$, then $h_2(t)$ coincides with $h_1(t)$ at all points t which are not discontinuity points of either $h_1(t)$ or $h_2(t)$. To prove this, we observe that the relation $Lh = (p-c_1) \int_0^{\infty} H_{c_1}(t) \exp\left[-(p-c_1)t\right]dt$, $c > c_1$, tells us that the Laplace Transform, over the half-plane $c > c_1$, of $H_{c_1}(t) \exp(c_1 t)$ is $(Lh)/(p-c_1)$. Since the convergence of the infinite integral $\int_0^{\infty} H_{c_1}(t) \exp\left[-(c-c_1)t\right]dt$, $c > c_1$, is absolute we may apply the Laplace version of the Fourier Integral Theorem to obtain

$$H_{c_1}(t) \exp(c_1 t) = \frac{1}{2\pi i} \int_{(c-i\infty)}^{(c+i\infty)} \frac{Lh}{p-c_1} \exp(pt)dp$$

this equality being valid at any continuity point of $h(t)$, since $H_{c_1}(t) \exp(c_1 t)$ is differentiable at any such continuity point. Thus $H_{c_1}(t)$ is unambiguously determined, at any continuity point of $h(t)$, by the values of Lh at the points of the complex p-plane whose real parts have any common value $c > c_1$ or, equivalently, by the values of Lh over the half-plane $c > c_1$. Since the derivative of $H_{c_1}(t)$, at any continuity point of $h(t)$, is $h(t) \exp(-c_1 t)$ it follows that $h(t)$ is unambiguously determined at any point where it is continuous by the values of Lh over the half-plane $c > c_1$. In particular, if $Lh = 0$ over the half-plane $c > c_1$ then $h(t) = 0$ at all its continuity points.

Note. It is not necessary, for the validity of this uniqueness theorem, that the Laplace Transforms, at the point $p = c_1$, of $h_1(t)$ and $h_2(t)$ be absolutely convergent.

Lecture 6

The Laplace Transform of $\exp(-t^2)$

We have seen that if the Laplace Transform, Lh, of a piecewise continuous right-sided function $h(t)$ exists at a point c_1 of the real axis of the complex p-plane then Lh exists, and is an analytic function of the complex variable p, over the half-plane $c > c_1$, so that Lh possesses a derivative with respect to p over this half-plane. If $h(t)$ is left-sided, instead of right-sided, $Lh = \int_{-\infty}^{0} h(t)\exp(-pt)\,dt = \int_0^{\infty} h(-t')\exp(pt')\,dt'$, $t' = -t$, $= \int_0^{\infty} h(-t')\exp(-p't')\,dt'$, $p' = -p$, $= \int_0^{\infty} h(-t)\exp(-p\ t)dt$ and $h(-t)$ is a right-sided function of t so that, if Lh exists at $p' = c_1' = -c_1$ it exists, and is an analytic function of the complex variable p', over the half-plane $c' > c_1'$ which is the same thing as saying that if Lh exists at $p = c_1$ it exists, and is an analytic function of the complex variable p, over the half-plane, $c < c_1$. If $h(t)$ is neither right-sided nor left-sided we may write it, since $u(t) + u(-t)$ is the constant function 1, as the sum of a right-sided and a left-sided function as follows: $h(t) = h(t)\,u(t) + h(t)\,u(-t)$. If, then, the Laplace Transform of the piecewise continuous right-sided function $h(t)\,u(t)$ exists at a point $p = c_1$ of the real axis in the complex p-plane and if the Laplace Transform of the piecewise continuous left-sided function $h(t)\,u(-t)$ exists at a point $p = c_2$ of this real axis, and if $c_2 > c_1$, Lh exists, and is an analytic function of the complex variable p, over the strip $c_1 < c < c_2$ parallel to the imaginary axis in the complex p-plane. For example, $\exp(-t^2)$ is the sum of the right-sided function $\exp(-t^2)\, u(t)$ and the left-sided function $\exp(-t^2)\, u(-t)$ and each of these functions possesses a Laplace Transform at any point c_1 of the real axis in the complex p-plane. Indeed,

$$\exp(-t^2)\exp(-c_1 t) = \exp \frac{c_1^2}{4} \exp(-v^2), \quad v = t + \frac{c_1}{2},$$

and so both of the infinite integrals

$$\int_0^\infty \exp(-t^2)\exp(-c_1 t)dt = \exp\left(\frac{c_1^2}{4}\right) \int_{(c_1)/2}^{\infty} \exp(-v^2)dv$$

$$\int_{-\infty}^0 \exp(-t^2)\exp(-c_1 t)dt = \exp\left(\frac{c_1^2}{4}\right) \int_{-\infty}^{(c_1)/2} \exp(-v^2)\, dv$$

exist, $\exp(-v^2)$ being dominated by $(1+v^2)^{-1}$ no matter what is the value of the real variable v. Thus the Laplace Transform of $\exp(-t^2)$ exists, and is an analytic function of the complex variable p, over any strip $c_1 < c < c_2$ parallel to the imaginary axis in the p-plane; in other words, the Laplace Transform of $\exp(-t^2)$ is an analytic function of p over the entire finite complex p-plane. At any point c_1 of the real axis in the complex p-plane this Laplace Transform has the value $A \exp\left(\frac{c_1^2}{4}\right)$ where

$$A = \int_{-\infty}^{(c_1)/2} \exp(-v^2)\, dv + \int_{(c_1)/2}^{\infty} \exp(-v^2)\, dv = \int_{-\infty}^{\infty} \exp(-v^2)dv.$$

In order to evaluate this infinite integral we observe that, if (r, θ) are plane polar coordinates, the double integral of $\exp(-r^2)$ over the circle of radius R with center at the origin is $\int \exp(-r^2)\, r dr d\theta = \pi \left\{ 1 - \exp(-R^2) \right\}$ and that the double integral of $\exp(-r^2)$ over the square of side 2b with center at the origin and with sides parallel to the coordinate axes is, since $r^2 = x^2+y^2$, the square of the integral $\int_{-b}^{b} \exp(-t^2)dt$. Since this square of side 2b is covered by the circle of radius R, if R is large enough, we know that the square of $\int_{-b}^{b} \exp(-t^2)dt < \pi \left\{1-\exp(-R^2)\right\}$, if R is large enough, and so the square of $\int_{-b}^{b} \exp(-t^2)\,dt$ is less than π, no matter what is the value of b. Hence the infinite integral $\int_{-\infty}^{\infty} \exp(-t^2)dt$ exists with a value $\leq \pi^{1/2}$. On the other hand, the square of side 2b covers the circle of radius R, if b is large enough, and this leads to the opposite inequality $\int_{-\infty}^{\infty} \exp(-t^2)dt \geq \pi^{1/2}$. Thus $\int_{-\infty}^{\infty} \exp(-t^2)dt = \pi^{1/2}$ so that the Laplace Transform of $\exp(-t^2)$ assumes the value $\pi^{1/2} \exp\left(\frac{c_1^2}{4}\right)$ at any point c_1 of the real axis in the complex p-plane. Hence it coincides with the analytic function $\pi^{1/2} \exp\left(\frac{p^2}{4}\right)$ on the real axis in the complex p-plane and this implies, since it is analytic over the entire finite complex p-plane, that it is $\pi^{1/2} \exp\left(\frac{p^2}{4}\right)$ over the entire finite complex p-plane:

$$\int_{-\infty}^{\infty} \exp(-t^2) \exp(-pt)dt = \pi^{1/2} \exp\left(\frac{p^2}{4}\right), \text{ p arbitrary.}$$

On setting $t = k^{1/2} t'$, $p = k^{-1/2}p'$, where k is any positive real number, we obtain $\int_{-\infty}^{\infty} \exp(-kt'^2) \exp-(p't')dt' = \left(\frac{\pi}{k}\right)^{1/2} \exp\left(\frac{p'^2}{4k}\right)$ so that the Laplace Transform of $\exp(-kt^2)$, $k>0$, is $\left(\frac{\pi}{k}\right)^{1/2} \exp\left(\frac{p^2}{4k}\right)$. In particular, on setting $k = \frac{1}{2}$, the Laplace Transform of $\exp\left(\frac{-t^2}{2}\right)$ is $(2\pi)^{1/2} \exp\left(\frac{p^2}{2}\right)$.

On evaluating the Laplace Transform of $\exp(-kt^2)$, $k > 0$, at any point $p = i\omega$ of the imaginary axis of the complex p-plane, we obtain $\int_{-\infty}^{\infty} \exp(-kt^2) \exp(-i\omega t)dt = \left(\frac{\pi}{k}\right)^{1/2} \exp\left(\frac{-\omega^2}{4k}\right)$ and it follows, on multiplication by $(2\pi)^{-1/2}$, that the Fourier Transform of $\exp(-kt^2)$ is $(2k)^{-1/2} \exp\left(\frac{-\omega^2}{4k}\right)$. In particular, on setting $k = \frac{1}{2}$, the Fourier Transform of $\exp(-\frac{1}{2} t^2)$ is $\exp(-\frac{1}{2}\omega^2)$; this result is expressed by the statement that $\exp(-\frac{1}{2} t^2)$ is its own Fourier Transform.

The Laplace Transform of the one-sided functions $\exp(-kt^2)\, u(t)$, $\exp(-kt^2)\, u(-t)$, $k > 0$, are not as simple as the Laplace Transform of their sum $\exp(-kt^2)$. For example, the Laplace Transform of $\exp(-kt^2)u(t)$, $k > 0$, is

$$\int_0^{\infty} \exp(-kt^2) \exp(-pt)dt = \exp\left(\frac{p^2}{4k}\right) \int_0^{\infty} \exp\left[-k(t+\frac{p}{2k})^2\right] dt =$$

$$k^{-1/2} \exp\left(\frac{p^2}{4k}\right) \int_{\frac{p}{2k^{1/2}}}^{\infty} \exp(-z^2)\, dz, \quad z = k^{1/2}(t + \frac{p}{2k}),$$ the

integration in the complex z-plane being along the ray of argument zero from $\frac{p}{2k^{1/2}}$ to ∞. Similarly, the Laplace Transform of $\exp(-kt^2)\, u(-t)$ is $k^{-1/2} \exp\left(\frac{p^2}{4k}\right) \int_{-\infty}^{\frac{p}{2k^{1/2}}} \exp(-z^2)dz$, the

integration being along the ray of argument zero from $-\infty = \infty \exp(i\pi)$ to $\frac{p}{2k^{1/2}}$. In particular, when $k = \frac{1}{4}$, the Laplace Transform of $\exp(-\frac{t^2}{4})\, u(t)$ is $2\exp(p^2) \int_p^{\infty} \exp(-z^2)dz$

and the Laplace Transform of $\exp(-\frac{t^2}{4})\, u(-t)$ is $2\exp(p^2) \int_{-\infty}^{p} \exp(-z^2)dz$.

If we are certain that it is permissible to differentiate with respect to k, under the sign of integration, the infinite integral $\int_{-\infty}^{\infty} \exp(-kt^2) \exp(-pt)\, dt$ which furnishes the Laplace Transform, $(\frac{\pi}{k})^{1/2} \exp\left(\frac{p^2}{4k}\right)$, of $\exp(-kt^2)$, $k > 0$, we may obtain the relation

$$\int_{-\infty}^{\infty} t^2 \exp(-kt^2) \exp(-pt)dt = \left(\frac{\pi}{k}\right)^{1/2} \exp\left(\frac{p^2}{4k}\right) \left[\frac{1}{2k} + \frac{p^2}{4k^2}\right]$$

which furnishes us with the Laplace Transform of $t^2\exp(-kt^2)$ and, continuing this process, we may obtain the Laplace Transform of the product of $\exp(-kt^2)$ by any even power of t (always provided that the differentiation of the infinite integral involved with respect to k, under the integral sign, is legitimate). Similarly, if we are certain that it is legitimate to differentiate, under the integral sign, the infinite integral $\int_{-\infty}^{\infty} \exp(-kt^2) \exp(-pt)dt$ with respect to p or, equivalently, with respect to the real part c of p, we may obtain the relation

$$\int_{-\infty}^{\infty} t \exp(-kt^2) \exp(-pt)dt = \left(\frac{\pi}{k}\right)^{1/2} \frac{p}{2k} \exp\frac{p^2}{4k}$$

which furnishes us with the Laplace Transform of $t \exp(-kt^2)$, $k > 0$, and continuing this process we may obtain, always under the same proviso, the Laplace Transform of the product of $\exp(-kt^2)$ by any positive integral power, odd or even, of t. In order to formulate, in as convenient a manner as possible, conditions which guarantee the validity of this differentiation of an infinite integral under the integral sign we shall consider the case where we propose to differentiate the infinite integral $\int_{-\infty}^{\infty} h(t) \exp(-pt)dt$, which furnishes the Laplace Transform of h(t), with respect to the real part c of the complex variable $p = c + i\omega$. The integrand, h(t) exp(-pt), of this infinite integral is a function F(t, c) of the two real variables (t, c), the imaginary part ω of p being supposed held constant, and the derivative, $F_c(t, c)$, of F(t, c) with respect to c, being -t h(t) exp(-pt), is since h(t) is, by hypothesis, a piecewise continuous function of the unrestricted real variable t, either a continuous function of the two variables (t, c) over any strip $-\infty < t < \infty$, $c_1 \leq c \leq c_2$ parallel to the t-axis in the (t, c)-plane, or else its points of discontinuity in any rectangle $a \leq t \leq b$, $c_1 \leq c \leq c_2$, where a and b are any two real numbers which are such that $a < b$ and c_1, c_2 are any two real numbers which are such that $c_1 < c_2$, lie on a finite number of lines parallel to the c-axis. We make now the following two additional hypotheses concerning the function $F_c(t, c)$ of the two real variables (t, c):

1) $F_c(t, c)$ is absolutely integrable with respect to t, for every value of c in a given closed interval $c_1 \leq c \leq c_2$, over $-\infty < t < \infty$.

2) The convergence of the infinite integral $\int_{-\infty}^{\infty} F_c(t, c)\, dt$ is uniform with respect to c over the interval $c_1 \leq c \leq c_2$

and a single hypothesis concerning F(t, c):

3) The infinite integral $\int_{-\infty}^{\infty} F(t, c_1) dt$ exists

and we shall show in the following paragraph that these three hypotheses are sufficient to guarantee the following three facts:

1') The infinite integral $\int_{-\infty}^{\infty} F(t, c)\, dt$ exists for each value of c in the interval $c_1 \leq c \leq c_2$.

2') $\int_{-\infty}^{\infty} F(t, c)\, dt$ is a differentiable function of c over the interval $c_1 \leq c \leq c_2$.

3') The derivative of $\int_{-\infty}^{\infty} F(t, c)\, dt$ with respect to c, where $c_1 \leq c \leq c_2$, is furnished by the formula $\left(\int_{-\infty}^{\infty} F(t, c) dt\right)_c = \int_{-\infty}^{\infty} F_c\,(t, c)\, dt$.

In other words, differentiation of the infinite integral $\int_{-\infty}^{\infty} F(t, c)\,dt$, with respect to c, under the integral sign is legitimate.

Writing $\int_{-\infty}^{\infty} F_c(t, c)\,dt$ in the form $\int_a^b F_c(t, c)\,dt + R_a^b(c)$, $a < b$, we know from 2), that $\left| R_a^b(c) \right|$ may be made arbitrarily small, say $< \epsilon$, for every c in the interval $c_1 \le c \le c_2$ by making -a and b positive and sufficiently large, the choice of a and b being independent of c. If, then, a and b are so chosen that $\left| R_a^b(c) \right| < \epsilon$, $c_1 \le c \le c_2$, and c and $c + \Delta c$ are any two values of c in the interval $c_1 \le c \le c_2$ we have $\left| \Delta \int_{-\infty}^{\infty} F_c(t, c)\,dt \right| \le \left| \Delta \int_a^b F_c(t,c)dt \right| + 2\epsilon$ where $\Delta \int_{-\infty}^{\infty} F_c(t,c)dt$ denotes $\left\{ \int_{-\infty}^{\infty} F_c(t, c+\Delta c)dt - \int_{-\infty}^{\infty} F_c(t,c)dt \right\}$ and similarly for $\Delta \int_a^b F_c(t,c)dt$. If $F_c(t, c)$ is a continuous function of the two variables (t, c) over the rectangle $a \le t \le b$, $c_1 \le c \le c_2$, it is a uniformly continuous function of the two variables (t, c) over this rectangle and so $\left| \Delta F_c(t,c) \right| = \left| F_c(t, c+\Delta c) - F_c(t,c) \right|$ may be made arbitrarily small, say $< \frac{\epsilon}{b-a}$, by making $\left| \Delta c \right|$ sufficiently small, the choice of Δc being independent of either t or c. Supposing Δc so chosen, we have $\left| \Delta \int_a^b F_c(t,c)dt \right| = \left| \int_a^b \Delta F_c(t,c)dt \right| \le \int_a^b \left| \Delta F_c(t,c) \right| dt < \epsilon$ which implies that $\left| \Delta \int_{-\infty}^{\infty} F_c(t,c)\,dt \right| < 3\epsilon$. Thus the two integrals $\int_a^b F_c(t,c)dt$, $\int_{-\infty}^{\infty} F_c(t,c)dt$ and, hence, their difference $R_a^b(c)$, are continuous functions of c over the interval $c_1 \leq c \leq c_2$. This conclusion remains true, by virtue of 1), when $F_c(t, c)$ fails to be continuous over the rectangle $a \leq t \leq b$, $c_1 \leq c \leq c_2$, since, by hypothesis, its points of discontinuity in this rectangle lie on a finite number of lines parallel to the c-axis in the (t, c)-plane. To prove this it suffices to treat the case where the points of discontinuity lie on a single line $t = d$, where $a < d < b$. We write $\int_a^b F_c(t, c)dt$ in the form $\int_a^{d-\delta_1} F_c(t, c)dt + \int_{d-\delta_1}^{d+\delta_2} F_c(t, c)dt + \int_{d+\delta_2}^{b} F_c(t, c)dt$, where δ_1 and δ_2 are positive numbers which are less than d-a and b-d, respectively. The first and third of these three integrals are continuous function of c over the interval $c_1 \leq c \leq c_2$, since $F_c(t, c)$ is a continuous function of the two variables (t, c) over the rectangles $a \leq t \leq d - \delta_1$, $c_1 \leq c \leq c_2$ and $d + \delta_2 \leq t \leq b$, $c_1 \leq c \leq c_2$, and we direct our attention to the second. The modulus of this second integral may be made arbitrarily small, say $< \epsilon$, by choosing δ_1 and δ_2 sufficiently small, the choice of δ_1 and δ_2 being, by virtue of 2), independent of c. Supposing δ_1 and δ_2 so chosen, we have $\left| \Delta \int_{d-\delta_1}^{d+\delta_2} F_c(t, c)dt \right| < 2\epsilon$ and it follows that, if Δc is chosen so small that $\left| \Delta \int_a^{d-\delta_1} F_c(t, c)dt \right| < \epsilon$ and $\left| \Delta \int_{d+\delta_2}^{b} F_c(t, c)dt \right| < \epsilon$ that $\left| \Delta \int_a^{b} F_c(t, c)dt \right| < 4\epsilon$ which implies that $\left|\Delta \int_{-\infty}^{\infty} F_c(t,c)dt\right| < 6\epsilon$ proving the continuity of the two integrals $\int_a^b F_c(t,c)dt$, $\int_{-\infty}^{\infty} F_c(t,c)dt$ and, hence, of their difference $R_a^b(c)$, over the interval $c_1 \leq c \leq c_2$. Hence the three functions $\int_a^b F_c(t,c)dt$, $\int_{-\infty}^{\infty} F_c(t,c)dt$ and $R_a^b(c)$ of c are integrable over the interval $c_1 \leq c \leq c_2$ and, if c' is any point of this interval, we have the relation

$$\int_{c_1}^{c'} \left\{ \int_{-\infty}^{\infty} F_c(t,c)dt \right\} dc = \int_{c_1}^{c'} \left\{ \int_a^b F_c(t,c)dt \right\} dc + \int_{c_1}^{c'} R_a^b(c)dc.$$

If $F_c(t,c)$ is a continuous function of the two variables (t,c) over the rectangle $a \leq t \leq b$, $c_1 \leq c \leq c_2$, the order of integration may be changed in the repeated integral on the right and the same argument as before shows that this remains true, by virtue of 1) and 2), when $F_c(t,c)$ fails to be continuous over the rectangle $a \leq t \leq b$, $c_1 \leq c \leq c_2$. Hence

$$\int_{c_1}^{c'} \left\{ \int_{-\infty}^{\infty} F_c(t,c)dt \right\} dc = \int_a^b \left\{ \int_{c_1}^{c'} F_c(t,c)dc \right\} dt + \int_{c_1}^{c'} R_a^b(c)\, dc$$ so that

$$\left| \int_{c_1}^{c'} \left\{ \int_{-\infty}^{\infty} F_c(t,c)dt \right\} dc - \int_a^b \left\{ \int_{c_1}^{c'} F_c(t,c)dc \right\} dt \right| \leq \epsilon(c'-c_1) \leq \epsilon(c_2-c_1)$$

proving the existence of the infinite integral $\int_{-\infty}^{\infty}\left\{\int_{c_1}^{c'} F_c(t,c)dc\right\} dt$ with the value $\int_{c_1}^{c'}\left\{\int_{-\infty}^{\infty} F_c(t,c)dt\right\} dc$. Since $\int_{c_1}^{c'} F_c(t,c)dc = F(t,c') - F(t,c_1)$ it follows, by virtue of 3), that the infinite integral $\int_{-\infty}^{\infty} F(t,c')\,dt$ exists, for every point c' of the interval $c_1 \leq c' \leq c_2$, its value being $\int_{c_1}^{c'}\left\{\int_{-\infty}^{\infty} F_c(t,c)dt\right\} dc + \int_{-\infty}^{\infty} F(t,c_1)dt$. The first of these two terms is a differentiable function of c', since $\int_{-\infty}^{\infty} F_c(t,c)dt$ is a continuous function of c, and the second is a constant function of c'. Hence the infinite integral $\int_{-\infty}^{\infty} F(t,c')dt$ is a differentiable function of c' over the interval $c_1 \leq c' \leq c_2$, its derivative being $\int_{-\infty}^{\infty} F_c(t,c')dt$. This completes the proof of the legitimacy of differentiating the infinite integral $\int_{-\infty}^{\infty} F(t,c)dt$ with respect to c under the sign of integration, when $F_c(t,c)$ satisfies conditions 1) and 2) and $F(t,c)$ satisfies condition 3).

Lecture 7

The Laplace Transform of the Product of a Right-sided Function by t and of the Integral of a Right-sided Function over the Interval $[0,t]$

We have seen that if a piecewise continuous right-sided function h(t) possesses a Laplace Transform Lh, which need not be absolutely convergent, at a point c_1 of the real axis in the complex p-plane then Lh may be written, over the half-plane $c > c_1$ in the form

$$Lh = (p-c_1)\int_0^\infty H_{c_1}(t)\exp\left[-(p-c_1)t\right]dt \quad \text{where } H_{c_1}(t)=\int_0^t h(s)\exp(-c_1 s)ds,$$

the convergence of the infinite integral which multiplies $p-c_1$ being absolute over this half-plane. The integrand, $H_{c_1}(t)\exp\left[-(p-c_1)t\right]$, of this infinite integral is a function $F(t,c)$, of the two real variables (t,c), where c is the real part of p, it being understood that the imaginary part ω of p is held constant, and the derivative of this function with respect to c, being the same as its derivative with respect to p, exists at every point p of the finite complex p-plane, with the value $-tH_{c_1}(t)\exp\left[-(p-c_1)t\right]$. Since $0 \le t\exp\left[-(c_2-c)t\right] < \dfrac{1}{c_2-c}$, if $c_2 > c$ and $t \ge 0$, $tH_{c_1}(t)\exp\left[-(c_2-c_1)t\right] =$ $tH_{c_1}(t)\exp\left[-(c_2-c)t\right]\exp\left[-(c-c_1)t\right]$, $c_1 < c < c_2$, is absolutely integrable over $0 \le t < \infty$ (since $H_{c_1}(t)\exp\left[-c-c_1)t\right]$, $c_1 < c < c_2$, is absolutely integrable over $0 \le t < \infty$). Thus the infinite integral $\int_0^\infty F_c(t,c)dt$ converges absolutely over the half-plane $c > c_1$. Moreover, the convergence of this infinite integral is uniform over the half-plane $c \ge c_1 + \delta$, where δ is any positive number, since, over this half-plane, $\left|\exp\left[-(p-c_1)t\right]\right| = \exp\left[-(c-c_1)t\right] \le \exp(-\delta t)$, and so it is permissible to differentiate with respect to c or, equivalently, with respect to p, under the sign of integration, the infinite integral $\int_0^\infty H_{c_1}(t) \exp\left[-(p-c_1)t\right] dt$, p being any point of the half-plane $c > c_1$; indeed, if $c > c_1$, we may set $\delta = \frac{1}{2}(c-c_1)$ and ensure that $c > c_1 + \delta$. Thus

$$(Lh)_p = \int_0^\infty H_{c_1}(t) \exp\left[-(p-c_1)t\right] dt - (p-c_1)\int_0^\infty tH_{c_1}(t)\exp\left[-(p-c_1)t\right]dt, c>c_1$$

$$= \int_0^\infty H_{c_1}(t) \left\{ t \exp\left[-(p-c_1)t\right]\right\}_t dt, \ c > c_1$$

and the right-hand side of this equation reduces, on integration by parts, since $H_{c_1}(\infty)$ exists and since $t \exp -(c-c_1)t$ tends to zero as $t \to \infty$, to $-\int_0^\infty t\, h(t) \exp(-pt)dt$. Thus we have the following useful result:

If the piecewise continuous right-sided function h(t) possesses, at a point c_1 of the real axis in the complex p-plane, a Laplace Transform, whose convergence need not be absolute, then the product, th(t) of h(t) by t possesses, over the half-plane $c > c_1$, the Laplace Transform $-(Lh)_p$

We express this result by the statement that multiplication of a piecewise continuous right-sided function by t is reflected, in the domain of Laplace Transforms, by differentiation with respect to p followed by a change of sign.

<u>Example.</u> $\exp(\alpha t)\, u(t)$ possesses, over the half-plane $c > \alpha_r$, where α_r is the real part of α, the Laplace Transform $\frac{1}{p-\alpha}$. Hence $t \exp(\alpha t)\, u(t)$ possesses, over the half-plane $c > \alpha_r$, the Laplace Transform $\frac{1}{(p-\alpha)^2}$. Continuing this process we see that, if n is any positive integer, $t^n \exp(\alpha t)\, u(t)$ possesses, over the half-plane $c > \alpha_r$, the Laplace Transform $\frac{n!}{(p-\alpha)^{n+1}}$.

Note. This is a special case of the result that $t^{\beta} \exp(\alpha t)\, u(t)$, where β is any complex number whose real part is > -1, possesses, over the half-plane $c > \alpha_r$, the Laplace Transform $\dfrac{\Gamma(\beta+1)}{(p-\alpha)^{\beta+1}}$, which result is an immediate consequence of an application of the Translation Theorem to the result that $t^{\beta} u(t)$ possesses, over the half-plane $c > 0$, the Laplace Transform $\dfrac{\Gamma(\beta+1)}{p^{\beta+1}}$.

Let us now consider a piecewise continuous right-sided function $h(t)$ which is such that the integral $\int_0^t h(s)ds = H_0(t)$, which we shall denote simply by $H(t)$, exists over $0 \leq t < \infty$. We do not assume the existence of $H(\infty)$, i.e., that $h(t)$ possesses at $p = 0$ a Laplace Transform, but we do assume the existence of a positive real number c_1 such that $h(t)$ possesses at $p = c_1$ a Laplace Transform which need not be absolutely convergent. $H(t)$ is an everywhere continuous right-sided function, which is, in addition, differentiable, with derivative $h(t)$, at the points of continuity of $h(t)$. The function $\phi(t) = \int_0^t H(v) \exp(-c_1 v)\, dv$ exists, since $H(t)$ is everywhere continuous, over $0 \leq t < \infty$ and is an everywhere differentiable function, its derivative being $H(t) \exp(-c_1 t)$. Since $H(0) = 0$, we obtain, on integration by parts,

$$\phi(t) = -\frac{H(t)\exp(-c_1 t)}{c_1} + \frac{1}{c_1}\int_0^t h(v)\exp(-c_1 v)\, dv$$

so that $\phi(t)\exp(c_1 t) = -\dfrac{H(t)}{c_1} + \dfrac{\exp(c_1 t)}{c_1}\displaystyle\int_0^t h(v)\exp(-c_1 v)\, dv.$

Hence, at the points of continuity of $h(t)$, $\left\{\phi(t)\exp(c_1 t)\right\}_t$

$$= -\frac{h(t)}{c_1} + \exp(c_1 t)\int_0^t h(v)\exp(-c_1 v)dv + \frac{h(t)}{c_1} = \exp(c_1 t)\int_0^t h(v)\exp(-c_1 v)dv$$

and this implies, since the derivative of $\phi(t)$ is continuous over $0 \leq t < \infty$, that $\left\{\phi(t)\exp(c_1 t)\right\}_t = \exp(c_1 t)\displaystyle\int_0^t h(v)\exp(-c_1 v)\, dv$ over $0 \leq t < \infty$. Since

$h(t)$ possesses, by hypothesis, a Laplace Transform at $p = c_1$, it follows that the quotient of $\left\{ \phi(t) \exp(c_1 t) \right\}_t$ by $\exp(c_1 t) = \left\{ \frac{1}{c_1} \exp(c_1 t) \right\}_t$ has, at $t = \infty$, the limit $(Lh)_{p=c_1}$, and we shall show in the next paragraph that this implies that the quotient of $\phi(t) \exp(c_1 t)$ by $\frac{1}{c_1} \exp(c_1 t)$ has, at $t = \infty$, the same limit, $(Lh)_{p=c_1}$. Assuming this, for the moment, it follows that $c_1 \phi(t) = - H(t) \exp(-c_1 t) + \int_0^t h(v) \exp(-c_1 v)\, dv$ has, at $t = \infty$, the limit $(Lh)_{p=c_1}$ and, since the second term on the right has, at $t = \infty$, the limit $(Lh)_{p=c_1}$, this implies that $H(t) \exp(-c_1 t)$ has at $t = \infty$, the limit zero. The existence of the limit, at $t = \infty$, of $\phi(t) = \int_0^t H(v) \exp(-c_1 v)\, dv$ assures us that $H(t)$ possesses, at $p = c_1$, a Laplace Transform and this implies that the Laplace Transform of $H(t)$ exists, and is an analytic function of the complex variable p, over the half-plane $c > c_1$. Since the limit, at $t = \infty$, of $\phi(t)$

$$= - \frac{H(t) \exp(-c_1 t)}{c_1} + \frac{1}{c_1} \int_0^t h(v) \exp(-c_1 v)\, dv \text{ is}$$

$$\frac{1}{c_1} \int_0^\infty h(v) \exp(-c_1 v)\, dv \text{ we see that } (LH)_{p=c_1} = \frac{1}{c_1} (Lh)_{p=c_1}.$$

The number c_1 may be replaced throughout the entire preceding argument by any real number $> c_1$, and so the value of (LH) at any point $p = c$ of the real axis in the complex p-plane which lies to the right of the point $p = c_1$ is the quotient of the value of Lh at $p = c$ by c. Since both $\frac{Lh}{p}$ and LH are analytic functions of the complex variable p over the half-plane $c > c_1$, it follows that $LH = \frac{1}{p} (Lh)$ over this half-plane. We express this result, which is the central one in the theory of Laplace Transforms, as follows:

Integration with respect to t, over the interval $[0, t]$, of a right-sided function is reflected, in the domain of Laplace Transforms, by division of p.

To complete the proof of this fundamental theorem let us denote $\phi(t) \exp(c_1 t)$ by f(t) and $\exp(c_1 t)$ by g(t). Since g(t) is monotone increasing and unbounded at $t = \infty$, we may associate with any positive real number t a real number $T \geq t$ such that, if $t' \geq T$,

$$\left|\frac{f(t)}{g(t')}\right| < \epsilon; \quad \left|\frac{g(t)}{g(t') - g(t)}\right| < \epsilon$$

where ϵ is an arbitrary positive number. Applying the Theorem of the Mean of differential calculus to the function

$f(s) - f(t) - \dfrac{f(t') - f(t)}{g(t') - g(t)} \left\{g(s) - g(t)\right\}$ of the real variable s, which function vanishes when s = t and when s = t', we see that

$$\frac{f_t(t'')}{g_t(t'')} = \frac{f(t') - f(t)}{g(t') - g(t)} = \left\{\frac{f(t')}{g(t')} - \frac{f(t)}{g(t')}\right\} \frac{g(t')}{g(t') - g(t)}$$

where t'' is some real number between t and t'. As $t \longrightarrow \infty$ so also do t' and t'' and so $\dfrac{f_t(t'')}{g_t(t'')}$ is of the form $\ell + \nu_1$, where $|\nu_1|$ is arbitrarily small, say $< \epsilon$, if t is sufficiently large, ℓ being the limit, at $t = \infty$, of $\dfrac{f_t(t)}{g_t(t)}$. Also $\dfrac{g(t')}{g(t')-g(t)} = 1 + \dfrac{g(t)}{g(t') - g(t)} =$ $1 + \nu_2$, where $|\nu_2| < \epsilon$ for every t, and $\dfrac{f(t)}{g(t')} = \nu_3$, where $|\nu_3| < \epsilon$ for every t, and so $\dfrac{f(t')}{g(t')} - \nu_3 = \dfrac{\ell + \nu_1}{1 + \nu_2} = \ell + \dfrac{\nu_1 - \ell\,\nu_2}{1 + \nu_2}$

so that $\left| \frac{f(t')}{g(t')} - \ell \right|$ is arbitrarily small if $t' \geq T$ is sufficiently large.

Hence $\frac{f(t)}{g(t)}$ has, at $t = \infty$, the limit ℓ.

The theorem which states that integration of piecewise continuous right-sided functions over the interval $[0, t]$ is reflected, in the domain of Laplace Transforms, by division by p may be presented in a different form which is useful in the application of the Laplace Transformation to differential equations. Let us suppose that the right-sided function $h(t)$ is continuous over $0 < t < \infty$, without being, necessarily, continuous at $t = 0$, so that $h(+0)$ may be different from zero, and that $h(t)$ possesses over $0 \leq t < \infty$ a piecewise continuous derivative $h_t(t)$. Writing $\int_0^t h_t(s)\, ds = h(t) - h(+0)$, $h(t) - h(+0)$ plays the role of $H(t)$, and so the mere assumption that $L(h_t)$ exists at a point c_1 of the positive part of the real axis in the complex p-plane guarantees that $L\left\{h(t) - h(+0)\right\}$ exists, and is an analytic function of the complex variable p, over the half-plane $c > c_1$ and, furthermore, that $L\left\{h(t) - h(+0)\right\} = \frac{L(h_t)}{p}$ over this half-plane. Since $L\left\{h(+0)\right\} = \frac{h(+0)}{p}$ over the half-plane $c > 0$, it follows that

$$L(h_t) = p\, Lh - h(+0) \qquad ; \quad c > c_1 > 0$$

We express this result by the statement that differentiation of a right-sided function $h(t)$ is reflected, in the domain of Laplace Transforms, by multiplication by p followed by subtraction of $h(+0)$.

Similarly, if h_t is continuous over $0 < t < \infty$ and h_{tt} is piecewise continuous and possesses a Laplace Transform at $p = c_1 > 0$, then

$$L(h_{tt}) = p\, L(h_t) - h_t(+0) = p^2\, Lh - p\, h(+0) - h_t(+0),$$

and so on.

Lecture 8

Functions of Exponential Type

The right-sided function $\exp(\alpha\, t)\, u(t)$, where α is any complex number, possesses, over the half-plane $c > \alpha_r$, where α_r is the real part of α, the Laplace Transform $\frac{1}{p-\alpha}$. This Laplace Transform is not only an analytic function of the complex variable p over the half-plane $c > \alpha_r$, but it possesses, in addition, the following two properties:

a) It is zero at $p = \infty$

b) It is analytic over the neighborhood $|p| > |\alpha|$ of $p = \infty$

It follows that any finite linear combination, $C_1 \exp(\alpha_1 t)\, u(t) + C_2 \exp(\alpha_2 t)\, u(t) + \ldots + C_n \exp(\alpha_n t)\, u(t)$, where $C_1, \ldots, C_n$ are any complex constants, of right-sided functions of the form $\exp(\alpha t)\, u(t)$ possesses the two additional properties a) and b). We term any piecewise continuous right-sided function h(t) a function of exponential type if it shares with any such finite linear combination of right-sided functions of the form $\exp(\alpha t)\, u(t)$ the following three properties:

1) Lh exists at some point c_1 of the real axis in the complex p-plane;

2) The analytic function, f(p), of the complex variable p which is furnished, over the half-plane $c > c_1$, by Lh is zero at $p = \infty$

3) f(p) is analytic over some neighborhood $|p| > R \geq 0$ of $p = \infty$

and we proceed to investigate what properties of a given piecewise continuous right-sided function h(t) are sufficient to ensure that h(t) is of exponential type. On denoting by $\frac{a_0}{p} + \frac{a_1}{p^2} + \ldots + \frac{a_n}{p^{n+1}} + \ldots$ the power series development of f(p) over the neighborhood $|p| > R$ of $p = \infty$ this infinite series converges absolutely at $p = R + \delta$ where δ is any positive number, and it may also converge (not, necessarily, absolutely) at p = R. For example, if $f(p) = (1+p^2)^{-1/2} = \frac{1}{p}\left(1 + \frac{1}{p^2}\right)^{-1/2} = \frac{1}{p}\left(1 - \frac{1}{2p^2} + \frac{1.3}{2.4p^4} - \ldots\right)$, a_n is zero when n is odd while $a_0 = 1$, $a_2 = -\frac{1}{2}$, $a_4 = \frac{1.3}{2.4}$, ... Thus

$$a_{2m} = (-1)^m \frac{2}{\pi} \int_0^{\pi/2} \cos^{2m}\theta \, d\theta$$ and $\frac{2}{\pi}\int_0^{\pi/2} \cos^{2m}\theta \, d\theta$ is an monotone decreasing function of m which has the limit zero at $m = \infty$ (since it is dominated by $\frac{2}{\pi}\left[\delta + (\cos\delta)^{2m}\left(\frac{\pi}{2} - \delta\right)\right]$ where δ is any positive number $< \frac{\pi}{2}$) so that the infinite series $1 - \frac{1}{2} + \frac{1.3}{2.4} - \ldots$, being alternating, is convergent. Thus the infinite series $\frac{1}{p} - \frac{1}{2p^3} + \frac{1.3}{2.4p^5} - \ldots$ which converges, with the sum $(1+p^2)^{-1/2}$, over the neighborhood $|p| > 1$ of $p = \infty$, also converges at p = 1. On the other hand, if $f(p) = (1-p^2)^{-1/2}$, the corresponding infinite series $\frac{1}{p} + \frac{1}{2p^3} + \frac{1.3}{2.4p^5} + \ldots$, which also converges, but with the sum $(1-p^2)^{-1/2}$, over the neighborhood $|p| > 1$ of $p = \infty$, fails to converge at p = 1; indeed $\frac{1.3\ldots(2m-1)}{2.4\ldots 2m} > \frac{2.4\ldots(2m-2)}{2.4\ldots 2m} = \frac{1}{2m}$ so that the infinite series $1 + \frac{1}{2} + \frac{1.3}{2.4} + \ldots$ does not converge. We use the symbol r to denote R, if the infinite series $\frac{a_0}{p} + \frac{a_1}{p^2} + \ldots$ converges at p = R, and to denote $R + \delta$, where δ is an

arbitrarily small positive number, otherwise. Thus the infinite series $\frac{a_0}{r} + \frac{a_1}{r^2} + \ldots$ converges and this implies that $\frac{|a_n|}{r^{n+1}}$ is arbitrarily small, say $< \epsilon$, if n is sufficiently large, say $\geq$ N. If, then, M is any positive number which dominates each of the N + 1 numbers $|a_0|$, $\frac{|a_1|}{r}$, . $\frac{|a_{N-1}|}{r^{N-1}}$, ϵr, we have $\frac{|a_n|}{r^n} \leq M$, $n = 0, 1, 2, \ldots$, and this implies that the two infinite power series in the complex variable z:

$$a_0 + a_1 z + \frac{a_2}{2!} z^2 + \ldots + \frac{a_{n-1}}{(n-1)!} z^{n-1} + \ldots; \quad |a_0| + |a_1| z + \frac{|a_2|}{2!} z^2 + \ldots + \frac{|a_{n-1}|}{(n-1)!} z^{n-1} + \ldots$$

converge over the entire finite complex z-plane, their sums at any point z of this plane being each dominated by M exp $(r\,|z|)$. Assigning real values to z we obtain two functions k(t), k*(t) of the unrestricted real variable t, where

$$k(t) = a_0\ a_1 t + \ldots + \frac{a_{n-1}}{(n-1)!} t^{n-1} + \ldots; k^*(t) = |a_0| + |a_1| t + \ldots + \frac{|a_{n-1}|}{(n-1)!} t^{n-1} + \ldots$$

and we know that both $|k(t)|$ and $|k^*(t)| \leq M \exp(r\,|t|)$. For example, when $f(p) = (1+p^2)^{-1/2}$, $k(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 . 4^2} - \ldots$ and $k^*(t) = 1 + \frac{t^2}{2^2} + \frac{t^4}{2^2 . 4^2} + \ldots$. In this case k(t) is known as the Bessel Function, $J_0(t)$, of the first kind, of index zero, and k*(t) is known as the modified Bessel Function, $I_0(t) = J_0(it)$, of the first kind, of index zero, and since r = 1 and we can take M = 1, we know that both $J_0(t)$ and $I_0(t)$ are dominated, over $-\infty < t < \infty$, by $\exp|t|$. We shall study the functions $J_0(t)$ and $I_0(t)$ in detail and shall see that the inequality $|J_0(t)| \leq \exp|t|$ is very crude for large values of $|t|$, it being possible to replace this inequality by the inequality $|J_0(t)| \leq 1$, but we cannot thus improve on the inequality $|I_0(t)| \leq \exp|t|$. The inequality $|J_0(t)| \leq 1$ assures us that the right-sided function $h_0(t) = J_0(t)\, u(t)$ possesses, over the half-plane $c > 0$, a Laplace Transform and we shall see that this Laplace Transform is $(1+p^2)^{-1/2}$. Similarly the right-sided function $h_0^*(t) = I_0(t)\, u(t)$ possesses, over the half-plane $c > 1$, the Laplace Transform $(1-p^2)^{-1/2}$. The fact that $h_0(t) = J_0(t)\, u(t)$ possesses a Laplace Transform over the half-plane $c > 0$, as contrasted with $h_0^*(t) = I_0(t)\, u(t)$ which does not possess a Laplace Transform over the half-plane $c > c_1$ if $c_1 < 1$, is due to the fact that $(1+p^2)^{-1/2}$ does not have a singular point in the half-plane $c > 0$ while $p = 1$ is a singular point of $(1-p^2)^{-1/2}$.

Since the two functions

$$k(t) = a_0 + a_1 t + \dots + \frac{a_{n-1}}{(n-1)!}\, t^{n-1} + \dots \; ; \; k^*(t) = |a_0| + |a_1|\, t + \dots$$

$$\frac{|a_{n-1}|}{(n-1)!}\, t^{n-1} + \dots$$ of the unrestricted real variable t are dominated, over $-\infty < t < \infty$, by $M \exp(r\,|t|)$ the two right-sided functions

$$h'(t) = k(t)\, u(t) \; ; \qquad h^*(t) = k^*(t)\, u(t)$$

the latter of which is a non-negative real-valued function of t, possess Laplace Transforms which are analytic functions of the complex variable p over the half-plane $c > r$ or, equivalently, since δ, if it is not zero, is arbitrarily small, over the half-plane $c > R$. If we denote by $h'_1(t)$ and $h'_2(t)$ the real and imaginary parts, respectively, of $h'(t)$ both $h'_1(t)$ and $h'_2(t)$, being dominated by $|h'(t)|$, also possess Laplace Transforms which are analytic functions of the complex variable p over the half-plane $c > R$ and this implies that each of the following four non-negative real-valued right-sided functions of the unrestricted real variable t, $h^*(t) \pm h'_1(t)$, $h^*(t) \pm h'_2(t)$, whose non-negativeness follows from the inequality $|h'(t)| \leq h^*(t)$, possesses a Laplace Transform which is an analytic function of the complex variable p over the half-plane $c > R$. Denoting by $A_0 + A_1 t + \dots \frac{A_{n-1}}{(n-1)!} t^{n-1} + \dots$ the everywhere convergent infinite series whose sum is $k^*(t) + k_1(t)$, for example, where $k_1(t)$ is the real part of $k(t)$, the coefficients A_0, A_1, ... of this series are all non-negative real numbers, since $|a_{n-1}|$ + the real part of a_{n-1}, $n = 1, 2, \dots$, is a non-negative real number, and we propose to show that the Laplace Transform, over the half-plane $c > R$, of $h^*(t) + h'_1(t)$ is furnished by the sum of the infinite series, $\frac{A_0}{p} + \frac{A_1}{p^2} + \dots$, the non-negativeness of the real numbers $A_0, A_1, \dots$ playing an essential role in our proof. A similar result holds for the other three non-negative real-valued right-sided functions $h^*(t) - h'_1(t)$, $h^*(t) + h'_2(t)$, $h^*(t) - h'_2(t)$ and, since $h'(t)$ is the linear combination $\frac{1}{2}\left\{h^*(t) + h'_1(t)\right\} - \frac{1}{2}\left\{h^*(t) - h'_1(t)\right\} + \frac{i}{2}\left\{h^*(t) + h'_2(t)\right\} - \frac{i}{2}\left\{h^*(t) - h'_2(t)\right\}$ of the four non-negative real-valued right-sided functions $h^*(t) \pm h'_1(t)$, $h^*(t) \pm h'_2(t)$, it follows that the Laplace Transform, over the half-plane $c > R$, of $h'(t)$ is the sum of the infinite power series $\frac{a_0}{p} + \frac{a_1}{p^2} + \ldots$. In other words, the two right-sided functions $h(t)$ and $h'(t)$ possess, over the half-plane $c > R$, coincident Laplace Transforms, and this implies that $h(t)$ coincides with $h'(t)$ at any point t which is a continuity point of both $h(t)$ and $h'(t)$. $h'(t)$ is everywhere continuous, save, possibly, at $t = 0$ and so $h(t)$ coincides with $h'(t)$ at every continuity point save, possibly, $t = 0$ of $h(t)$. Hence $h(t+0)$ and $h(t-0)$ exist, with the common value $h'(t)$, at any discontinuity point, if one exists, of $h(t)$ and, since we have agreed to set $h(t) = \frac{1}{2}\left\{h(t+0) + h(t-0)\right\}$ at any discontinuity point of $h(t)$ where both $h(t+0)$ and $h(t-0)$ exist, it follows that $h(t)$ is coincident with $h'(t) = k(t)\, u(t)$.

Once, then, we shall have proved that the Laplace Transform, over the half-plane $c > R$, of $h^*(t) + h'_1(t)$ is $\frac{A_0}{p} + \frac{A_1}{p^2} + \ldots$ we shall know that $h(t)$ is of the form $k(t)\, u(t)$ where $k(t)$ is the sum of an everywhere convergent power series in t, this power series being such that both $|k(z)|$ and $|k^*(z)|$, where z is an arbitrary complex number and $k^*(z)$ is the sum of the power series obtained from the power series $a_0 + a_1 z + \frac{a_2}{2!} z^2 + \ldots$ whose sum is $k(z)$ by replacing each of its coefficients by its absolute value, are dominated, over the entire finite complex z-plane, by a constant times $\exp r|z|$, r being $= R$ if the power series $\frac{a_0}{R} + \frac{a_1}{R^2} + \ldots$ converges and being $= R + \delta$, where δ is an arbitrary positive number, otherwise. For example, the function of exponential type whose Laplace Transform, over the half-plane $c > 1$, is $(1+p^2)^{-1/2}$, is $J_0(t)\,u(t)$ and the function of exponential type whose Laplace Transform, over the half-plane $c > 1$, is $(1 - p^2)^{-1/2}$ is $J^*_0(t)\,u(t) = I_0(t)u(t)$ and $J_0(z)$ is dominated, over the entire finite complex z-plane, by a constant (actually 1) times $\exp|z|$ while $I_0(z)$ is dominated, over the entire finite complex z-plane, by a constant times $\exp(1+\delta)|z|$, δ any positive number. Actually, since $I_0(z) = J_0(iz)$, $I_0(z)$ is dominated, over the entire finite complex z-plane, by $\exp|z|$. We shall prove in our next lecture that the Laplace Transform, over the half-plane $c > R$, of $h^*(t) + h'_1(t)$ is $\frac{A_0}{p} + \frac{A_1}{p^2} + \ldots$ and, furthermore, that if $h(t)$ is the product of $u(t)$ by the sum, $k(t)$, of an everywhere convergent infinite series $a_0 + a_1 t + \frac{a_2}{2!} t^2 + \ldots$ which is such that $k(z) = a_0 + a_1 z + \frac{a_2}{2!} z^2 + \ldots$ is dominated, over the entire finite complex z-plane, by a constant times $\exp(r|z|)$, where r is a positive real number, then $h(t)$ is of exponential type. In other words, this property of right-sided functions is characteristic of functions of exponential type; every function of exponential type possesses it and every right-sided function which possesses it is of exponential type.

We conclude with the observation that the convergence of the Laplace Transform, over the half-plane $c > R$, of a function $h(t)$ of exponential type is absolute, since $|h(t)| \leq h^*(t)$ and $h^*(t)$ possesses, over the half-plane $c > R$, the Laplace Transform $\frac{|a_0|}{p} + \frac{|a_1|}{p^2} + \ldots$.

Lecture 9

The Characterization of Functions of Exponential Type

Our first task is to show that, if the product of u(t) by the sum of an everywhere convergent infinite series $A_0 + A_1 t + \ldots + \frac{A_{n-1}}{(n-1)!} t^{n-1} + \ldots$ with non-negative coefficients, $A_0, A_1, \ldots$, possesses, over a half-plane $c > R \geq 0$, a Laplace Transform, this Laplace Transform is the sum of the infinite series $\frac{A_0}{p} + \frac{A_1}{p^2} + \ldots + \frac{A_{n-1}}{p^n} + \ldots$. To do this we observe that the infinite integral $\int_0^\infty \left\{ A_0 + A_1 t + \ldots + \frac{A_{n-1} t^{n-1}}{(n-1)!} + \ldots \right\} \exp(-ct)\, dt$ which furnishes, at the point $p = c > R$ of the positive real axis in the complex p-plane, the Laplace Transform in question is the sum by columns $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_m^n$ of the double series of non-negative terms which is defined by the formula

$$u_m^n = \int_m^{m+1} \frac{A_n}{n!} t^n \exp(-ct)dt; \; m=0, 1, 2, \ldots, n=0, 1, 2, \ldots$$

(the sum of the infinite series which is furnished by the (m+1)st column of the $\infty \times \infty$ matrix which has u_m^n as the element in its (m+1)st column and (n+1)st row being $\int_m^{m+1} \left\{ A_0 + A_1 t + \ldots + \frac{A_{n-1}}{(n-1)!} t^{n-1} + \ldots \right\}$ $\exp(-ct)dt$). Denoting by s this sum by columns of the non-negative double series we write $\sigma_j = \sum_{m=0}^{j} \sum_{n=0}^{j} u_m^n$, $j = 0, 1, 2, \ldots$, so that the sequence $\sigma_0, \sigma_1, \sigma_2 \ldots$ is monotone non-decreasing and σ_j is dominated by s, no matter what is the value of j. Thus the monotone sequence $\sigma_0, \sigma_1, \ldots$ has a limit σ, its least upper bound, and $\sigma \leq s$. From the definition of s we know that the non-negative number $s - \sum_{m=0}^{j} \sum_{n=0}^{\infty} u_m^n$ may be made arbitrarily small, say $< \epsilon$, by choosing j sufficiently large and, once j has been so chosen, each of the j+1 non-negative numbers $\sum_{n=0}^{\infty} u_m^n - \sum_{n=0}^{k} u_m^n$, $m = 0, 1, \ldots, j$, may be made arbitrarily small, say $< \frac{\epsilon}{j+1}$, by choosing k sufficiently large. Hence the non-negative number $s - \sum_{m=0}^{j} \sum_{n=0}^{k} u_m^n$ is less than 2ϵ if j and k are large enough so that, in particular, on denoting by p the larger of the two numbers j and k, the non-negative number $s - \sigma_p$ is less than 2ϵ if p is sufficiently large, proving that $\sigma = s$. Turning now to the rows of our $\infty \times \infty$ matrix we observe that any partial sum $\sum_{m=0}^{j} u_m^n$ of the infinite series furnished by the elements in the (n+1) st row of this matrix is dominated by σ_p where p is the greater of the two integers n and j and this implies that $\sum_{m=0}^{j} u_m^n$ is dominated by $\sigma = s$ no matter what are the integers n and j so that the infinite series $\sum_{m=0}^{\infty} u_m^n$ is convergent, with sum $\leq s$, no matter what is the value of $n = 0, 1, 2, \ldots$. The argument given above tells us that the non-negative number $\sum_{n=0}^{k} \sum_{m=0}^{\infty} u_m^n - \sum_{n=0}^{k} \sum_{m=0}^{j} u_m^n$ is arbitrarily small if j is large enough and, since the two finite sums $\sum_{m=0}^{j} \sum_{n=0}^{k} u_m^n$ and $\sum_{n=0}^{k} \sum_{m=0}^{j} u_m^n$ are equal, it follows that the

difference between s and $\sum_{n=0}^{k} \sum_{m=0}^{\infty} u_m^n$ is arbitrarily small if k is sufficiently large. In other words, the sum by rows $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_m^n$, of the non-negative double series $\{u_m^n\}$ exists with the same value, s, as its sum by columns $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_m^n$. Since $u_m^n = \int_m^{m+1} \frac{A_n}{n!} t^n \exp(-ct)dt$, the sum of the infinite series $\sum_{m=0}^{\infty} u_m^n$, being the infinite integral $\int_0^{\infty} \frac{A_n}{n!} t^n \exp(-ct)dt$, is $\frac{A_n}{c^{n+1}}$ (the Laplace Transform of t^n u(t) being $\frac{n!}{p^{n+1}}$) and so we have proved that the Laplace Transform of $\left\{A_0 + A_1 t + \ldots + \frac{A_{n-1}}{(n-1)!} t^{n-1} + \ldots\right\} u(t)$ is furnished, over the part of the positive real axis in the complex p-plane which is covered by the half-plane $c > R > 0$, by the sum of the convergent infinite series $\frac{A_0}{c} + \frac{A_1}{c^2} + \ldots + \frac{A_{n-1}}{c^n} + \ldots$.
Since both the Laplace Transform of $\left\{A_0 + A_1 t + \ldots + \frac{A_{n-1}}{(n-1)!} t^{n-1} + \ldots\right\} u(t)$ and the sum of the infinite series $\frac{A_0}{p} + \frac{A_1}{p^2} + \ldots + \frac{A_{n-1}}{p^n} + \ldots$ are analytic functions of the complex variable p over the half-plane $c > R$ it follows that the Laplace Transform of $\left\{A_0 + A_1 t + \ldots \frac{A_{n-1}}{(n-1)!} t^{n-1} + \ldots\right\} u(t)$ is furnished, over the half-plane $c > R$, by the sum of the convergent infinite series $\frac{A_0}{p} + \frac{A_1}{p^2} + \ldots + \frac{A_{n-1}}{p^n} + \ldots$. For example, once we are assured that $I_0(t) = 1 + \frac{t^2}{2^2} + \frac{t^4}{2^2 . 4^2} + \ldots$ possesses a Laplace Transform over the half-plane $c > 1$, the result we have just proved tells us that this Laplace Transform is $\frac{1}{p} + \frac{1}{2p^3} + \frac{1.3}{2.4p^5} + \ldots$ $= (1-p^2)^{-1/2}$

We have now completed the proof of the theorem stated in our last lecture, namely, that, if a piecewise continuous right-sided function h(t) is of exponential type i.e., if it possesses a Laplace Transform which is zero at $p = \infty$ and is an analytic function $f(p) = \frac{a_0}{p} + \frac{a_1}{p^2} + \dots$ of the complex variable p over a neighborhood $|p| > R$ of $p = \infty$, then h(t) is the product of u(t) by the sum of the everywhere convergent infinite series $a_0 + a_1 t + \dots + \frac{a_{n-1}}{(n-1)!} t^{n-1} + \dots$.

We now propose to show, conversely, that if the sum $k(z) = a_0 + a_1 z + \dots + \frac{a_{n-1}}{(n-1)!} z^{n-1} + \dots$ of an everywhere convergent power series in a complex variable z is such that $|k(z)| \leq M \exp(R|z|)$, where M and R are positive real constants, then the right-sided function $h(t) = k(t)\, u(t)$, $-\infty < t < \infty$, is of exponential type. Since k(z) is analytic over the entire finite complex z-plane the integral of $k(z) / z^{n+1}$ around the circle $|z| = b$ in the positive sense, b being any positive number, is $2\pi i \frac{a_n}{n!}$ and, since $|k(z)| \leq M \exp(Rb)$ at all points of the circle $|z| = b$, we have $|a_n| \leq \frac{n!\ M \exp(Rb)}{b^n}$ no matter what is the positive number b. Setting $b = n / R$ we obtain $|a_n| \leq \frac{n!\ M \exp(n) R^n}{n^n}$ or, equivalently, $\log |a_n| \leq \log(n!) + \log M + n + n \log R - n \log n$. To appraise the expression on the right-hand side of this inequality we consider the curve $y = \log x$, $x > 0$. Since $\int_a^b \frac{dx}{x} = \log b - \log a$, $0 < a < b$, and since $\frac{1}{x} < \frac{1}{a}$ over the interval $a < x \leq b$, we have $\frac{\log b - \log a}{b - a} < \frac{1}{a}$ so that the secant of the curve $y = \log x$, which passes through the two points P_a: (a, log a), P_b: (b, log b), is less steep than the tangent at P_a to the curve and, similarly, this secant is steeper than the tangent at P_b to the curve. If follows that the secant in question does not intersect the curve in more than two points; for, if P_a, P_b, P_c, where $a < b < c$, were three collinear points of the curve $y = \log x$ the secant $P_a P_c$ would be at once steeper than, and less steep than, the tangent to the curve at P_b. Over the interval $a \leq x \leq b$ the ordinate of the curve $y = \log x$ is $\geq$ the ordinate of the secant $P_a P_b$, the equality holding only at the end points of the interval, and so $\int_a^b (\log x)\, dx > \frac{1}{2} (\log a + \log b)(b-a)$ or, equivalently, $b \log b - b - a \log a + a > \frac{1}{a} (\log a + \log b)(b-a)$. Setting $b = a+1$ and then assigning to a, in turn, the values 1, ..., n we obtain the following n inequalities, n being any positive integer,

$$2 \log 2 - 1 > \frac{1}{2} \log 2$$

$$3 \log 3 - 2 \log 2 - 1 > \frac{1}{2} \log 2 + \frac{1}{2} \log 3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(n+1) \log(n+1) - n\log n - 1 > \frac{1}{2} \log n + \frac{1}{2} \log (n+1)$$

and these yield, on addition, the inequality $(n+1) \log(n+1) - n > \log(n!) + \frac{1}{2} \log(n+1)$ or, equivalently, $(n+ \frac{1}{2}) \log(n+1) - n > \log(n!)$. Hence $\log |a_n| < (n+\frac{1}{2}) \log(n+1) + \log M + n \log R - n \log n$ so that $\log(|a_n|^{1/n}) < \frac{1}{2n} \log(n+1) + \log(1+\frac{1}{n}) + \log R + \frac{1}{n} \log M$. If, then, n is sufficiently large, $\left| \log(|a_n|^{1/n}) - \log R \right|$ is arbitrarily small, say $< \log(1+\epsilon)$, where ϵ is an arbitrary positive number, so that $|a_n| < \left\{R(1+\epsilon)\right\}^n$ if n is sufficiently large. Thus there exists a positive number M' such that the quotient of $|a_n|$ by $\left\{R(1+\epsilon)\right\}^n$ is less than M' for all non-negative integral values 0, 1, 2, ... of n and this implies that the sum, $k^*(z)$, of the everywhere convergent infinite series $|a_0| + |a_1| z + \frac{|a_2|}{2!} z^2 + \ldots$ is dominated over the entire finite complex z-plane by $M' \exp\left[R(1+\epsilon)|z|.\right]$ Hence, by the argument of the preceding paragraph, $h(t) = k(t)\, u(t)$ possesses, over the half-plane $c > R$, the Laplace Transform $\frac{a_0}{p} + \frac{a_1}{p^2} + \ldots$ which is zero at $p = \infty$ and also, by virtue of the inequality $|a_n| < M' \left\{R(1+\epsilon)\right\}^n$, an analytic function of the complex variable p over the neighborhood $|p| > R$ of $p = \infty$. In other words, h(t) is of exponential type.

If the coefficient a_0 of $\frac{1}{p}$ in the expansion, over the neighborhood $|p| > R$ of $p = \infty$, of f(p) as a power series in $\frac{1}{p}$ is zero we may integrate f(p) from any point p for which $|p| > R$ to $p = \infty$, obtaining the new function $\int_p^\infty f(q)\, dq = \frac{a_1}{p} + \frac{a_2}{2p^2} + \frac{a_3}{3p^3} + \ldots$ which is the Laplace Transform, over the half-plane $c > R$, of the product of $k(t)/t = a_1 + \frac{a_2}{2!} t + \frac{a_3}{3!} t^2 + \ldots$ by u(t) where $k(t)\, u(t) = (a_1 t + \frac{a_2}{2!} t^2 + \ldots)\, u(t)$ is the right-sided function whose Laplace Transform, over the half-plane $c > R$, is $f(p) = \frac{a_1}{p^2} + \frac{a_2}{p^3} + \ldots$. Thus, when a function of exponential type is zero at $t = 0$ its quotient by t is also of exponential type and division by t is reflected, in the domain of Laplace Transforms, by integration from the point p whose real part c is $> R$ to $p = \infty$. For instance, $(\sin t)\, u(t)$ is a function of exponential type which is zero at $t = 0$, its Laplace Transform, over the half-plane $c > 0$, being $(1+p^2)^{-1}$ which is an analytic function of the complex variable p over the neighborhood $|p| > 1$ of $p = \infty$.

Hence the Laplace Transform of $\frac{\sin t}{t}$ u(t), over the half-plane $c > 1$, is $\int_p^\infty \frac{dq}{1+q^2} = \int_0^{1/p} \frac{ds}{1+s^2}$, $s = \frac{1}{q}$, $=$ arc tan $\frac{1}{p}$.

We have already seen that the Laplace Transform of $\frac{\sin t}{t}$ u(t) exists at $p = 0$ with the value $\frac{\pi}{2}$ and so the Laplace Transform of $\frac{\sin t}{t}$ u(t) exists, and is an analytic function of the complex variable p, over the half-plane $c > 0$. Since arc tan $\frac{1}{p}$ is also an analytic function of p over the half-plane $c > 0$ it follows that the Laplace Transform of $\frac{\sin t}{t}$ u(t), over the half-plane $c > 0$, is arc tan $\frac{1}{p}$.

Exercise. Show that, if the real part α_r of α is positive, the Laplace Transform of $\frac{\exp(\alpha t) - 1}{t}$ u(t), over the half-plane $c > \alpha_r$, is $-\log(1-\frac{\alpha}{p})$. Show, also, that the Laplace Transform of $\frac{\exp(\alpha t) - 1 - \alpha t}{t^2}$ u(t), over the half-plane $c > \alpha_r$, is $\alpha - (\alpha - p)\log(1-\frac{\alpha}{p})$.

Denoting by f(p) the function $\frac{a_0}{p} + \frac{a_1}{p^2} + \ldots$ which is analytic over the neighborhood $|p| > R$ of $p = \infty$ and which is furnished by the Laplace Transform of a function $(a_0 + a_1 t + \frac{a_2}{2!}t^2 + \ldots)$ u(t) $= k(t)$ u(t) of exponential type let us consider the integral $\oint_C f(p)\exp(pt)\,dp$, where C is any simple closed curve, of finite length l, all of whose points lie in the region $|p| > R$ of the complex p-plane and which encircles the circle $|p| = R$ and, hence, all the singular points of f(p). The infinite series $\frac{a_0}{p} + \frac{a_1}{p^2} + \ldots$ converges uniformly along C and so it may be integrated along C, after multiplication by exp(pt), term -by-term. Since $\oint_C \frac{\exp(pt)}{p^{n+1}} dp = 2\pi i \frac{t^n}{n!}$ it follows that $\frac{1}{2\pi i} \oint_C f(p) \exp(pt) dp = a_0 + a_1 t + \frac{a_2}{2!} t^2 + \ldots = k(t)$. We already know, from the Laplace version of the Fourier Integral Theorem, that $\frac{1}{2\pi i} \int_{(c-i\infty)}^{(c+i\infty)} f(p) \exp(pt)\, dp = k(t)\, u(t)$, $c > R$, the convergence of the infinite integral which furnishes, over the half-plane $c > R$, the Laplace Transform of $k(t)\, u(t)$ being absolute, and so $\frac{1}{2\pi i}$ times the integral of $f(p) \exp(pt)$ around the circle $|p| = r > R$ in the positive sense from $c + i(r^2 - c^2)^{1/2}$ to $c - i(r^2 - c^2)^{1/2}$, where $R < c < r$, has the limit $k(t) - k(t)u(t) = k(t)u(-t)$ as $r \longrightarrow \infty$. Indeed, the integral of $f(p) \exp(pt)$ around the circle $|p| = r$ in the positive sense from $c - i(r^2 - c^2)^{1/2}$ to $c + i(r^2 - c^2)^{1/2}$ has the limit $\int_{(c-i\infty)}^{(c+i\infty)} f(p) \exp(pt) dt = 2\pi i\, k(t)u(t)$ as $r \longrightarrow \infty$, this integral being the same as the integral of $f(p) \exp(pt)$ along the segment of the straight line: real part of $p = c$, from $c - i(r^2 - c^2)^{1/2}$ to $c + i(r^2 - c^2)^{1/2}$.

Lecture 10

The Polynomials of Laguerre

We are now ready to study the application of Laplace Transforms to ordinary linear differential equations with variable coefficients and we begin with the equation

$$tx_{tt} + (1-t)x_t + \lambda x = 0; \quad \lambda \text{ a real constant}$$

which occurs in the wave-mechanical treatment of the hydrogen atom. We assume that there exists a right-sided function h(t) which satisfies this differential equation save, possibly, at $t = 0$, where h(t) may not be differentiable. Furthermore, we assume that $L(h_{tt})$ exists at a point $p = c_1$ of the positive real axis of the complex p-plane. Then all three of the right-sided functions h_{tt}, h_t and h possess Laplace Transforms which are analytic functions of the complex variable p over the half-plane $c > c_1$ and $L(h_t) = p\,Lh - h(+0)$, $L(h_{tt}) = p^2Lh - ph(+0) - h_t(+0)$. We denote Lh simply by f and observe that $L(th_t) = -\left\{ L(h_t) \right\}_p = -f - pf_p$, $L(th_{tt}) = -2pf - p^2f_p + h(+0)$, over the half-plane $c > c_1$. Since, by hypothesis, $th_{tt} + (1-t)\,h_t + \lambda h = 0$, save, possibly, at $t = 0$, it follows that f satisfies, over the half-plane $c > c_1$, the first-order linear differential equation

$$p(p-1)f_p + (p-1-\lambda)f = 0$$

Writing this equation in the form $\dfrac{f_p}{f} = \dfrac{\lambda}{p(p-1)} - \dfrac{1}{p} = \dfrac{\lambda}{p-1} - \dfrac{\lambda+1}{p}$

we see that f is a constant times $(p-1)^{\lambda}/p^{\lambda+1} = \dfrac{1}{p}\left(1-\dfrac{1}{p}\right)^{\lambda}$ so that f is zero at $p = \infty$ and is an analytic function of the complex variable p over the neighborhood $|p| > 1$ of $p = \infty$. Thus h(t) is a function of exponential type which is dominated over $0 \leq t < \infty$ by a constant times $\exp\left[(1+\delta)t\right]$, where δ is an arbitrary positive number. The power series development of $\frac{1}{p}(1-\frac{1}{p})^{\lambda}$, over the neighborhood $|p| > 1$ of $p = \infty$, is $\frac{1}{p} - \frac{\lambda}{p^2} + \frac{\lambda(\lambda-1)}{2!p^3} - \frac{\lambda(\lambda-1)(\lambda-2)}{3!\,p^4} + \ldots$ and so h(t) is a constant times the product of u(t) by the sum $k_{\lambda}(t)$ of the everywhere convergent infinite series

$$1 - \lambda t + \frac{\lambda(\lambda-1)}{(2!)^2}t^2 - \frac{\lambda(\lambda-1)(\lambda-2)}{(3!)^2}t^3 + \ldots$$

It is now easy to justify our hypotheses concerning the existence of h(t) by verifying that $k_{\lambda}(t)\,u(t)$ satisfies these hypotheses with c_1 any number > 1. Since each term of $k_{\lambda}(t)$ is dominated by a constant times the corresponding term of $\exp\left[(1+\delta)t\right]$ and since a power series may be differentiated term-by-term the second derivative of $k_{\lambda}(t)$ is dominated, over $0 \leq t < \infty$, by a constant times $\exp\left[(1+\delta)t\right]$ (the constant being the product of the previous constant by $(1+\delta)^2$) so that $\left\{k_{\lambda}(t)\,u(t)\right\}_{tt}$ possesses a Laplace Transform at $p = 1 + \delta'$, $\delta' > \delta$. It remains only to verify that $x = k_{\lambda}(t)\,u(t)$ satisfies, for every value of $t \neq 0$, the differential equation $t x_{tt} + (1-t)\,x_t + \lambda x = 0$. To do this we avail ourselves of the relations $L(tx_{tt}) = -2pf - p^2 f_p + x(+0)$, $L(x_t) = pf - x(+0)$, $L(t\,x_t) = -f - pf_p$ which are valid, over the half-plane $c > 1$, since $Lx = f$ over this half-plane. It follows, since $p(p-1)f_p + (p-1-\lambda)f = 0$, that the Laplace Transform of $tx_{tt} + (1-t)x_t + \lambda x$, over the half-plane $c > 1$, is zero and this implies that $tx_{tt} + (1-t)x_t + \lambda x = 0$, $-\infty < t < \infty$. In particular, $t\,(k_\lambda)_{tt} + (1-t)\,(k_\lambda)_t + \lambda k_\lambda = 0$, $0 \leq t < \infty$, and, since the left-hand side of this equation is the sum of an everywhere convergent power series in t, it follows that this equation is valid over the extended range $-\infty < t < \infty$. Indeed, if we replace t, in the power series whose sum if $k_\lambda(t)$, by an arbitrary complex number z, we obtain a function $k_\lambda(z) = 1 - \lambda z + \frac{\lambda(\lambda-1)}{(2!)^2} z^2 - \ldots$ of the complex variable z which is analytic over the entire finite complex z-plane and we know that $z\left\{k_\lambda(z)\right\}_{zz} + (1-z)\left\{k_\lambda(z)\right\}_z + \lambda k_\lambda(z)$ is zero over the positive part of the real axis in the complex z-plane. However, this implies that it is zero over the entire finite complex z-plane and, in particular, over the part $-\infty < t \leq 0$ of the real axis in the complex z-plane. Thus $x = k_\lambda(t)$ is a solution, over $-\infty < t < \infty$, of the differential equation $tx_{tt} + (1-t)x_t + \lambda x = 0$ and we know that x is dominated, over $-\infty < t < \infty$, by a constant times $\exp\left[(1+\delta)|t|\right]$. We now raise the following question: For what values of λ, if any, is x dominated, over $-\infty < t < \infty$, by a constant times $\exp(\alpha|t|)$ where α is any positive number less than 1. For this to be the case $k_\lambda(t)u(t)$ must possess, over the half-plane $c > \alpha$, and not merely over the half-plane $c > 1$, a Laplace Transform and so $\frac{1}{p}(1-\frac{1}{p})^\lambda$ can have no singularities in the half-plane $c > \alpha$. Hence λ must be a non-negative integer for, if not, $p = 1$ would be a singular point of $\frac{1}{p}(1-\frac{1}{p})^\lambda$. For example, if $\lambda = -1$, $p = 1$ is a pole of $\frac{1}{p}(1-\frac{1}{p})^\lambda$ while, if

$\lambda = \frac{1}{2}$, p = 1 is a branch point of $\frac{1}{p}(1 - \frac{1}{p})^{\lambda}$, the function $\frac{1}{p}(1 - \frac{1}{p})^{1/2}$ of the complex variable p not being uniform over any neighborhood of p = 1. When λ is a non-negative integer n = 0, 1, 2, ..., the power series whose sum is $k_{\lambda}(t)$ is a constant times the polynomial function of t, of degree n, $1 - nt + \frac{n(n-1)}{(2!)^2} t^2 - \ldots + (-1)^n \frac{t^n}{n!}$. Choosing the, as yet undetermined, multiplicative constant to be n!, so that the coefficient of t^n in this polynomial function becomes $(-1)^n$, we obtain the following sequence of polynomial functions of t:

$$L_n(t) = (-1)^n \left\{ t^n - n^2 t^{n-1} + \frac{n^2(n-1)^2}{2!} t^{n-2} - \frac{n^2(n-1)^2(n-2)^2}{3!} t^{n-3} + \ldots + (-1)^n n! \right\}, \quad n = 0, 1, 2, \ldots$$

For example, $L_0(t) = 1$, $L_1(t) = -(t-1)$, $L_2(t) = t^2 - 4t + 2$

$$L_3(t) = -(t^3 - 9t^2 + 18t - 6)$$

$$L_4(t) = t^4 - 16t^3 + 72t^2 - 96t + 24$$

and so on. These polynomials are known as the polynomials of Laguerre and the differential equation $tx_{tt} + (1-t)x_t + nx = 0$ is known as Laguerre's differential equation of index n. The Laplace Transform, over the half-plane $c > 0$, of $L_n(t)\,u(t)$ is $\frac{n!}{p}(1 - \frac{1}{p})^n$. The restriction of λ to the non-negative integers 0, 1, 2, ..., which is imposed by the requirement that $e^{-\alpha|t|} k_{\lambda}(t)$, $0 < \alpha < 1$, be bounded over $-\infty < t < \infty$, furnishes the quantisation of the radial coordinate in the wave-mechanical theory of the hydrogen atom.

It is easy to show, on availing ourselves of the fact that the Laplace Transform of $L_n(t)\,u(t)$, over the half-plane $c > 0$, is $\frac{n!}{p}(1-\frac{1}{p})^n$ that the function $x_n = L_n(t)$ of the unrestricted real variable t satisfies the linear second-order difference equation

$$x_{n+1} + (t-2n-1)x_n + n^2 x_{n-1} = 0, \quad n = 1, 2, 3, \ldots$$

Indeed the Laplace Transform of $tL_n(t)\,u(t)$, over the half-plane $c > 0$, is $\frac{n!}{p^2}(1-\frac{1}{p})^n - \frac{n(n!)}{p^3}(1-\frac{1}{p})^{n-1}$ and this appears, on writing $\frac{1}{p^2} = \frac{1}{p}\left\{1-(1-\frac{1}{p})\right\}$, $\frac{1}{p^3} = \frac{1}{p}\left\{1-2(1-\frac{1}{p})+(1-\frac{1}{p})^2\right\}$, in the form $-\frac{(n+1)!}{p}(1-\frac{1}{p})^{n+1} + \frac{(2n+1)(n!)}{p}(1-\frac{1}{p})^n$ $-\frac{n^2\,(n-1)!}{p}(1-\frac{1}{p})^{n-1}$ which is the Laplace Transform, over the half-plane $c > 0$, of $-L_{n+1}(t)\,u(t) + (2n+1)\,L_n(t)\,u(t) - n^2L_{n-1}(t)\,u(t)$. Hence, by virtue of the uniqueness theorem, $tL_n(t)\,u(t) = -L_{n+1}(t)\,u(t)$ $+ (2n+1)\,L_n(t)\,u(t) - n^2L_{n-1}(t)\,u(t)$ or, equivalently, $tL_n(t) = -L_{n+1}(t) + (2n+1)\,L_n(t) - n^2L_{n-1}(t)$, $0 \leq t < \infty$. Since this equation connecting polynomials is valid for more than a finite number of values of t it must be an identity so that, on collecting terms,

$$L_{n+1}(t) + (t-2n-1)\,L_n(t) + n^2L_{n-1}(t) = 0, \quad -\infty < t < \infty$$

Similarly, we can show that $L_n(t)$ is the product of the nth derivative, $D^n\left[t^n \exp(-t)\right]$, of $t^n \exp(-t)$ by exp t. Indeed the Laplace Transform, over the half-plane $c > -1$, of $\exp(-t)\,u(t)$ is $\frac{1}{p+1}$ and so the Laplace

Transform, over the half-plane $c > -1$, of $t^n \exp(-t)\, u(t)$ is $\frac{n!}{(p+1)^{n+1}}$.
Since $t^n \exp(-t)$ is zero, together with its derivatives up to the order $n - 1$, inclusive, at $t = 0$, if follows that the Laplace Transform, over the half-plane $c > -1$, of $D^n\left[t^n \exp(-t)\right] u(t)$ is $\frac{n!p^n}{(p+1)^{n+1}}$ so that, by virtue of the Translation Theorem, the Laplace Transform, over the half-plane $c > 0$, of $\exp(t)\, D^n\left[t^n \exp(-t)\right] u(t)$ is $\frac{n!(p-1)^n}{p^{n+1}} = \frac{n!}{p}(1 - \frac{1}{p})^n$. Hence $\exp(t)\, D^n\left[t^n \exp(-t)\right] = L_n(t)$, $0 \leq t < \infty$, which implies, since both sides of the equation are polynomial functions of t, that

$$L_n(t) = \exp(t)\, D^n\, t^n \exp(-t) \quad , \; -\infty < t < \infty$$

Let us denote by $L^*_n(t)$ the polynomial function of t, of degree n, obtained by replacing each coefficient of $L_n(t)$ by its absolute value. For example,

$$L_0^*(t) = 1, \; L_1^*(t) = t + 1, \; L_2^*(t) = t^2 + 4t+2$$

$$L^*_3(t) = t^3 + 9t^2 + 18t + 6$$

and so on, $L^*_n(t)$ being $L_n(-t)$, $n = 0, 1, 2, \ldots$ Then the Laplace Transform of $L^*_n(t)\, u(t)$, over the half-plane $c > 0$, being obtained from the Laplace Transform of $L_n(t)\, u(t)$ by replacing each coefficient in the development of this latter as a power series in $\frac{1}{p}$ by its absolute value, is $\frac{n!}{p}(1 + \frac{1}{p})^n$. Since $L^*_n(t) = L_n(-t)$, $x^*_n = L^*_n(t)$ satisfies the differential equation

$$t(x^*_n)_{tt} + (1+t)\,(x^*_n)_t - n\, x^*_n = 0$$

which we term the modified Laguerre differential equation of index n.

Similarly x^*_n satisfies the linear second order difference equation

$$x^*_{n+1} - (t + 2n + 1)\, x^*_n + n^2\, x^*_{n-1} = 0$$

and

$$L^*_n(t) = \exp(-t)\, D^n\, (t^n \exp t), \quad n = 0,\ 1,\ 2,\ \ldots$$

Let x be any non-negative real number less than 1 and let us consider the non-negative double series (u_j^k) where

$$u_j^k = \binom{k}{j} \frac{t^j x^k}{j!}, \quad 0 \leq t < \infty,\ 0 \leq x < 1,\ j = 0,\ 1,\ 2, \ldots,\ k = 0,\ 1,\ 2, \ldots$$

Then the $\infty \times \infty$ matrix which has u_j^k as the element in its (k+1)st row and (j+1)st column is triangular with zeros above the diagonal, the binomial coefficient $\binom{k}{j}$ being zero if $k < j$. The sum of the elements in the (k+1)st row of this $\infty \times \infty$ triangular matrix is $\frac{x^k}{k!} L^*_k(t)$ while the sum of the elements in the (j+1)st column is $\frac{t^j x^j}{j!} \left\{ 1 + (j+1)\, x + \frac{(j+2)\,(j+1)}{2!}\, x^2 + \ldots \right\} = \frac{t^j x^j}{j!(1-x)^{j+1}}$. Hence the sum by columns of the non-negative double series (u^k_j) exists with the value $\frac{1}{1-x} \exp \frac{tx}{1-x}$ and this implies that the sum by rows, namely, $\sum_{k=0}^{\infty} \frac{x^k}{k!} L_k^*(t)$, exists with the same value. Thus

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} L_k^*(t) = \frac{1}{1-x} \exp \frac{tx}{1-x}, \quad 0 \leq x < 1,\ 0 \leq t < \infty$$

If we multiply u^k_j by $(-1)^j$ we obtain a new double series (v^k_j), where $v^k_j = \binom{k}{j} \frac{(-t)^j x^k}{j!}$, $j = 0,\ 1,\ 2,\ \ldots,\ k = 0,\ 1,\ 2,\ \ldots,$

This double series is no longer non-negative over $0 \leq t < \infty$ but each of the two double series $(u_j^k \pm v_j^k)$ is non-negative and the sum by columns of each of these two non-negative double series exists. Since $v_j^k = \frac{1}{2}(u_j^k + v_j^k) - \frac{1}{2}(u_j^k - v_j^k)$ it follows that, despite the fact that the double series (v_j^k) is not non-negative, its sum by rows exists and has the same value as its sum by columns.

Thus

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} L_k(t) = \frac{1}{1-x} \exp \frac{-tx}{1-x}$$

which is the same thing as saying that the previous relation

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} L^*_k(t) = \frac{1}{1-x} \exp \frac{tx}{1-x}$$

is valid over the entire t - axis, $-\infty < t < \infty$. The same argument shows that we may change the sign of x and so

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} L_k{}^*(t) = \frac{1}{1-x} \exp \frac{tx}{1-x}, \quad -1 < x < -1, \quad -\infty < t < \infty$$

or, equivalently,

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} L_k(t) = \frac{1}{1-x} \exp \frac{-tx}{1-x}, \quad -1 < x < -1, \quad -\infty < t < \infty.$$

We conclude with the remark that Laguerre's differential equation $t\,x_{tt} + (1-t)x_t + nx = 0$ possesses a second solution, linearly independent of $L_n(t)$, but this solution does not have as much importance as $L_n(t)$ in applications to physical problems since it possesses a logarithmic singularity at $t = 0$. For example, when $n = 0$, this second solution may be taken to be the indefinite integral $\int^t \frac{\exp s}{s} ds$, of $\frac{\exp t}{t}$. If this second solution is required it may be obtained by writing $x = yL_n(t)$ in Laguerre's differential equation; on doing this we find that y_t is a constant times the quotient of exp t by $t\ L_n^2(t)$.

Lecture 11

Bessel's Differential Equation

As a second application of the Laplace Transformation to ordinary linear differential equations with variable coefficients we consider the second-order linear differential equation

$$t^2 x_{tt} + t x_t + (t^2 - n^2)\, x = 0$$

which is known as Bessel's differential equation, of index n, $n = 0, 1, 2, \ldots$, being a non-negative integer. It is not difficult to see that $x_n(t) = \frac{1}{\pi} \int_0^\pi \cos(t \sin\theta - n\theta)\, d\theta$ is a solution of this differential equation. Indeed,

$$\left\{x_n(t)\right\}_t = -\frac{1}{\pi} \int_0^\pi \sin(t \sin\theta - n\theta)(\sin\theta)\, d\theta$$

$$\left\{x_n(t)\right\}_{tt} = -\frac{1}{\pi} \int_0^\pi \cos(t \sin\theta - n\theta)(\sin^2\theta)\, d\theta$$

so that $\left\{x_n(t)\right\}_{tt} + x_n(t) = \frac{1}{\pi} \int_0^\pi \cos(t \sin\theta - n\theta)(\cos^2\theta)\, d\theta$. A simple integration by parts yields

$$\left\{x_n(t)\right\}_t = \frac{1}{\pi} \sin(t \sin\theta - n\theta)\cos\theta \Big|_0^\pi - \frac{1}{\pi}\int_0^\pi \cos(t\sin\theta - n\theta)(\cos\theta)(t\cos\theta - n)\, d\theta$$

$$= -\frac{t}{\pi}\int_0^\pi \cos(t\sin\theta - n\theta)(\cos^2\theta)\, d\theta + \frac{n}{\pi}\int_0^\pi \cos(t\sin\theta - n\theta)(\cos\theta)\, d\theta$$

so that

$$t\left\{x_n(t)\right\}_{tt} + \left\{x_n(t)\right\}_t + t\, x_n(t) = \frac{n}{\pi}\int_0^\pi \cos(t\sin\theta - n\theta)(\cos\theta)\, d\theta$$

Now, $\int_0^{\pi} \left\{\cos(t\sin\theta - n\theta)\right\} (t\cos\theta - n)d\theta = \sin(t\sin\theta - n\theta) \Big|_0^{\pi} = 0$

and so $t\int_0^{\pi}\left\{\cos(t\sin\theta - n\theta)\right\}(\cos\theta)d\theta = n\int_0^{\pi}\left\{\cos(t\sin\theta - n\theta)\right\}d\theta = n\pi\, x_n(t)$,

so that

$$t^2\left\{x_n\right\}_{tt} + t\left\{x_n(t)\right\}_t + t^2 x_n(t) = n^2 x_n(t)$$

which proves that $x_n(t)$ is a solution of Bessel's equation of index n. We denote this solution by $J_n(t)$ and observe that $\left|J_n(t)\right| \leq 1$, $\left|\left\{J_n(t)\right\}_t\right| \leq 1$, $\left|\left\{J_n(t)\right\}_{tt}\right| \leq 1$ so that the right-sided function $h_n(t) = J_n(t)\, u(t)$ is a solution, save, possibly, at $t = 0$, where $h_n(t)$ may not be differentiable, of Bessel's differential equation of index n, this solution being such that the three piecewise continuous right-sided functions $h_n(t)$, $\left\{h_n(t)\right\}_t$, $\left\{h_n(t)\right\}_{tt}$ all possess Laplace Transforms which are analytic functions of the complex variable p over the half-plane $c > 0$. On denoting the Laplace Transform of $h_n(t)$, over the half-plane $c > 0$, by f we have $|f| \leq \int_0^{\infty} \exp(-ct)dt = \frac{1}{c}$, $c > 0$, so that $|f|$ tends to zero as $c \to \infty$. Furthermore, $L\left[\left\{h_n(t)\right\}_t\right] = pf - h_n(+0)$, $L\left[\left\{h_n(t)\right\}_{tt}\right] = p^2 f - p h_n(+0) - \left\{h_n\right\}_t(+0)$, $L(th_n) = -f_p$, $L(t^2 h_n) = f_{pp}$, $L\left[t\left\{h_n(t)\right\}_t\right] = -f - pf_p$, $L\left[t\left\{h_n\right\}_{tt}\right] = -p^2 f_p - 2pf + h_n(+0)$, $L\left[t^2\left\{h_n(t)\right\}_{tt}\right] = p^2 f_{pp} + 4pf_p + 2f$, all these equations being valid over the half-plane $c > 0$. Since $t^2\left\{h_n(t)\right\}_{tt} + t\left\{h_n(t)\right\}_t + (t^2 - n^2)\, h_n(t) = 0$, $t \neq 0$, it follows that, over the half-plane $c > 0$,

$$(1 + p^2) f_{pp} + 3pf_p + (1-n^2) f = 0$$

This homogeneous second-order linear differential equation is readily solved by setting $p = \sinh z$ so that $f_z = (\cosh z) f_p$, $f_{zz} = (\cosh^2 z) f_{pp} + (\sinh z) f_p$. Thus $f_p = (\operatorname{sech} z) f_z$, $(1+p^2) f_{pp} = f_{zz} - (\tanh z) f_z$, so that $f_{zz} + 2(\tanh z) f_z + (1-n^2) f = 0$. Writing, finally, $(\cosh z) f = f'$, we have $(\cosh z) f_z + (\sinh z) f = f'_z$, $(\cosh z) f_{zz} + 2(\sinh z) f_z + (\cosh z) f = f'_{zz}$ so that $f'_{zz} = n^2 (\cosh z) f = n^2 f'$. If, then, $n = 1, 2, 3, \ldots$, f' is a linear combination, with constant coefficients, of $\exp nz$ and $\exp(-nz)$. If x and y are the real and imaginary parts, respectively, of z, $c = \sinh x \cos y$ and so, if $-\frac{\pi}{2} < y < \frac{\pi}{2}$, $c \longrightarrow \infty$ as $x \longrightarrow \infty$. Furthermore, the quotient of $\exp(nx)$ by $\cosh x \longrightarrow 2$ as $x \longrightarrow \infty$, if $n = 1$, and $\longrightarrow \infty$, as $x \longrightarrow \infty$, if $n = 2, 3, \ldots$, while the quotient of $\exp(-nx)$ by $\cosh x \longrightarrow 0$ as $x \longrightarrow \infty$, $n = 1, 2, 3, \ldots$ Since $f = f'/\cosh z$ tends to zero as $c \longrightarrow \infty$ it follows that the coefficient of $\exp(nz)$ in the linear combination of $\exp(nz)$ and $\exp(-nz)$ which furnishes f' must be zero. Thus, if $n = 1, 2, 3, \ldots$, $f(p)$ is a constant times $\exp(-nz)$ divided by $\cosh z$, i.e., a constant times $(\cosh z - \sinh z)^n = \left\{ (1+p^2)^{1/2} - p \right\}^n$ divided by $(1+p^2)^{1/2}$. On the other hand, if $n = 0$, f' is a linear function of z so that f is of the form $\frac{a + bz}{\cosh z}$ where a and b are constants. Hence at any point $c = \sinh x$ of the positive real axis in the complex p-plane, $cf = \dfrac{c(a+bx)}{\cosh x} = \tanh x\,(a+bx)$ and this is not bounded at $x = \infty$ unless $b = 0$ (since $\tanh x \longrightarrow 1$ as $x \to \infty$). Thus, for all non-negative integral values, 0, 1, 2, ..., of n, f is a constant times exp(-nz) divided by cosh z, i.e., a constant, C, times $\left\{ (1+p^2)^{1/2} - p \right\}^n / (1+p^2)^{1/2}$. Since this function of the complex variable p is zero at $p = \infty$ and analytic over the neighborhood $|p| > 1$ of $p = \infty$, its development as a power series in $\frac{1}{p}$ starting out with $\dfrac{C}{2^n p^{n+1}}$ ($((1+p^2)^{1/2} - p$ being $p(1+p^{-2})^{1/2} - p = \dfrac{1}{2p} + \dots$) $h_n(t)$ is a function of exponential type, namely, the product of u(t) by an everywhere convergent power series in t which starts out with the term $C\dfrac{t^n}{2^n . n!}$. Thus $J_n(t)$ vanishes, together with its derivatives up to the (n-1)st, inclusive, at $t = 0$, while the nth derivative of $J_n(t)$ does not vanish at $t = 0$.

The nth derivative of $J_n(t) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos(t\sin\theta - n\theta)\,d\theta$, at $t = 0$, is $\dfrac{(-1)^k}{\pi}\displaystyle\int_0^{\pi} \sin(n\theta)\sin^n\theta\,d\theta$, if $n = 2k+1$ is odd, and $\dfrac{(-1)^k}{\pi}\displaystyle\int_0^{\pi} \cos(n\theta)\sin^n\theta\,d\theta$ is $n = 2k$ is even. In the first case, $\sin^n\theta = (2i)^{-n}\left\{\exp(i\theta) - \exp(-i\theta)\right\}^n$ is a linear combination of sines of odd integral multiples of θ, the coefficient of $\sin n\theta$ in this linear combination being $(2i)^{-n+1} = \dfrac{(-1)^k}{2^{n-1}}$ while, in the second case, $\sin^n\theta$ is a linear combination of cosines of even integral multiples of θ, the coefficient of $\cos n\theta$ in this linear combination being $2(2i)^{-n} = \dfrac{(-1)^k}{2^{n-1}}$. Now $\dfrac{1}{\pi}\displaystyle\int_0^{\pi} (\sin n\theta)(\sin m\theta)\,d\theta = 0$, if the odd numbers m and n are unequal, and $\dfrac{1}{\pi}\displaystyle\int_0^{\pi} (\cos n\theta)(\cos m\theta)\,d\theta = 0$, if the even numbers m and n are unequal, while both $\frac{1}{\pi}\int_0^{\pi}\left\{\sin n\theta\right\}^2 d\theta$, $n = 1, 3, 5, \ldots$ and $\frac{1}{\pi}\int_0^{\pi}\left\{\cos n\theta\right\}^2 d\theta$, $n = 2, 4, \ldots$ have the common value $\frac{1}{2}$. Hence, if n is any positive integer, the nth derivative of $J_n(t)$ has, at $t = 0$, the value $\frac{1}{2^n}$ and this is also true when $n = 0$, the value of $J_0(t) = \frac{1}{\pi}\int_0^{\pi}\cos(t\sin\theta)d\theta$ at $t = 0$ being 1. Thus the multiplicative constant C is 1 and

$$L\left\{J_n(t)\, u(t)\right\} = \frac{\exp(-nz)}{\cosh z}, \quad p = \sinh z,\ n = 0, 1, 2, \ldots, c > 0$$

$$= \frac{\left\{(1+p^2)^{1/2} - p\right\}^n}{(1+p^2)^{1/2}}, \quad c > 0$$

In particular, $L\left\{J_0(t)\, u(t)\right\} = \frac{1}{(1+p^2)^{1/2}}$, $L\left\{J_1(t)\, u(t)\right\} = 1 - \frac{p}{(1+p^2)^{1/2}}, c > 0.$

Since $J_1(0) = 0$, it follows that $L\left\{t\, J_1(t)\, u(t)\right\} = \left\{\frac{p}{(1+p^2)^{1/2}}\right\}_p$

$= \frac{1}{(1+p^2)^{3/2}}$, $c > 0$. In general, the Laplace Transform, over the half-plane $c > 0$, of $t^n J_n(t)\, u(t)$ is simpler, when $n > 0$, than that of $J_n(t)\, u(t)$. To obtain this Laplace Transform we first set up the differential equation which is satisfied by $y = t^n x$, where $x = J_n(t)$. Thus $x = t^{-n}y$, $x_t = t^{-n}y_t - nt^{-n-1}y$, $x_{tt} = t^{-n}y_{tt} - 2nt^{-n-1}y_t + n(n+1)t^{-n-2}y$ and, since $t^2x_{tt} + tx_t + (t^2 - n^2)x = 0$, it follows that $ty_{tt} - (2n-1)y_t + ty = 0$. Denoting $L\left\{y\, u(t)\right\}$ by g and noting that $y(0) = 0$, since $n > 0$, we have $L\left[\left\{y\, u(t)\right\}_t\right] = pg$, $L\left[\left\{y\, u(t)\right\}_{tt}\right] = p^2g - y_t(0)$, so that $L\left[t\left\{y\, u(t)\right\}_{tt}\right] = -p^2g_p - 2pg$ and, since $L\left\{t\, y\, u(t)\right\} = -g_p$, we obtain $(1+p^2)g_p + (2n+1)pg = 0$ so that g is a constant times $(1+p^2)^{-n-(1/2)}$

and, since the power series development of y starts out with the term $\frac{t^{2n}}{2^n(n!)}$, the multiplying constant is $\frac{(2n)!}{2^n(n!)} = (2n-1)(2n-3)\dots 3.1$:

$$L\left\{t^n J_n(t)\, u(t)\right\} = \frac{(2n-1)(2n-3)\dots \quad 3.1}{(1+p^2)^{n+(1/2)}}, \; c > 0, \; n = 0, 1, 2\dots$$

(it being understood that, when $n = 0$, $(2n-1)\dots \; 3.1$ is replaced by 1). Since the development of $\frac{(2n-1)\cdots \; 3.1}{(1+p^2)^{n+(1/2)}}$ as a power series in $\frac{1}{p}$ is $(2n-1)\cdots \; 3.1\left\{\frac{1}{p^{2n+1}} - \frac{n+\frac{1}{2}}{p^{2n+3}} + \frac{(n+\frac{1}{2})(n+\frac{3}{2})}{2!\,p^{2n+5}} - \dots\right\}$

the development of $t^n J_n(t)$ as a power series in t is

$$(2n-1)\dots \; 3.1\left\{\frac{t^{2n}}{(2n)!} - \frac{(n+\frac{1}{2})\,t^{2n+2}}{(2n+2)!} + \frac{(n+\frac{1}{2})(n+\frac{3}{2})\,t^{2n+4}}{(2n+4)!\,2!} - \dots\right\}$$

and so the development of $J_n(t)$ as a power series in t is

$$\frac{1}{n!}\left(\frac{t}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{t}{2}\right)^{n+2} + \frac{1}{(n+2)!\,2!}\left(\frac{t}{2}\right)^{n+4} - \frac{1}{(n+3)!\,3!}\left(\frac{t}{2}\right)^{n+6} + \dots$$

For example, $J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \dots$

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^2.4} + \frac{t^5}{2^2.4^2.6} - \dots$$

so that $J_1(t)$ is the negative of the derivative of $J_0(t)$ with respect to t; this result could have been predicted, without computation, from the fact that $L\left\{J_1(t)\, u(t)\right\}$, namely $1 - \frac{p}{(1+p^2)^{1/2}}$, is the negative of $p\, L\left\{J_0(t)\, u(t)\right\} - J_0(0)$.

Exercise. Show that the product of $u(t)$ by $J_n(t)/t^n$ is a function of exponential type whose Laplace Transform, over the half-plane $c > 0$, is $\frac{(\cosh z)^{2n-1}}{2^{n-1}(n-1)!}\int_z^{\infty} (\operatorname{sech} v)^{2n}\, dv$, $p = \sinh z$, and, in particular, that the Laplace Transform, over the half-plane $c > 0$, of $\frac{J_1(t)}{t}$ u(t) is $\cosh z - \sinh z = (1+p^2)^{1/2} - p$.

Hint. The differential equation satisfied by $w = t^{-n} x$, where $x = J_n(t)$, is $tw_{tt} + (2n+1)w_t + tw = 0$ and it follows, since $w(0) = \frac{1}{2^n . n!}$ that the differential equation satisfied by $L(w\, u(t)) = g'$ is $(1+p^2)g'_p - (2n-1)pg' = -\frac{1}{2^{n-1}.(n-1)!}$. The solution of this differential equation which is zero at $p = \infty$ is $g' = (1+p^2)^{n - \frac{1}{2}} s$, where $(1+p^2)^{n + \frac{1}{2}} s_p = -\frac{1}{2^{n-1}.(n-1)!}$ and s is zero at $p = \infty$. Thus $s = \frac{1}{2^{n-1}.(n-1)!} \int_p^\infty \frac{dq}{(1+q^2)^{n\ 1/2}} = \frac{1}{2^{n-1}.(n-1)!} \int_z^\infty (\operatorname{sech} v)^{2n} dv$, $q = \sinh v$, $p = \sinh z$.

If we replace each coefficient of the everywhere convergent power series in t whose sum is $J_n(t)$ by its absolute value we obtain a new everywhere-convergent power series in t whose sum, $J_n^*(t)$, is the product of $J_n(it)$ by $(-i)^n$. $J_n^*(t)$ is termed the modified Bessel function, of the first kind, of index n and is usually denoted by $I_n(t)$ but we shall use, for the present, the notation $J_n^*(t)$ to recall the manner in which the modified Bessel function, $J_n^*(t)$ or $I_n(t)$, is derived from $J_n(t)$. The expansion, over the neighborhood $|p| > 1$ of $p = \infty$, of the Laplace Transform of $J_n^*(t)$ u(t) as a power series in $\frac{1}{p}$ is obtained from the corresponding expansion of the Laplace Transform of $J_n(t)$ u(t) by replacing each coefficient of the latter expansion by its absolute value.

Now the expansion of the Laplace Transform of $J_n(t)\,u(t)$ as a power series in $\frac{1}{p}$ is of the form $f_n(p) = \frac{a_n}{p^{n+1}} + \frac{a_{n+2}}{p^{n+3}} + \frac{a_{n+4}}{p^{n+5}} + \dots$ where $a_n = \frac{1}{2^n}$ and the coefficients $a_n, a_{n+2}, \dots$ are alternately positive and negative. Hence $f_n(ip) = (-i)^{n+1}\left\{\frac{a_n}{p^{n+1}} - \frac{a_{n+2}}{p^{n+3}} + \frac{a_{n+4}}{p^{n+5}} - \dots\right\} = (-i)^{n+1}\left\{\frac{|a_n|}{p^{n+1}} + \frac{|a_{n+2}|}{p^{n+3}} + \dots\right\}$ so that $f_n^*(p)$, the sum, over $|p| > 1$, of the power series $\frac{|a_n|}{p^{n+1}} + \frac{|a_{n+2}|}{p^{n+3}} + \dots$, is $i^{n+1} f_n(ip) = \frac{i^{n+1}\left\{(1-p^2)^{1/2} - ip\right\}^n}{(1-p^2)^{1/2}}$. Writing $(1-p^2)^{1/2} = i(p^2-1)^{1/2}$ this takes the form $\frac{\left\{p - (p^2-1)^{1/2}\right\}^n}{(p^2-1)^{1/2}}$, which is $\frac{\exp(-nz^*)}{\sinh z^*}$ where $\cosh z^* = p$. Thus, side by side with the result

$$L\left\{J_n(t)\,u(t)\right\} = \frac{\exp(-nz)}{\cosh z}, \quad p = \sinh z,\ c > 0$$

we have the result

$$L\left\{J_n^*(t)\,u(t)\right\} = \frac{\exp(-nz^*)}{\sinh z^*}, \quad p = \cosh z^*,\ c > 0.$$

<u>Exercise 1.</u> Show that $x_n^* = J_n^*(t) = I_n(t)$ satisfies the modified Bessel equation, of index n, $t^2(x_n^*)_{tt} + t(x_n^*)_t - (t^2 + n^2)\,x_n^* = 0$

<u>Hint.</u> Denoting $J_n(t)$ by $x_n(t)$, $x_n(z)$, where z is a complex variable, satisfies the differential equation $z^2(x_n)_{zz} + z(x_n)_z + (z^2 - n^2)\,x_n = 0$. Thus along the imaginary axis $z = iy$ in the complex z-plane, along which $(x_n)_y = i(x_n)_z$, $(x_n)_{yy} = i^2(x_n)_{zz}$,

we have $y^2(x_n)_{yy} + y(x_n)_y - (y^2 + n^2)\, x_n = 0$ and, since $x_n(iy)$ is a constant times $x_n^*(y)$ it follows that $y^2(x_n^*)_{yy} + y(x_n^*)_y - (y^2+n^2)\, x_n^* = 0$, $-\infty < y < \infty$.

Exercise 2. Show that if $n = 1, 2, \ldots$ both $|J_n(z)|$ and $|J_n^*(z)|$ are dominated over the entire complex z-plane by a constant times $|z|^{-n} \exp\left[(1+\delta)\,|z|\right]$ where δ is an arbitrary positive number.

Hint. $L\left\{t^n J_n(t)\, u(t)\right\} = \dfrac{(2n-1)\ldots 3.1}{(1+p^2)^{n+(1/2)}}$ and the development of $(1+p^2)^{-n-\frac{1}{2}}$ as a power series in $\frac{1}{p}$ converges at $p = 1 + \delta$.

Note. If we substitute $p = 1$ in this series we obtain a series whose terms steadily increase in numerical value and which, therefore, fails to converge.

Exercise 3. Show that both $|J_0(z)|$ and $|J_0^*(z)| = |I_0(z)|$ are dominated, over the entire complex z-plane by $\exp|z|$.

Hint. The development of $(1+p^2)^{-1/2}$ as a power series in $\frac{1}{p}$ reduces, when $p = 1$, to $1 - \frac{1}{2} + \frac{1.3}{2.4} - \ldots$ which converges; furthermore each of its terms is numerically ≤ 1.

Lecture 12

The Recurrence, and other, Relations Connecting Bessel Functions

The Laplace Transforms, over the half-plane $c > 0$, of the two right-sided functions $t^{\pm\frac{n}{2}} J_n(2t^{1/2})u(t)$, each of which is of exponential type, are particularly simple. To obtain them we set $2t^{1/2} = s$, $t \geq 0$, and we denote $J_n(s)$ by $x(s)$ and write $x(s) = y(t)$. Then $y_t = x_s t^{-1/2}$ so that $x_s = t^{1/2} y_t = \frac{1}{2} s y_t$ and $x_{ss} = \frac{1}{2} y_t + \frac{1}{4} s^2 y_{tt} = \frac{1}{2} y_t + t y_{tt}$. Since $s^2 x_{ss} + s x_s + (s^2 - n^2) x = 0$ it follows that $t^2 y_{tt} + t y_t + (t - \frac{n^2}{4}) y = 0$. We next set $y = t^{-\frac{n}{2}} v$ so that $y_t = t^{-\frac{n}{2}} v_t - \frac{n}{2} t^{-\frac{n}{2} - 1} v$, $y_{tt} = t^{-\frac{n}{2}} v_{tt} - n t^{-\frac{n}{2} - 1} v_t + \frac{n}{2}(\frac{n}{2} + 1) v$ and find that v satisfies the differential equation $t v_{tt} + (1-n) v_t + v = 0$. All three of the right-sided functions $v(t)\, u(t)$, $v_t(t)\, u(t)$, $v_{tt}(t) u(t)$ possess Laplace Transforms over the half-plane $c > 0$ and, on denoting by g the first of these three Laplace Transforms, we have $L\left[v_t(t)\, u(t)\right] = pg - v(0)$, $L\left[v_{tt}(t)\, u(t)\right] = p^2 g - p v(0) - v_t(0)$ where $v(0) = 0$ if $n = 1, 2, 3, \ldots$, while $v(0) = 1$ if $n = 0$. Since $L\left[t v_{tt}(t)\, u(t)\right] = -2pg - p^2 g_p + v(0)$, g satisfies the differential equation $p^2 g_p + \left\{(n+1) p - 1\right\} g - n v(0) = 0$ or, equivalently, since $n v(0) = 0$, $n = 0, 1, 2, \ldots$, $p^2 g_p + \left\{(n+1) p - 1\right\} g = 0$. Thus g is a constant times $\exp(-\frac{1}{p})/p^{n+1}$ and, since the development of $t^{\frac{n}{2}} J_n(2t^{1/2})$ as a power series in t starts out with the term $\frac{1}{n!} t^n$, the multiplying constant is 1. Thus

$$L\left[t^{n/2} J_n(2t^{1/2})\, u(t)\right] = \frac{\exp(-\frac{1}{p})}{p^{n+1}} \ , \ c > 0,\ n = 0,\ 1,\ 2,\ \ldots$$

From this we obtain the Laplace Transform, over the half-plane $c > 0$, of $L\left[t^{-\frac{n}{2}} J_n(2t^{1/2})\, u(t)\right]$ by integrating n times, with respect to p, from p to ∞ ($t^{-n/2} J_n(2t^{1/2})$ being the quotient of $t^{n/2} J_n(2t^{1/2})$ by t^n). Writing

$$\frac{\exp(-\frac{1}{p})}{p^{n+1}} = \frac{1}{p^{n+1}} - \frac{1}{p^{n+2}} + \frac{1}{2!p^{n+3}} - \ldots$$

we obtain $\frac{1}{n!p} - \frac{1}{(n+1)!p^2} + \frac{1}{(n+2)!p^3} - \ldots$, which is the part involving negative powers of p in the Laurent development of $(-1)^n p^{n-1} \exp(-\frac{1}{p})$ over $0 < |z| < \infty$, i.e., the finite complex z-plane punctured at the origin:

$$L\left[t^{-n/2} J_n(2t^{1/2})\, u(t)\right] = \frac{1}{n!p} - \frac{1}{(n+1)!p^2} + \frac{1}{(n+2)!p^3} - \ldots,$$

$$c > 0,\ n = 0,\ 1,\ 2,\ \ldots$$

Thus, multiplication of the Laplace Transform of $t^{n/2} J_n(2t^{1/2})\, u(t)$, $n = 1, 2, 3, \ldots$, by p is equivalent to replacing n by n - 1 while multiplication of the Laplace Transform of $t^{-\frac{n}{2}} J_n(2t^{1/2}) u(t)$, $n = 0, 1, 2, \ldots$, by p, followed by subtraction of $\frac{1}{n!}$, the value of $t^{-n} J_n(2t^{1/2})$ at $t = 0$, is equivalent to replacing n by n+1 and changing the sign of the Laplace Transform. This implies, by virtue of the uniqueness theorem, that

a) The derivative of $t^{n/2} J_n(2t^{1/2})$, $n > 0$, with respect to t is $t^{\frac{n-1}{2}} J_{n-1}(2t^{1/2})$

and

b) The derivative of $t^{-\frac{n}{2}} J_n(2t^{1/2})$, $n \geq 0$, with respect to t is $-t^{-\frac{n+1}{2}} J_{n+1}(2t^{1/2})$.

Since differentiation with respect to t is the same as differentiation with respect to $s = 2t^{1/2}$ followed by multiplication by $t^{-1/2} = \frac{2}{s}$, these results can be expressed as follows:

a') The derivative of $(\frac{s}{2})^n J_n(s)$, $n = 1, 2, 3, \ldots,$ with respect to s, is $(\frac{s}{2})^n J_{n-1}(s)$

b') The derivative of $(\frac{s}{2})^{-n} J_n(s)$, $n = 0, 1, 2, \ldots,$ with respect to s, is $-(\frac{s}{2})^{-n} J_{n+1}(s)$.

These results have been proven only for non-negative values of s but they remain valid if s is replaced by an arbitrary complex number z since each side of each of the resulting equations is an analytic function of the complex variable z over the entire finite complex z-plane. In this way we obtain the following two sequences of relations

$$\left.\begin{aligned} z\left\{J_n(z)\right\}_z + n\,J_n(z) &= z\,J_{n-1}(z), \quad n = 1, 2, 3, \ldots \\ z\left\{J_n(z)\right\}_z - n\,J_n(z) &= -z\,J_{n+1}, \quad n = 0, 1, 2, \ldots \end{aligned}\right\} \text{; z arbitrary}$$

and these yield, on addition and subtraction, the sequences of relations,

$$2\left\{J_n(z)\right\}_z = J_{n-1}(z) - J_{n+1}(z),\ n = 1, 2, 3, \ldots$$
$$2n\, J_n(z) = z\left\{J_{n-1}(z) + J_{n+1}(z)\right\},\ n = 1, 2, 3, \ldots \quad ;\ z \text{ arbitrary.}$$

The second of these two sequences of relations furnishes what are known as the recurrence relations connecting the sequence $J_0(z)$, $J_1(z)$, $J_2(z)$, ... of Bessel functions of the first kind. These recurrence relations express the fact that $J_n(z)$ is a solution of the linear second-order difference equation

$$J_{n+1}(z) - \frac{2n}{z} J_n(z) + J_{n-1}(z) = 0;\ n = 1, 2, 3, \ldots;\ z \text{ arbitrary.}$$

Exercise 1. Show that the sequence of modified Bessel functions $I_n(z) = J_n{}^*(z)$, satisfies the two sequences of relations

$$2\left\{I_n(z)\right\}_z = I_{n-1}(z) + I_{n+1}(z),\ 2nI_n(z) = z\left\{I_{n-1}(z) - I_{n+1}(z)\right\},$$
$n = 1, 2, 3, \ldots$.

Exercise 2. Show that the Laplace Transforms, over the half-plane $c > 1$, of $t^{\frac{n}{2}} I_n(2t^{1/2})\, u(t)$ and $t^{-n/2} I_n(2t^{1/2}) u(t)$ are $\exp(\frac{1}{p})/p^{n+1}$ and $\frac{1}{n!p} + \frac{1}{(n+1)!p^2} + \frac{1}{(n+2)!p^3} + \ldots$, respectively.

We have seen that the Laplace Transform, over the half-plane $c > 1$, of $J_n{}^*(t)\, u(t) = I_n(t)\, u(t)$ is $\exp(-nz^*)/\sinh z^*$, where $p = \cosh z^*$. On taking the real and imaginary parts of the equation $p = \cosh z^*$, we obtain $c = (\cosh x^*)\cos y^*$, $\omega = (\sinh x^*)\sin y^*$ where c, ω are the real and imaginary parts, respectively, of p and x^*, y^* are the real and imaginary parts, respectively, of z^*. Thus the relation

$p = \cosh z^*$ maps the strip $-\frac{\pi}{2} < y^* < \frac{\pi}{2}$ in the complex z^*-plane onto the half-plane $c > 0$, the points of this strip in the complex z^*-plane for which x^* has any given value other than zero mapping into the points of the ellipse $\frac{c^2}{(\cosh x^*)^2} + \frac{\omega^2}{(\sinh x^*)^2} = 1$ in the complex p-plane, and the points of the strip for which $x^* = 0$ mapping into the line segment $0 < c \leq 1$, $\omega = 0$. Thus the relation $p = \cosh z^*$ furnishes a one-to-one mapping of the positive half, $x^* > 0$, of the strip $-\frac{\pi}{2} < y^* < \frac{\pi}{2}$ onto the half-plane $c > 0$ with the line-segment $0 < c \leq 1$, $\omega = 0$, removed. Over the positive half, $x^* > 0$, of the strip $-\frac{\pi}{2} < y < \frac{\pi}{2}$, $\left|\exp(-z^*)\right| = \exp(-x^*)$ is < 1 and so, θ being any real number, $\left|\exp\left[-(z^*-i\theta)\right]\right| < 1$ so that the infinite series $1 + 2\exp\left[-2(z^*-i\theta)\right] + 2\exp\left[-4(z^*-i\theta)\right] + 2\exp\left[-6(z^*-i\theta)\right] + \ldots$ converges, its sum being $1 + \frac{2\exp\left[-2(z^*-i\theta)\right]}{1-\exp\left[-2(z^*-i\theta)\right]} = \frac{\cosh(z^*-i\theta)}{\sinh(z^*-i\theta)}$. If $z^* = x^*$ is real and positive it follows, on taking the real parts of the terms of the infinite series, that

$$1 + 2(\cos 2\theta)\exp(-2x^*) + 2(\cos 4\theta)\exp(-4x^*) + \ldots = \frac{1}{2}\left\{\frac{\cosh(x^*-i\theta)}{\sinh(x^*-i\theta)} + \frac{\cosh(x^*+i\theta)}{\sinh(x^*+i\theta)}\right\} = \frac{\sinh 2x^*}{2\left\{\sinh^2 x^* \cos^2\theta + \cosh^2 x^* \sin^2\theta\right\}}$$

$$= \frac{\sinh x^* \cosh x^*}{\cosh^2 x^* - \cos^2\theta}$$

When $z = x^*$ is real and positive, $p = c$ is real and > 1 and

$\exp(-nx^*)/\sinh x^*$ is the value at $p = c$ of the Laplace Transform, $f_n^*(p)$, of $J_n^*(t)\,u(t)$ and so we find, on division by $\sinh x^*$, that

$$f_0^*(c) + 2(\cos 2\theta)\, f_2^*(c) + 2(\cos 4\theta)\, f_4^*(c) + \dots \;=\; \frac{c}{c^2-\cos^2\theta}$$

In particular, when $\theta = 0$,

$$f_0^*(c) + 2f_2^*(c) + 2f_4^*(c) + \dots \;=\; \frac{c}{c^2-1}$$

The coefficients of the development of $f_n^*(c)$ as a power series in $\frac{1}{c}$ are non-negative real numbers and we construct the non-negative double series (u^k_j) where u^k_j, $j = 0, 1, 2, \dots$, $k = 0, 1, 2, \dots$ is the term involving $\frac{1}{c^{2j+1}}$ in $\delta_k f_{2k}^*(c)$, δ_k being 2 if $k = 1, 2, \dots$ while $\delta_0 = 1$. The $\infty \times \infty$ matrix which has u^k_j as the element in its (j+1)st column and (k+1)st row is triangular, with zeros below the diagonal, since $f_{2k}^*(c)$ starts out with the $\frac{1}{c^{2k+1}}$ term. The sum of the elements in the (k+1)st row of the matrix is $\delta_k\, f_{2k}^*(c)$ and so we know that the sum by rows of the non-negative double series (u^k_j) exists with the value $\frac{c}{c^2-1}$ and this implies that the sum by columns exists with the same value. The sum by columns is a power series in $\frac{1}{c}$ whose sum, over the part $c > 1$ of the real axis in the complex p-plane, is $\frac{1}{c} + \frac{1}{c^3} + \dots$ and so the sum of the elements in the (j+1)st column is $\frac{1}{c^{2j+1}}$. Let us now consider the non-negative double series (v^k_j), where $v^k_j = c^{2j+1} \frac{t^{2j}}{(2j)!} u^k_j$, t any real number; the sum of the elements in the (j+1)st column of the $\infty \times \infty$ matrix which has v^k_j as the element in its (j+1) st column and (k+1)st row, $j = 0, 1, 2, \ldots$; $k = 0, 1, 2, \ldots$; is $\frac{t^{2j}}{(2j)!}$ so that the sum by columns of the non-negative double series (v^k_j) exists with the value cosh t and this implies that the sum by rows of the non-negative double series (v^k_j) exists with the value cosh t. The sum of the elements in the (k+1)st row is $\delta_k J^*_{2k}(t)$ and so we have the relation

$$J^*_0(t) + 2 J^*_2(t) + 2 J^*_4(t) + \ldots \quad = \cosh t, \ -\infty < t < \infty$$

which implies, in particular, that $J^*_0(t) \leq \cosh t$ over $-\infty < t < \infty$. We may now apply the argument just given, which depended only on the non-negativeness of δ_k, $k = 0, 1, 2, \ldots$, to the two series

$$f_0^*(c) + \left\{1 + (\cos 2\theta)\right\} f_2^*(c) + \left\{1 + (\cos 4\theta)\right\} f_4^*(c) + \ldots$$

$$\left\{1 - (\cos 2\theta)\right\} f_2^*(c) + \left\{1 - (\cos 4\theta)\right\} f_4^*(c) + \ldots$$

whose sums are $\frac{1}{2}\left\{\frac{c}{c^2-1} + \frac{c}{c^2-\cos^2\theta}\right\}$ and $\frac{1}{2}\left\{\frac{c}{c^2-1} - \frac{c}{c^2-\cos^2\theta}\right\}$, respectively. We find that the two series

$$J_0^*(t) + \left\{1 + (\cos 2\theta)\right\} J_2^*(t) + \left\{1 + (\cos 4\theta)\right\} J_4^*(t) + \ldots$$

$$\left\{1 - (\cos 2\theta)\right\} J_2^*(t) + \left\{1 - (\cos 4\theta)\right\} J_4^*(t) + \ldots$$

converge over $-\infty < t < \infty$, their sums being

$\frac{1}{2}\left\{\cosh t + \cosh(t\cos\theta)\right\}$ and $\frac{1}{2}\left\{\cosh t - \cosh(t\cos\theta)\right\}$, respectively, the Laplace Transform of $\cosh(t\cos\theta)\, u(t)$, at $p = c > 1$, being $\frac{c}{c^2-\cos^2\theta}$. Hence, on subtraction,

$$J_0^*(t) + 2(\cos 2\theta)\, J_2^*(t) + 2(\cos 4\theta)\, J_4^*(t) + \ldots$$
$$= \cosh(t\cos\theta), \quad -\infty < t < \infty, \quad -\infty < \theta < \infty$$

Exercise 3. Show that $(\cos\theta)\, J_1^*(t) + (\cos 3\theta)\, J_3^*(t) + \ldots$
$= \frac{1}{2} \sinh(t\cos\theta), \ -\infty < t < \infty, \ -\infty < \theta < \infty.$

Hint. The infinite series, $\exp\left[-(x^*-i\theta)\right] + \exp\left[-3(x^*-i\theta)\right] + \ldots$, is convergent, with the sum $\frac{1}{2\sinh(x^*-i\theta)}$, if $x^* > 0$.

The facts that $J_{2k}^*(t)$, $k = 0, 1, 2, \ldots$, is a non-negative continuous function of the unrestricted real variable t and that the sum, $\cosh t$, of the everywhere convergent infinite series $J_0^*(t) + 2J_2^*(t) + \ldots$ is everywhere continuous assure us that the convergence of this infinite series is uniform over the closed interval $0 \leq t \leq T$, where T is an arbitrary positive number. Indeed, the remainder, $R_n(t)$, after n terms of this infinite series possesses the following two properties:

1) It is continuous over the interval $0 \leq t \leq T$

2) $0 \leq R_{n'}(t) \leq R_n(t), \ -\infty < t < \infty, \ n' > n$

By virtue of 1) $R_n(t)$ assumes its maximum value, over the interval $0 \leq t \leq T$, at some point, t_n say, of this interval and the infinite sequence of numbers $t_1, t_2, \ldots$ possesses at least one accumulation point, $\bar{t}$ say, in the interval $0 \leq t \leq T$. Since $R_n(t)$, $n = 1, 2, \ldots$, is continuous at $\bar{t}$

we know that $\left| R_n(t) - R_n(\bar{t}) \right|$ is arbitrarily small, say $< \epsilon$, if $|t - \bar{t}|$ is sufficiently small, say $< \delta_n$. Since the infinite series $J_0^*(t) + 2J_2^*(t) + \ldots$ is convergent at $\bar{t}$, $0 \leq R_n(\bar{t}) < \epsilon$ if n is sufficiently large and we denote by N any such sufficiently large value of n so that $0 \leq R_N(\bar{t}) < \epsilon$. Then $0 \leq R_N(t) < 2\epsilon$ if $|t - \bar{t}| < \delta_N$ and this implies that $0 \leq R_n(t) < 2\epsilon$ if $n > N$ and $|t - \bar{t}| < \delta_N$. There exists, since $\bar{t}$ is an accumulation point of the sequence of numbers $t_1, t_2, \ldots$, an n, say n', $> N$ such that $\left| t_{n'} - \bar{t} \right| < \delta_N$ and so $0 \leq R_{n'}(t_{n'}) < 2\epsilon$ which implies, by virtue of the definition of $t_{n'}$, that $0 \leq R_{n'}(t) < 2\epsilon$, $0 \leq t \leq T$, and, hence, that $0 \leq R_n(t) < 2\epsilon$ over $0 \leq t \leq T$ if $n \geq n'$, the choice of n' being independent of t. In other words, the convergence, over the interval $0 \leq t \leq T$, of the infinite series $J_0^*(t) + 2J_2^*(t) + 2J_4^*(t) + \ldots$ is uniform. If z is any complex number, $J_{2k}^*(z)$, $k = 0, 1, 2, \ldots$, is dominated by $J_{2k}^*(|z|)$ and, so the infinite series $J_0^*(z) + 2(\cos\theta) J_2^*(z) + 2(\cos 4\theta) J_4^*(z) + \ldots$, being dominated by the infinite series $J_0^*(|z|) + 2J_2^*(|z|) + \ldots$ converges uniformly over the disc, $0 \leq |z| \leq T$, with center at the origin, in the complex z-plane. Each term of this infinite series is an analytic function of the complex variable z over the disc and so the sum of the infinite series is an analytic function of z over the disc as is also $\cosh(z \cos\theta)$.

Since these two analytic functions of z coincide over the diameter $-T \leq t \leq T$ of the disc they coincide over the entire disc and this implies, since the positive number T is arbitrary, that they coincide over the entire finite complex z-plane. Thus $J_0^*(z) + 2(\cos 2\theta)\, J_2^*(z) + 2(\cos 4\theta)\, J_4^*(z) + \ldots = \cosh(z\cos\theta)$, z arbitrary, $-\infty < \theta < \infty$. Assigning purely imaginary values it to z we obtain

$$J_0(t) - 2(\cos 2\theta)\, J_2(t) + 2(\cos 4\theta)\, J_4(t) - \ldots = \cos(t \cos \theta),$$

$$-\infty < t < \infty, \quad -\infty < \theta < \infty$$

and, in particular, on setting $\theta = 0$,

$$J_0(t) - 2\, J_2(t) + 2\, J_4(t) - \ldots \quad = \cos t, \quad -\infty < t < \infty.$$

On setting $\theta = \frac{\pi}{3}$ in the relation $J_0^*(t) + 2(\cos 2\theta)J_2^*(t) + \ldots = \cosh(t\cos\theta)$ we obtain $J_0^*(t) - J_2^*(t) - J_4^*(t) + 2J_6^*(t) - \ldots = \cosh\left(\frac{t}{2}\right)$ and, on combining this relation with the relation $J_0^*(t) + 2\, J_2^*(t) + \ldots = \cosh t$, we obtain

$$J_0^*(t) + 2\, J_6^*(t) + 2\, J_{12}^*(t) + \ldots \quad = \frac{1}{3}\left\{\cosh t + 2\cosh\left(\frac{t}{2}\right)\right\}, \quad -\infty<t<\infty$$

Thus $\frac{1}{3}\left\{\cosh t + 2\cosh\left(\frac{t}{2}\right)\right\}$ is an upper bound, over $-\infty < t < \infty$, for $J_0^*(t)$, the excess of this upper bound over $J_0^*(t)$ being twice the sum of the infinite series $J_6^*(t) + J_{12}^*(t) + \ldots$.

If $|t| \leq 5$ and $n \geq 6$, $J_n^*(t) = \frac{1}{n!}\left(\frac{t}{2}\right)^n + \frac{1}{(n+1)!}\left(\frac{t}{2}\right)^{n+2}$

$+ \frac{1}{(n+2)!\,2!}\left(\frac{t}{2}\right)^{n+4} + \ldots$ is less than

$$\frac{1}{n!}\left(\frac{t}{2}\right)^n \left\{1 + \frac{1}{n+1}\left(\frac{t}{2}\right)^2 + \frac{1}{(n+1)^2}\left(\frac{t}{2}\right)^4 + \ldots\right\} = \frac{1}{n!}\left(\frac{t}{2}\right)^n \left\{1 - \frac{1}{n+1}\left(\frac{t}{2}\right)^2\right\}^{-1}.$$

For example, when $t = 1$, $J_6^*(1) < (2.3)10^{-5}$, $J_{12}^*(1) < 6.10^{-13}$, and so on, so that $\frac{1}{3}\left\{\cosh 1 + 2\cosh\frac{1}{2}\right\}$ is greater than $J_0^*(1)$, the excess being less than $(4.7)10^{-5}$. Actually, $\frac{1}{3}\left\{\cosh 1+2\cosh\frac{1}{2}\right\}$ $= 1.26611$, $J_0^*(1) = 1.26607$, both to 5 decimals.

On replacing θ by $\theta - \frac{\pi}{2}$ in the relation $J_0^*(t) + 2(\cos 2\theta)\, J_2^*(t)+\ldots$ $= \cosh(t\cos\theta)$, we obtain $J_0^*(t) - 2(\cos 2\theta)\, J_2^*(t) + 2(\cos 4\theta)\, J_4^*(t)-\ldots$ $= \cosh(t\sin\theta)$ and on setting, in turn, $\theta = 0$ and $\theta = \frac{\pi}{3}$ in this relation we obtain the two relations

$$J_0^*(t) - 2J_2^*(t) + 2J_4^*(t) - \ldots = 1$$

$$J_0^*(t) + J_2^*(t) - J_4^*(t) - 2J_6^*(t) - \ldots = \cosh\left(\frac{3^{1/2}t}{2}\right)$$

On combining these two relations we obtain

$$J_0^*(t) - 2J_6^*(t) + 2J_{12}^*(t) - \ldots = \frac{1}{3}\left\{1 + 2\cosh\left(\frac{3^{1/2}t}{2}\right)\right\}$$

Now it follows from the recurrence relation $2n\, J_n^*(t) = t\left\{J_{n-1}^*(t) - J_{n+1}^*(t)\right\}$ $n = 1, 2, 3, \ldots$, that $J_{n-1}^*(t) \geq J_{n+1}^*(t)$, the equality holding only when $n > 1$ and $t = 0$. Thus $J_6^*(t) \geq J_8^*(t) \geq J_{10}^*(t) \geq J_{12}^*(t) \geq \ldots$, so that the sum of the infinite series $J_6^*(t) - J_{12}^*(t) + \ldots$ is non-negative. Hence $\frac{1}{3}\left\{1 + 2\cosh\left(\frac{3^{1/2}t}{2}\right)\right\}$ is a lower bound, over $-\infty < t < \infty$, for $J_0^*(t)$, the numerical value of the difference between $J_0^*(t)$ and this lower bound being twice the sum of the infinite series $J_6^*(t) - J_{12}^*(t) + \ldots$.

We have, then, obtained an upper and a lower bound, over $-\infty < t < \infty$, for $J_0^*(t)$ and we know that the mean of these two bounds, namely, $\frac{1}{3}\left\{\frac{1+\cosh t}{2} + \cosh\left(\frac{t}{2}\right) + \cosh\left(\frac{3^{1/2}t}{2}\right)\right\}$ is an upper bound, over $-\infty < t < \infty$, for $J_0^*(t)$, the excess of this upper bound over $J_0^*(t)$ being twice the sum of the infinite series $J_{12}^*(t) + J_{24}^*(t) + \ldots$.

On combining the two relations $J_0^*(t) + 2(\cos 2\theta) J_2^*(t) + \dots = \cosh(t\cos\theta)$, $J_0^*(t) - 2(\cos 2\theta) J_2^*(t) + \dots = \cosh(t\sin\theta)$, we obtain the relation

$$J_0^*(t) + 2(\cos 4\theta) J_4^*(t) + \dots = \frac{1}{2}\left\{\cosh(t\cos\theta) + \cosh(t\sin\theta)\right\}$$

and from this we deduce that $\frac{1}{3}\left\{\cosh(t\cos\frac{\pi}{12}) + \cosh(t\sin\frac{\pi}{12}) + \cosh(2^{-1/2}t)\right\}$ is a lower bound, over $-\infty < t < \infty$, for $J_0^*(t)$, the difference between $J_0^*(t)$ and this lower bound being twice the sum of the infinite series $J_{12}^*(t) - J_{24}^*(t) + \dots$. Hence $\frac{1}{6}\left\{\frac{1+\cosh t}{2} + \cosh(t\cos\frac{\pi}{12}) + \cosh(t\sin\frac{\pi}{12}) + \cosh(\frac{t}{2}) + \cosh(\frac{3^{1/2}t}{2}) + \cosh(2^{-1/2}t)\right\}$ is an upper bound, over $-\infty < t < \infty$, for $J_0^*(t)$ the excess of this upper bound over $J_0^*(t)$ being twice the sum of the infinite series $J_{24}^*(t) + J_{48}^*(t) + \dots$. If $|t| \leq 9$ this excess is less than 10^{-7}. Continuing this process one step further we see that

$$\frac{1}{12}\left\{\frac{\cosh t+1}{2} + \cosh(t\cos\frac{\pi}{24}) + \cosh(t\sin\frac{\pi}{24}) + \cosh(t\cos\frac{\pi}{12}) + \cosh(t\sin\frac{\pi}{12}) + \cosh(t\cos\frac{\pi}{8}) + \cosh(t\sin\frac{\pi}{8}) + \cosh(\frac{t}{2}) + \cosh(\frac{3^{1/2}t}{2}) + \cosh(t\cos\frac{5\pi}{24}) + \cosh(t\sin\frac{5\pi}{24}) + \cosh(2^{-1/2}t)\right\}$$

is an upper bound, over $-\infty < t < \infty$, for $J_0^*(t)$ the excess of this upper bound over $J_0^*(t)$ being twice the sum of the infinite series $J_{48}^*(t) + J_{96}^*(t) + \dots$. If $|t| \leq 10$ this excess is less than 1.2×10^{-28}.

The same argument may be applied to $J_0(t)$, the hyperbolic cosines being replaced by ordinary cosines, the only difference being that we cannot say, since $J_{2k}(t)$, $k = 1, 2, \dots$, may assume negative values, that our approximations are upper, or lower, bounds as the case may be. For example, $\frac{1}{3}\left\{\cos t + 2\cos(\frac{t}{2})\right\}$ is an approximation, over $-\infty < t < \infty$, to $J_0(t)$, the difference $\frac{1}{3}\left\{\cos t + 2\cos(\frac{t}{2})\right\} - J_0(t)$

being the product of the sum of the infinite series $J_6(t) - J_{12}(t) + \ldots$ by -2. Since $|J_n(t)| \leq |J_n^*(t)|$, $n = 0, 1, 2, \ldots$, the approximations to $J_0(t)$ which we obtain in this way are at least as good as the approximations we have obtained to $J_0^*(t)$. When $t = \frac{\pi}{2}$ the approximation to $J_0(t)$ which is furnished by $\frac{1}{3}\left\{\cos t + 2\cos(\frac{t}{2})\right\}$ is $\frac{1}{3}.2^{1/2} = 0.4714$, to 4 decimals, while $J_0(\frac{\pi}{2}) = 0.4720$, to 4 decimals. When $t = \frac{\pi}{4}$, the approximation to $J_0(\frac{\pi}{4})$ is 0.85162, to 5 decimals, while $J_0(\frac{\pi}{4}) = 0.85163$, to 5 decimals.

Exercise 4. Show that $J_1^*(t) + (\cos 4\theta)\left[J_3^*(t) + J_5^*(t)\right]$

$$+ (\cos 8\theta)\left[J_7^*(t) + J_9^*(t)\right] + \ldots = \frac{1}{2}\left\{\cos\theta \sinh(t\cos\theta) + \sin\theta \sinh(t\sin\theta)\right\}$$

and deduce that

$$J_1^*(t) + \left[J_{11}^*(t) + J_{13}^*(t)\right] + \left[J_{23}^*(t) + J_{25}^*(t)\right] + \ldots = \frac{1}{3}\left[\frac{\sinh t}{2} + \frac{3^{1/2}}{2}\sinh\left(\frac{3^{1/2}t}{2}\right) + \frac{1}{2}\sinh\left(\frac{t}{2}\right)\right]$$

$$J_1^*(t) - \left[J_{11}^*(t) + J_{13}^*(t)\right] + \left[J_{23}^*(t) + J_{25}^*(t)\right] - \ldots$$

$$= \frac{1}{3}\left[(\cos\tfrac{\pi}{12})\sinh(t\cos\tfrac{\pi}{12}) + (\sin\tfrac{\pi}{12})\sinh(t\sin\tfrac{\pi}{12}) + 2^{-1/2}\sinh(2^{-1/2}t)\right]$$

Exercise 5. Show that $\frac{1}{6}\left[\frac{\sinh t}{2} + (\cos\frac{\pi}{12})\sinh(t\cos\frac{\pi}{12}) + (\sin\frac{\pi}{12})\sinh(t\sin\frac{\pi}{12})\right.$ $\left. + \frac{3^{1/2}}{2}\sinh\left(\frac{3^{1/2}t}{2}\right) + \frac{1}{2}\sinh\left(\frac{t}{2}\right) + 2^{-1/2}\sinh(2^{-1/2}t)\right]$ is an upper bound, over $0 \leq t < \infty$, for $J_1^*(t)$, the excess of this upper bound over $J_1^*(t)$ being the sum of the infinite series

$$\left[J_{23}^*(t) + J_{25}^*(t)\right] + \left[J_{47}^*(t) + J_{49}^*(t)\right] + \ldots$$

Exercise 6. Show that $\frac{1}{6}\Big[(\cos\frac{\pi}{24})\sinh(t\cos\frac{\pi}{24}) + (\sin\frac{\pi}{24})\sinh(t\sin\frac{\pi}{24}) + (\cos\frac{\pi}{8})\sinh(t\cos\frac{\pi}{8}) + (\sin\frac{\pi}{8})\sinh(t\sin\frac{\pi}{8}) + (\cos\frac{5\pi}{24})\sinh(t\cos\frac{5\pi}{24}) + (\sin\frac{5\pi}{24})\sinh(t\sin\frac{5\pi}{24})\Big]$ is a lower bound, over $0 \leq t < \infty$, for $J_1{}^*(t)$, the numerical value of the difference between $J_1{}^*(t)$ and this lower bound being the sum of the infinite series

$$\left[J_{23}{}^*(t) + J_{25}{}^*(t)\right] - \left[J_{47}{}^*(t) + J_{49}{}^*(t)\right] + \cdots$$

Exercise 7. Show that the mean of the upper and lower bounds for $J_1{}^*(t)$, given in Exercises 5 and 6, respectively, is an upper bound, over $0 \leq t < \infty$, for $J_1{}^*(t)$ the excess of this upper bound over $J_1{}^*(t)$ being the sum of the infinite series

$$\left[J_{47}{}^*(t) + J_{49}{}^*(t)\right] + \left[J_{95}{}^*(t) + J_{97}{}^*(t)\right] + \cdots$$

Exercise 8. Write down approximations to $J_1(t)$ analogous to the approximations to $J_1{}^*(t)$ which are furnished by Exercises 5, 6, and 7.

Lecture 13

The Problem of the Vibrating String

We shall discuss in this lecture the application of the Laplace Transformation to the problem of a vibrating string, of length 1, with fixed end-points. This is one of the simplest instances of what is known as a boundary-value and initial-condition problem. In the first place, the transverse displacement $d = d(x, t)$, $0 \leq x \leq 1$, $0 \leq t < \infty$, must satisfy the linear second-order partial differential equation with constant coefficients:

$$D: \quad a^2 d_{xx} - d_{tt} = 0; \; 0 \leq x \leq 1, \; 0 \leq t < \infty, \; a > 0$$

In the second place the boundary values of d, i.e., the values of d when $x = 0$ and when $x = 1$, are specified as follows:

$$B: \quad d(0, t) = 0; \; d(1, t) = 0, \; 0 \leq t < \infty$$

Finally, d must satisfy the following initial conditions:

$$I: \quad d(x, 0) = \phi(x); \qquad d_t(x, 0) = v(x), \; 0 \leq x \leq 1$$

$\phi(x)$ and $v(x)$ being given continuous functions of x over the interval $0 \leq x \leq 1$. We assume that this boundary-value and initial-condition problem possesses a solution $d(x, t)$ with the following properties:

1) d_{tt} is piecewise continuous, for all values of x in the interval $0 \leq x \leq 1$, over $0 \leq t < \infty$ and the Laplace Transform of

$d_{tt}\, u(t)$ exists at a point $p = c_1$ of the positive real axis in the complex p-plane.

2) The infinite integral $\int_0^\infty d(x, t) \exp(-pt)dt$ which furnishes, over the half-plane $c > c_1$, the Laplace Transform $f = f(x, p)$ of $d(x, t)\, u(t)$ is twice differentiable with respect to x under the integral sign so that $d_{xx}\,(x, t)\, u(t)$ possesses, over the half-plane $c > c_1$, the Laplace Transform f_{xx}.

Since $L\left[d_t(x, t)\, u(t)\right] = pf - \phi$, $L\left[d_{tt}(x, t)\, u(t)\right] = p^2 f - p\phi - v$, $c > c_1$, and since $a^2 d_{xx} - d_{tt} = 0$, $0 \leq t < \infty$, f must satisfy the non-homogeneous second-order ordinary linear differential equation

$$D': \quad a^2 f_{xx} - p^2 f = -p\phi - v; \quad 0 \leq x \leq 1$$

p playing the role of a constant parameter. Furthermore, the boundary values $f(0, p)$ and $f(1, p)$ are zero for all points p in the half-plane $c > c_1$:

$$B': \quad f(0, p) = 0; \quad f(1, p) = 0; \quad c > c_1.$$

Thus the boundary-value and initial-condition problem D, B, I, is replaced by the boundary-value problem D', B'. We shall solve this simpler problem and shall then determine a function $d(x, t)$ which is such that the Laplace Transform of $d(x, t)\, u(t)$, over some half-plane $c > c_1 > 0$, is the solution, $f = f(x, p)$, of the boundary-value problem D', B' which we have obtained. All that remains, then, is to verify that $d(x, t)$ is a solution of the boundary-value and initial- condition problem D, B, I, and to show that this problem does not possess any other solution.

Our first step in solving the boundary-value problem D', B', is to consider the associated homogeneous boundary-value problem D", B', where D" is the homogeneous second-order linear differential equation, with constant coefficients, $a^2 k_{xx} - p^2 k = 0$. D" does not have, no matter what is the value of $c_1 > 0$, a non-trivial solution, i.e., a solution which does not vanish identically, which satisfies the boundary conditions B'. Indeed $\sinh(q\,x)$ and $\sinh\left[q\,(1-x)\right]$, $q = \frac{p}{a}$, are two linearly independent solutions of D" so that the general solution of D" is $A \sinh(q\,x) + A' \sinh\left[q(1-x)\right]$ where A and A' are undetermined constants; for this to be zero at $x = 0$ we must have $A' = 0$, since $\sinh(q\,1) \neq 0$, and $A \sinh(q\,1)$ is not zero if A is not zero. In order to avoid this dilemma of the non-existence of a non-trivial solution of the homogeneous boundary-value problem D", B' we lighten our requirements on the function $k(x, p)$ as follows: we do not insist that k satisfy D'' at <u>all</u> points of the interval $0 \leq x \leq 1$; we require merely that, in addition to satisfying the boundary conditions B', it satisfy D" at all the points of this interval save one, s say, at which it does not possess a second derivative, s being an interior point of the interval, so that $0 < s < 1$. We do require that k be continuous at $x = s$ and that it possess both a right-hand derivative,

$k_x(s+0)$ and a left-hand derivative, $k_x(s-0)$. In order to completely determine this function k of x, which depends on the parameter s and which we denote by $\Gamma\binom{x}{s}$, we prescribe the difference $k_x(s-0) - k_x(s+0) = \Gamma_x\binom{s-0}{s} - \Gamma_x\binom{s+0}{s}$ to be $\frac{1}{a^2}$, the reciprocal of the value at s of the coefficient of k_{xx} in D'' (this coefficient being actually, in the particular problem we are discussing, independent of s).

Over the interval $0 \leq x \leq s$, $\Gamma\binom{x}{s}$ is a linear combination, $A_1 \sinh(q\,x) + A'_1 \sinh\left[q(l-x)\right]$, with constant coefficients, of the two linearly independent solutions $\sinh(q\,x)$, $\sinh\left[q(l-x)\right]$ of D'', and we denote this linear combination by $\Gamma_1\binom{x}{s}$. Similarly, over the interval $s \leq x \leq l$, $\Gamma\binom{x}{s}$ is of the form $\Gamma_2\binom{x}{s} = A_2 \sinh(q\,x) + A'_2 \sinh\left[q(l-x)\right]$. Since $\Gamma_1\binom{0}{s} = 0$, $A'_1 = 0$, and, since $\Gamma_2\binom{l}{s} = 0$, $A_2 = 0$ and, finally, since $\Gamma_1\binom{s}{s} = \Gamma_2\binom{s}{s}$, $A_1 \sinh(qs) = A'_2 \sinh\left[q(l-s)\right]$. Thus $\Gamma_1\binom{x}{s} = A \sinh\left[q(l-s)\right] \sinh q\,x$, $\Gamma_2\binom{x}{s} = A \sinh(q\,s) \sinh\left[q(l-x)\right]$, where A is an undetermined function of s. Then $\Gamma_x\binom{s-0}{s} = \left[\left\{\Gamma_1\binom{x}{s}\right\}_x\right]_{x=s} = Aq \sinh\left[q(l-s)\right]\cosh(qs)$ and, similarly, $\Gamma_x\binom{s+0}{s} = -Aq \sinh(qs) \cosh\left[q(l-s)\right]$ so that $Aq\left\{\sinh\left[q(l-s)\right] \cosh(qs) + \sinh(qs) \cosh\left[q(l-s)\right]\right\}$, i.e., $Aq \sinh ql$, $= \frac{1}{a^2}$. Thus $A = (a^2 q \sinh ql)^{-1}$ and

$$\Gamma_1\binom{x}{s} = \frac{\sinh q(l-s) \sinh(qx)}{a^2 q \sinh(ql)} \;;\; \Gamma_2\binom{x}{s} = \frac{\sinh(qs) \sinh q(l-x)}{a^2 q \sinh ql}$$

$0 \leq x \leq l$; $0 < s < l$. We define $\Gamma\binom{x}{s}$ when $s = 0$ and $s = l$ by the requirement that $\Gamma\binom{x}{s}$ be a continuous function of s at s = 0 and s = 1; thus $\Gamma\binom{x}{0} = \Gamma_2\binom{x}{0} = 0$ and, similarly, $\Gamma\binom{x}{1} = \Gamma_1\binom{x}{1} = 0$. We observe that $\Gamma_2\binom{x}{s}$ may be obtained from $\Gamma_1\binom{x}{s}$ by merely interchanging x and s and this implies that $\Gamma\binom{s}{x} = \Gamma\binom{x}{s}$, i.e., that $\Gamma\binom{x}{s}$ is a symmetric function of the two real variables (x, s) over the square $0 \leq x \leq 1$, $0 \leq s \leq 1$. Indeed, if $x \leq s$, $\Gamma\binom{s}{x} = \Gamma_2\binom{s}{x}$, while $\Gamma\binom{x}{s} = \Gamma_1\binom{x}{s}$; similarly, if $x \geq s$, $\Gamma\binom{s}{x} = \Gamma_1\binom{s}{x}$ and $\Gamma\binom{x}{s} = \Gamma_2\binom{x}{s}$. The function $\Gamma\binom{x}{s}$ of the two real variables (x, s) is the second of two functions known as the Green's functions of the boundary-value problem D'', B'; the first of these two Green's functions, with which we shall not be concerned since $p \neq 0$, is the function $G\binom{x}{s}$ to which $\Gamma\binom{x}{s}$ reduces when p = 0:

$$G\binom{x}{s} = G_1\binom{x}{s} = \frac{(1-s)\,x}{a^2 1}\ ;\ 0 \leq x \leq s$$

$$G\binom{x}{s} = G_2\binom{x}{s} = \frac{s(1-x)}{a^2 1}\ ,\ s \leq x \leq 1$$

We now turn to the non-homogeneous differential equation

$$D':\quad a^2 f_{xx} - p^2 f = -p\phi - v,\ 0 \leq x \leq 1$$

On combining this with the homogeneous equation

$$D'':\quad a^2 \Gamma_{xx} - p^2 \Gamma = 0; \qquad x \neq s$$

in such a way as to eliminate the undifferentiated functions f and Γ we obtain

$$a^2\left[\Gamma\binom{x}{s} f_{xx} - f(x)\left\{\Gamma\binom{x}{s}\right\}_{xx}\right] = -p\Gamma\binom{x}{s}\phi(x) - \Gamma\binom{x}{s} v(x),$$

$x \neq s$.

Now $\Gamma\binom{x}{s} f_{xx} - f(x)\left\{\Gamma\binom{x}{s}\right\}_{xx} = \left[\Gamma\binom{x}{s} f_x - f(x)\left\{\Gamma\binom{x}{s}\right\}_x\right]_x$, $x \neq s$, and so the integral of $\Gamma\binom{x}{s} f_{xx} - f(x)\left\{\Gamma\binom{x}{s}\right\}_{xx}$ over the interval $0 \leq x \leq 1$ is $\left[\Gamma\binom{x}{s} f_x - f(x)\left\{\Gamma\binom{x}{s}\right\}_x\right]\Big|_0^s$

$+ \left[\Gamma\binom{x}{s} f_x - f(x)\left\{\Gamma\binom{x}{s}\right\}_x\right]_s^1$, it being necessary to break the interval of integration into the two parts $0 \leq x \leq s$, $s \leq x \leq 1$, since $\left\{\Gamma\binom{x}{s}\right\}_x$ is discontinuous at $x = s$. Regarding $\Gamma\binom{x}{s}$ as an integral operator we denote $\int_0^1 \Gamma\binom{x}{s} h(s)\, ds$, where $h(x)$ is any function which is integrable over $0 \leq x \leq 1$, simply by Γh. Then $\int_0^1 \Gamma\binom{x}{s} \phi(x)\, dx$, being the same as $\int_0^1 \Gamma\binom{s}{x} \phi(x)\, dx$, is the value of $\Gamma\phi$ at $x = s$ and, similarly, $\int_0^1 \Gamma\binom{x}{s} v(x)\, dx$ is the value of Γv at $x = s$. Since both Γ and f are zero at $x = 0$ and at $x = 1$ the expression $(\Gamma f_x - f\Gamma_x)\Big|_0^s + (\Gamma f_x - f\Gamma_x)\Big|_s^1$ reduces to $-f\,\Gamma_x\Big|_{s+0}^{s-0}$ $= -f(s)/a^2$ and so

$$f(s) = p(\Gamma\phi)(s) + (\Gamma v)(s);$$

it was to obtain this simple expression that the particular value $\frac{1}{a^2}$ of the discontinuity $\Gamma_x\Big|_{s+0}^{s-0}$ in the first derivative of Γ at $x = s$ was prescribed. Since s is any point of the open interval $0 < s < 1$ we may write the result just obtained in the form $f(x) = p\,\Gamma\phi + \Gamma v$, $0 < x < 1$, and, since $f(x)$, $\Gamma\phi$, and Γv are all continuous at $x = 0$ and at $x = 1$, we have

$$f = p\,\Gamma\phi + \Gamma v,\quad 0 \leq x \leq 1$$

What we have proved so far is a uniqueness theorem; granting the existence of a solution of the boundary-value problem D', B', this solution is unambiguously determined by the formula $f = p\Gamma\phi + \Gamma v$. We must now remove the existence hypothesis by verifying that $p\Gamma\phi + \Gamma v$ is actually a solution of the boundary-value problem D', B'. To do this we first observe that the continuity of $\phi(x)$ and of $v(x)$ over the interval $0 \leq x \leq 1$ assures us that $f = p\Gamma\phi + \Gamma v$ is twice differentiable over this interval. Indeed, on writing $\Gamma\phi$, for example, as $\int_0^x \Gamma_2\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)\phi(s)ds + \int_x^1 \Gamma_1\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)\phi(s)ds$ we see that $\Gamma\phi$ is differentiable over the interval $0 \leq x \leq 1$, its derivative, $(\Gamma\phi)_x$, being furnished by

$$\int_0^x \left\{\Gamma_2\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)\right\}_x \phi(s)ds + \int_x^1 \left\{\Gamma_1\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)\right\}_x \phi(s)ds + \left\{\Gamma_2\left(\begin{smallmatrix} x \\ x \end{smallmatrix}\right) - \Gamma_1\left(\begin{smallmatrix} x \\ x \end{smallmatrix}\right)\right\}\phi(x)$$

which reduces to $\int_0^x \left\{\Gamma_2\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)\right\}_x \phi(s)\,ds + \int_x^1 \left\{\Gamma_1\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)\right\}_x \phi(s)\,ds$ since $\Gamma_2\left(\begin{smallmatrix} x \\ x \end{smallmatrix}\right) = \Gamma_1\left(\begin{smallmatrix} x \\ x \end{smallmatrix}\right)$, by virtue of the continuity of $\Gamma\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)$ at $x = s$. Hence $(\Gamma\phi)_x$ is differentiable over $0 \leq x \leq 1$, its derivative, $(\Gamma\phi)_{xx}$, being furnished by $\int_0^x \left\{\Gamma_2\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)\right\}_{xx} \phi(s)ds$

$$+ \int_x^1 \left\{\Gamma_1\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)\right\}_{xx} \phi(s)ds + \left[\left\{\Gamma_2\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)\right\}_x - \left\{\Gamma_1\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)\right\}_x\right]_{s=x} \phi(x)$$

$$= \int_0^1 \left\{\Gamma\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)\right\}_{xx} \phi(s)\,ds - \frac{1}{a^2}\phi(x) = \frac{p^2}{a^2}\Gamma\phi - \frac{1}{a^2}\phi.$$

Applying this result to the function $f = p\Gamma\phi + \Gamma v$ we see that f is twice differentiable over the interval $0 \leq x \leq 1$, its second derivative, f_{xx}, being furnished by the formula

$$f_{xx} = \frac{p^3}{a^2} \Gamma\phi - \frac{p}{a^2}\phi + \frac{p^2}{a^2}\Gamma v - \frac{1}{a^2} v$$

$$= \frac{p^2}{a^2} f - \frac{p}{a^2}\phi - \frac{1}{a^2} v, \quad 0 \leq x \leq 1$$

Thus $a^2 f_{xx} - p^2 f = - p\phi - v$ so that f satisfies the differential equation D'. That $f = p\Gamma\phi + \Gamma v$ satisfies the boundary conditions B' is evident since $\Gamma\binom{0}{s} = 0$, $\Gamma\binom{1}{s} = 0$, $0 < s < 1$ so that, if h(x) is any function which is integrable over $0 \leq x \leq 1$, Γh is zero at $x = 0$ and at $x = 1$. Thus we have the following definitive result:

The unambiguously determinate solution, $f = f(x, p)$, of the boundary-value problem

D': $\quad a^2 f_{xx} - p^2 f = -p\phi - v$

B': $\quad f(0, p) = 0; \; f(1, p) = 0$

is

$$f = p\Gamma\phi + \Gamma v$$

where

$$\Gamma\binom{x}{s} = \frac{\sinh q(1-s) \sinh qx}{a^2 q \sinh q1}, \quad q = \frac{p}{a}, \quad 0 \leq x \leq s$$

$$= \frac{\sinh qs \sinh q(1-x)}{a^2 q \sinh q1}, \quad s \leq x \leq 1.$$

Lecture 14

The Solution of the Problem of the Vibrating String

We have seen that the unambiguously determinate solution of the boundary-value problem:

$$D': \quad a^2 f_{xx} - p^2 f = -p\phi - v; \ 0 \leq x \leq l$$

$$B': \quad f(0, p) = 0; \ f(l, p) = 0$$

is $f = p\Gamma\phi + \Gamma v$, where the integral operator Γ is furnished by the formulas

$$\Gamma\binom{x}{s} = \Gamma_1\binom{x}{s} = \frac{\sinh q(l-s)\ \sinh(q\,x)}{a^2 q \sinh(ql)}, \quad q = \frac{p}{a}, \quad 0 \leq x \leq s$$

$$\Gamma\binom{x}{s} = \Gamma_2\binom{x}{s} = \frac{\sinh(qs)\sinh\ q\,(l-x)}{a^2 q \sinh(ql)}, \quad s \leq x \leq l$$

and our first task now is the determination of a function d(x, t) which is such that the product of d(x, t) by u(t) has, over some half-plane $c > c_1$, the Laplace Transform $p\Gamma\phi + \Gamma v$. If we regard x as fixed, $\Gamma\binom{x}{s}$ is a function of s and p which is analytic save at the points $\frac{n\pi a}{l}i$, $n = 0, \pm 1, \pm 2, \ldots$, on the imaginary axis in the complex p-plane, at which $\sinh ql = \sinh \frac{pl}{a}$ is zero, and so $c_1 \geq 0$. If d(x, t) is, for each value of x in the interval $0 \leq x \leq l$, bounded over $0 \leq t < \infty$, d(x, t) u(t) possesses a Laplace Transform over the half-plane $c > 0$ and we set $c_1 = 0$. We first examine in detail the first term

$$p \Gamma\phi = p \int_0^x \Gamma_2\left(\frac{x}{s}\right) \phi(s)\, ds + p \int_x^1 \Gamma_1\left(\frac{x}{s}\right) \phi(s)\, ds$$

$$= \frac{1}{a \sinh ql}\left[\int_0^x \sinh\left[q(l-x)\right] \sinh(qs)\phi(s)ds \right.$$

$$\left. + \int_x^1 \sinh (qx) \sinh\left[q(l-s)\right]\phi(s)\, ds\right]$$

of $f = p\Gamma\phi + \Gamma v$. On writing $\sinh q(l-x) = \frac{1}{2} \exp(ql)\left\{\exp(-qx)-\exp-q(2l-x)\right\}$, $\sinh(qs) = \frac{1}{2}\left\{\exp(qs) - \exp(-qs)\right\}$, and so on, we obtain

$$p\Gamma\phi = \frac{\exp(ql)}{4a\sinh(ql)}\left[\int_0^x \left\{\exp\left[-q(x-s)\right] - \exp\left[-q(x+s)\right] - \exp\left[-q(2l-x-s)\right] + \exp\left[-q(2l-x+s)\right]\right\}\phi(s)ds \right.$$

$$\left. + \int_x^1 \left\{\exp\left[-q(s-x)\right] - \exp\left[-q(s+x)\right] - \exp\left[-q(2l-s-x)\right] + \exp\left[-q(2l-s+x)\right]\right\}\phi(s)\, ds\right]$$

$$= \frac{\exp(ql)}{4a\sinh(ql)}\left[\int_0^x \left\{\exp\left[-q(x-s)\right] + \exp\left[-q(2l-x+s)\right]\right\}\phi(s)\, ds \right.$$

$$- \int_0^1 \left\{\exp\left[-q(x+s)\right] + \exp\left[-q(2l-x-s)\right]\right\}\phi(s)\, ds$$

$$\left. + \int_x^1 \left\{\exp\left[-q(s-x)\right] + \exp\left[-q(2l-s+x)\right]\right\}\phi(s)\, ds\right]$$

This complicated expression takes a simpler appearance if we extend the range of definition of $\phi(x)$ from the interval $0 \leq x \leq l$ to the entire x - axis, $-\infty < x < \infty$, by saying that $\phi(x)$ is an odd periodic function of period $2l$, it being permissible to do this since $\phi(0) = 0$ and $\phi(l) = 0$. The oddness of $\phi(x)$ furnishes the values of $\phi(x)$ over the interval $-l \leq x \leq 0$, since its values over the interval $0 \leq x \leq l$ are known, and the periodicity, with period $2l$, of $\phi(x)$ furnishes the values of $\phi(x)$ over $-\infty < x < \infty$ since its values over the interval $-l \leq x \leq l$

are known. In the various integrals which appear in the expression furnishing $p\Gamma\phi$ we make changes, of the type $s = \alpha + \beta t$, of the variable of integration from s to t in such a way that the exponential factor in each of the transformed integrals is $\exp(-pt) = \exp(-aqt)$. For example, in the integral $\int_0^x \exp\left[-q(x-s)\right]\phi(s)ds$ we write $s = x - at$ so that it appears as $a\int_0^{x/a} \exp(-pt)\,\phi(x-at)dt$; in the integral $\int_0^x \exp\left[-q(2l-x+s)\right]\phi(s)ds$ we write $s = x - 2l + at$ so that it appears as $\int_{\frac{2l-x}{a}}^{\frac{2l}{a}} \exp(-pt)\,\phi\,(x-2l+at)dt = \int_{\frac{2l-x}{a}}^{\frac{2l}{a}} \exp(-pt)\phi(x+at)dt$; in the integral $\int_0^l \exp\left[-q(x+s)\right]\phi(s)ds$ we write $s = -x + at$ so that it appears as

$$\int_{\frac{x}{a}}^{\frac{l+x}{a}} \exp(-pt)\phi(at - x)dt = -\int_{\frac{x}{a}}^{\frac{l+x}{a}} \exp(-pt)\phi(x-at)dt,$$

and so on. Continuing in this way we obtain

$$p\Gamma\phi = \frac{\exp(ql)}{4\sinh(ql)} \int_0^{\frac{2l}{a}} \left\{\phi(x-at) + q(x+at)\right\}\exp(-pt)dt$$

Now $\dfrac{\exp(ql)}{2\sinh(ql)} = \left\{1 - \exp(-2ql)\right\}^{-1}$ and, since the real part of $q = p/a$ is positive, this may be written as the sum of the convergent infinite series $1 + \exp(-2ql) + \exp(-4ql) + \ldots$. The product of

$\int_0^{\frac{2l}{a}} \left\{\phi(x-at) + \phi(x+at)\right\}\exp(-pt)dt$ by $\exp(-2kql)$, $k = 1, 2, \ldots,$ appears, on writing $t = s - 2k\frac{l}{a}$, as $\int_{2k\frac{l}{a}}^{2(k+1)\frac{l}{a}} \left\{\phi(x-as+2kl)\right.$

$\left. + \phi(x+as+2kl)\right\}\exp(-ps)ds$ and this is the same, since $\phi(x)$ is periodic with period 2l, as $\displaystyle\int_{2k\frac{l}{a}}^{2(k+1)\frac{l}{a}} \left\{ \phi(x-at) + \phi(x+at) \right\} \exp(-pt)dt$

Since $\left\{ \phi(x-at) + \phi(x+at) \right\} \exp(-pt)$, being continuous, is bounded over the interval $0 \leq t \leq \dfrac{2l}{a}$ the infinite series obtained by multiplying each term of the infinite series $1 + \exp(-2ql) + \exp(-4ql) + \ldots$ by $\left\{ \phi(x-at) + \phi(x+at) \right\} \exp(-pt)$ is uniformly convergent over the interval $0 \leq t \leq \dfrac{2l}{a}$ and so term-by-term integration over the interval is legitimate. Hence

$$p\Gamma\phi = \frac{1}{2} \sum_{k=0}^{\infty} \int_{2k\frac{l}{a}}^{2(k+1)\frac{l}{a}} \left\{ \phi(x-at) + \phi(x+at) \right\} \exp(-pt)dt$$

$$= \frac{1}{2} \int_0^{\infty} \left\{ \phi(x-at) + \phi(x+at) \right\} \exp(-pt)dt.$$

so that the Laplace Transform, over the half-plane $c > 0$, of $\frac{1}{2}\left\{ \phi(x-at) + \phi(x+at) \right\} u(t)$ is $p\Gamma\phi$. Similarly the Laplace Transform, over the half-plane $c > 0$, of $\frac{1}{2}\left\{ v(x-at) + v(x+at) \right\} u(t)$ is $p\Gamma v$, it being understood that $v(x)$ is an odd periodic function, with period 2l, of the unrestricted real variable x, and this implies that the Laplace Transform, over the half-plane $c > 0$, of $\left[\frac{1}{2} \int_0^t \left\{ v(x-as) + v(x+as) \right\} ds \right] u(t)$ is Γv. Hence the Laplace Transform, over the half-plane $c > 0$, of the product of $u(t)$ by $\frac{1}{2} \left[\phi(x-at) + \phi(x+at) + \int_0^t \left\{ v(x-as) + v(x+as) \right\} ds \right]$ is $p\Gamma\phi + \Gamma v$.

We now proceed to show that the function

$$d(x, t) = \frac{1}{2} \left[\phi(x-at) + \phi(x+at) + \int_0^t \left\{ v(x-as) + v(x+as) \right\} ds \right],$$

$0 \leq x \leq l$, $-\infty < t < \infty$, is a solution of the boundary-value and initial-condition problem D, B, I and that this problem possesses no other solution. On introducing the variables $\xi = x - at$, $\tau = x+at$, D takes the form $d_{\xi\tau} = 0$ and d(x, t) becomes $\frac{1}{2}\left[\phi(\xi) + \phi(\tau) + \frac{1}{a}\int_{\xi}^{\tau} v(\sigma)\,d\sigma\right]$ so that d_{ξ} is a function of ξ alone; hence d(x, t) is a solution of the differential equation D. That d(0, t) is 0 over $0 \leq t < \infty$ and, indeed, over $-\infty < t < \infty$, is an immediate consequence of the fact that $\phi(t)$ and v(t) are, after their range of definition has been extended, odd functions of the unrestricted real variable t. Similarly, since $\phi(t)$ and v(t) are not only odd but also periodic functions, with period 2l, d(l, t) = 0 over $-\infty < t < \infty$; indeed, $\phi(l-at) = \phi(-l-at) = -\phi(l+at)$ and $v(l-as) = -v(l+as)$. Thus d(x, t) satisfies the boundary conditions B. Finally, $d(x, 0) = \phi(x)$ and $d_t(x, 0) = \frac{1}{2}\left[-a\phi_x(x) + a\phi_x(x) + 2v(x)\right] = v(x)$ so that d(x, t) satisfies the initial conditions I. Thus d(x, t) is a solution of the boundary-value and initial-condition problem D, B, I. If this problem possessed two different solutions their difference $\Delta(x, t)$ would be a solution of the associated homogeneous problem D, B, I' where

I': $d(x, 0) = 0$; $d_t(x, 0)$

Being a solution of the differential equation D, which appears, when written in terms of the variables ξ, τ, as $d_{\xi\tau} = 0$, $\Delta(x, t)$ is of the form $F(\xi) + G(\tau)$ and the initial conditions I' yield $F(x) + G(x) = 0$, $F_x(x) - G_x(x) = 0$. On differentiating the first of these two relations and combining the result with the second we see that F and G are constant functions of their arguments so that $\Delta(x, t)$ is a constant function of the two variables (x, t). Being zero when $t = 0$ it is identically zero. Thus we have the following result:

The unambiguously determinate solution of the problem of the vibrating string is $d(x, t) = \frac{1}{2}\left\{\phi(x-at) + \phi(x+at)\right\} + \frac{1}{2}\int_0^t \left\{v(x-as) + v(x+as)\right\} ds = \frac{1}{2}\left\{\phi(x-at) + \phi(x+at)\right\} + \frac{1}{2a}\int_{x-at}^{x+at} v(\sigma)d\sigma$.

If we denote $\int_0^x v(s)\,ds$ by $V(x)$ it is clear that $V(x)$ is an even periodic function, with period $2l$, of the unrestricted real variable x. Indeed $V(x) - V(-x) = \int_{-x}^{x} v(s)\,ds$ is zero by virtue of the oddness of $v(x)$ and $V(x+2l) - V(x) = \int_x^{x+2l} v(s)\,ds = \int_x^l v(s)ds + \int_l^{x+2l} v(s)ds = \int_x^l v(s)ds + \int_{-l}^{x} v(\sigma)d\sigma$, $\sigma = s-2l$, $= \int_{-l}^{l} v(s)ds = 0$.

The unambiguously determinate solution, $d(x,t)$, of the boundary-value and initial-condition problem may be written as the sum of the following two functions of $x - at = \xi$ and $x + at = \tau$, respectively,

$$d_1(x, t) = \frac{1}{2}\phi(x-at) - \frac{1}{2a}V(x-at)$$
$$d_2(x, t) = \frac{1}{2}\phi(x+at) + \frac{1}{2a}V(x+at)$$

and we observe that $d_2(-x, t) = -d_1(x, t)$, $d_2(l-x, t) = -d_1(l+x, t)$. $d_1(x-at)$ is constant so long as $x-at$ remains constant and we say that it represents a wave travelling, in the direction of the positive x-axis, with velocity a. When x attains the value l and begins to assume values $> l$ we must replace $d_1(l+x, t)$ by $-d_2(l-x, t)$ which represents a wave travelling, in the direction of the negative x-axis, with velocity a. When x, in the expression $d_1(x, t)$, attains the value $2l$ and begins to assume values $> 2l$, $l-x$, in the expression $d_2(l-x, t)$, attains the value 0 and begins to assume negative values and we must replace $d_2(l-x, t)$ by $d_1(x-l, t)$ and so on. We express this result by the statement that the solution of the problem of the vibrating string is the sum of two waves, one travelling with velocity a in the direction of the positive x-axis and the other with velocity a in the direction of the negative x-axis, these waves being subjected to continual reflections at the ends of the string.

The level curves of the functions $\xi = x - at$, $\tau = x + at$ play a dominant role in the theory of the partial differential equation

$$D: \quad a^2 d_{xx} - d_{tt} = 0$$

and they are known as the characteristics of this differential equation. Let us suppose that the values of the two first-order derivatives, d_x and d_t, of $d(x, t)$ are assigned, as continuously differentiable functions of a parameter α, along some smooth curve $x = x(\alpha)$, $t = t(\alpha)$ in the (x, t)-plane. Then $(d_x)_\alpha = d_{xx} x_\alpha + d_{xt} t_\alpha$, $(d_t)_\alpha = d_{tx} x_\alpha + d_{tt} t_\alpha$ along this curve and these relations, together with the relations $a^2 d_{xx} - d_{tt} = 0$, $d_{tx} = d_{xt}$ enable us to unambiguously determine the three second-order derivatives, d_{xx}, d_{xt}, d_{tt}, of $d(x, t)$

along the curve at any point of the curve at which the 3 x 3 matrix

$$\begin{pmatrix} x_\alpha & t_\alpha & 0 \\ 0 & x_\alpha & t_\alpha \\ a^2 & 0 & -1 \end{pmatrix}$$

is non-singular. Since the determinant of this matrix is $a^2(t_\alpha)^2 - (x_\alpha)^2 = (at_\alpha - x_\alpha)(at_\alpha + x_\alpha)$ we see that d_{xx}, d_{xt}, d_{tt} are unambiguously determinate at any point of the curve $x = x(\alpha)$, $t = t(\alpha)$, at which this curve is not tangent to any characteristic of the differential equation D, i. e., to any member of either of the two families of straight lines $x - at = \text{const}$, $x + at = \text{constant}$.

Lectures on Applied Mathematics

Lecture 15

The Generalized Vibrating String Problem

The simplest generalization of the partial differential equation $a^2 d_{xx} - d_{tt} = 0$, which occurs in the theory of the vibrating string, is the partial differential equation $a^2 d_{xx} - d_{tt} + p d_x + q d_t + r d = 0$, where $a > 0$ and p, q, r are given constants. Setting $d = d' \exp(\alpha x + \beta t)$, where α and β are undetermined constants, we have

$d_x = (d_x{}' + \alpha d') \exp(\alpha x + \beta t)$; $d_t = (d_t{}' + \beta d') \exp(\alpha x + \beta t)$

$d_{xx} = (d'_{xx} + 2\alpha d'_x + \alpha^2 d') \exp(\alpha x + \beta t)$; $d_{tt} = (d'_{tt} + 2\beta d'_t + \beta^2 d')\exp(\alpha x + \beta t)$

so that d' satisfies the partial differential equation

$$a^2 d'_{xx} - d'_{tt} + (2a^2\alpha + p)\, d'_x + (q - 2\beta) d'_t + (a^2\alpha^2 - \beta^2 + p\alpha + q\beta + r) d' = 0$$

Setting $\alpha = -p/2a^2$, $\beta = q/2$ the terms involving the first-order derivatives of d' disappear so that $d' = d\left[\exp\left(\frac{px}{2a^2} - \frac{q}{2} t\right)\right]$ is a solution of the partial differential equation

$$a^2 d'_{xx} - d'_{tt} + \left(r - \frac{p^2}{4a^2} + \frac{q^2}{4}\right) d' = 0$$

If $r = \frac{1}{4}\left(\frac{p^2}{a^2} - q^2\right)$ this is the partial differential equation which we have already met in the theory of the vibrating string. Assuming that $r - \frac{p^2}{4a^2} + \frac{q^2}{4}$ is not zero we denote its absolute value by k^{-2}, $k > 0$, and we write $x = kx'$, $t = kt'$, so that $d'_{x'} = k\, d'_x$, $d'_{x'x'} = k^2 d'_{xx}$ and, similarly, $d'_{t't'} = k^2 d'_{tt}$. Thus $a^2 d'_{x'x'} - d'_{tt} \pm d' = 0$, the upper, or lower, sign being used according as $r - \frac{p^2}{4a^2} + \frac{q^2}{4}$ is positive, or negative, respectively. Dropping the primes attached to x, t and d we are confronted by one or other of the two partial differential equations $a^2 d_{xx} - d_{tt} \pm d = 0$ and we first consider the equation

$$D: \quad a^2 d_{xx} - d_{tt} + d = 0$$

We take the boundary and initial conditions to be the same as in the problem of the vibrating string, namely,

$$B: \quad d(0, t) = 0; \qquad d(l, t) = 0; \; 0 \le t < \infty$$

$$I: \quad d(x, 0) = \phi(x); \qquad d_t(x, 0) = v(x); \; 0 \le x \le l$$

and we term the boundary-value and initial condition problem D, B, I, the generalized vibrating string problem.

Proceeding as in the case of the vibrating string problem we encounter the second-order ordinary differential equation $a^2 f_{xx} - (p^2-1) f = -p\phi - v$, rather than $a^2 f_{xx} - p^2 f = -p\phi - v$ and we set $q = \frac{1}{a}(p^2-1)^{1/2}$, rather than $q = \frac{p}{a}$. Thus the boundary-value problem D', B' has the same formal appearance as in the case of the vibrating string problem, the difference between the two problems lying entirely in the definition of q as a function of p. The integral operator $\Gamma = \Gamma\left(\begin{smallmatrix} x \\ s \end{smallmatrix}\right)$ which we encounter is, then, the same function of q as before but this implies that it is a different function of p. Its singularities, instead of lying on the imaginary axis in the complex p-plane, are the points p for which $p^2 - 1 = -\frac{n^2\pi^2 a^2}{l^2}$,

$n = 0, 1, 2, \ldots$, so that $p = 1$, corresponding to $n = 0$, is a singular point of Γ. Thus the half-plane over which $d(x, t)\, u(t)$ possesses, we assume, a Laplace Transform f cannot be, as it was in the problem of the vibrating string, the half-plane $c > 0$; since Γ does not possess any singularities in the half-plane $c > 1$ we assume that $d(x, t)\, u(t)$ possesses, over the half-plane $c > 1$, a Laplace Transform f. The same argument as in the problem of the vibrating string shows that $aq \Gamma \phi = \frac{1}{2} \int_0^\infty \left\{ \phi(x-at) + \phi(x+at) \right\} \exp(-aqt) dt$ provided that the real part of q is > 0 and that the range of definition of $\phi(x)$ has been extended from the interval $0 \leq x \leq 1$ to the entire x-axis by the statement that $\phi(x)$ is an odd periodic function, with period 21, of x. Since aq is no longer p, the integral $\int_0^\infty \left\{ \phi(x-at)+\phi(x+at) \right\} \exp(-aqt)dt$ is no longer the Laplace Transform of $\left\{ \phi(x-at) + \phi(x+at) \right\} u(t)$ and we proceed as follows. Writing

$$\Gamma\phi = \frac{1}{2a} \int_0^\infty \left\{ \phi(x-at) + \phi(x+at) \right\} \frac{\exp(-aqt)}{q} dt$$

$$= \frac{1}{2a^2} \int_0^\infty \left\{ \phi(x-s) + \phi(x+s) \right\} \frac{\exp(-sq)}{q} ds; \; s = at$$

we try to determine an integral operator, $K = K(\frac{t}{s})$, which is such that $K(\frac{t}{s})\, u(t)$, s any non-negative constant, possesses, over the half-plane $c > 1$, the Laplace Transform $\frac{\exp(-sq)}{q}$, $q = \frac{(p^2-1)^{1/2}}{a}$.

Once we have determined K we may write $\Gamma\phi$ in the form

$$\Gamma\phi = \frac{1}{2a^2} \int_0^\infty \left(\left[\phi(x-s) + \phi(x+s)\right] \left\{ \int_0^\infty K\left(\frac{t}{s}\right) \exp(-pt)dt \right\} \right) ds$$

and, provided that the order of integration in this repeated infinite integral may be changed, it follows that

$$\Gamma\phi = \frac{1}{2a^2} \int_0^\infty \left\{ \int_0^\infty K\left(\frac{t}{s}\right) \left[\phi(x-s) + \phi(x+s)\right] ds \right\} \exp(-pt)dt$$

We shall see, when we determine K, that $K\left(\frac{t}{s}\right)$ is zero if $s > at$ so that $\Gamma\phi$ is the Laplace Transform, over the half-plane $c > 1$, of $\frac{1}{2a^2} \int_0^{at} K\left(\frac{t}{s}\right) \left[\phi(x-s) + \phi(x+s)\right] ds$ and this implies that the Laplace Transform, over the half-plane $c > 1$, of

$$\frac{1}{2a} K\left(\frac{t}{at-0}\right) \left\{ \phi(x-at) + \phi(x+at) \right\} + \frac{1}{2a^2} \int_0^{at} K_t\left(\frac{t}{s}\right) \left[\phi(x-s) + \phi(x+s) \right] ds$$

is $p\Gamma\phi$.

We turn, now, to the determination of the integral operator K. Setting, in the relation $q = \frac{1}{a}(p^2-1)^{1/2}$, $p = \cosh z^*$, we have $q = \frac{1}{a}\sinh z^* = \frac{1}{a}\left[p - \exp(-z^*)\right]$ and so

$$\begin{aligned} \exp(-sq) &= \exp\left(-\frac{s}{a}p\right) \exp\left[\frac{s}{a}\exp(-z^*)\right] \\ &= \exp\left(-\frac{s}{a}p\right)\left[1 + \frac{s}{a}\exp(-z^*) + \frac{s^2}{2!a^2}\exp(-2z^*) + \dots\right] \end{aligned}$$

Now $\exp(-nz^*)/\sinh z^* = f_n^*(p)$ is the Laplace Transform, over the half-plane $c > 1$, of $J_n^*(t)\,u(t) = I_n(t)\,u(t)$ and, since the coefficients of the infinite series $f_0^*(p) + \frac{s}{a} f_1^*(p) + \frac{s^2}{2!a^2} f_2^*(p) + \dots$ are all non-negative, s being, by hypothesis, non-negative, it follows that the Laplace Transform, over the half-plane $c > 1$, of the product of the sum of the everywhere convergent infinite series $I_0(t) + \frac{s}{a} I_1(t) + \frac{s^2}{2!a^2} I_2(t) + \ldots$ by $u(t)$ is $\exp\left[\frac{s}{a}\exp(-z^*)\right] / \sinh z^*$. We shall show in the next paragraph that the sum of this infinite series is $\left[I_0 \, (1 + \frac{2s}{at})^{1/2} \, t\right]$; admitting this, for the moment, $\exp(-sq)/q$ is, over the half-plane $c > 1$, the product of the Laplace Transform of $aI_0\left[(1+\frac{2s}{at})^{1/2} \, t\right] u(t)$ by $\exp(-\frac{s}{a}p)$:

$$\exp(-sq)/q = a \exp(-\frac{s}{a} p) \int_0^{\infty} I_0\left[(1+\frac{2s}{at})^{1/2} \, t\right] \exp(-pt)dt$$

$$= a \int_{\frac{s}{a}}^{\infty} I_0\left[(\mathcal{T}^2 - \frac{s^2}{a^2})^{1/2}\right] \exp(-p\,\mathcal{T})d\mathcal{T}, \quad \mathcal{T} = t + \frac{s}{a}$$

In other words, $K(\begin{smallmatrix} t \\ s \end{smallmatrix}) = a \, I_0\left[(t^2 - \frac{s^2}{a^2})^{1/2}\right] u(t - \frac{s}{a})$ so that, in particular, $K(\begin{smallmatrix} t \\ s \end{smallmatrix}) = 0$ if $s > at$. Since the derivative of $I_0(t)$ with respect to t is $I_1(t)$, $K_t(\begin{smallmatrix} t \\ s \end{smallmatrix}) = \frac{at}{(t^2 - \frac{s^2}{a^2})^{1/2}} \, I_1\left[(t^2 - \frac{s^2}{a^2})^{1/2}\right] u(t - \frac{s}{a})$

and so, since $K(\begin{smallmatrix} t \\ at-0 \end{smallmatrix}) = a$, the Laplace Transform, over the half-plane $c > 1$, of $\frac{1}{2}\left[\phi(x-at) + \phi(x+at)\right]$

$$+ \frac{1}{2a} \int_0^{at} \frac{t}{(t^2 - \frac{s^2}{a^2})^{1/2}} \, I_1\left[(t^2 - \frac{s^2}{a^2})^{1/2}\right] \left\{\phi(x-s) + \phi(x+s)\right\} ds \text{ is}$$

$p\,\Gamma\phi$. Thus the solution of the boundary-value and initial condition problem D, B, I which is suggested by the application of the Laplace transformation is

$$d(x,t) = \frac{1}{2}\left[\phi(x-at) + \phi(x+at)\right]$$

$$+ \frac{1}{2a}\int_0^{at} \frac{t}{(t^2 - \frac{s^2}{a^2})^{1/2}} I_1\left[(t^2 - \frac{s^2}{a^2})^{1/2}\right]\left\{\phi(x-s)+\phi(x+s)\right\}ds$$

$$+ \frac{1}{2a}\int_0^{at} I_0\left[(t^2 - \frac{s^2}{a^2})^{1/2}\right]\left\{v(x-s) + v(x+s)\right\} ds.$$

We must, before verifying that the function $d(x,t)$ furnished by this formula is a solution of the boundary-value and initial condition problem D, B, I, justify the statement, made in the preceding paragraph, that the sum of the everywhere convergent infinite series $I_0(t) + \frac{s}{a} I_1(t) + \frac{s^2}{2!a^2} I_2(t) + \ldots$ is $I_0\left[(1 + \frac{2s}{at})^{1/2} t\right]$. To do this we recall that the Laplace Transform, over the half-plane $c > 0$, of $t^{\frac{n}{2}} I_n(2t^{1/2})\, u(t)$, $n = 0, 1, 2, \ldots,$ is $\exp(\frac{1}{p})/p^{n+1}$. On replacing t by $(1+\alpha)t'$ and p by $p'/(1+\alpha)$, where α is any positive real number, in the relation $\frac{1}{p^{n+1}} \exp(\frac{1}{p}) = \int_0^{\infty} t^{\frac{n}{2}} I_n(2t^{1/2}) \exp(-pt)dt$, $c > 0$, and then dropping the primes we obtain

$$\frac{(1+\alpha)^{\frac{n}{2}}}{p^{n+1}} \exp\left(\frac{1+\alpha}{p}\right) = \int_0^{\infty} t^{\frac{n}{2}} I_n\left[2(1+\alpha)^{1/2}t^{1/2}\right] \exp(-pt)dt,$$

$c > 0$, so that the Laplace Transform, over the half-plane $c > 0$, of $(1+\alpha)^{-\frac{n}{2}} t^{n/2} I_n\left[2(1+\alpha)^{1/2}t^{1/2}\right] u(t)$ is $\frac{1}{p^{n+1}} \exp \frac{1+\alpha}{p}$

$$= \frac{1}{p^{n+1}} \exp\left(\frac{1}{p}\right)\left\{1 + \frac{\alpha}{p} + \frac{\alpha^2}{2!p^2} + \ldots\right\}$$

Since the coefficients of the infinite series on the right-hand side are all non-negative, α being, by hypothesis, non-negative the infinite series $t^{n/2} I_n(2t^{1/2}) + \alpha t^{\frac{n+1}{2}} I_{n+1}(2t^{1/2})$ $+ \frac{\alpha^2}{2!} t^{\frac{n+2}{2}} I_{n+2}(2t^{1/2}) + \ldots$ is everywhere convergent and the product of its sum by u(t) has, over the half-plane $c > 0$, the Laplace Transform $\frac{1}{p^{n+1}} \exp \frac{1+\alpha}{p}$. Hence, by the uniqueness theorem,

$$t^{n/2} I_n(2t^{1/2}) + \alpha t^{\frac{n+1}{2}} I_{n+1}(2t^{1/2}) + \frac{\alpha^2}{2!} t^{\frac{n+2}{2}} I_{n+2}(2t^{1/2})$$
$$+ \ldots = (1+\alpha)^{-\frac{n}{2}} t^{\frac{n}{2}} I_n\left[2(1+\alpha)^{1/2} t^{1/2}\right] \quad 0 \leq t < \infty,$$

$$0 \leq \alpha < \infty$$

On dividing through by $t^{n/2}$ and writing $2t^{1/2} = t'$, and then dropping the prime, we obtain

$$I_n(t) + \frac{\alpha t}{2} I_{n+1}(t) + \frac{1}{2!} \left(\frac{\alpha t}{2}\right)^2 I_{n+2}(t) + \ldots$$
$$= (1+\alpha)^{-\frac{n}{2}} I_n\left[(1+\alpha)^{1/2} t\right], \quad 0 \leq t < \infty, \ 0 \leq \alpha < \infty$$

For any given value of $t \geq 0$ the left-hand side of this equation is a power series in α which converges for every positive value of α and so, if we regard α as a complex, rather than a real, variable the sum of this power series is an analytic function of α over the entire finite complex α-plane. The right-hand side, $(1+\alpha)^{-n/2} I_n\left[(1+\alpha)^{1/2} t\right]$, of our equation is also, being the sum of an everywhere convergent power series in $(1+\alpha)$, an analytic function of α over the entire finite complex α-plane and, since these two analytic functions of α coincide over the non-negative real axis in the complex α-plane, they coincide over the entire finite complex α-plane. In the same way we see that, α being any given complex number, $I_n(z) + \frac{\alpha z}{2} I_{n+1}(z) + \frac{1}{2!}\left(\frac{\alpha z}{2}\right)^2 I_{n+2}(z) + \ldots$ is an analytic function of the complex variable z which coincides over the entire finite complex z-plane with $(1+\alpha)^{-\frac{n}{2}} I_n((1+\alpha)^{1/2} z)$. Thus

$$I_n(z) + \frac{\alpha z}{2} I_{n+1}(z) + \frac{1}{2!}\left(\frac{\alpha z}{2}\right)^2 I_{n+2}(z) + \ldots$$
$$= (1+\alpha)^{-\frac{n}{2}} I_n\left[(1+\alpha)^{1/2} z\right]$$

where α and z are arbitrary complex numbers. On setting $z = iz'$ and then dropping the prime we see that this relation is equivalent to the relation

$$J_n(z) - \frac{\alpha z}{2} J_{n+1}(z) + \frac{1}{2!}\left(\frac{\alpha z}{2}\right)^2 J_{n+2}(z) - \ldots$$
$$= (1+\alpha)^{-\frac{n}{2}} J_n\left[(1+\alpha)^{1/2} z\right]$$

On setting $n = 0$, $z = t$, $\alpha = \frac{2s}{at}$, we obtain

$$I_0(t) + \frac{s}{a} I_1(t) + \frac{1}{2!}\frac{s^2}{a^2} I_2(t) + \ldots = I_0\left[\left(1+\frac{2s}{at}\right)^{1/2} t\right]$$

which is the relation we wished to prove.

Exercise 1. Show that $I_n(z) - \frac{z}{2} I_{n+1}(z) + \frac{1}{2!}\left(\frac{z}{2}\right)^2 I_{n+2}(z) - \ldots$

$$= \frac{z^n}{2^n \cdot n!}, \quad n = 0, 1, 2, \ldots, \quad z \text{ an arbitrary complex number.}$$

Hint. Set $\alpha = -1$.

Exercise 2. Show that $J_n(z) = I_n(z) - z I_{n+1}(z) + \frac{z^2}{2!} I_{n+2}(z) - \ldots$, $n = 0, 1, 2, \ldots$, z an arbitrary complex number, and deduce that $I_n(z) = J_n(z) + zJ_{n+1}(z) + \frac{z^2}{2!} J_{n+2}(z) + \ldots$, $n = 0, 1, 2, \ldots$, z an arbitrary complex number.

Hint. Set $\alpha = -2$.

Exercise 3. Show that $J_0(3^{1/2}z) = J_0(z) - zJ_1(z) + \frac{z^2}{2!} J_2(z) - \frac{z^3}{3!} J_3(z) + \ldots$, z an arbitrary complex number.

Exercise 4. Show that

$$J_0(2^{1/2}z) = J_0(z) - \frac{z}{2} J_1(z) + \frac{1}{2!}\left(\frac{z}{2}\right)^2 J_2(z) - \ldots$$

$$J_1(2^{1/2}z) = 2^{1/2}\left\{ J_1(z) - \frac{z}{2} J_2(z) + \frac{1}{2!}\left(\frac{z}{2}\right)^2 J_3(z) - \ldots \right\}$$

$$J_2(2^{1/2}z) = 2\left\{ J_2(z) - \frac{z}{2} J_3(z) + \frac{1}{2!}\left(\frac{z}{2}\right)^2 J_4(z) - \ldots \right\}$$

and so on, z an arbitrary complex number.

In our next lecture we shall verify that the function $d(x, t)$ which we have obtained in this lecture is a solution of the boundary value problem D, B, I and shall show that this boundary value problem does not possess any other solution.

Lecture 16

The Solution of the Generalized Vibrating String Problem

The solution, d(x, t), of the generalized vibrating string problem which has been suggested by applying the Laplace transformation may be written in the form $d_1(x,t) + d_2(x,t)$ where

$$d_1(x,t) = \frac{1}{2}\left\{\phi(x-at) + \phi(x+at)\right\} + \frac{t}{2a}\int_0^{at} \frac{I_1(\alpha)}{\alpha}\left\{\phi(x-s) + \phi(x+s)\right\} ds$$

$$d_2(x,t) = \frac{1}{2a}\int_0^{at} I_0(\alpha)\left\{v(x-s) + v(x+s)\right\} ds; \quad \alpha = \left(t^2 - \frac{s^2}{a^2}\right)^{1/2}$$

In order to verify that d(x, t) is a solution of the partial differential equation $a^2 d_{xx} - d_{tt} + d = 0$ it is sufficient to verify that $d_2(x,t)$ is a solution of this differential equation. Indeed, $d_1(x,t)$ is the derivative with respect to t of $\frac{1}{2a}\int_0^{at} I_0(\alpha)\left\{\phi(x-s) + \phi(x+s)\right\} ds$ and if $d_2(x,t)$ is a solution of the differential equation $a^2 d_{xx} - d_{tt} + d = 0$ so also is $\frac{1}{2a}\int_0^{at} I_0(\alpha)\left\{\phi(x-s) + \phi(x+s)\right\} ds$ and this implies that the derivative of this expression with respect to t, namely $d_1(x,t)$, is a solution of the differential equation $a^2 d_{xx} - d_{tt} + d = 0$. On making the substitution $s = x-s'$ in the integral $\int_0^{at} I_0(\alpha)\, v(x-s)ds$, and the substitution $s = s'-x$ in the integral $\int_0^{at} I_0(\alpha)\, v(x+s)ds$, and then dropping the prime $d_2(x,t)$ appears in the form

$$d_2(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} I_0(\beta)\, v(s)ds; \quad \beta = \left\{t^2 - \frac{(x-s)^2}{a^2}\right\}^{1/2}$$

Upon introducing as new independent variables the two functions $\xi = x-at$, $\tau = x+at$ of x and t whose level curves are the characteristics of the differential equation

$$D: \quad a^2 d_{xx} - d_{tt} + d = 0$$

D appears in the form

$$D^*: \quad 4a^2 d_{\xi\tau} + d = 0$$

and we have to verify that

$$\psi(\xi,\tau) = \int_{\xi}^{\tau} I_0(\beta)\, v(s)\, ds; \quad \beta = (\tau - s)^{1/2}(s-\xi)^{1/2}/a$$

is a solution of the partial differential equation D^*. Since $\beta = 0$ when $s = \tau$,

$$\psi_\tau = \frac{1}{2a}\int_{\xi}^{\tau} I'_0(\beta)\,(\tau-s)^{-1/2}(s-\xi)^{1/2}\, v(s)ds + v(\tau)$$

where the prime attached to I_0 denotes differentiation with respect to its argument β. On differentiating ψ_τ with respect to ξ we obtain

$$\psi_{\tau\xi} = -\frac{1}{4a^2}\int_{\xi}^{\tau}\left\{I''_0(\beta) + aI'_0(\beta)\,(\tau-s)^{-1/2}(s-\xi)^{-1/2}\right\}v(s)ds$$

$$= -\frac{1}{4a^2}\int_{\xi}^{\tau}\left\{I''_0(\beta) + \frac{1}{\beta}I'_0(\beta)\right\}v(s)ds$$

and it follows, since $I''_0(\beta) + \frac{1}{\beta}I'_0(\beta) = I_0(\beta)$, that $\psi_{\tau\xi} = -\frac{1}{4a^2}\psi$ which proves that $\psi(\xi,\tau)$ is a solution of the partial differential equation D^*. This completes the proof of the fact that $d(x,t) = d_1(x,t) + d_2(x,t)$ is a solution of the partial differential equation D.

That $d(x, t)$ satisfies the boundary conditions B and the initial conditions is proved in the same way as in the problem of the vibrating string. Thus $d_1(0, t) = 0$, since $\phi(x)$ is an odd function of the unrestricted real variable x and $d_2(0, t) = 0$ since $v(x)$ is also an odd function of the unrestricted real variable x; similarly $d_1(l, t) = 0$, $d_2(l, t) = 0$ since $\phi(x)$ and $v(x)$ are not only odd functions of x but are also periodic with period 2l. $d_1(x, 0)$ is evidently $\phi(x)$ and $d_2(x, 0)$ is evidently 0 so that $d(x, 0) = \phi(x)$. Finally, $\left[d_1(x, t)\right]_t$ is zero when $t = 0$ and $\left[d_2(x, t)\right]_t$ is $v(x)$ when $t = 0$ so that $d_t(x, 0) = v(x)$. This completes the proof of the fact that $d(x, t)$ is a solution of the generalized vibrating string boundary-value and initial condition problem.

The proof that the generalized vibrating string boundary-value and initial-condition problem does not possess a solution differing from $d(x, t)$ is not as simple as the proof of the corresponding uniqueness theorem for the vibrating string boundary-value and initial-condition problem. We first observe that, if (ξ_1, τ_1) is any point in the (ξ, τ) - plane, the function $w = I_0(\delta)$, where $\delta = (\xi - \xi_1)^{1/2}(\tau - \tau_1)^{1/2}/a$, of the two variables (ξ, τ) satisfies the differential equation $4a^2 w_{\xi\tau} + w = 0$. Indeed, on denoting differentiation with respect to δ by a prime,

$$w_\xi = \frac{1}{2a} I'_0 (\xi - \xi_1)^{-1/2} (\tau - \tau_1)^{1/2}, \quad w_{\xi\tau} = \frac{1}{4a^2} I''_0$$

$$+ \frac{1}{4a} I'_0 (\xi - \xi_1)^{-1/2} (\tau - \tau_1)^{-1/2} = \frac{1}{4a^2}\left\{I''_0 + \frac{1}{\delta} I'_0\right\} = \frac{1}{4a^2} I_0 = \frac{1}{4a^2} w.$$

On combining the two differential equations $4a^2 d_{\xi\tau} + d = 0$, $4a^2 \omega_{\xi\tau} + w = 0$ in such a way as to eliminate the undifferentiated functions d and w we obtain $wd_{\xi\tau} - dw_{\xi\tau} = 0$ or, equivalently, $(wd_{\xi})_{\tau} = (dw_{\tau})_{\xi}$. If, then, C is any piecewise smooth closed curve in the (ξ , τ)-plane the integral of wd_{ξ} with respect to ξ around C is the negative of the integral of $d\omega_{\tau}$ with respect to τ around C, both integrals being taken in the positive sense. Now $w = 1$ when $\delta = 0$, i.e., when $\xi = \xi_1$ or $\tau = \tau_1$ and $w_{\tau} = \frac{1}{2a} I'_0(\xi - \xi_1)^{\frac{1}{2}} (\tau - \tau_1)^{-1/2}$ is zero when $\xi = \xi_1$ and so, if C consists partly of segments of the lines $\xi = \xi_1$ and $\tau = \tau_1$ and, if we denote the remainder of C by Γ, we have

$$d(P_1) - d(\xi_1, \tau_1) + \int_{P_1}^{P_2} wd_{\xi}\, d\xi + \int_{P_1}^{P_2} dw_{\tau}\, d\tau = 0$$

where P_1 and P_2 are the points where the lines $\tau = \tau_1$ and $\xi = \xi_1$, respectively, intersect the curve Γ and both the integrals from P_1 to P_2 are taken along the curve Γ. Thus $d(\xi_1, \tau_1)$ is unambiguously determined by the values of d and of d_{ξ} along the curve Γ. If the curve Γ is a segment of the line $\xi - \tau = 0$, which corresponds to $t = 0$, d and $d_{\xi} = \frac{1}{2}(d_x - \frac{1}{a} d_t) = \frac{1}{2}(\phi_x - \frac{1}{a} v)$ are given along Γ and we see that $d(\xi_1, \tau_1)$ is unambiguously determinate. If the curve Γ is not a segment of the line $\xi - \tau = 0$ we have to deal with the phenomenon of reflection at the ends $x = 0$ and $x = 1$ and it is not hard to see that d_{ξ} is known along the lines $\xi + \tau = 0$ and

$\xi + \tau = 2l$ which correspond to $x = 0$ and $x = l$, respectively. It will suffice to consider the first reflection at the end $x = 0$ for which Γ consists of a segment, lying in the second quadrant and ending at the origin, of the line $\xi + \tau = 0$ and a segment, lying in the first quadrant and beginning at the origin, of the line $\xi - \tau = 0$. We take ξ_1 to be in the interval $0 < \xi < l$ and move the point (ξ_1, τ_1) towards the point P_2 along the line $\xi = \xi_1$. Observing that $d\xi = -d\tau$ along the line $\xi + \tau = 0$ we obtain

$$d(P_1) + d_\tau(P_1) - d_\xi(P_1) - d_\tau(P_2) + w(P_1)\, d_\xi(P_1) - d(P_1) w_\tau(P_1) = 0$$

In our boundary-value and initial-condition problem $d(x, t)$ is identically zero when $x = 0$ and so both d and $d_t = \frac{1}{a}(d_\tau - d_\xi)$ are zero along the line $\xi + \tau = 0$. Hence $w(P_1)\, d_\xi(P_1) = d_\tau(P_2)$ and since w is zero only at (ξ_1, τ_1) which is distinct from P_1, the ξ-coordinate of P_1 being negative, it follows that $d_\xi(P_1)$ is the quotient of $d_\tau(P_2) = \frac{1}{2}\left\{ d_x(P_2) + \frac{1}{a}\, d_t(P_2)\right\} = \frac{1}{2}\left\{\phi_x(x) + \frac{1}{a}\, v(x)\right\}$ by $w(P_1)$. Thus d_ξ is known along Γ and the solution of our boundary-value and initial-condition problem which is furnished by an application of the Laplace transformation is the only one which exists.

The modifications necessary to deal with the differential equation $a^2 d_{xx} - d_{tt} - d = 0$, rather than $a^2 d_{xx} - d_{tt} + d = 0$, the boundary and initial conditions being the same as before, are minor. Setting $q = \frac{1}{a}(1+p^2)^{1/2}$, instead of $\frac{1}{a}(1-p^2)^{1/2}$ as before, the boundary-value problem D', B, which we encounter has the same formal appearance as before, the difference between the two problems residing entirely in the different definitions of q as a function of p. All the singularities of sinh ql, regarded as a function of p, being furnished by the formula, $p^2 = -1 - \frac{n^2\pi^2a^2}{l^2}$, n = 0, 1, 2, ..., lie on the imaginary axis in the complex p-plane, and so the half-plane over which $d_{tt}\, u(t)$ is supposed to possess a Laplace Transform is the half-plane $c > 0$, rather than the half-plane $c > 1$ as before. In order to find the integral operator $K = K(\overset{t}{s})$, $s > 0$, which is such that the Laplace Transform, over the half-plane $c > 0$, of $K(\overset{t}{s})\, u(t)$ is $\exp(-qs)/q$ we set $p = \sinh z$ so that $aq = \cosh z = p + \exp(-z)$ and $\exp(-qs) = \exp(-\frac{sp}{a})\left\{1 - \frac{s}{a}\exp(-z) + \frac{s^2}{2!a^2}\exp(-2z) - \ldots\right\}$. Since $\exp(-nz)/\cosh z$, n = 0, 1, 2, ..., is the Laplace Transform, over the half-plane $c > 0$, of $J_n(t)\, u(t)$ it follows that $\exp(-qs)/q$ is the product of the Laplace Transform, over the half-plane $c > 0$, of $\left\{J_0(t) - \frac{s}{a} J_1(t) + \frac{s^2}{2!a^2} J_2(t) - \ldots\right\} u(t)$ by $a \exp(-\frac{s}{a} p)$. The sum of the infinite series $J_0(t) - \frac{s}{a} J_1(t) + \frac{s^2}{2!a^2} J_2(t) - \ldots$ is $J_0\left[(1 + \frac{2s}{at})^{1/2}\, t\right]$ and it follows that

$$K(\overset{t}{s}) = a\, J_0\left(t^2 - \frac{s^2}{a^2}\right)^{1/2} u(t - \frac{s}{a}),\ s \geq 0$$

and, since the derivative of $J_0(t)$ with respect to t is $-J_1(t)$, this implies that the Laplace Transform, over the half-plane $c > 0$, of $\frac{1}{2}\left[\phi(x-at) + \phi(x+at)\right] - \frac{t}{2a}\int_0^{at} \frac{1}{(t^2 - \frac{s^2}{a^2})^{1/2}} J_1\left[(t^2 - \frac{s^2}{a^2})^{1/2}\right]\left\{\phi(x-s) + \phi(x+s)\right\} ds$

is $p\Gamma\phi$. Thus the solution of our boundary-value and initial-condition problem which is suggested by the application of the Laplace Transformation is

$$d(x,t) = \frac{1}{2}\left[\phi(x-at) + \phi(x+at)\right] - \frac{t}{2a} \int_0^{at} \frac{1}{\left(t^2 - \frac{s^2}{a^2}\right)^{1/2}} I_1\left[\left(t^2 - \frac{s^2}{a^2}\right)^{1/2}\right]$$

$$\left\{\phi(x-s) + \phi(x+s)\right\} ds + \frac{1}{2a} \int_0^{at} J_0\left[\left(t^2 - \frac{s^2}{a^2}\right)^{1/2}\right]\left\{v(x-s)\right.$$

$$\left. + v(x+s)\right\} ds$$

it being understood that the range of definition of $\phi(x)$ and $v(x)$ is extended by the statement that both $\phi(x)$ and $v(x)$ are odd periodic functions, with period 2l, of the unrestricted real variable x.

The verification that this function of x and t actually is a solution of our boundary-value and initial-condition problem and the proof of the fact that this problem does not possess more than one solution are the same as before (the function $J_0(\delta)$, $\delta = (\xi - \xi_1)^{1/2}(\tau - \tau_1)^{1/2}/a$, playing, in the proof of the uniqueness theorem, the role previously played by $I_0(\delta)$.

Lecture 17

The Asymptotic Series for $\int_p^\infty \exp(-z^2)dz$

Let h(t) be a piecewise continuous right-sided function which is zero over the interval $0 \leq t \leq \delta$, where δ is any positive number, and let h(t) possess a Laplace Transform at a point $p = c_1 > 0$ of the positive real axis in the complex p-plane. We suppose, further, that $H(t) = \int_0^t h(s)ds$ is defined over $0 \leq t < \infty$ but we do not assume the existence of the infinite integral $\int_0^\infty h(s)ds$. We have seen that $H(t) \exp(-c_1 t)$ has the limit 0 at $t = \infty$ and this implies, since $H(t) \exp(-c_1 t)$ is everywhere continuous, that $H(t) \exp(-c_1 t)$ is bounded over $0 \leq t < \infty$, i.e., that there exists a positive constant M such that $|H(t)| \exp(-c_1 t) \leq M$ for every non-negative value of t. Thus H(t) possesses, over the half-plane $c > c_1$, an absolutely convergent Laplace Transform and $Lh = p(LH)$, $c > c_1$. Since H(t) is, like h(t), zero over the interval $0 \leq t \leq \delta$ we have

$$LH = \int_\delta^\infty H(t) \exp(-pt)dt = \int_\delta^\infty H(t) \exp(-c_1 t) \exp\left[-(p-c_1)t\right] dt$$

so that

$$|LH| \leq M \int_\delta^\infty \exp\left[-(c-c_1)t\right] dt = \frac{M}{c-c_1} \exp\left[-(c-c_1)\delta\right]$$

and the right-hand side of this inequality $\leq \frac{2M}{c} \exp\left[-(c-c_1)\delta\right]$ if $c \geq 2c_1$. On denoting arg p by θ, so that $|p| = c(\sec\theta)$, it follows that $|L(h)| \leq 2M(\sec\theta)\exp\left[-(c-c_1)\delta\right]$ or, equivalently, that

$$|Lh| \exp(c\,\delta) \leq 2M\,(\sec\theta)\exp(c_1\delta)$$

so that $|Lh|\exp(c\,\delta)$ is bounded as $p \longrightarrow \infty$ along the ray $0 \longrightarrow p$. If p lies in the sector $-\frac{\pi}{2} + \beta \leq \theta \leq \frac{\pi}{2} - \beta$, $0 < \beta < \frac{\pi}{2}$, $\sec\theta \leq \sec\beta$ and $|Lh|\exp(c\,\delta)$ is bounded over the part of the half-plane $c > c_1$ which is covered by the sector $-\frac{\pi}{2} + \beta \leq \theta \leq \frac{\pi}{2} - \beta$. Now the product of any positive power of p by $\exp(-c\,\delta)$ tends to zero as $p \longrightarrow \infty$ along any curve which lies in the sector $-\frac{\pi}{2} + \beta \leq \theta \leq \frac{\pi}{2} - \beta$, the convergence to zero being uniform over the sector, and so we have the following result:

The product of Lh by any positive power of p tends to zero as $p \longrightarrow \infty$ along any curve which lies in the sector $-\frac{\pi}{2} + \beta \leq \theta \leq \frac{\pi}{2} - \beta$, the convergence to zero being uniform over this sector.

We next consider a piecewise-continuous right-sided function h(t) which possesses a Laplace Transform at $p = c_1$ and which, while not zero over any interval $0 \leq t \leq \delta$, can be written, if δ is sufficiently small, in the form $t^{\alpha}\left\{A + \epsilon(t)\right\}$ where

1) α is a constant whose real part α_r is > -1 and A is any constant

2) $\epsilon(t)$ is continuous over $0 \leq t \leq \delta$ and arbitrarily small, say $|\epsilon(t)| < \epsilon$, if δ is sufficiently small, say $\delta \leq \delta_1$.

The right-sided function $h_1(t)$ which $= t^{\alpha}\left\{A + \epsilon(t)\right\}$ over the interval $0 \leq t \leq \delta_1$ and $= 0$ if $t > \delta_1$ possesses, over the half-plane $c > 0$, the Laplace Transform $\int_0^{\delta_1} t^{\alpha}\left\{A + \epsilon(t)\right\}\exp(-pt)dt$ and we may write this in the form $A\int_0^{\infty} t^{\alpha}\exp(-pt)dt$ $- A\int_{\delta_1}^{\infty} t^{\alpha}\exp(-pt)dt + \int_0^{\delta_1} t^{\alpha}\ \epsilon(t)\exp(-pt)dt = \dfrac{A\Gamma(\alpha+1)}{p^{\alpha+1}}$ $+ I_2 + I_3$, say. I_2 is the Laplace Transform of a piecewise continuous right-sided function which is zero over the interval $0 \leq t \leq \delta_1$ and so the product of $|I_2|$ by any positive power of p tends to zero as $p \longrightarrow \infty$ along any curve which lies in the sector $-\frac{\pi}{2} + \beta \leq \theta \leq \frac{\pi}{2} - \beta$.

$$|I_3| \leq \epsilon\int_0^{\delta_1} t^{\alpha_r}\exp(-ct)dt < \epsilon\int_0^{\infty} t^{\alpha_r}\exp(-ct)dt$$

$= \epsilon\,\dfrac{\Gamma(\alpha_r+1)}{c^{\alpha_r+1}}$ and, since $p^{\alpha+1} = \exp\left[(\alpha+1)\log p\right]$ so that

$$\left|p^{\alpha+1}\right| = \exp\left\{(\alpha_r + 1)\log|p| - \alpha_i\,\theta\right\} \leq (c\sec\beta)^{\alpha_r+1}\ \exp(|\alpha_i|\tfrac{\pi}{2}),$$

it follows that $p^{\alpha+1} I_3$ is arbitrarily small if δ_1 is sufficiently small. Since both $h(t)$ and $h_1(t)$ possess Laplace Transforms at $p = c_1$ so also does their difference $h_2(t) = h(t) - h_1(t)$ and, since $h_2(t)$ is zero over the interval $0 \leq t \leq \delta_1$, the product of Lh_2 by any positive power of p tends to zero as $p \longrightarrow \infty$ along any curve which lies in the sector $-\frac{\pi}{2} + \beta \leq \theta \leq \frac{\pi}{2} - \beta$. Since $Lh = Lh_1 + Lh_2$ it follows that $p^{\alpha+1}(Lh) - A\Gamma(\alpha+1)$ is arbitrarily small if 1) δ_1 is sufficiently small and 2) c is sufficiently large. Since $p^{\alpha+1}(Lh) - A\Gamma(\alpha+1)$

is independent of δ_1 the proviso 1) may be omitted and so we have the following important result:

$p^{\alpha+1}(Lh) - A\Gamma(\alpha+1)$ tends to zero as $p \longrightarrow \infty$ along any curve which lies in the sector $-\frac{\pi}{2} + \beta \leq \theta \leq \frac{\pi}{2} - \beta$, the convergence to zero being uniform over this sector.

The right-sided function $h(t) = \exp\left(-\frac{t^2}{4}\right) u(t)$ possesses, at any point p of the complex p-plane, the Laplace Transform $2 \exp(p^2) \int_p^\infty \exp(-z^2)\, dz$, the integral being extended along the ray, from p to ∞ in the complex z-plane, whose angle is zero. We denote by $s_n(t)$ the sum of the first n terms of the power series development of $\exp\left(-\frac{t^2}{4}\right)$ near $t = 0$:

$$s_n(t) = 1 - \left(\frac{t}{2}\right)^2 + \frac{1}{2!}\left(\frac{t}{2}\right)^4 - \ldots + (-1)^{n-1} \frac{1}{(n-1)!} \left(\frac{t}{2}\right)^{2n-2}$$

and observe that $s_n(t)\, u(t)$ possesses, over the half-plane $c > 0$, the Laplace Transform

$$L(s_n(t)\, u(t)) = \frac{1}{p} - \frac{1}{2p^3} + \frac{1.3}{2^2 p^5} - \ldots + (-1)^{n-1} \frac{1.3\ldots(2n-3)}{2^{n-1} p^{2n-1}}$$

The right-sided function $h_n(t) = \left\{\exp\left(-\frac{t^2}{4}\right) - s_n(t)\right\} u(t)$ is of the form $t^{2n}\left\{\frac{(-1)^n}{n!\,2^{2n}} + \epsilon(t)\right\} u(t)$ where $\epsilon(t)$ is continuous over any interval $0 \leq t \leq \delta$ and, furthermore, $|\epsilon(t)|$ is arbitrarily small if δ is sufficiently small, by virtue of the continuity, at $t = 0$, of $\exp\left(-\frac{t^2}{4}\right) - s_n(t)$ and the fact that the value, at $t = 0$, of $\exp\left(-\frac{t^2}{4}\right) - s_n(t)$ is 0. Hence $p^{2n+1} L(h_n(t)) - (-1)^n \frac{(2n)!}{n!\, 2^{2n}}$ tends to zero as $p \longrightarrow \infty$ along any curve which lies in the sector $-\frac{\pi}{2} + \beta \leq \theta \leq \frac{\pi}{2} - \beta$, $0 < \beta < \frac{\pi}{2}$. The Laplace Transform, over the half-plane $c > 0$, of $h_n(t)$ is $2\exp(p^2)\int_p^\infty \exp(-z^2)dz$

$$- \left\{ \frac{1}{p} - \frac{1}{2p^3} + \frac{1.3}{2^2.p^5} - \ldots + (-1)^{n-1} \frac{1.3\ldots(2n-3)}{2^{n-1}p^{2n-1}} \right\}$$

and so the product of

$$\Delta_{n+1} = 2(\exp p^2)\int_p^\infty \exp(-z^2)dz - \left\{ \frac{1}{p} - \frac{1}{2p^3} + \frac{1.3}{2^2p^5} - \ldots + (-1)^n \frac{1.3\ldots(2n-1)}{2^n.\ p^{2n+1}} \right\}$$

by p^{2n+1} tends to zero as $p \longrightarrow \infty$ along any curve which lies in the sector $-\frac{\pi}{2} + \beta \leq \theta \leq \frac{\pi}{2} - \beta$. We express this result by the statement that the infinite series $\frac{1}{p} - \frac{1}{2p^3} + \frac{1.3}{2^2p^5} - \frac{1.3.5}{2^3p^7} + \ldots$, which fails to converge at any point p of the finite complex p-plane, is an asymptotic series, over the sector $-\frac{\pi}{2} < \arg p < \frac{\pi}{2}$, for the function $2\exp(p^2)\int_p^\infty \exp(-z^2)dz$ of the complex variable p and we write

$$2\exp(p^2)\int_p^\infty \exp(-z^2)dz \sim \frac{1}{p} - \frac{1}{2p^3} + \frac{1.3}{2^2p^5} - \ldots,$$

$$-\frac{\pi}{2} < \arg p < \frac{\pi}{2}$$

The sector over which this asymptotic series is valid may be enlarged to $-\frac{3\pi}{4} < \arg p < \frac{3\pi}{4}$. To see this, let α be any number in the interval $0 < \alpha < \frac{\pi}{4}$ and set $t = v\exp(-i\alpha)$ in the infinite integral $\int_0^\infty \exp\left(-\frac{t^2}{4}\right)\exp(-pt)\,dt$ which defines $L(\exp(-\frac{t^2}{4})\,u(t))$; then $L(\exp(-\frac{t^2}{4})\,u(t))$ appears in the form

$$\exp(-i\alpha)\int_0^{\infty} \exp\left[-\frac{v^2}{4}\exp(-2i\alpha)\right]\exp\left[-p\exp(-i\alpha)v\right]dv,$$ the integral being along the ray from 0 to ∞ in the complex v-plane whose angle is α. The modulus of the integrand of this integral, at any point v whose modulus and argument are R and ϕ, respectively, is $\exp -\left[\frac{R^2}{4}\cos 2(\alpha-\phi) + |p| R\cos(\alpha-\phi-\theta)\right]$, where θ is the argument of p, and, if $0 \leq \phi \leq \alpha$, so that $\cos 2(\alpha-\phi)$ is positive, the product of this modulus by R tends to zero as $R \longrightarrow \infty$, the convergence being uniform with respect to ϕ. Hence the integral of $\exp\left[-\frac{v^2}{4}\exp(-2i\alpha)\right]\exp\left[-p\exp(-i\alpha)\ v\right]$ along the arc of the circle $|v| = R$ from $v = R$ to $v = R\exp(i\alpha)$ tends to zero as $R \longrightarrow \infty$ and this implies that the integral of $\exp\left[-\frac{v^2}{4}\exp(-2i\alpha)\right]\exp\left[-p\exp(-i\alpha)v\right]$ along the ray from 0 to ∞, in the complex v-plane, whose angle is α is the same as the integral of the same integrand along the positive real axis in the complex v-plane. Thus $2\exp(p^2)\int_p^{\infty}\exp(-z^2)dz$, which is the Laplace Transform of $\exp\left(-\frac{t^2}{4}\right)u(t)$, may be written in the form $\exp(-i\alpha)\int_0^{\infty}\exp\left[-\frac{t^2}{4}\exp(-2i\alpha)\right]\exp\left[-p\exp(-i\alpha)t\right]dt$ where $0 < \alpha < \frac{\pi}{4}$ and the same argument shows that it may be written in this same form if $-\frac{\pi}{4} < \alpha < 0$ so that

$$2\exp(p^2)\int_p^{\infty}\exp(-z^2)dz =$$

$$= \exp(-i\alpha)\int_0^{\infty}\exp\left[-\frac{t^2}{4}\exp(-2i\alpha)\right]\exp\left[-p\exp(-i\alpha)t\right]dt;$$

$$-\frac{\pi}{4} < \alpha < \frac{\pi}{4}.$$

On denoting by $s'_n(t)$ the sum of the first n terms of the power series development of $\exp\left[-\frac{t^2}{4}\exp(-2i\alpha)\right]$ near $t = 0$:

$$s'_n(t) = 1 - \exp(-2i\alpha)\left(\frac{t}{2}\right)^2 + \frac{\exp(-4i\alpha)}{2!}\left(\frac{t}{2}\right)^4 - \ldots$$

$$+ (-1)^{n-1}\frac{\exp-(2n-2)\alpha}{(n-1)!}\left(\frac{t}{2}\right)^{2n-2}$$

and denoting $p\exp(-i\alpha)$ by p' we have

$$\exp(-i\alpha)\int_0^\infty s'_n(t)\exp(-p't)dt = \frac{\exp(-i\alpha)}{p'} - \frac{1}{2}\frac{\exp(-3i\alpha)}{(p')^3} + \ldots$$

$$+ (-1)^{n-1}\frac{1.3\ldots(2n-3)\exp-(2n-1)\alpha}{2^{n-1}(p')^{2n-1}} = \frac{1}{p} - \frac{1}{2p^3} + \ldots$$

$$+ (-1)^{n-1}\frac{1.3\ldots 2n-3}{2^{n-1}p^{2n-1}}$$

provided that $-\frac{\pi}{2} < \arg p' < \frac{\pi}{2}$ or, equivalently, that $-\frac{\pi}{2} + \alpha < \arg p < \frac{\pi}{2} + \alpha$. Since α may be assigned any value in the interval $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$ it follows, by a repetition of the argument already given in the case $\alpha = 0$, that the asymptotic series $\frac{1}{p} - \frac{1}{2p^3} + \frac{1.3}{2^2p^5} - \ldots$ for $2\exp(p^2)\int_p^\infty \exp(-z^2)dz$ is valid over the sector $-\frac{3\pi}{4} < \arg p < \frac{3\pi}{4}$.

In order to deal with the sector $\frac{3\pi}{4} \leq \arg p \leq \frac{5\pi}{4}$ of the complex p-plane we set $z = -z'$ in the infinite integral $\int_p^\infty \exp(-z^2)dz$ and then drop the prime:

$$\int_p^\infty \exp(-z^2)dz = \int_{-\infty}^{-p}\exp(-z^2)dz = \int_{-\infty}^{\infty}\exp(-z^2)dz$$

$$- \int_{-p}^{\infty}\exp(-z^2)dz$$

Now $\int_{-\infty}^{\infty} \exp(-z^2)\, dz = \exp(y^2) \int_{-\infty}^{\infty} \exp(-x^2) \exp(-2iyx)dx$,

$z = x+iy$, and $\int_{-\infty}^{\infty} \exp(-x^2) \exp(-2iyx)dx$, being the Laplace Transform at $p = 2iy$ of $\exp(-t^2)$, is $\pi^{1/2} \exp(-y^2)$ so that $\int_{-\infty}^{\infty} \exp(-z^2)dz = \pi^{1/2}$. When $\frac{3\pi}{4} \leq \arg p \leq \frac{5\pi}{4}$,

$-\frac{\pi}{4} \leq \arg(-p) \leq \frac{\pi}{4}$ and so we may use the asymptotic series we have already obtained for $2 \exp(-p)^2 \int_{-p}^{\infty} \exp(-z^2)dz$: to obtain the result

$$2 \exp(p^2) \int_{p}^{\infty} \exp(-z^2)dz \sim 2\pi^{1/2} \exp(p^2) - \left[\frac{1}{-p} - \frac{1}{2(-p)^3} + \dots\right]$$

$$= 2\pi^{1/2} \exp(p^2) + \frac{1}{p} - \frac{1}{2p^3} + \frac{1.3}{2^2p^5} - \dots; \frac{3\pi}{4} \leq \arg p \leq \frac{5\pi}{4}.$$

We may use this trick if $\frac{\pi}{2} \leq \arg p < \frac{3\pi}{4}$ or if $-\frac{5\pi}{4} < \arg p \leq -\frac{\pi}{2}$ and so we obtain two different asymptotic expressions for $2 \exp(p^2) \int_{p}^{\infty} \exp(-z^2)dz$ over these sectors, the difference being that one of the asymptotic expressions contains the additive term $2\pi^{1/2} \exp(p^2)$. When p lies in either of these two sectors the real part of p^2 is negative and so the product of $\exp(p^2)$ by any positive power of p tends to zero as $p \longrightarrow \infty$ along any curve which lies in the sector. Thus we see that we may use, for the function $2 \exp(p^2) \int_{p}^{\infty} \exp(-z^2)dz$, the asymptotic series $\frac{1}{p} - \frac{1}{2p^3} + \frac{1.3}{2^2p^3} - \dots$, with or without the additive term $2\pi^{1/2}\exp(p^2)$, over the sectors $\frac{\pi}{4} < \arg p < \frac{3\pi}{4}$, $-\frac{3\pi}{4} < \arg p < -\frac{\pi}{4}$ while over the sector $\frac{3\pi}{4} \leq \arg p \leq \frac{5\pi}{4}$ the additive term must be used and over the sector $-\frac{\pi}{4} \leq \arg p \leq \frac{\pi}{4}$ it must not be used. The fact that $2\exp(p^2)\int_p^\infty \exp(-z^2)dz$ has different asymptotic expressions over different sectors of the complex p-plane is an instance of what is known as Stokes' phenomenon.

In order to appraise, when $-\frac{\pi}{2} < \arg p < \frac{\pi}{2}$, the difference Δ_{n+1} between $2\exp(p^2)\int_p^\infty \exp(-z^2)dz$ and the sum of the first n+1 terms of the asymptotic series $\frac{1}{p} - \frac{1}{2p^3} + \frac{1.3}{2^2p^5} - \ldots$ we observe that, if τ is any real number, $\exp\tau = 1 + \tau + \frac{\tau^2}{2!} + \ldots \frac{\tau^n}{n!} + \exp(\theta\tau)\frac{\tau^{n+1}}{(n+1)!}$ where $0 < \theta < 1$, θ varying with τ. Setting $\tau = -\frac{t^2}{4}$ we obtain

$$\exp\left(-\frac{t^2}{4}\right) = 1 - \left(\frac{t}{2}\right)^2 + \frac{1}{2!}\left(\frac{t}{2}\right)^4 - \ldots + (-1)^n\frac{1}{n!}\left(\frac{t}{2}\right)^{2n}$$

$$+ (-1)^{n+1}\frac{\exp\left(-\frac{\theta t^2}{4}\right)}{(n+1)!}\left(\frac{t}{2}\right)^{2n+2}$$ so that Δ_{n+1}, which is the Laplace Transform, over the half-plane $c > 0$, of $\left[\exp\left(-\frac{t^2}{4}\right) - \left\{1 - \left(\frac{t}{2}\right)^2 + \ldots + (-1)^n\frac{1}{n!}\left(\frac{t}{2}\right)^{2n}\right\}\right]u(t)$ may be written as $\frac{(-1)^{n+1}}{2^{2n+2}.(n+1)!}\int_0^\infty \exp\left(-\frac{\theta t^2}{4}\right)t^{2n+2}\exp(-pt)dt.$

Since $0 < \theta < 1$, $0 < \exp\left(-\frac{\theta t^2}{4}\right) \leq 1$, no matter what is the value of t, and so $|\Delta_{n+1}| \leq \frac{1}{2^{2n+2}.(n+1)!}\int_0^\infty t^{2n+2}\exp(-ct)dt$ $= \frac{1.3.\ldots.(2n+1)}{2^{n+1}c^{2n+3}}$. Thus Δ_{n+1} is dominated by the first term omitted of the asymptotic series. Furthermore, if $p = c > 0$ is real, Δ_{n+1} has the sign of this first term omitted since $\int_0^\infty \exp\left(-\frac{\theta t^2}{4}\right) t^{2n+2} \exp(-ct)dt$ is positive. For example, $2\exp(c^2) \int_c^\infty \exp(-t^2)dt$, $c > 0$, lies between $\frac{1}{c}$ and $\frac{1}{c} - \frac{1}{2c^3}$ so that if we use $\frac{\exp(-c^2)}{2c}$ as an approximation to $\int_c^\infty \exp(-t^2)dt$, $c > 0$, this approximation is in excess by less than $\frac{100}{2c^2-1}$ per cent; if we use $\exp(-c^2)\left\{\frac{1}{2c} - \frac{1}{4c^3}\right\}$ as an approximation to $\int_c^\infty \exp(-t^2)dt$, $c > 0$, this approximation is too small by less than $\frac{150}{c^2(2c^2-1)}$ per cent and so on. If $c = 1$, the term of the asymptotic series $\frac{1}{c} - \frac{1}{2c^3} + \frac{1.3}{2^2c^5} - \ldots$ whose numerical value is least is the second term and the asymptotic series cannot guarantee a better approximation than that given by its first term; if $c = 2$, the term whose numerical value is least is the fifth and the asymptotic series cannot guarantee a better approximation than that given by the sum of its first 4 terms. In general, if c is an integer, the asymptotic series cannot guarantee a better approximation than that given by the sum of its first c^2 terms.

Lecture 18

The Asymptotic Series for $(2\pi p)^{1/2} \exp(-p)\, I_n(p), |\arg p| < \frac{\pi}{2}$, The Hankel Functions

If p is any complex number, the infinite series $I_0(p) + 2(\cos\theta)\, I_2(p) + 2(\cos 4\theta)\, I_4(p) + \dots$ converges, with the sum $\cosh(p\cos\theta)$, and, since this infinite series is dominated by the infinite series $I_0(|p|) + 2\, I_2(|p|) + \dots$ which converges, with the sum $\cosh|p|$, the convergence is uniform over any closed interval. Hence the infinite series may be integrated term-by-term, after multiplication by $\cos 2m\theta$, $m = 0, 1, 2, \dots,$ over the interval $0 \leq \theta \leq \pi$ so that

$$I_{2m}(p) = \frac{1}{\pi} \int_0^{\pi} \cosh(p\cos\theta)\,(\cos 2m\theta)\, d\theta, \quad m = 0, 1, 2, \dots$$

If, then, $f(\theta)$ is any linear combination, $c_0 + c_1 \cos 2\theta + \dots + c_n \cos 2n\theta$, of the functions $\cos 2m\theta$, $m = 0, 1, \dots, n$, we have

$$\frac{1}{\pi} \int_0^{\pi} \cosh(p\cos\theta)\, f(\theta)\, d\theta = c_0 I_0(p) + c_1 I_2(p) + \dots + c_n I_{2n}(p)$$

Now, $(-1)^n 2^{2n-1} \sin^{2n}\theta = \frac{1}{2}\left\{\exp(i\theta) - \exp(-i\theta)\right\}^{2n}$, $n = 1, 2, \dots,$ is such a linear combination, the coefficients $c_n, c_{n-1}, \dots, c_1$ being the first n coefficients of the binomial expansion $(1-x)^{2n} = 1 - 2nx + \binom{2n}{2} x^2 - \dots$ and

$c_0 = (-1)^n \frac{1}{2} \binom{2n}{n}$ being one-half the (n+1)st term of this binomial expansion. For example, when $n = 1$, $-2 \sin^2\theta = \cos 2\theta - 1$; when $n = 2$, $2^3 \sin^4 \theta = \cos 4\theta - 4 \cos 2\theta + 3$, and so on. Hence

$$\frac{(-1)^n 2^{2n-1}}{\pi} \int_0^\pi \cosh(p \cos \theta) \sin^{2n}\theta \, d\theta = c_n I_{2n}(p)$$

$$+ c_{n-1} I_{2n-2}(p) + \dots + c_0 I_0(p)$$

where $c_n, \dots, c_1$ are the first n coefficients of the binomial expansion of $(1-x)^{2n}$ and c_0 is one-half the (n+1)st coefficient of this expansion. Setting $n = 1$, we obtain

$$-\frac{2}{\pi} \int_0^\pi \cosh(p \cos \theta) (\sin^2\theta) \, d\theta = I_2(p) - I_0(p) = -\frac{2}{p} I_1(p)$$

so that

$$\frac{1}{\pi} \int_0^\pi \cosh(p \cos \theta) (\sin^2\theta) \, d\theta = \frac{1}{p} I_1(p)$$

Setting $n = 2$, we obtain

$$\frac{2^3}{\pi} \int_0^\pi \cosh(p \cos \theta) (\sin^4\theta) \, d\theta = I_4(p) - 4 I_2(p) + 3 I_0(p)$$

Now, $I_4(p) = I_2(p) - \frac{6}{p} I_3(p) = I_2(p) - \frac{6}{p} \left\{ I_1(p) - \frac{4}{p} I_2(p) \right\}$

$$= I_2(p) + 3 \left\{ I_2(p) - I_0(p) \right\} + \frac{24}{p^2} I_2(p)$$

so that $I_4(p) - 4 I_2(p) + 3 I_0(p) = \frac{24}{p^2} I_2(p)$. Hence

$$\frac{1}{\pi} \int_0^\pi \cosh (p \cos \theta) (\sin^4\theta) \, d\theta = \frac{3}{p^2} I_2(p).$$

These two results are special cases, corresponding to $n = 1$ and $n = 2$, of the general formula

$$\frac{1}{\pi}\int_0^\pi \cosh(p\cos\theta)\,(\sin^{2n}\theta)\,d\theta = \frac{A_n}{p^n}\,I_n(p)$$

where $A_n = 1.3.5\ldots(2n-1)$, $n = 1, 2, \ldots$. Setting $A_0 = 1$ this formula covers the case $n = 0$ so that

$$\frac{1}{\pi}\int_0^\pi \cosh(p\cos\theta)\,(\sin^{2n}\theta)\,d\theta = \frac{A_n}{p^n}\,I_n(p),\ n = 0, 1, 2, \ldots$$

This general formula may be easily verified by multiplying the infinite series $I_0(p) + 2(\cos 2\theta)\,I_2(p) + \ldots$, whose sum is $\cosh(p\cos\theta)$, by $\sin^{2n}\theta$ and integrating term-by-term. We prefer, however, to derive it by an extension of the method by which we proved it in the case $n = 2$ since we obtain in this way a useful generalization of the recurrence relation, $I_n(p) = I_{n-2}(p) - \frac{2(n-1)}{p}\,I_{n-1}(p)$, $n = 2, 3, \ldots$. Replacing n by n-1 in this relation we obtain $I_{n-1}(p) = I_{n-3}(p) - \frac{2(n-2)}{p}\,I_{n-2}(p)$, $n = 3, 4, \ldots$, so that

$$I_n(p) = I_{n-2}(p) - \frac{2(n-1)}{p}\,I_{n-3}(p) + \frac{2^2(n-1)(n-2)}{p^2}\,I_{n-2}(p),\ n=3, 4, \ldots$$

and since $\frac{2(n-3)}{p}\,I_{n-3}(p) = I_{n-4}(p) - I_{n-2}(p)$, $n = 4, 5, \ldots$, the right-hand side of the relation just derived may be replaced, when $n = 4, 5, \ldots$, by $I_{n-2}(p) - \frac{n-1}{n-3}\left\{I_{n-4}(p) - I_{n-2}(p)\right\} + \frac{2^2(n-1)(n-2)}{p^2}\,I_{n-2}(p)$. Thus

$$I_n(p) - 2\,\frac{n-2}{n-3}\,I_{n-2}(p) + \frac{n-1}{n-3}\,I_{n-4}(p) = \frac{2^2(n-1)(n-2)}{p^2}\,I_{n-2}(p),\ n=4, 5, \ldots$$

This is our first extension of the recurrence relation $I_n(p) - I_{n-2}(p) = -\frac{2(n-1)}{p}\,I_{n-1}(p)$, $n = 2, 3, \ldots$; it yields, when $n = 4$, the relation

$$I_4(p) - 4\,I_2(p) + 3\,I_0(p) = \frac{2^3.3}{p^2}\,I_2(p)$$

On setting $I_{n-2}(p) = I_{n-4}(p) - \frac{2(n-3)}{p} I_{n-3}(p)$ on the right-hand side of this extension of the recurrence relation and using the fact that

$$I_{n-2}(p) - 2\frac{n-4}{n-5} I_{n-4}(p) + \frac{n-3}{n-5} I_{n-6}(p) = \frac{2^2(n-3)(n-4)}{p^2} I_{n-4}(p), \; n=6,7,\ldots,$$

we obtain

$$I_n(p) - 2\frac{n-2}{n-3} I_{n-2}(p) + \frac{n-1}{n-3} I_{n-4}(p) = \frac{(n-1)(n-2)}{(n-3)(n-4)} \left\{ I_{n-2}(p) - 2\frac{n-4}{n-5} I_{n-4}(p) + \frac{n-3}{n-5} I_{n-6}(p) \right\} - \frac{2^3(n-1)(n-2)(n-3)}{p^3} I_{n-3}(p)$$

or, equivalently,

$$I_n(p) - \frac{3(n-2)}{n-4} I_{n-2}(p) + \frac{3(n-1)}{n-5} I_{n-4}(p) - \frac{(n-1)(n-2)}{(n-4)(n-5)} I_{n-6}(p) = -\frac{2^3}{p^3}(n-1)(n-2)(n-3) I_{n-3}(p), \; n=6,\ 7,\ 8,\ \ldots.$$

This is our second extension of the recurrence relation $I_n(p) - I_{n-2}(p) = -2\frac{n-1}{p} I_{n-1}(p)$; it yields, when $n = 6$, the relation

$$I_6(p) - 6\, I_4(p) + 15\, I_2(p) - 10\, I_4(p) = -\frac{2^5.3.5}{p^3} I_3(p)$$

On setting $n = 3$ in the relation $(-1)^n \frac{2^{2n-1}}{\pi} \int_0^{\pi} \cosh(p\cos\theta)\,(\sin^{2n}\theta)\,d\theta = c_n I_{2n}(p) + \ldots + c_0 I_0(p)$ we obtain

$$\frac{1}{\pi} \int_0^{\pi} \cosh(p\cos\theta)\,(\sin^6\theta)\,d\theta = \frac{3.5}{p^3} I_3(p) = \frac{A_3}{p^3} I_3(p)$$

Proceeding in this way, we obtain the following generalized recurrence relation whose validity may be verified by an induction proof:

$$I_m(p) - n\,\alpha_n^1 I_{m-2}(p) + \binom{n}{2} \alpha_n^2 I_{m-4}(p) - \ldots + (-1)^n \alpha_n^n I_{m-2n}(p) = (-1)^n \frac{2(m-1)(m-2)\ldots(m-n)}{p^n} I_{m-n}(p)$$

$n = 1, 2, \ldots, m = 2n, 2n+1, \ldots$. The coefficient, α_n^k, of

$(-1)^k \binom{n}{k} I_{m-2k}(p)$ on the left-hand side of this relation is a fraction whose denominator is $(m-n-1) \ldots (m-n-k)$ and whose numerator is independent of n, this numerator being determined by the fact that $\alpha_k^k = \dfrac{(m-1) \ldots (m-k+1)}{(m-k-1) \ldots (m-2k+1)}$ if $k = 2, 3, \ldots,$ while $\alpha_1^1 = 1$. Thus $\alpha_n^1 = \dfrac{m-2}{m-n-1}$ and $\alpha_n^k = \dfrac{(m-1)(m-2) \ldots (m-k+1)(m-2k)}{(m-n-1) \ldots \quad (m-n-k)}$, $k = 2, 3, \ldots, n$. When $m = 2n$, this generalized recurrence formula yields the relation

$$I_{2n}(p) - 2n\, I_{2n-2}(p) + \binom{2n}{2} I_{2n-4}(p) - \ldots + (-1)^n \frac{1}{2} \binom{2n}{n} I_0(p)$$
$$= (-1)^n \frac{2^{2n-1}}{p^n} A_n I_n(p)$$

and this implies that $\dfrac{1}{\pi} \displaystyle\int_0^{\pi} \cosh(p \cos \theta)(\sin^{2n}\theta) d\theta = \frac{A_n}{p^n} I_n(p)$, $n = 0, 1, 2, \ldots$. On replacing $\cosh(p \cos \theta)$ by $\frac{1}{2}\left\{\exp(p \cos \theta) + \exp(-p \cos \theta)\right\}$ and observing that $\displaystyle\int_0^{\pi} \exp(p \cos \theta)(\sin^{2n}\theta)\, d\theta = \int_0^{\pi} \exp(-p \cos \theta')(\sin^{2n} \theta')\, d\theta'$, $\theta' = \pi - \theta$, we obtain

$$\pi A_n p^{-n} I_n(p) = \int_0^{\pi} (\sin^{2n}\theta) \exp(-p \cos \theta)\, d\theta$$
$$= \int_{-1}^{1} (1-v^2)^{n-(1/2)} \exp(-pv) dv, \quad v = \cos \theta$$
$$= (\exp p) \int_0^{2} t^{n-(1/2)} (2-t)^{n-(1/2)} \exp(-pt) dt, \quad t = v+1$$

so that $\pi A_n p^{-n} \exp(-p) I_n(p)$ is the Laplace Transform, over the entire finite complex p-plane, of the right-sided function which $= t^{n-(1/2)}(2-t)^{n-(1/2)}$ over the interval $0 < t < 2$ and which $= 0$, if $t > 2$.

If $n = 0$, this right-sided function is unbounded at $t = 0$ and $t = 2$.

Over the interval $0 < t < 2$, $t^{n-(1/2)}(2-t)^{n-(1/2)}$ may be written in the form

$$2^{n-(1/2)} t^{n-(1/2)} \left[1 - (n - \frac{1}{2}) \frac{t}{2} + \frac{(n - \frac{1}{2})(n - \frac{3}{2})}{2!} (\frac{t}{2})^2 - \dots + (-1)^{k-1} \frac{(n - \frac{1}{2}) \dots (n - \frac{2k-3}{2})}{(k-1)!} (\frac{t}{2})^{k-1}\right]$$

$$+ t^{n+k-(1/2)}\left[(-1)^k \frac{2^{n-k+(1/2)}(n - \frac{1}{2}) \dots (n - \frac{2k-1}{2})}{k!} + \epsilon(t)\right]$$ where $\epsilon(t)$

is continuous over $0 \leq t < 2$ and is arbitrarily small over $0 \leq t \leq \delta$, if δ is sufficiently small. Hence, if the real part c of p is positive, the product of $\pi A_n p^{-n} \exp(-p) I_n(p)$

$$- 2^{n-(1/2)}\left[\frac{\Gamma(n + \frac{1}{2})}{p^{n+(1/2)}} - \frac{(n - \frac{1}{2})\Gamma(n + \frac{3}{2})}{2p^{n+(3/2)}} + \dots + (-1)^k \frac{(n - \frac{1}{2}) \dots (n - \frac{2k-1}{2})}{2^k . k!} \frac{\Gamma(n + k + \frac{1}{2})}{p^{n+k+(1/2)}}\right]$$ by $p^{n+k+(1/2)}$

tends to zero as $p \to \infty$ along the ray $0 \to p$, the convergence being uniform over the sector $-\frac{\pi}{2} + \beta \leq \arg p < \frac{\pi}{2} - \beta$, where β is any positive number less than $\frac{\pi}{2}$. Since $\Gamma(n + \frac{1}{2}) = (n - \frac{1}{2}) \Gamma(n - \frac{1}{2})$ $= \dots = (n - \frac{1}{2})(n - \frac{3}{2}) \dots \frac{1}{2} \pi^{1/2} = 2^{-n} A_n \pi^{1/2}$ it follows, on multiplication by $2^{1/2} \pi^{-1/2} A_n^{-1} p^{n+(1/2)}$, that the product of

$$\Delta_{k+1} = (2\pi p)^{1/2} \exp(-p) I_n(p) - \left[1 - \frac{4n^2-1}{8p} + \frac{(4n^2-1)(4n^2-3^2)}{2!(8p)^2} - \dots + (-1)^k \frac{(4n^2-1)(4n^2-3^2) \dots \{4n^2-(2k-1)^2\}}{k!(8p)^k}\right]$$

by p^k tends to zero as $p \longrightarrow \infty$ along any curve which is covered by the sector $-\frac{\pi}{2} + \beta \leq \arg p < \frac{\pi}{2} - \beta$ so that the infinite series $1 - \frac{4n^2-1}{8p} + \frac{(4n^2-1)(4n^2-3^2)}{2!(8p)^2} - \ldots$, which fails to converge at any point of the finite complex p-plane, is an asymptotic series, over the sector $-\frac{\pi}{2} + \beta \leq \arg p \leq \frac{\pi}{2} - \beta$, for the function $(2\pi p)^{-1/2} \exp(-p)\, I_n(p)$ of the complex variable p.

The asymptotic series which we have just obtained for $(2\pi p)^{-1/2} \exp(-p)\, I_n(p)$ is not valid when p = it is a pure imaginary and so it fails to provide an asymptotic series for $J_n(t)$. To obtain an asymptotic series useful in the calculation of $J_n(t)$ we proceed as follows. Let v be a complex variable and let the complex v-plane be cut along the segment $0 < v < 2$ of its real axis so as to make $v^{n-(1/2)}(2-v)^{n-(1/2)} \exp(-pv)$ uniform over the two-sheeted Riemann surface so obtained, the value of this function at any point of the lower sheet of this Riemann surface being the negative of its value at the corresponding point of the upper sheet. Let us consider the closed curve C on this two-sheeted Riemann surface which consists of the following four parts:

1) The line segment from the point $2-\delta$ in the lower sheet to the point δ in the lower sheet, $0 < \delta < 1$.

2) The circumference $|v| = \delta$ from the point δ in the lower sheet to the point δ in the upper sheet.

3) The line segment from the point δ in the upper sheet to the point $2-\delta$ in the upper sheet.

4) The circumference $|v - 2| = \delta$ from the point $2 - \delta$ in the upper sheet to the point $2 - \delta$ in the lower sheet.

The integral of $v^{n-(1/2)}(2-v)^{n-(1/2)}\exp(-pv)$ along C is independent of δ and as $\delta \longrightarrow 0$ the contributions from the parts 2) and 4) of C tend to 0 while the contributions from the parts 1) and 3) of C tend to $\int_0^2 t^{n-(1/2)}(2-t)^{n-(1/2)}\exp(-pt)dt$ so that the integral of $v^{n-(1/2)}(2-v)^{n-(1/2)}\exp(-pv)$ along C has the value $2\pi A_n p^{-n}\exp(-p)I_n(p)$.

We next consider the closed curve C' on our two-sheeted Riemann surface which consists of the following five parts:

1') The line segment from the point $R\exp(i\alpha)$ in the lower sheet to the point δ in the lower sheet where R is any positive number, α is any number which is such that $|\alpha + \arg p| < \frac{\pi}{2}$ and δ is any positive number less than 1

2') The circumference $|v| = \delta$ from the point δ in the lower sheet to the point δ in the upper sheet.

3') The line segment from the point δ in the upper sheet to the point $2 - \delta$ in the upper sheet

4') The circumference $|v - 2| = \delta$ from the point $2 - \delta$ in the upper sheet to the point $2 - \delta$ in the lower sheet

5') The line segment from the point $2 - \delta$ in the lower sheet to the point $R\exp(i\alpha)$ in the lower sheet

The integral of $v^{n-(1/2)}(2-v)^{n-(1/2)}\exp(-v)$ along C' is the same as the integral of the same integrand along C and is independent of R, α and δ. As $\delta \longrightarrow 0$ the contributions to this integral from the parts 2') and 4') of C' tend to 0 while the contribution from the part 3') of C' tends to $\int_0^2 t^{n-(1/2)}(2-t)^{n-(1/2)}\exp(-pt)dt$ $= \pi A_n p^{-n}\exp(-p)\, I_n(p)$. As $R \longrightarrow \infty$ the contribution from the part 1') of C' tends to $\int_0^\infty v^{n-(1/2)}(2-v)^{n-(1/2)}\exp(-pv)dv$, the integral being extended along the ray of angle α from 0 to ∞ (this infinite integral existing since $|\alpha + \arg p| < \frac{\pi}{2}$) and the contribution from the part 5') of C' tends to $-\int_2^\infty v^{n-(1/2)}(2-v)^{n-(1/2)}\exp(-pv)dv$, the integral being extended along the ray of angle α from 2 to ∞.

Thus

$$\pi A_n p^{-n}\exp(-p)\, I_n(p) = \int_0^{\infty \exp i\alpha} v^{n-(1/2)}(2-v)^{n-(1/2)}\exp(-pv)dv$$

$$- \int_2^{\infty \exp i\alpha} v^{n-(1/2)}(2-v)^{n-(1/2)}\exp(-pv)dv = I_1 - I_2, \text{ say.}$$

If $\frac{\pi}{2} \leq \arg p < \pi$ we set $\alpha = -\frac{\pi}{2}$ and, if $-\pi < \arg p \leq -\frac{\pi}{2}$ we set $\alpha = \frac{\pi}{2}$. Treating the first case, we set $v = it$ in I_1, and $v = 2-it$ in I_2 so that, in each of the two integrals, $0 \leq t < \infty$. I_1 appears as $2^{n-(1/2)}(-i)^{n+(1/2)}\int_0^\infty t^{n-(1/2)}(1+\frac{it}{2})^{n-(1/2)}\exp(-p't)dt$, where $p' = -ip$ so that the real part of p' is positive, and I_2 appears as

$-2^{n-(1/2)}\, i^{n+(1/2)}\int_0^\infty t^{n-(1/2)}(1-\frac{it}{2})^{n-(1/2)}\exp(-p't)dt$ times $\exp(-2p)$.

On multiplying through by exp p we obtain $\pi A_n (ip')^{-n} I_n(ip')$

$$= 2^{n-(1/2)}(-i)^{n+(1/2)}\exp(ip')\int_0^\infty t^{n-(1/2)}(1+\frac{it}{2})^{n-(1/2)}\exp(-p't)dt$$

$$+ 2^{n-(1/2)} i^{n+(1/2)}\exp(-ip')\int_0^\infty t^{n-(1/2)}(1-\frac{it}{2})^{n-(1/2)}\exp(-p't)dt$$

or, equivalently, since $i^{-n} I_n(ip') = J_n(p')$, we have $2J_n(p')$

$$= 2^{n+(1/2)}\pi^{-1} A_n^{-1}(p')^n \exp\left[i(p'-(n+\frac{1}{2})\frac{\pi}{2})\right]\int_0^\infty t^{n-(1/2)}(1+\frac{it}{2})^{n-(1/2)}\exp(-p't)dt$$

$$+ 2^{n+(1/2)}\pi^{-1} A_n^{-1}(p')^n \exp\left[-i(p'-(n+\frac{1}{2})\frac{\pi}{2})\right]\int_0^\infty t^{n-(1/2)}(1-\frac{it}{2})^{n-(1/2)}\exp(-p't)dt$$

$$= H_n^{(1)}(p') + H_n^{(2)}(p')$$

where $H_n^{(1)}(p')$ is the first member, and $H_n^{(2)}(p')$ is the second member, on the right-hand side and $0 \leq \arg p' < \frac{\pi}{2}$. $H_n^{(1)}(p')$ and $H_n^{(2)}(p')$ are known as Hankel Functions. A similar argument shows that the relation $2 J_n(p') = H_n^{(1)}(p') + H_n^{(2)}(p')$ remains valid when $-\frac{\pi}{2} < \arg p' < 0$, in which case $-\pi < \arg p < -\frac{\pi}{2}$ and we set $\alpha = \frac{\pi}{2}$. We shall derive in our next lecture, from this representation of $J_n(p')$ as the mean of the two Hankel functions, $H_n^{(1)}(p')$ and $H_n^{(2)}(p')$, a formula furnishing $J_n(t)$, wnen t is real and positive, as the sum of two asymptotic series, each of which has the convenient property that the error made in stopping at any term has the same sign as the next term and is dominated by this term.

Lecture 19

The Asymptotic Series for $P_n(c)$ and $Q_n(c)$

If p is any complex number whose real part c is positive the Hankel Function $H_n^{(1)}(p)$ is defined by the formula

$$H_n^{(1)}(p) = 2^{n+(1/2)}\pi^{-1}A_n^{-1}p^n \exp\left[i(p-(n+\frac{1}{2})\frac{\pi}{2})\right]\int_0^\infty t^{n-(1/2)}(1+\frac{it}{2})^{n-(1/2)}\exp(-pt)dt$$

In particular, when p = c is real and positive, we have, on making the substitution $t = t'/c$ and then dropping the prime,

$$H_n^{(1)}(c) = 2^{n+(1/2)}\pi^{-1}A_n^{-1}c^{-1/2}\exp\left[i(c-(n+\frac{1}{2})\frac{\pi}{2})\right]\int_0^\infty t^{n-(1/2)}(1+\frac{it}{2c})^{n-(1/2)}\exp(-t)dt$$

Similarly,

$$H_n^{(2)}(c) = 2^{n+(1/2)}\pi^{-1}A_n^{-1}c^{-1/2}\exp\left[-i(c-(n+\frac{1}{2})\frac{\pi}{2})\right]\int_0^\infty t^{n-(1/2)}(1-\frac{it}{2c})^{n-(1/2)}\exp(-t)dt$$

so that

$$J_n(c) = \frac{1}{2}\left\{H_n^{(1)}(c) + H_n^{(2)}(c)\right\} = (\frac{2}{\pi c})^{1/2}\left\{P_n(c)\cos\left[c-(n+\frac{1}{2})\frac{\pi}{2}\right]\right.$$
$$\left. - Q_n(c)\sin\left[c-(n+\frac{1}{2})\frac{\pi}{2}\right]\right\}$$

where

$$P_n(c) = \frac{2^{n-1}}{\pi^{1/2}A_n}\int_0^\infty t^{n-(1/2)}\left\{(1+\frac{it}{2c})^{n-(1/2)} + (1-\frac{it}{2c})^{n-(1/2)}\right\}\exp(-t)dt$$

$$; n = 0,\ 1,\ 2,\ \ldots$$

$$Q_n(c) = \frac{2^{n-1}}{\pi^{1/2}A_n i}\int_0^\infty t^{n-(1/2)}\left\{(1+\frac{it}{2c})^{n-(1/2)} - (1-\frac{it}{2c})^{n-(1/2)}\right\}\exp(-t)dt$$

If $n = 1, 2, \ldots,$ $J_{n-1}(c) = \left(\frac{2}{\pi c}\right)^{\frac{1}{2}} \left\{ -P_{n-1}(c) \sin\left[c - (n+\frac{1}{2})\frac{\pi}{2}\right] \right.$

$\left. - Q_{n-1}(c) \cos\left[c - (n+\frac{1}{2})\frac{\pi}{2}\right] \right\}$ and, if $n = 0, 1, 2, \ldots,$

$$J_{n+1}(c) = \left(\frac{2}{\pi c}\right)^{1/2} \left\{ P_{n+1}(c) \sin\left[c-(n+\frac{1}{2})\frac{\pi}{2}\right] + Q_{n+1}(c) \cos\left[c-(n+\frac{1}{2})\frac{\pi}{2}\right] \right\}$$

and it follows, since $J_{n-1}(c) - J_{n+1}(c) = \frac{2n}{c} J_n(c)$, $n = 1, 2, \ldots,$ that

$$P_{n+1}(c) - P_{n-1}(c) = -\frac{2n}{c} Q_n(c); \; Q_{n+1}(c) - Q_{n-1}(c) = \frac{2n}{c} P_n(c);$$

$n = 1, 2, \ldots$ We shall obtain asymptotic series for $P_0(c)$, $Q_0(c)$, $P_1(c)$, $Q_1(c)$ and shall deduce from these, by means of the recurrence relations just derived, asymptotic series for $P_n(c)$ and $Q_n(c)$, $n = 2, 3, \ldots$.

In order to obtain asymptotic series for $P_0(c)$ and $Q_0(c)$ we observe that, if v is any non-real complex number,

$(1-v)^{-1/2} = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\phi}{1 - v \sin^2\phi}.$ Indeed,

$$\int_0^{\pi/2} \frac{d\phi}{1 - v \sin^2\phi} = \int_0^{\frac{\pi}{2}} \frac{2d\phi}{2 - v + v \cos 2\phi} = \int_0^{\pi} \frac{d\theta}{2 - v + v \cos \theta}, \quad \theta = 2\phi,$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta}{2 - v + v \cos \theta}$$

Setting $\exp i\theta = z$, so that $d\theta = \frac{dz}{iz}$, $\frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta}{2 - v + v \cos \theta}$

$= \frac{1}{iv} \oint_C \frac{dz}{z^2 + 2(\frac{2}{v} - 1)z + 1}$ where C is the circumference $|z| = 1$.

The two zeros, z_1 and z_2, of the quadratic polynomial $z^2 + 2\left(\frac{2}{v} - 1\right)z+1$ are such that $z_1 z_2 = 1$ and, since their sum, $2\left(1 - \frac{2}{v}\right)$, is not real, neither can be a complex number of unit modulus for, if $|z_1| = 1$, for example, the reciprocal z_2 of z_1 would be its conjugate, $\bar{z}_1$, and $z_1 + z_2$ would be real. Writing $z_1 = 1 - \frac{2}{v} + \frac{2}{v}(1 - v)^{1/2}$ we see that, when $|v| < 1$, $z_1 = -\frac{1}{4}v + \dots$ tends to zero with v. Hence $|z_1| < 1$ if $|v|$ is sufficiently small and this implies, since $|z_1|$ is never 1, that $|z_1| < 1$ no matter what is the non-real complex number v and this implies that $|z_2| > 1$ no matter what is the non-real complex number v so that z_1 is the only zero of $z^2 + 2\left(\frac{2}{v} - 1\right)z+1$ which lies inside C. The coefficient of $\frac{1}{z-z_1}$ in the development of $\frac{1}{(z-z_1)(z-z_2)}$ as an infinite series of the type $\frac{c_{-1}}{z-z_1} + c_0 + c_1(z-z_1) + \dots$ is $\frac{1}{z_1-z_2} = \frac{v}{4}(1-v)^{-1/2}$ and so

$$\oint_C \frac{dz}{z^2+2\left(\frac{2}{v} - 1\right)z+1} = \frac{i\pi v}{2}(1-v)^{-1/2},$$

proving that $(1-v)^{-1/2} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\phi}{1-v\sin^2\phi}$. The usefulness of this result lies in the fact that is provides us with a convenient expression for the remainder in the binomial expansion of $(1-v)^{-1/2}$. Writing $(1- v\sin^2\phi)^{-1}$

$$= 1 + v\sin^2\phi + \dots + v^{2m-1}\sin^{4m-2}\phi + \frac{v^{2m}\sin^{4m}\phi}{1 - v\sin^2\phi}, \quad m = 1,2,\dots,$$

we obtain $(1-v)^{-1/2} = 1 + \frac{1}{2} v + \frac{1.3}{2.4} v^2 + \ldots + \frac{1.3\ldots(4m-3)}{2.4\ldots(4m-2)} v^{2m-1}$

$+ \frac{2}{\pi} v^{2m} \int_0^{\pi/2} \frac{(\sin^{4m}\phi)d\phi}{1-v\sin^2\phi}$. On changing the sign of v we obtain, by addition and subtraction, the relations

$$\frac{1}{2}\left\{(1-v)^{-1/2} + (1+v)^{-1/2}\right\} = 1 + \frac{1.3}{2.4} v^2 + \ldots + \frac{1.3\ldots(4m-5)}{2.4\ldots(4m-4)} v^{2m-2}$$
$$+ \frac{2}{\pi} v^{2m} \int_0^{\pi/2} \frac{\sin^{4m}\phi}{1-v^2\sin^4\phi} d\phi$$

$$\frac{1}{2}\left\{(1-v)^{-1/2} - (1+v)^{-1/2}\right\} = \frac{1}{2} v + \frac{1.3.5}{2.4.6} v^3 + \ldots + \frac{1.3\ldots(4m-3)}{2.4\ldots(4m-2} v^{2m-1}$$
$$+ \frac{2}{\pi} v^{2m+1} \int_0^{\pi/2} \frac{\sin^{4m+2}\phi}{1-v^2\sin^4\phi} d\phi$$

Setting $v = \frac{it}{2c}$, $0 \leq t < \infty$, we obtain

$$\frac{1}{2}\left\{(1-\frac{it}{2c})^{-1/2} + (1+\frac{it}{2c})^{-1/2}\right\} = 1 - \frac{1.3}{2!} \frac{t^2}{(4c)^2} + \ldots$$
$$+ (-1)^{m-1} \frac{1.3.\ldots.(4m-5)}{(2m-2)!} \frac{t^{2m-2}}{(4c)^{2m-2}}$$
$$+ (-1)^m \frac{2}{\pi} (\frac{t}{2c})^{2m} \int_0^{\frac{\pi}{2}} \frac{\sin^{4m}\phi}{1+\frac{t^2}{4c^2}\sin^4\phi} d\phi$$

$$\frac{1}{2i}\left\{(1-\frac{it}{2c})^{-\frac{1}{2}} - (1+\frac{it}{2c})^{-\frac{1}{2}}\right\} = \frac{t}{4c} - \frac{1.3.5}{3!} \frac{t^3}{(4c)^3} + \ldots$$
$$+ (-1)^{m-1} \frac{1.3\ldots(4m-3)}{(2m-1)!} \frac{t^{2m-1}}{(4c)^{2m-1}}$$
$$+ (-1)^m \frac{2}{\pi}(\frac{t}{2c})^{2m+1} \int_0^{\pi/2} \frac{\sin^{4m+2}\phi}{1+\frac{t^2}{4c^2}\sin^4\phi} d\phi$$

Since $1 + \frac{t^2}{4c^2}\sin^4\phi \geq 1$, the integral $\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^{4m}\phi}{1+\frac{t^2}{4c^2}\sin^4\phi}\, d\phi$, which is positive, is dominated by $\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^{4m}\phi d\phi = \frac{1.3...(4m-1)}{2.4...(4m)}$ and, similarly, the integral $\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^{4m+2}\phi}{1+\frac{t^2}{4c^2}\sin^4\phi}\, d\phi$, which is also positive, is dominated by $\frac{1.3...(4m+1)}{2.4...(4m+2)}$. On denoting by $\theta(t)$ any positive function of t which is dominated by 1 we have the following two equations which we shall refer to as the equations A_0:

$$\frac{1}{2}\left\{(1 - \frac{it}{2c})^{-1/2} + (1 + \frac{it}{2c})^{-\frac{1}{2}}\right\} = 1 - \frac{1.3}{2!}\frac{t^2}{(4c)^2} + \dots$$

$$+ (-1)^{m-1}\frac{1.3...(4m-5)}{(2m-2)!}\frac{t^{2m-2}}{(4c)^{2m-2}}$$

$$+ (-1)^m\frac{1.3...(4m-1)}{(2m)!}\frac{t^{2m}}{(4c)^{2m}}\theta(t)$$

A_0:

$$\frac{1}{2i}\left\{(1 - \frac{it}{2c})^{-1/2} - (1 + \frac{it}{2c})^{-1/2}\right\} = \frac{t}{4c} - \frac{1.3.5}{3!}\frac{t^3}{(4c)^3} + \dots$$

$$+ (-1)^{m-1}\frac{1.3...(4m-3)}{(2m-1)!}\frac{t^{2m-1}}{(4c)^{2m-1}}$$

$$+ (-1)^m\frac{1.3...(4m+1)}{(2m+1)!}\frac{t^{2m+1}}{(4c)^{2m+1}}\theta(t)$$

where the positive function $\theta(t)$ which appears on the right-hand side of each of these two equations is not the same in the two equations. Upon integrating the equations A_0 over the interval $[0, t]$ and using the fact that the integral of $t^{2m}\theta(t)$ over this interval is positive and is dominated by $\frac{t^{2m+1}}{2m+1}$ we obtain the following two equations which we shall refer to as the equations A_1:

$$\frac{1}{2i}\left\{(1+\frac{it}{2c})^{1/2} - (1-\frac{it}{2c})^{1/2}\right\} = \frac{t}{4c} - \frac{1.3}{3!}\frac{t^3}{(4c)^3} + \dots$$

$$+ (-1)^{m-1}\frac{1.3\ldots(4m-5)}{(2m-1)!}\frac{t^{2m-1}}{(4c)^{2m-1}}$$

$$+ (-1)^{m}\frac{1.3\ldots(4m-1)}{(2m+1)!}\frac{t^{2m+1}}{(4c)^{2m+1}}\theta(t)$$

A_1:

$$\frac{1}{2}\left\{(1+\frac{it}{2c})^{1/2} + (1-\frac{it}{2c})^{1/2}\right\} = 1 + \frac{1}{2!}\frac{t^2}{(4c)^2} - \frac{1.3.5}{4!}\frac{t^4}{(4c)^4} + \dots$$

$$+ (-1)^{m-1}\frac{1.3\ldots(4m-3)}{(2m)!}\frac{t^{2m}}{(4c)^{2m}}$$

$$+ (-1)^{m}\frac{1.3\ldots(4m+1)}{(2m+2)!}\frac{t^{2m+2}}{(4c)^{2m+2}}\theta(t)$$

where the positive function $\theta(t)$ which appears on the right-hand side of each of the equations A_1 is not the same in each of the two equations nor the same as the positive function $\theta(t)$ which appeared on the right-hand side of the equations A_0.

Upon multiplying the equations A_0 by the non-negative function $t^{-1/2}\exp(-t)$ and integrating over the positive real axis we obtain, on denoting by θ any positive number which is dominated by 1,

$$P_0(c) = \frac{1}{\pi^{1/2}}\left[\Gamma\left(\frac{1}{2}\right) - \frac{1.3}{2!}\frac{\Gamma\left(\frac{5}{2}\right)}{(4c)^2} + \dots\right.$$

$$+ (-1)^{m-1}\frac{1.3\dots(4m-5)}{(2m-2)!}\frac{\Gamma\left(2m-\frac{3}{2}\right)}{(4c)^{2m-2}}$$

$$\left.+ (-1)^m\frac{1.3\dots(4m-1)}{(2m)!}\frac{\Gamma\left(2m+\frac{1}{2}\right)}{(4c)^{2m}}\theta\right] = 1 - \frac{1^2.3^2}{2!(8c)^2} + \dots$$

$$+ (-1)^{m-1}\frac{1^2.3^2\dots(4m-5)^2}{(2m-2)!(8c)^{2m-2}} + (-1)^m\frac{1^2.3^2\dots(4m-1)^2}{(2m)!\,(8c)^{2m}}\theta$$

$$Q_0(c) = -\frac{1}{\pi^{1/2}}\left[\frac{\Gamma\left(\frac{3}{2}\right)}{4c} - \frac{1.3.5}{3!}\frac{\Gamma\left(\frac{7}{2}\right)}{(4c)^3} + \dots\right.$$

$$+ (-1)^{m-1}\frac{1.3\dots(4m-3)}{(2m-1)!}\frac{\Gamma\left(2m-\frac{1}{2}\right)}{(4c)^{2m-1}}$$

$$\left.+ (-1)^m\frac{1.3\dots(4m+1)}{(2m+1)!}\frac{\Gamma\left(2m+\frac{1}{2}\right)}{(4c)^{2m+1}}\theta\right] = -\frac{1}{8c} + \frac{1^2.3^2.5^2}{3!(8c)^3}$$

$$- \dots + (-1)^m\frac{1^2.3^2\dots(4m-3)^2}{(2m-1)!(8c)^{2m-1}} + (-1)^{m+1}\frac{1^2.3^2\dots(4m+1)^2}{(2m+1)!(8c)^{2m+1}}\theta$$

where the positive number θ which appears on the right-hand side of each of these two equations is not the same in the two equations.

Neither of the two infinite series

$$1 - \frac{1^2.3^2}{2!(8c)^2} + \frac{1^2.3^2.5^2.7^2}{4!(8c)^4} - \dots, \quad -\frac{1}{8c} + \frac{1^2.3^2.5^2}{3!(8c)^3} - \frac{1^2.3^2.5^2.7^2.9^2}{5!(8c)^5} + \dots$$

converges for any finite value of c but the first of these two infinite series is an asymptotic series for $P_0(c)$ and the second is an asymptotic series for $Q_0(c)$. Each of these two asymptotic series possesses the property that the error made in stopping at any term has the sign of the next term and is dominated by this term.

Upon multiplying the equations A_1 by the non-negative function $t^{1/2}\exp(-t)$ and integrating over the interval $[0, t]$ we obtain, similarly,

$$Q_1(c) = \frac{3}{8c} - \frac{3^2.5.7}{3!(8c)^3} + \dots + (-1)^{m-1}\frac{3^2\dots(4m-5)^2(4m-3)(4m-1)}{(2m-1)!\,(8c)^{2m-1}}$$

$$+ (-1)^m \frac{3^2\dots(4m-1)^2(4m+1)(4m+3)}{(2m+1)!\,(8c)^{2m+1}}\,\theta$$

$$P_1(c) = 1 + \frac{3.5}{2!(8c)^2} - \frac{3^2.5^2.7.9}{4!(8c)^4} + \dots + (-1)^{m-1}\frac{3^2.5^2\dots(4m-3)^2(4m-1)(4m+1)}{(2m)!\,(8c)^{2m}}$$

$$+(-1)^m \frac{3^2.5^2\dots(4m+1)^2(4m+3)(4m+5)}{(2m+2)!\,(8c)^{2m+2}}\,\theta$$

where, again, the positive number θ which appears on the right-hand side of these equations is not the same in each of the two equations. Thus the two infinite series

$$1 + \frac{3.5}{2!(8c)^2} - \frac{3^2.5^2.7.9}{4!(8c)^4} + \frac{3^2.5^2.7^2.9^2.11.13}{6!(8c)^6} - \dots$$

$$\frac{3}{8c} - \frac{3^2.5.7}{3!(8c)^3} + \frac{3^2.5^2.7^2.9.11}{5!(8c)^5} - \dots$$

each of which fails to converge for any finite value of c, are asymptotic series for $P_1(c)$ and $Q_1(c)$, respectively. The asymptotic series for $Q_1(c)$ is alternating while the asymptotic series for $P_1(c)$ is alternating if we remove its first term. The asymptotic series for $Q_1(c)$ possesses the same property as the asymptotic series for $P_0(c)$ and $Q_0(c)$: The error made in stopping at any term has the sign of the next term and is dominated by this next term. On the other hand, the asymptotic series for $P_1(c)$ does not, necessarily, possess this property if we stop at the first term; for this asymptotic series all we can claim is that: The error made in stopping at any term, after the first, has the sign of the next term and is dominated by this next term.

The asymptotic series which we have obtained for $P_0(c)$ and $P_1(c)$ are special cases, corresponding to n = 0 and n = 1, respectively, of the series

$$1 - \frac{(4n^2-1^2)(4n^2-3^2)}{2!(8c)^2} + \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)(4n^2-7^2)}{4!(8c)^4}$$

and the asymptotic series which we have obtained for $Q_0(c)$ and $Q_1(c)$ are special cases, corresponding to n = 0 and n = 1, respectively, of the series

$$\frac{(4n^2-1^2)}{8c} - \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{3!(8c)^3} + \ldots$$

We proceed to show that these series are asymptotic series for $P_n(c)$ and $Q_n(c)$, respectively, and, furthermore, that, if n = 2k is even, these asymptotic series possess the property that the error made in stopping at any term after the kth has the sign of the next term and is dominated by this next term; if $n = 2k + 1$ is odd the error made in stopping at any term after the $(k + 1)$st of the asymptotic series for $P_n(c)$, and at any term after the kth of the asymptotic series for $Q_n(c)$, has the sign of the next term and is dominated by this term. We have shown that these statements are true when $n = 0$ and when $n = 1$ and, assuming that they are true for any two consecutive values, $j-1$ and j, of n we proceed to show that this assumption implies that they are true for the next consecutive value, $j+1$, of n. Thus we assume that

$$P_n(c) = 1 - \frac{(4n^2-1^2)(4n^2-3^2)}{2!(8c)^2} + \dots + (-1)^{m-1} \frac{(4n^2-1^2)\dots\{4n^2-(4m-5)^2\}}{(2m-2)!(8c)^{2m-2}}$$

$$+ (-1)^m \frac{(4n^2-1^2)\dots\{4n^2-(4m-1)^2\}}{(2m)!\,(8c)^{2m}}\,\theta$$

if $n = j-1$ and m is sufficiently large and that

$$Q_n(c) = \frac{(4n^2-1^2)}{8c} - \dots + (-1)^m \frac{(4n^2-1^2)\dots\{4n^2-(4m-7)^2\}}{(2m-3)!(8c)^{2m-3}}$$

$$+ (-1)^{m+1} \frac{(4n^2-1^2)\dots\{4n^2-(4m-3)^2\}}{(2m-1)!(8c)^{2m-1}}\,\theta'$$

if $n = j$ and m is sufficiently large, both the positive numbers θ and θ' being dominated by 1. The coefficient of $(-1)^r\left\{(2r)!(8c)^{2r}\right\}^{-1}$, $r = 0, 1, \dots, m=1$, in the asymptotic series for $P_{j-1}(c)$ is the product $(2j-4r-1)\ldots(2j+4r-3)$ of all the odd integers, not necessarily positive, beginning with $2j-4r-1$ and ending with $2j+4r-3$ and the coefficient of $(-1)^r\left\{(2r)!(8c)^{2r}\right\}^{-1}$, $r = 1, \ldots, m-1$, in the product of the asymptotic series for $Q_j(c)$ by $-\frac{2j}{c}$ is $32jr$ times the product $(2j-4r+3)\ldots(2j+4r-3)$ of all the odd integers beginning with $2j-4r+3$ and ending with $2j+4r-3$. Since $(2j-4r-1)(2j-4r+1) + 32jr = (2j+4r-1)(2j+4r+1)$ the coefficient of $(-1)^r\left\{(2r)!\,(8c)^{2r}\right\}^{-1}$, $r = 1, \ldots, m-1$, in the result of subtracting $\frac{2j}{c}$ times the asymptotic series for $Q_j(c)$ from the asymptotic series for $P_{j-1}(c)$ is the product, $(2j-4r+3)\ldots(2j+4r+1)$, of all the odd integers beginning with $2j-4r+3$ and ending with $(2j+4r+1)$ and this product is the value, when $n = j+1$, of $(4n^2-1^2)\ldots\left\{4n^2-(4r-1)^2\right\}$. Since $P_{j+1}(c) = P_{j-1}(c) - \frac{2j}{c}Q_j(c)$ it follows that, when $n = j+1$, $P_n(c)$ is

$$1 - \frac{(4n^2-1^2)(4n^2-3^2)}{2!(8c)^2} + \ldots + (-1)^{m-1}\frac{(4n^2-1^2)\ldots\left\{4n^2-(4m-5)^2\right\}}{(2m-2)!(8c)^{2m-2}}$$

plus a remainder term, this remainder term being $(-1)^m\left\{(2m)!(8c)^{2m}\right\}^{-1}$ times the product of $(2j-4m-1)(2j-4m+1)\,\theta + 32jm\,\theta'$ by the product, $(2j-4m+3)\ldots(2j+4m-3)$, of all the odd integers beginning with $2j-4m+3$ and ending with $2j+4m-3$. Since $(2j-4m-1)(2j-4m+1) + 32jm = (2j+4m-1)(2j+4m+1)$, $(2j-4m-1)(2j-4m+1)\,\theta + 32jm\,\theta'$ is of the form $(2j+4m-1)(2j+4m+1)\,\theta''$ where θ'' is positive and dominated by 1 provided that m is large enough to make 2j-4m+1 negative (so that (2j-4m-1)(2j-4m+1) is positive). Thus the remainder term is the value, when $n = j+1$, of $(-1)^m \dfrac{(4n^2-1^2)\ldots\{4n^2-(4m-1)^2\}}{(2m)!(8c)^{2m}}\theta''$ which proves the validity, when $n = j+1$, of the statement made concerning the asymptotic series for $P_n(c)$. In the same way we prove the validity, when $n = j+1$, of the statement made concerning the asymptotic series for $Q_n(c)$. This completes the proof, by mathematical induction, of the validity, for all non-negative integral values of n, of the statements made concerning the asymptotic series for $P_n(c)$ and $Q_n(c)$.

Now the product of any term, say the rth, of the asymptotic series for $Q_j(c)$ by $-\frac{2j}{c}$ becomes, by virtue of the relation $P_{j+1}(c) = P_{j-1}(c) - \frac{2j}{c}Q_j(c)$, part of the (r+1)st term of the asymptotic series for $P_{j+1}(c)$ while the product of the rth term of the asymptotic series for $P_j(c)$ becomes, by virtue of the relation $Q_{j+1}(c) = Q_{j-1}(c) + \frac{2j}{c}P_j(c)$ part of the rth term of the asymptotic series for $Q_{j+1}(c)$. Thus, in order to be assured that the error made in stopping at any term of the asymptotic series in question has the sign of the next term and is dominated by this next term we must take

1) more than 1 term of the asymptotic series for $P_2(c)$, $Q_2(c)$ and $Q_3(c)$

2) more than 2 terms of the asymptotic series for

$P_3(c)$, $P_4(c)$, $Q_4(c)$ and $Q_5(c)$

and so on. In general, if $n = 2k$ is even, we must take more than k terms of the asymptotic series for $P_n(c)$ and $Q_n(c)$ while, if $n = 2k+1$ is odd, we must take more than $k+1$ terms of the asymptotic series for $P_n(c)$ and more than k terms of the asymptotic series for $Q_n(c)$.

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