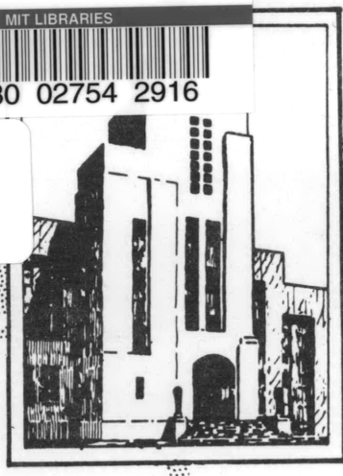

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STATISTICAL MECHANICS OF TWO-DIMENSIONAL WAVES WITH FINITE ENERGY

by

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APPLIED MATHEMATICS LABORATORY
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ABSTRACT

Classical methods of the Statistical Mechanics of mechanical systems with a finite number of degrees of freedom have been applied to study the case of two-dimensional gravity waves with finite energy.

1. - Statistical models for gravity waves have been recently given by G. Birkhoff and Jack Kotik[1] (1951), W. J. Pierson, Jr.[12] (1955), and M. Rosenblatt[13] (1957); we approach here the statistics of waves by a different way, extending to a continuous medium the classical methods of the Statistical Mechanics of mechanical systems with a finite number of degrees of freedom.

The state of a mechanical system with k degrees of freedom is represented by a point ω in the 2k dimensional space R^{2k} where $q_1, \dots q_k$ and $p_1, \dots p_k$ are the "coordinates" and the "conjugate moments"; the set $\Omega \subset R^{2k}$ corresponding to all possible states defines the "phase space." Through the canonical equations:

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \qquad j = 1, 2 \dots, k.$$

($H(p_j, q_j)$ = Hamiltonian), the motion of the system is represented by a trajectory Γ in Ω ; according to the existence and uniqueness theorem, to every point $\omega \in \Omega$ corresponds one and only one Trajectory Γ , which defines the motion of the system for $-\infty < t < +\infty$, when the initial state is represented by ω . Now, the Statistical Mechanics of Gibbs is essentially based on the assumption that the initial state ω is chosen at random in Ω according to some probability law; namely, one supposes that a measure μ is defined on a σ -algebra (Borel field) S of subsets of Ω and that for every $A \in S$:

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[1]References are listed on page 15.

$$\text{Prob}\left[\omega \in A\right] = \mu(A).*$$

Every function, which depends of the state ω_t of the system at the time t, say $f(\omega_t)$, becomes a random function of the time t; when the initial state $\omega_0 = \omega$ has been chosen, the values of $f(\omega_t)$ for $-\infty < t < +\infty$, define a "sample" of the random function.

It has been shown elsewhere[2] that for a continuous medium, a Statistical Mechanics can be built exactly on the same general outline, when one replaces the 2k dimensional phase space Ω, by some suitable function space, the state of a continuous medium being defined by a set of functions. One is lead to consider random integrals of the partial differential equations defining the motion of the medium, the "random" being introduced, as in Gibbs Statistical Mechanics, by the hypothesis that the initial state is chosen in Ω, according to some given probability law.

As an example, we have sketched[3] a Statistical Mechanics of a vibrating string with fixed ends; this example uses random integrals of the one-dimensional wave equation. We have also studied random integrals of the heat equation in an infinite rod[4, 5, 6] and, random integrals of the Laplace equation in the unit circle.[5]

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* We recall that a probability measure μ is a set function such that:

(a) $\mu(A) \geqslant 0$, for every $A \in S$

(b) $\mu(\Omega) = 1$

(c) $\mu\left[\bigcup_1^{+\infty} A_n\right] = \Sigma_1^{+\infty} \mu(A_n)$

for every sequence of disjoint sets $A_n \in S$.

One can say that, if the motion of the system is defined by linear partial differential equations, with constant coefficients, the random integrals (corresponding to random initial values) are easy to define and thus the corresponding Statistical Mechanics can be built on these foundations.

Of course, even for systems with a finite number of degrees of freedom, there is not a unique Statistical Mechanics, the choice of the probability measure μ being, to a large extent, arbitrary; this is obviously the same for continuous systems. In fact, in the finite case the phase space Ω is completely defined by the conditions imposed to the coordinates q_j, p_j by the structure of the system itself; on the contrary, in general, for a continuous system the function space is not so clearly imposed and depends of what class of functions one would think a priori as suitable for "regular integrals" of the partial differential equations.

Thus, not only the measure on Ω , but also Ω itself has to be selected for more or less plausible physical reasons; in some cases, like the heat conduction in an infinite rod, one can choose between at least three function spaces, which all lead to a theory perfectly coherent from an abstract point of view.

2. - The motion of gravity surface waves in the linearized theory being defined by a linear partial differential equation is one interesting case where the ideas we have outlined apply very well.

To reduce as far as possible the unessential computational difficulties we will consider here only a particular case: two-dimensional gravity waves, with finite energy. Any other problem could be dealt with exactly along the same lines.

We consider two-dimensional motions in water of infinite depth; the fluid occupies the half plane:

$$\{x, y: -\infty < x < +\infty, y \geqslant 0\}$$

We assume that the physical units have been so chosen that the density of the fluid is $\rho = 1$ and the acceleration of gravity $g = 1$. The velocity potential $\varphi(x, y, t)$ satisfies the equations:

$$(1) \qquad \varphi_{xx} + \varphi_{yy} = 0$$

$$(2) \qquad \varphi_{tt} = \varphi_y$$

in the domain

$$\{x, y, t: -\infty < x < +\infty, y > 0, -\infty < t < +\infty\}$$

From the velocity potential $\varphi(x, y, t)$ the velocities u, v in the x, y directions are given by:

$$(3) \qquad u(x, y, t) = -\varphi_x(x, y, t), \quad v(x, y, t) = -\varphi_y(x, y, t)$$

and the elevation of the free surface above the mean level $y = 0$ is given by

$$(4) \qquad \eta(x, t) = -\varphi_t(x, 0, t).$$

Now, in this general frame, we will consider the motion corresponding to a given free surface, the fluid being at rest, at the time $t = 0$.

Initial conditions:

1^o The initial impulse applied to the free surface is zero:

$$(5) \qquad \varphi(x, 0, 0) = 0$$

2^o The initial shape of the free surface is given:

$$(6) \qquad \varphi_t(x, 0, 0) = F(x)$$

the function $F(x)$ satisfying:

$H_1)$ $F(x) \in L^2$

$H_2)$ $F(x)$ absolutely continuous

$H_3)$ $F'(x) \in L^2$

$H_4)$ $F(-x) = F(x)$;

H_1 states that the potential energy

$$\frac{1}{2} \int_{-\infty}^{+\infty} \eta(x, 0)^2 \, dx$$

due to the initial elevation is finite; H_3 implies that $F(x)$ is bounded and vanishes at infinity; H_2 that the free surface admits a tangent almost everywhere; H_4 is an inessential simplification made to avoid considering an integral similar to the integral (7) where $\cos \lambda x$ is replaced by $\sin \lambda x$.

It has been proved by J. Kampé de Fériet and Jack Kotik[10] that there is one and only one velocity potential satisfying the equations (1) and (2) and the initial conditions (5) and (6); this velocity potential is given by:

$$(7) \qquad \varphi(x, y, t) = \int_0^{+\infty} e^{-\lambda y} \frac{\sin \sqrt{\lambda}\, t}{\sqrt{\lambda}} \cos \lambda x \; f(\lambda) \, d\lambda$$

where $f(\lambda)$ is the Fourier - Plancherel transform of the given function $F(x)$:

$$(8) \qquad f(\lambda) = \underset{N \to +\infty}{\mathrm{l.i.m.}} \frac{2}{\pi} \int_0^N \cos \lambda x \; F(x) \, dx \qquad ;$$

the elevation of the free surface is:

$$(9) \qquad \eta(x, t) = - \int_0^{+\infty} \cos \sqrt{\lambda}\, t \quad \cos \lambda x \; f(\lambda) \, d\lambda \, .$$

From the hypothesis H_1, H_2, and H_3 we have:

$$(10) \qquad f(\lambda) \in L^2 \quad \text{and} \quad \lambda f(\lambda) \in L^2$$

there is a one-to-one correspondence (up to an equivalence in L^2) between the function $f(\lambda)$ satisfying (10) and the function $F(x)$ satisfying H_1, H_2, H_3, and H_4. If one takes an arbitrary function $f(\lambda)$ satisfying (10), then $F(x)$ is defined by:

$$F(x) = \int_0^{+\infty} \cos \lambda x \; f(\lambda) \, d\lambda \tag{11}$$

and

$$F'(x) = - \underset{N \to +\infty}{\mathrm{l.i.m.}} \int_0^N \sin \lambda x \; \lambda f(\lambda) \, d\lambda. \tag{12}$$

Let us define a function $\omega(\lambda)$ on $(0, +\infty)$ by:

$$\begin{aligned} \omega(\lambda) &= f(\lambda) & 0 \leqslant \lambda \leqslant 1. \\ \omega(\lambda) &= \lambda f(\lambda) & 1 \leqslant \lambda < +\infty \end{aligned} \tag{13}$$

To each $f(\lambda)$ satisfying (10) corresponds by (13) one and only one function $\omega(\lambda) \in L^2 \; [0, +\infty]$; this is proved by the simple remarks:

$$f(\lambda) \in L^2 \Rightarrow f(\lambda) \in L^2 \; [0, 1] \Rightarrow \omega(\lambda) \in L^2 \; [0, 1]$$

$$\lambda f(\lambda) \in L^2 \Rightarrow \lambda f(\lambda) \in L^2 \; [1, +\infty] \Rightarrow \omega(\lambda) \in L^2 \; [1, +\infty]$$

and

$$\omega(\lambda) \in L^2 \; [0, +\infty] \Rightarrow \omega(\lambda) \in L^2 \; [0, 1] \Rightarrow f(\lambda) \in L^2 \; [0, 1]$$

$$\lambda f(\lambda) \in L^2 \; [0, 1]$$

$$\omega(\lambda) \in L^2 \; [0, +\infty] \Rightarrow \omega(\lambda) \in L^2 \; [1, +\infty] \Rightarrow \lambda f(\lambda) \in L^2 \; [1, +\infty]$$

$$f(\lambda) \in L^2 \; [1, +\infty]$$

Thus there is a one-to-one correspondence (up to an equivalence in L^2) between the motions of the fluid defined by the velocity potential $\varphi(x, y, t)$ and the initial states corresponding to $\omega(\lambda) \in L^2 \; [0, +\infty]$, the function $f(\lambda)$ in (7) being deduced from $\omega(\lambda)$ by (13).

A Statistical Mechanics of our two-dimensional waves with finite energy, in the spirit of Gibbs Statistical Mechanics, shall be based on the assumption that the initial state is chosen at random: This means that <u>the point</u> $\omega = \omega(\lambda)$ <u>is chosen in</u> $\Omega \equiv L^2 \left[0, +\infty\right]$ <u>according to a given probability law</u>.

3. - To define a probability measure on $L^2 \left[0, +\infty\right]$ we will use the method described in Reference 9 which gives, not only one, but a large class of measures; all of them are, according to the terminology of Miss E. Mourier, L - measures; every bounded linear functional is measurable with respect to any one of these measures. This condition is a very natural one in our problem, because, as will be shown later, all the salient functions $\varphi(x, y, t)$, $v(x, y, t)$ $u(x, y, t)$, $\eta(x, t)$ are, for fixed (x, y, t), bounded linear functional on $L^2 \left[0, +\infty\right]$.

We start by defining a class of measures ν on the particular Hilbert-space l^2. Let us call R^∞ the space, whose points ξ have a countable number of coordinates

$$\xi = \left[\xi_1, \ldots, \xi_n \ldots\ldots\right] \qquad \xi_n \in R$$

and $R, R^2, \ldots R^n$ its finite dimensional sub-spaces; then l^2 is the subset of R^∞ for which:

$$\Sigma_1^{+\infty} \xi_n^2 < +\infty \qquad ;$$

it becomes an Hilbert-space when the norm is defined by:

$$\|\xi\| = \left[\Sigma_1^{+\infty} \xi_n^2\right]^{\frac{1}{2}}$$

According to a celebrated theorem of Kolmogoroff (Reference 11, pp. 24-29) a probability-measure ν in R^∞ is completely determined by any sequence of

Lebesgue-Stieltjes measures $\nu_1, \dots, \nu_n, \dots$ on R, $R^2, \dots, R^n$ satisfying the following conditions:

(a) any Borel set $B^n \subset R^n$ is ν_n - measurable

(b) $\nu_n(R^n) = 1$

(c) if C^{n+1} is a cylinder set in R^{n+1} having as basis the Borel set B^n in R^n:

$$C^{n+1} = \left\{ (\xi_1, \dots, \xi_n) \in B^n, \quad -\infty < \xi_{n+1} < +\infty \right\}$$

Then:

$$\nu_{n+1}(C^{n+1}) = \nu_n(B^n)$$

Let us assume that the sequence of measures $\nu_1, \dots, \nu_n, \dots$ satisfying (a), (b), (c) satisfies the condition:

$$\Sigma_1^{+\infty} \int_{R^n} \xi_n^2 \, d\nu_n < +\infty \quad . \tag{14}$$

An immediate consequence of Fatou's-lemma proves that

$$\Sigma_1^{+\infty} \xi_n^2 < +\infty$$

except at most on a set of ν - measure 0; thus:

$$\nu(l^2) = 1$$

and ν defines a probability-measure on the Hilbert-space l^2. Using the probability language, the preceding result amounts to say that ν corresponds to any sequence of random variables:

$$\xi_1, \dots, \xi_n, \dots$$

such that:*

$$\Sigma_1^{+\infty} \overline{\xi_n^2} < +\infty \tag{15}$$

---

* For a random variable X, we denote by $\overline{X}$ the mathematical expectation or mean value.

Then:

(16) $\quad$ Prob $\left[\Sigma_1^{+\infty} \xi_n^2 < +\infty\right] = 1.$

Now, coming back to $L^2 \left[0, +\infty\right]$, let us assume that we know a complete orthonormal basis $e_n(\lambda)$ in this Hilbert-space:*

$$\int_0^{+\infty} e_m(\lambda)\, e_n(\lambda)\, d\lambda = \delta_{m,n}$$

Then the equations

$$(17) \qquad \xi_n = \int_0^{+\infty} e_n(\lambda)\, \omega(\lambda)\, d\lambda$$

establish a one-to-one correspondence between l^2 and $L^2 \left[0, +\infty\right]$, which is even an isomorphism if in $L^2 \left[0, +\infty\right]$ the norm is defined as usual by:

$$\|\omega\| = \left[\int_0^{+\infty} \omega(\lambda)^2\, d\lambda\right]^{\frac{1}{2}}.$$

The probability-measure ν on l^2 induces, by this isomorphism, a probability-measure μ on $L^2 \left[0, +\infty\right]$ if one puts:

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* From the Laguerre polynomials:

$$L_n(\lambda) = e^{\lambda} \frac{d^n}{d\lambda^n} \left(\lambda^n e^{-\lambda}\right)$$

$$= (-1)^n \left[\lambda^n - \frac{n^2}{1!} \lambda^{n-1} + \frac{n^2 (n-1)^2}{2!} \lambda^{n-2} - \dots\right]$$

such a basis is defined, as example, by

$$e_n(\lambda) = \frac{1}{(n-1)!} e^{-\lambda/2} L_{n-1}(\lambda) \qquad n = 1, 2, \dots\dots$$

$$\mu(A) = \nu(B) \qquad A \longleftrightarrow B$$

for any measurable $B \subset l^2$; one has:

$$\mu(\Omega) = \nu(l^2) = 1$$

In fact, if one goes back to the definition of ν, the measure μ is such that, for any Borel-set $B^n \subset R^n$ and any n:

$$\text{Prob}\left[\left(\int_0^{+\infty} e_1(\lambda)\,\omega(\lambda)\,d\lambda, \ldots \int_0^{+\infty} e_n(\lambda)\,\omega(\lambda)\,d\lambda\right) \in B^n\right] = \nu_n(B^n)$$

The measure μ is an L - measure, because any bounded linear functional on $L^2\left[0, +\infty\right]$ is represented by a series which converges for almost all ω

$$\omega^*(\omega) = \Sigma_1^{+\infty} \xi_n \alpha_n \qquad \Sigma_1^{+\infty} \alpha_n^2 < +\infty \quad ;$$

each ξ_n being, by definition μ - measurable, so is $\omega^*(\omega)$.

4. - Let us first consider the elevation $\eta(x, t)$; for each fixed (x, t) the elevation is measurable; i.e., is a well-defined random variable; to prove this we have only to remark that, using (13), (9) can be written:

$$\eta(x, t) = \int_0^{+\infty} G(\lambda, x, t)\,\omega(\lambda)\,d\lambda. \tag{18}$$

where:

$$G(\lambda, x, t) = -\cos\sqrt{\lambda}\,t \quad \cos\lambda x \qquad 0 \leqslant \lambda \leqslant 1.$$

$$= -\frac{\cos\sqrt{\lambda}\,t \quad \cos\lambda x}{\lambda} \qquad 1 \leqslant \lambda < +\infty \quad ;$$

thus: $\quad G(\lambda, x, t) \in L^2\left[0, +\infty\right]$ for all (x, t)

Thence $\eta(x, t)$ is, for each (x, t), a bounded linear functional.

In fact, if one puts:

$$\eta_n(x, t) = \int_0^{+\infty} G(\lambda, x, t)\,e_n(\lambda)\,d\lambda, \tag{19}$$

the elevation is defined by the series:

(20) $$\eta(x, t) = \Sigma_1^{+\infty} \xi_n \eta_n(x, t).$$

which converges with probability one; i.e., for almost all $\omega(\lambda) \in L^2[0, +\infty]$.

5. - The case where the random variables $\xi_1, \ldots \xi_n, \ldots\ldots$ are independent is particularly simple and important; let us suppose

$$\overline{\xi_n} = 0 \qquad \overline{\xi_m \xi_n} = 0 \quad m \neq n \qquad \overline{\xi_n^2} = \sigma_n^2$$

the σ_n satisfying by hypothesis the condition

(15') $$\Sigma_1^{+\infty} \sigma_n^2 < +\infty$$

From (20) we obtain for the mean and variance of $\eta(x, t)$:

(21) $$\overline{\eta(x, t)} = 0$$

(22) $$\overline{\eta(x, t)^2} = \Sigma_1^{+\infty} \sigma_n^2 \eta_n(x, t)^2$$

The covariance of η at two points x', x'' and two times t', t'' is given by:

(22') $$\overline{\eta(x', t') \eta(x'', t'')} = \Sigma_1^{+\infty} \sigma_n^2 \eta_n(x' t') \eta_n(x'', t'')$$

Both series (22) and (22') converge for all (x, t) and (x'. t') (x'', t'').

In order that the elevation $\eta(x, t)$ be a normal (Gaussian) random variable, what is more or less suggested by the experiments (Ref. 12, p. 157), one has only to suppose that the ξ_n are <u>independent</u> <u>normal</u> <u>variables</u>:

$$\text{Prob}\left[\xi_n < a\right] = \frac{1}{\sqrt{2\pi}\,\sigma_n} \int_{-\infty}^{a} e^{-\frac{s^2}{2\sigma_n^2}} ds$$

Thus the series (20) defines a normal random variable with mean zero and variance given by (22); in that case in formula (22) only the values of the σ_n have to be obtained from the experiments.

6. - Like the elevation, all the salient functions:

velocity potential φ (x, y, t) and velocities u(x, y, t) , v(x, y, t) are bounded linear functionals for any fixed (x, y, t) in

$$\left\{ x,\ y,\ t:\ -\infty < x < +\infty\ ,\ y > 0,\ -\infty < t < +\infty \right\}$$

This can be seen readily if one writes:

$$\varphi(x,\ y,\ t) = \int_0^{+\infty} G^{(1)}\ (\lambda,\ x,\ y,\ t)\ \omega(\lambda)\ d\lambda$$

$$u\ (x,\ y,\ t) = \int_0^{+\infty} G^{(2)}\ (\lambda,\ x,\ y,\ t)\ \omega(\lambda)\ d\lambda$$

$$v\ (x,\ y,\ t) = \int_0^{+\infty} G^{(3)}\ (\lambda,\ x,\ y,\ t)\ \omega(\lambda)\ d\lambda$$

where the functions $G^{(i)}$ are given by:

$$G^{(1)} = e^{-\lambda y} \frac{\sin \sqrt{\lambda}\, t}{\sqrt{\lambda}} \quad \cos \lambda x \qquad 0 \leqslant \lambda \leqslant 1$$

$$= \frac{1}{\lambda}\ e^{-\lambda y} \frac{\sin \sqrt{\lambda}\, t}{\sqrt{\lambda}} \quad \cos \lambda x \qquad 1 \leqslant \lambda < +\infty$$

$$G^{(2)} = e^{-\lambda y} \sin \sqrt{\lambda}\, t.\ \sqrt{\lambda} \quad \sin \lambda x$$

$$= e^{-\lambda y} \frac{\sin \sqrt{\lambda}\, t}{\sqrt{\lambda}} \quad \sin \sqrt{\lambda}\, x$$

$$G^{(3)} = e^{-\lambda y} \sin \sqrt{\lambda}\, t \quad \sqrt{\lambda} \quad \cos \lambda x.$$

$$= e^{-\lambda y} \frac{\sin \sqrt{\lambda}\, t}{\sqrt{\lambda}} \quad \cos \lambda x.$$

obviously all the $G^{(i)}$ belong to $L^2\ [0,\ +\infty]$.

If one puts:

$$(23) \qquad \varphi_n\ (x,\ y,\ t) = \int_0^{+\infty} G^{(1)}\ (\lambda,\ x,\ y,\ t)\ e_n\ (\lambda)\ d\lambda$$

(24) $$u_n(x, y, t) = \int_0^{+\infty} G^{(2)}(\lambda, x, y, t)\, e_n(\lambda)\, d\lambda,$$

(25) $$v_n(x, y, t) = \int_0^{+\infty} G^{(3)}(\lambda, x, y, t)\, e_n(\lambda)\, d\lambda,$$

then one has:

(26) $$(x, y, t) = \Sigma_1^{+\infty} \xi_n \; \varphi_n(x, y, t)$$

(27) $$u(x, y, t) = \Sigma_1^{+\infty} \xi_n \; u_n(x, y, t)$$

(28) $$v(x, y, t) = \Sigma_1^{+\infty} \xi_n \; v_n(x, y, t)$$

These convergent series define φ, u, v for each fixed (x, y, t) as a random variable.

Assuming, as in § 5, that the ξ_n are independent, we have for the mean and variances:

(29) $$\overline{\varphi(x, y, t)} = 0$$
$$\overline{\varphi(x, y, t)^2} = \Sigma_1^{+\infty} \sigma_n^2 \; \varphi_n(x, y, t)^2$$

(30) $$\overline{u(x, y, t)} = 0$$
$$\overline{u(x, y, t)^2} = \Sigma_1^{+\infty} \sigma_n^2 \; u_n(x, y, t)^2$$

(31) $$\overline{v(x, y, t)} = 0$$
$$\overline{v(x, y, t)^2} = \Sigma_1^{+\infty} \sigma_n^2 \; v_n(x, y, t)^2$$

The correlation between the two components of the velocity is given by:

(32) $$\overline{u(x, y, t)\, v(x, y, t)} = \Sigma_1^{+\infty} \sigma_n^2 \; u_n(x, y, t)\, v_n(x, y, t)$$

For each choice of the probability measure μ on the space $\Omega = L^2 [0, +\infty]$ corresponding to all possible initial conditions (in our particular problem) the formulas (22), (29), (30), and (31) completely determine the statistical properties, up to the second order, of the velocity potential, the velocities and the elevation; if one makes the plausible assumption that all these functions are for fixed (x, y, t) normal random variables, the numerical values of $\sigma_1, \ldots \sigma_n, \ldots,$ subjected to condition (15'), are the only empirical parameters.

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