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THE CONVERGING FACTOR FOR THE EXPONENTIAL INTEGRAL

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ABSTRACT

The term "converging factor," first explicitly used by J.R. Airey in 1937, is generally defined as the factor by which the final term of a truncated series must be multiplied to yield the remainder of the series. In this report the converging factor associated with the asymptotic series for the exponential integral $Ei(x)$ of both positive and negative real argument x is discussed in detail, and numerical values thereof for integral arguments are tabulated to 45 or more decimal places. Auxiliary tables are presented to permit the evaluation of this factor to comparable accuracy for intermediate values of the argument. Asymptotic series for the converging factor are rigorously developed, and the exact (rational) values of the first 21 coefficients therein are presented. As a byproduct the first 20 nontrivial coefficients of Stirling's asymptotic series for the factorial function are deduced. A method of J.P. Gram for the evaluation of the exponential integral is presented in detail, and original tables of values of the exponential integral are given to 44 significant figures for integral values of x extending from 5 through 20 and to 50 decimal places for integral values of x ranging from -5 to -20, inclusive.

INTRODUCTION

A study of the exponential integral $Ei\ (x)$ should appropriately include references to the logarithmic integral $Li\ (x)$, when x is positive, inasmuch as the fundamental relation $Li(x) = Ei\ (\log_e x)$ exists between these functions.

Indeed, the logarithmic integral was historically the precursor of the exponential integral, having been studied and tabulated by J. von Soldner[1] in 1809, whereas the exponential integral appears to have been first treated similarly by C.A. Bretschneider[2] in 1843.

The principal motivation for the study and evaluation of the logarithmic integral by such mathematicians as F.W. Bessel, James Glaisher, C.F. Gauss, and D.N. Lehmer has been its importance in the theory of the asymptotic distribution of prime numbers.

In a letter to Encke in 1849, Gauss conjectured that the number $\pi(x)$ of primes less than x can be approximated by $Li\ (x)$, in the sense that their ratio differs from unity by an amount that is arbitrarily small numerically when x is sufficiently large. Chebyshev[3] proved in 1851 that if a limiting value of this ratio exists, it must be unity.

In 1896, J. Hadamard[4] and C. de la Vallee Poussin,[5] following research of B. Riemann, succeeded independently in discovering rigorous proofs of the Prime Number Theorem, which implies the truth of the Gauss-Chebyschev conjecture. More recently, J. Littlewood[6] has

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[1]References are listed on page 98.

proved that the difference $Li\,(x) - \pi\,(x)$ changes sign infinitely often. However, within the range of existing numerical information, the preceding difference is always positive, and the first value of x for which it becomes negative is known only to be extremely large.

The exponential integral was investigated and tabulated concurrently with the foregoing number-theoretic studies. Bretschneider[7] extended in 1861 his earlier table of 1843. Additional calculations of values of this function were performed to 11 and 18 decimal places by J.W.L. Glaisher[8] in 1870 and to 20 decimals by J.P. Gram[9] in 1884. More recently, extensive tables have been prepared by the British Association for the Advancement of Science[10] in 1927 and 1931 (10 and 11 significant figures) and by the New York Work Progress Administration[11] in 1940 (9 and 10 decimal places).

Abridged tables of the exponential integral, giving values ranging from 10 to 18 significant figures have been published by C.A. Coulson and W.E. Duncanson[12] in 1942, M. Kotani[13] in 1955, F.E. Harris[14] in 1957, and J. Miller and R.P. Hurst[15] in 1958.

As explicitly noted by recent table-makers, the exponential integral has assumed an increasingly important role in applied mathematics because of its occurrence in the modern theory of molecular structure, which requires more accurate numerical values of this function than those provided by the standard tables.

This need for greater accuracy has led to further research in improved techniques in the numerical evaluation of the exponential integral to high precision over the complete range of the argument.

One of the most promising approaches to this problem of attaining higher precision is that of converging factors, which were first introduced by J.R. Airey[16] in 1937 to improve the accuracy obtainable by means of asymptotic expansions. Subsequently this technique has been studied by a number of mathematicians, including J.B. Rosser[17] in 1951, J.C.P. Miller[18] in 1952, and R.B. Dingle[19] in 1958. This method was anticipated substantially by T.J. Stieltjes[20] in his doctoral thesis, which was published in 1886.

Airey's study of the converging factor considered, as a special case, the determination of that factor for the exponential integral of negative argument, for which the asymptotic series is alternating. His analysis did not permit the consideration of this integral when the argument is positive. Miller[18] has pointed out the difficulties inherent in the determination of the converging factor in the latter case.

As Miller[18] has observed, the converging factor for the exponential integral satisfies both a differential equation and a difference equation. By solving numerically the differential equation, and then using the difference equation, one may determine this converging factor to any desired degree of accuracy.

In this report, we tabulate the values of this factor $C_n(x)$ to 45 decimal places, corresponding to positive integral values of the argument x between 5 and 20 inclusive, exceeding in each the subscript n by unity. When the argument x is a negative integer we give the converging factor to 50 decimal places for x between 5 and 10, inclusive, and to 48 decimal places for x between 11 and 20, inclusive. We also present tables that permit the evaluation to comparable accuracy of the converging factor for nonintegral values of the argument.

The first 23 coefficients of the asymptotic series for the converging factor for negative integral argument, are tabulated, as well as the first 21 coefficients of the corresponding series for positive integral values of the argument. Airey[16] published 22 coefficients of the former series, and Dingle[19] gave the first four coefficients of the latter series.

As a byproduct of this research, we have obtained the first 20 nontrivial coefficients of Stirling's asymptotic series for the Gamma (or factorial) function. Heretofore, to the best of our knowledge, only seven of these coefficients have been published.[21]

Finally, we furnish close approximations to the higher coefficients of Stirling's series and of the asymptotic series for the converging factor of the exponential integral of positive argument.

THE EXPONENTIAL INTEGRAL

The exponential integral $Ei\, x$ is defined for all real values of x, other than zero, by the formula

$$Ei\, x = \int_{-\infty}^{x} \frac{\exp t}{t}\, dt,$$

it being understood that, when x is positive, the Cauchy principal value of the integral is taken. When the argument of the exponential integral is negative, we denote it by $-x$, so that x is positive, and $Ei\,(-x) = \int_{-\infty}^{-x} \frac{\exp t}{t}\, dt = -\int_{x}^{\infty} \frac{\exp(-\tau)}{\tau}\, d\tau,\ t = -\tau$. Thus

$$-Ei(-x) = \int_{x}^{\infty} \frac{\exp(-\tau)}{\tau}\, d\tau = \int_{x}^{1} \frac{\exp(-\tau)}{\tau}\, d\tau + \int_{1}^{\infty} \frac{\exp(-\tau)}{\tau}\, d\tau$$

$$= \int_{1}^{x} \frac{1-\exp(-\tau)}{\tau}\, d\tau - \log_e x + \int_{1}^{\infty} \frac{\exp(-\tau)}{\tau}\, d\tau,$$

and we deduce that

$$-Ei(-x) = \int_{0}^{x} \frac{1-\exp(-\tau)}{\tau} - \log_e x - \gamma = x - \frac{x^2}{2.2!} + \frac{x^3}{3.3!} - \cdots - \log_e x - \gamma,$$

where $\gamma = \int_{0}^{1} \frac{1-\exp(-\tau)}{\tau}\, d\tau - \int_{1}^{\infty} \frac{\exp(-\tau)}{\tau}\, d\tau$ is Euler's constant, whose value, to 50 decimal places, is 0.57721|56649 01532 86060 65120 90082 40243 10421 59335 93992. When the argument of the exponential integral is positive, $Ei\, x$ is the limit, as $\delta > 0$ tends to zero, of

$\int_{-\infty}^{-x} \frac{\exp t}{t}\, dt + \int_{-x}^{-\delta} \frac{\exp t}{t}\, dt + \int_{\delta}^{x} \frac{\exp t}{t}\, dt$. Writing $t = -\tau$ in the first two of these three integrals, we see that $Ei\, x$ is the limit, as $\delta > 0$ tends to zero, of $-\int_{x}^{\infty} \frac{\exp(-\tau)}{\tau}\, d\tau - \int_{\delta}^{x} \frac{\exp(-\tau)}{\tau}\, d\tau$ $+ \int_{\delta}^{x} \frac{\exp t}{t}\, dt = -\int_{x}^{\infty} \frac{\exp(-\tau)}{\tau}\, d\tau + \int_{\delta}^{x} \frac{1-\exp(-\tau)}{\tau}\, d\tau + \int_{\delta}^{x} \frac{\exp t - 1}{t}\, dt$. Since $\frac{1-\exp(-\tau)}{\tau}$ and $\frac{\exp t - 1}{t}$ are continuous at $\tau = 0$ and $t = 0$, respectively, this limit is $-\int_{x}^{\infty} \frac{\exp(-\tau)}{\tau}\, d\tau$

$$+\int_0^x \frac{1-\exp(-\tau)}{\tau}\,d\tau + \int_0^x \frac{\exp t - 1}{t}\,dt.$$ Since

$$\int_0^x \frac{1-\exp(-\tau)}{\tau}\,d\tau = \int_0^1 \frac{1-\exp(-\tau)}{\tau}\,d\tau + \log_e x - \int_1^\infty \frac{\exp(-\tau)}{\tau}\,d\tau + \int_x^\infty \frac{\exp(-\tau)}{\tau}\,d\tau,$$

we have the relation

$$Ei\,x = \gamma + \log_e x + \int_0^x \frac{\exp t - 1}{t}\,dt = \gamma + \log_e x + \left(x + \frac{x^2}{2.2!} + \frac{x^3}{3.3!} + \cdots\right)$$

Thus the formula

$$Ei\,x = \gamma + \log_e |x| + \left(x + \frac{x^2}{2.2!} + \frac{x^3}{3.3!} + \cdots\right)$$

is valid for both positive and negative values of the argument x.

For values of x numerically greater than 5 the evaluation of the infinite series $\sum_1^\infty \frac{x^k}{k.k!}$ to an accuracy of, say, 50 or more decimal places is onerous. For example, when $|x| = 20$, 144 terms of the series must be taken to ensure an accuracy of 64 decimal places, the resulting values being Ei 20 = 256 15652.66405 65888 20481 12080 40980 71829 38269 83577 40342 60637 41233 85463 5090, $-Ei(-20) = 0.0_{10}$ 98355 25290 64988 16903 96987 10889 47760 74356 32407 31695 2391. The difference in the number of significant figures, for positive and negative values of the argument when the number of decimal places is fixed, is striking when the numerical value of the argument is greater than 10, say. Thus, when $|x| = 20$ and the number of decimal places is 64, we obtain 72 significant figures of Ei 20 but only 54 significant figures of $Ei(-20)$. It is to avoid the very tedious calculations just referred to that the concept of the converging factor has been introduced. Owing to the fact that, when the argument of the exponential integral is positive, we must take the Cauchy principal value of the integral which defines $Ei\,x$, the cases where x is positive and where x is negative must be treated separately.

THE CONVERGING FACTOR WHEN THE ARGUMENT OF THE EXPONENTIAL INTEGRAL IS POSITIVE

Writing $t = x - u$ in the integral $\int_{-\infty}^x \frac{\exp t}{t}\,dt$, which defines $Ei\,x$, we see that

$Ei\, x = \dfrac{\exp x}{x}\, C(x)$, where $C(x) = \displaystyle\int_0^\infty \frac{\exp(-u)}{1-\frac{u}{x}}\, du$. Since $\dfrac{1}{1-\frac{u}{x}} = 1 + \dfrac{u}{x} + \cdots + \dfrac{u^{n-1}}{x^{n-1}}$ $+ \dfrac{u^n}{x^n\left(1-\frac{u}{x}\right)}$, where n is any positive integer, and since $\displaystyle\int_0^\infty \exp(-u)\, u^k\, du = k!$, $k = 0, 1, 2,$ $\ldots$, we have the relation

$$C(x) = 1 + \frac{1}{x} + \frac{2!}{x^2} + \cdots + \frac{(n-1)!}{x^{n-1}} + C_n(x)\,\frac{n!}{x^n},$$

where

$$C_n(x) = \frac{1}{n!}\int_0^\infty \frac{\exp(-u)\, u^n}{1-\frac{u}{x}}\, du.$$

$C_n(x)$ is the factor by which the nth term of the nowhere convergent infinite series $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{x^k}$ must be multiplied so that, when the product is added to the sum of the first n terms of this infinite series, the result will be $C(x)$. It is known as the "converging factor" for $C(x)$ or, equivalently, for $Ei\, x$, $x > 0$.

On writing $Ei\, x = Ei\, 1 + \displaystyle\int_1^x \frac{\exp t}{t}\, dt$ and setting $t = x - u$, we see that $C(x) = \dfrac{x\, Ei\, 1}{\exp x} + \displaystyle\int_0^{x-1} \frac{\exp(-u)}{1-\frac{u}{x}}\, du$. Upon integration by parts, we obtain the relation $\displaystyle\int_0^{x-1} \frac{\exp(-u)}{1-\frac{u}{x}}\, du = 1 - \frac{x\, e}{\exp x} + \frac{1}{x}\int_0^{x-1} \frac{\exp(-u)}{\left(1-\frac{u}{x}\right)^2}\, du$, so that

$$C(x) = 1 + \frac{1}{x}\left\{\int_0^{x-1} \frac{\exp(-u)}{\left(1-\frac{u}{x}\right)^2}\, du + \frac{x^2(Ei\, 1 - e)}{\exp x}\right\}$$

Thus $C_1(x) = \displaystyle\int_0^{x-1} \frac{\exp(-u)}{\left(1-\frac{u}{x}\right)^2}\, du + \frac{x^2(Ei\, 1 - e)}{\exp x}$ A second integration by parts yields the relation

$$C(x) = 1 + \frac{1}{x} + \frac{2!}{x^2}\int_0^{x-1} \frac{\exp(-u)}{\left(1-\frac{u}{x}\right)^3}\, du + \frac{x\{Ei\ 1-(1+1)\,e\}}{\exp x}$$

so that

$$C_2(x) = \int_0^{x-1} \frac{\exp(-u)}{\left(1-\frac{u}{x}\right)^3}\, du + \frac{x^3\{Ei\ 1-(1+1)\,e\}}{2!\exp x}$$

Continuing to integrate by parts we find that, if n is any positive integer,

$$C_n(x) = \int_0^{x-1} \frac{\exp(-u)}{\left(1-\frac{u}{x}\right)^{n+1}}\, du + \frac{x^{n+1}\{Ei\ 1-c_n e\}}{n!\exp x},$$

where $c_n = 1 + 1 + 2! + \ldots + (n-1)!$. Setting $u = x - t$, we obtain the formula

$$C_n(x) = \frac{x^{n+1}}{\exp x}\left\{\int_1^x \frac{\exp t}{t^{n+1}} + \frac{Ei\ 1-c_n e}{n!}\right\},$$

from which it follows that

$$\frac{d}{dx}\, C_n(x) = \left(\frac{n+1}{x} - 1\right) C_n(x) + 1.$$

The terms of the nowhere convergent infinite series $1 + \frac{1}{x} + \frac{2!}{x^2} + \ldots$ initially decrease, if $x>1$, the ratio of the $(n+1)$ *st* term to the nth term being $\frac{n}{x}$. Thus, if x is an integer, the xth term and the $(x+1)$ *st* term are equal, all other terms of the series being greater than these. If x is not an integer, the least term is the nth, where n is the least integer greater than x. We find it convenient, when x is an integer, to take n to be $x-1$, and we observe that $\frac{d}{dx}\, C_n(x)$ has, at $x=n+1$, the value 1. In general, we write $x=n+1+h$, and note that, for any given x, n can be chosen so that $-\frac{1}{2} \le h < \frac{1}{2}$. Thus, $h=0$ when x is an integer, and $C_n(x) = C_n(n+1+h)$ has in a neighborhood of $h=0$ the Taylor development

$$C_n(n+1+h) = C_n(n+1) + d_1 h + d_2 h^2 + \ldots,$$

where the coefficients $d_1, d_2, \ldots$ are defined by the relations

$$d_1 = \left\{\frac{d}{dh}\, C_n(n+1+h)\right\}_{h=0};\ d_2 = \frac{1}{2!}\left\{\frac{d^2}{dh^2}\, C_n(n+1+h)\right\}_{h=0};\ \ldots$$

Writing the relation $\frac{d}{dx}\, C_n(x) = \left(\frac{n+1}{x} - 1\right) C_n(x) + 1$ in the form $(n+1+h)\,\frac{d}{dh}\, C_n(n+1+h) = -h\, C_n(n+1+h) + n+1+h$, we see that $d_1 = 1$, no matter what the value of n is. Upon differentiating with respect to h the relation just written, we obtain the relation $(n+1+h)\,\frac{d^2}{dh^2}\, C_n(n+1+h) = -(1+h)\,\frac{d}{dh}\, C_n(n+1+h) - C_n(n+1+h) + 1$, from which it follows that $2(n+1)d_2 = -\, C_n(n+1)$. Again differentiating with respect to h, we obtain the relation $(n+1+h)\,\frac{d^3}{dh^3}\, C_n(n+1+h) = -(2+h)\,\frac{d^2}{dh^2}\, C_n(n+1+h) - 2\,\frac{d}{dh} C_n\,(n+1+h)$, from which it follows that $3(n+1)d_3 = -(2d_2 + d_1)$. Continuing in this way, we obtain the recurrence relation

$$k(n+1)d_k = -\{(k-1)d_{k-1} + d_{k-2}\},\ k = 3, 4, \ldots$$

It follows that $(n+1)\,|d_k| \leq \frac{(k-1)\,|d_{k-1}| + |d_{k-2}|}{k}$, so that $(n+1)\,|d_k|$ is dominated by the greater of the two numbers $|d_{k-1}|$ and $|d_{k-2}|$. Denoting, for a moment, by M the greater of the two numbers 1 and $|d_2|$, we have $|d_3| \leq \frac{M}{n+1} < M$, so that $|d_4| \leq \frac{M}{(n+1)}$, which implies that $|d_5| \leq \frac{M}{(n+1)^2} < \frac{M}{n+1}$. Hence, $|d_6| \leq \frac{M}{(n+1)^2}$, so that $|d_7| \leq \frac{M}{(n+1)^3}$, and so on. In general, $|d_{2k}| \leq \frac{M}{(n+1)^{k-1}}$ and $|d_{2k+1}| \leq \frac{M}{(n+1)^k}$, $k = 1, 2, \ldots$, so that the power series $C_n(n+1) + d_1 h + d_2 h^2 + \ldots$ converges if $|h|^2 < (n+1)$. Owing to alternations in sign, the d's tend much more rapidly to zero than is indicated by the rough appraisals $|d_{2k}| \leq \frac{M}{(n+1)^{k-1}}$, $|d_{2k+1}| \leq \frac{M}{(n+1)^k}$, and the convergence of the power series $C_n(n+1) + d_1 h + d_2 h^2 + \ldots$ is quite rapid for values of h which are such that $|h| \leq \frac{1}{2}$, and it is good even when $h = \pm 1$.

As an example we carry out the calculation for $n = 10$ to 10 decimal places. Since $C_{19}(20) = \frac{20^{19}}{19!}\left\{C(20) - 1 - \left|\frac{1}{20} - \frac{2!}{(20)^2} - \ldots - \frac{18!}{(20)^{18}}\right.\right\}$, where $C(20) = \frac{20\, Ei\, 20}{\exp 20}$, and since $\frac{20^{19}}{19!}$ is approximately equal to 4.10^7, we determine $C(20)$ to 18 decimal places to ensure that $C_{19}(20)$ is correct to 10 decimal places. Since $\dfrac{20}{\exp 20}$ is approximately 4.10^{-8}, it suffices to start with the value 256 15652.66405 65888 2 of Ei 20, correct to 11 decimal places. We find that $C(20) = 1.05595\ 59055\ 92962\ 645$, to 18 decimal places and that $C_{19}(20) = 0.66815\ 49634$, to 10 decimal places. Hence,

$$d_2 = -\frac{1}{40}\ C_{19}(20) = -0.01670\ 38741; \quad d_3 = -\frac{1}{60}\ (2d_2+1) = -0.01610\ 98709$$

$$d_4 = -\frac{1}{80}\ (3d_3+d_2) = +0.\ 0_3 81\ 29186; \quad d_5 = -\frac{1}{100}\ (4d_4+d_3) = +0.\ 0_3 12\ 85820$$

$$d_6 = -\frac{1}{120}\ (5d_5+d_4) = -0.\ 0_4 1\ 21319; \quad d_7 = -\frac{1}{140}\ (6d_6+d_5) = -0.\ \quad 0_6 3985$$

$$d_8 = -\frac{1}{160}\ (7d_7+d_6) = +0.\ \quad 0_7 933$$

Setting $h = \pm 0.1$ in the formula $C_{19}(20+h) = (C_{19}(20) + d_2h^2 + d_4h^4 + \dots) + (h + d_3h^3 + \dots)$, we find that $C_{19}(20 \pm 0.1) = 0.66798\ 80060 \pm 0.09998\ 38914$, so that $C_{19}(20.1) = 0.76797\ 18974$, $C_{19}(19.9) = 0.56800\ 41146$. Similarly, setting $h = \pm 0.2$, we find that $C_{19}(20.2) = 0.86735\ 92704$, $C_{19}(19.8) = 0.46761\ 69462$, and setting, in turn, $h = \pm 0.3, \pm 0.4, \pm 0.5$, we find that $C_{19}(20.3) = 0.96622\ 63484$, $C_{19}(19.7) = 0.36709\ 00326$, $C_{19}(20.4) = 1.06714\ 60645$, $C_{19}(19.6) = 0.26920\ 54959$, $C_{19}(20.5) = 1.16201\ 98943$, $C_{19}(19.5) = 0.16603\ 93319$. Setting $h = \pm 1$, we obtain the values of $C_{19}(21)$ and $C_{19}(19)$, but we now need the values, $-0.0_8 19$ and $-0.0_9 4$, of d_9 and d_{10}, respectively. We find that

$C_{19}(21) = 1.63627\ 02796$, $C_{19}(19) = -0.33176\ 63418$. Since $C(x) = 1 + \dfrac{1}{x} + \dfrac{2!}{x^2} + \dots$ $+ \dfrac{(n-1)!}{x^{n-1}} + C_n(x)\dfrac{n!}{x^n} = 1 + \dfrac{1}{x} + \dots + \dfrac{n!}{x^n} + C_{n+1}(x)\dfrac{(n+1)!}{x^{n+1}}$, we have the relation $C_n(x)\dfrac{n!}{x^n} = \dfrac{n!}{x^n} + C_{n+1}(x)\dfrac{(n+1)!}{x^{n+1}}$, so that $C_n(x) = 1 + \dfrac{n+1}{x}\ C_{n+1}(x)$. In particular, $C_n(n+1) = 1 + C_{n+1}(n+1)$ and $C_n(n+2) = 1 + \dfrac{n+1}{n+2}\ C_{n+1}(n+2)$ or, equivalently, $C_{n+1}(n+2) = \dfrac{n+2}{n+1}\ (C_n(n+2) - 1)$. Thus $C_{18}(19) = 0.66823\ 36582$; $C_{20}(21) = 0.66808\ 37936$.

We have carried out these calculations to 45 decimal places, starting with the value 0.66815 49634 35226 78819 83897 04143 02536 96920 53972 of $C_{19}(20)$, this value having been determined from the value of Ei 20 given in Section 1. Doing this, we find that the values of $d_2, \dots, d_{10}$ and of $C_{18}(19)$ and $C_{19}(20)$ calculated, as an example, to 10 decimal places in the preceding paragraph are all correct to the last decimal place. Indeed, the rounding errors balance more or less when we work downwards from $C_n(n+1)$ to $C_{n-1}(n)$. For example, when we work downwards from $C_{19}(20)$ to $C_9(10)$ we obtain the value 0.66965 59030 74791 50274 69509 06002 39100 44134 63910 for the latter converging factor.

On calculating $Ei\ 10$ from the formula $Ei\ 10 = \gamma + \log_e 10 + \sum_{k=1}^{\infty} \frac{10^k}{k.\,k!}$, we obtain the value $Ei\ 10 = 2492.22897\ 62418\ 77759\ 13844\ 01439\ 98524\ 84898\ 96471\ 01430\ 94234\ 53879$, and from this we determine $C_9(10)$, the resulting value differing from the one just given only in that the last figure is a 4 rather than a zero. Since $d_2 = -\frac{C_n(n+1)}{2(n+1)}$, an error of a few units in the last decimal place does not show up in d_2; nor does it show up in the remaining coefficients d_k, $k=3, \ldots,$ since $d_3 = -\frac{1}{3(n+1)}(1+2d_2)$, and so on. Thus all 45 decimals of the coefficients d_k, for the various values of n from 19 to 5, which we give are correct, while the last decimal place of the converging factors $C_n(n+1)$, from $n=18$ to $n=5$, should be treated as a guard figure. As a second illustration of the balancing of rounding errors, we observe that when we start with the adjusted value of $C_9(10)$ and work down to $C_4(5)$, we obtain the value 0.67268 95170 64739 00786 17640 74824 64036 16676 57502 for the latter converging factor. On calculating $Ei\ 5 = \gamma + \log_e 5 + \sum_{k=1}^{\infty} \frac{5^k}{k.\,k!}$, we obtain the value $Ei\ 5 = 40.18527\ 53558\ 03177\ 45509\ 14217\ 93795\ 86709\ 54190\ 87399\ 19593\ 04339$, and from this we determine $C_4(5)$, the resulting value differing from the one just given only in that the last figure is a 4 rather than a 2. We give, in Table 1, the values of $C_n(n+1)$ for values of n from 4 to 20, inclusive.

TABLE 1

n	$C_n(n+1)$								
4	0.67268	95170	64739	00786	17640	74824	64036	16676	57502
5	0.67167	41597	23592	90621	95365	26431	71574	70413	39625
6	0.67095	13224	50164	67304	61715	04486	00544	04599	27971
7	0.67041	06361	50233	20732	48806	52522	58909	47249	09018
8	0.66999	10087	76206	46897	98473	55147	74517	86251	13002
9	0.66965	59030	74791	50274	69509	06002	39100	44134	63910
10	0.66938	21330	06108	66931	32580	83438	23344	00650	58853
11	0.66915	42786	33243	76931	23142	24278	12072	08024	08638
12	0.66896	16871	06122	82925	20774	85568	66422	65282	79016
13	0.66879	67632	76254	16516	75869	82985	99144	33040	86388
14	0.66865	39464	12270	33310	40940	04574	12613	90972	13591
15	0.66852	90719	66716	05797	90869	97889	23880	02248	96284
16	0.66841	89593	70525	45706	35900	03013	07512	75110	46446
17	0.66832	11376	92053	06079	09122	85663	76541	04682	94792
18	0.66823	36582	16749	55546	74074	39908	71627	59446	40700
19	0.66815	49634	35226	78819	83897	04143	02536	96920	53972
20	0.66808	37935	81999	53426	39404	80412	44492	86773	05041

We give, in Appendix A, for each value of n from $n=5$ to $n=19$, inclusive, those of the coefficients $d_2, d_3, \ldots$ of the formula $C_n(n+1+h) = C_n(n+1) + h + d_2 h^2 + d_3 h^3 + \ldots$ which are $>5.10^{-46}$. Setting $h = \pm \frac{1}{2}$, we determine, from these coefficients, $C_n\left(n \pm \frac{1}{2}\right)$. We give, in Table 2, the values of $C_n\left(n+\frac{1}{2}\right)$ for values of n from 5 to 20, inclusive (the value of $C_{20}(20.5)$ being determined from the formula $C_{19}(20.5) = 1 + \frac{20}{(20.5)} C_{20}(20.5)$). The relation $C_n\left(n+\frac{1}{2}\right) = \frac{2n+1}{2n}\left\{C_{n-1}\left(n+\frac{1}{2}\right) - 1\right\}$ is a valuable check on the accuracy of the calculation of $C_n\left(n+\frac{1}{2}\right)$.

TABLE 2

n	$C_n\left(n+\frac{1}{2}\right)$								
5	0.16435	81787	32824	63356	24811	10356	03926	17601	21766
6	0.16472	79184	59369	57258	48457	83181	63680	84212	61611
7	0.16499	59048	79195	19111	23616	36303	04427	79758	93422
8	0.16519	89725	53858	68381	43177	01915	63027	97752	33222
9	0.16535	81207	01555	91649	22637	86862	23315	17773	85394
10	0.16548	61859	19084	74190	39375	64255	33025	08750	70403
11	0.16559	14525	23073	06583	46611	66205	55839	98883	00910
12	0.16567	95041	85508	53801	31280	54885	06834	82801	53054
13	0.16575	42405	31555	51032	76432	11682	28997	12029	68983
14	0.16581	84670	19345	05171	40102	04695	81856	07425	33746
15	0.16587	42532	73479	26557	24502	27424	07426	75750	35418
16	0.16592	31593	37322	07115	37202	20350	62852	72203	73929
17	0.16596	63833	26044	64441	80190	60953	55049	21332	79815
18	0.16600	48605	77667	52445	91440	08227	61412	86253	88695
19	0.16603	93319	16784	90147	73212	83531	12998	63887	59720
20	0.16607	03916	98012	61120	08635	46571	23841	37800	17186

THE CONVERGING FACTOR WHEN THE ARGUMENT OF THE EXPONENTIAL INTEGRAL IS NEGATIVE

Setting $t = -\tau$ in the integral $\int_{-\infty}^{-x} \frac{\exp t}{t}\, dt$, which defines $Ei(-x)$, where $x > 0$, we see that

$$-Ei(-x) = \int_{x}^{\infty} \frac{\exp(-\tau)}{\tau}\, d\tau = \frac{\exp(-x)}{x} \int_{0}^{\infty} \frac{\exp(-u)}{1 + \frac{u}{x}}\, du, \quad \tau = x + u.$$

Thus $-Ei(-x) = \frac{\exp(-x)}{x}\, \Gamma(x)$, where $\Gamma(x) = \int_{0}^{\infty} \frac{\exp(-u)}{1 + \frac{u}{x}}\, du$

Since $\frac{1}{1 + \frac{u}{x}} = 1 - \frac{u}{x} + \ldots + (-1)^{n-1} \frac{u^{n-1}}{x^{n-1}} + (-1)^n \frac{u^n}{x^n \left(1 + \frac{u}{x}\right)}$, where n is any positive integer, we have the relation

$$\Gamma(x) = 1 - \frac{1}{x} + \frac{2!}{x^2} - \ldots + (-1)^{n-1} \frac{(n-1)!}{x^{n-1}} + (-1)^n\, \Gamma_n(x) \frac{n!}{x^n}, \quad n = 1, 2, 3, \ldots$$

where

$$\Gamma_n(x) = \frac{1}{n!} \int_{0}^{\infty} \frac{\exp(-u) u^n}{1 + \frac{u}{x}}\, du, \quad n = 1, 2, \ldots$$

$\Gamma_n(x)$ is the converging factor for $\Gamma(x)$ or, equivalently, for $-Ei(-x)$, $x > 0$.
Since $\Gamma(x) = 1 - \frac{1}{x} + \frac{2!}{x^2} - \ldots + (-1)^{n-1} \frac{(n-1)!}{x^{n-1}} + \Gamma_n(x)(-1)^n \frac{n!}{x^n} = 1 - \frac{1}{x} + \frac{2!}{x^2} - \ldots + (-1)^n \frac{n!}{x^n} + \Gamma_{n+1}(x)(-1)^{n+1} \frac{(n+1)!}{x^{n+1}}$, $\Gamma_n(x)$ satisfies the recurrence relation

$$\Gamma_n(x) = 1 - \frac{n+1}{x}\, \Gamma_{n+1}(x)$$

Upon writing $\Gamma_n(x)$ in the form $-\frac{1}{n!} \int_{u=0}^{u=\infty} \frac{u^n}{\left(1 + \frac{u}{x}\right)}\, d(\exp -u)$ and integrating by parts, we see that

$$\Gamma_n(x) = \Gamma_{n-1}(x) - \frac{1}{n!x} \int_{0}^{\infty} \frac{\exp(-u)\, u^n}{\left(1 + \frac{u}{x}\right)^2}\, du$$

Also $\frac{d\Gamma_n}{dx} = \frac{1}{n!x}\int_0^\infty \frac{\exp(-u)u^n \cdot \frac{u}{x}}{\left(1+\frac{u}{x}\right)^2} du = \frac{1}{n!x}\int_0^\infty \frac{\exp(-u)u^n}{1+\frac{u}{x}} du - \frac{1}{n!x}\int_0^\infty \frac{\exp(-u)u^n}{\left(1+\frac{u}{x}\right)^2} du,$

so that $-\frac{1}{n!x}\int_0^\infty \frac{\exp(-u)u^n}{\left(1+\frac{u}{x}\right)^2} du = \frac{d\Gamma_n}{dx} - \frac{1}{x}\Gamma_n$. Since $\Gamma_{n-1}(x) = 1 - \frac{n}{x}\Gamma_n(x)$, $n=1, 2, \ldots$,

$\Gamma_0(x)$ denoting $\Gamma(x)$, it follows that $\Gamma_n(x)$ satisfies the differential equation

$$\frac{d\Gamma_n}{dx} = \left(\frac{n+1}{x} + 1\right)\Gamma_n - 1$$

Writing, as before, $x = n+1+h$, where $-\frac{1}{2} \leq h \leq \frac{1}{2}$, the solution of this differential equation has the power series development

$$\Gamma_n(x) = \Gamma_n(n+1) + \delta_1 h + \delta_2 h^2 + \delta_3 h^3 + \ldots$$

In contrast to the development of $C_n(x)$, δ_1 no longer has the constant value unity, its value being $2\Gamma_n(n+1) - 1$. Upon writing the differential equation satisfied by $\Gamma_n(x)$ in the form $x\frac{d\Gamma_n}{dx} = (n+1+x)\Gamma_n - x$ and differentiating it, we see that $x\frac{d^2\Gamma_n}{dx^2} = (n+x)\frac{d\Gamma_n}{dx} + \Gamma_n - 1$, and upon evaluating this equation at $h=0$, so that $x=n+1$, we see that $2(n+1)\delta_2 = (2n+1)\delta_1 + \Gamma_n(n+1) - 1$. A second differentiation with respect to x shows that $x\frac{d^3\Gamma_n}{dx^3} = (n+x-1)\frac{d^2\Gamma_n}{dx^2} + 2\frac{d\Gamma_n}{dx}$, so that $3(n+1)\delta_3 = 2n\delta_2 + \delta_1$. Similarly, $4(n+1)\delta_4 = (2n-1)\delta_3 + \delta_2$, and generally

$$k(n+1)\delta_k = (2n+3-k)\delta_{k-1} + \delta_{k-2}, \quad k=3, 4, 5, \ldots$$

Denoting, for a moment, by M the greater of the two numbers $|\delta_{k-1}|$ and $|\delta_{k-2}|$, it follows from this recurrence relation that, if $k \leq 2n+3$, $|\delta_k| \leq \frac{2n+4-k}{k(2n+1)} M < \frac{M}{k}$ and that, if $k > 2n+3$, $|\delta_k| \leq \frac{k-2-2n}{k(n+1)} M < \frac{M}{n+1}$. Thus the power series $\Gamma_n(n+1) + \delta_1 h + \delta_2 h^2 + \delta_3 h^3 + \ldots$ converges if $|h| < n+1$. The convergence is very rapid, all the δ's which carry an odd subscript being positive, while all which carry an even subscript are negative. On setting $h = \pm 1$, we obtain the two relations

$$\Gamma_n(n+2) = \Gamma_n(n+1) + \delta_1 + \delta_2 + \delta_3 + \ldots,$$

$$\Gamma_n(n) = \Gamma_n(n+1) - \delta_1 + \delta_2 - \delta_3 + \ldots,$$

and from these we deduce, using the recurrence relation $\Gamma_n(x) = 1 - \frac{n+1}{x}\Gamma_{n+1}(x)$, that

$$\Gamma_{n+1}(n+2) = \frac{n+2}{n+1}\{1-(\Gamma_n(n+1) + \delta_1 + \delta_2 + \dots\quad)\}$$

$$\Gamma_{n-1}(n) \quad = 1 - \Gamma_n(n+1) + \delta_1 - \delta_2 + \delta_3 - \dots$$

An error in δ_1, such as a rounding error, induces approximately the same error in δ_2 and δ_3, but when k is greater than about 6 the error induced by this error in δ_1, if it is not a gross one, disappears in δ_k. In view of the permanence in signs in the formula furnishing $\Gamma_{n+1}(n+2)$, and the alternation in signs in the formula furnishing $\Gamma_{n-1}(n)$, it is better to work downwards from n to $n-1$ than upwards from n to $n+1$. When we do this, the rounding errors practically cancel.

As an example, we show the calculation, to 10 decimal places, for $n=19$. Since $\Gamma_{19}(20) = -\frac{20^{19}}{19!}\left\{\Gamma_{20} - \left(1 - \frac{1}{20} + \frac{2!}{(20)^2} - \dots + \frac{18!}{(20)^{18}}\right)\right\}$, where $\Gamma(20) = -20\,(\exp 20) \times Ei(-20)$, and since $\frac{20^{19}}{19!}$ is approximately 4.10^7, we must determine $\Gamma(20)$ to 18 decimal places to ensure that $\Gamma_{19}(20)$ is correct to 10 decimal places. Since 20 exp 20 is nearly 10^{10}, we must determine $-Ei(-20)$ to 28 decimal places to ensure that $\Gamma(20)$ is correct to 18 decimal places. Conversely, if $\Gamma_{19}(20)$ is known to 10 decimal places, we can calculate $-Ei(-20)$ to 28 decimal places (or 27 decimal places if we regard the last decimal place as a guard figure). From the value of $-Ei(-20)$ given in Section 1 we find that $\Gamma_{19}(20) = 0.50617\ 10499$ to 10 decimal places. Hence,

$$\delta_1 = 2\Gamma_{19}(20) - 1 = 0.01234\ 20998; \quad \delta_2 = \frac{39\delta_1 - \{1 - \Gamma_{19}(20)\}}{40} = -0.00031\ 21764$$

$$\delta_3 = \frac{\delta_1 + 38\delta_2}{60} = 0.00000\ 79899; \quad \delta_4 = \frac{\delta_2 + 37\delta_3}{80} = -0.00000\ 02069$$

$$\delta_5 = \frac{\delta_3 + 36\delta_4}{100} = 0.00000\ 00054; \quad \delta_6 = \frac{\delta_4 + 35\delta_5}{120} = -0.00000\ 00001$$

Note how much more rapidly the coefficients δ_i decrease in numerical magnitude than the coefficients d_i, which arose when discussing the converging factor when the argument of the exponential integral is positive. For example, $|\delta_6| = 10^{-10}$, while $|d_6|$ is greater than 10^{-5}. Setting $h = \pm 0.1$, we find that $\Gamma_{19}(20 \pm 0.1) = 0.50616\ 79281 \pm 0.00123\ 42180$, so that $\Gamma_{19}(20.1) = 0.50740\ 21461$ and $\Gamma_{19}(19.9) = 0.50493\ 37101$. Setting $h = \pm 0.2$, we find that $\Gamma_{19}(20 \pm 0.2) = 0.50615\ 85625 \pm 0.00246\ 84839$, so that $\Gamma_{19}(20.2) = 0.50862\ 70464$ and $\Gamma_{19}(19.8) = 0.50369\ 00786$. Similarly, $\Gamma_{19}(20 \pm 0.3) = 0.50614\ 29524 \pm 0.00370\ 28456$, so that $\Gamma_{19}(20.3) =$

0.50984 57980 and $\Gamma_{19}(19.7) = 0.50244\ 01068$; $\Gamma_{19}(20 \pm 0.4) = 0.50612\ 10964 \pm 0.00493\ 73514$, so that $\Gamma_{19}(20.4) = 0.51105\ 84478$ and $\Gamma_{19}(19.6) = 0.50118\ 37450$; $\Gamma_{19}(20 \pm 0.5) = 0.50609\ 29929 \pm 0.00617\ 20488$, so that $\Gamma_{19}(20.5) = 0.51226\ 50417$ and $\Gamma_{19}(19.5) = 0.49992\ 09441$. Setting $h = \pm 1$, we find that $\Gamma_{19}(21) = 0.51820\ 87616$ and $\Gamma_{19}(19) = 0.49350\ 85714$; since $\Gamma_{19}(21) = 1 - \frac{20}{21}\Gamma_{20}(21)$ and $\Gamma_{18}(19) = 1 - \Gamma_{19}(19)$, it follows that

$$\Gamma_{20}(21) = 0.50588\ 08003; \quad \Gamma_{18}(19) = 0.50649\ 14286.$$

We have carried out the calculations to 48 decimal places, starting with the value 0.50617 10498 71381 60623 62865 49103 98633 92600 67424 949 of $\Gamma_{19}(20)$; to obtain this accuracy it was necessary to calculate $-Ei(-20)$ to 65 decimal places. The correct values of $\Gamma_{20}(21)$ and $\Gamma_{18}(19)$, to 10 decimal places turn out to be 0.50588 08006 and 0.50649 14287, showing clearly that it is better to determine $\Gamma_{n-1}(n)$ from $\Gamma_n(n+1)$ than to determine $\Gamma_{n+1}(n+2)$ from $\Gamma_n(n+1)$. When we work down from $\Gamma_{19}(20)$ to $\Gamma_9(10)$ we obtain the value 0.51218 19943 76050 59331 41598 75483 65226 30795 54973 214 for the latter converging factor; the value of $\Gamma_9(10)$ obtained by first calculating $-Ei(-10)$ differs from this only in that the last two figures are 22 instead of 14, which is a striking illustration of how the rounding errors balance out when we work down from larger to lesser values of n. The value of $\Gamma_9(10)$ to 50 decimals, obtained by calculating the value of $-Ei(-10)$ to 60 decimals, is 0.51218 19943 76050 59331 41598 75483 65226 30795 54973 22154. Starting with this value and working down to $\Gamma_4(5)$, we find that $\Gamma_4(5) = 0.52372\ 08704\ 07838\ 77766\ 75769\ 75593\ 86660\ 48727\ 78580\ 43665$. The correct value of $\Gamma_4(5)$, obtained by first calculating $-Ei(-5)$, differs from this only in that the last figure is a 6 instead of a 5. We give, in Table 3, the values of $\Gamma_n(n+1)$ to 50 decimal places, for values of n from 3 to 9 inclusive and to 48 decimal places for values of n from 10 to 19, inclusive. We also give $\Gamma_{20}(21)$ to 47 decimal places. The last two decimals may be treated as guard figures.

We give, in Appendix B, for each value of n from $n=4$ to $n=19$, inclusive, those of the coefficients $\delta_1, \delta_2, \ldots$ of the formula $\Gamma_n(n+1+h) = \Gamma_n(n+1) + \delta_1 h + \delta_2 h^2 + \ldots$ which are $> 5.10^{-51}$, if $4 \le n \le 9$, and $> 5.10^{-49}$ if $10 \le n \le 19$, the last two decimal places being treated as guard figures. Setting $h = \pm\frac{1}{2}$, we determine, from these coefficients, $\Gamma_n\left(n \pm \frac{1}{2}\right)$. We give, in Table 4, the values of $\Gamma_n\left(n + \frac{1}{2}\right)$ for values of n from 4 to 20, inclusive, these values being given to 50 decimals for $4 \le n \le 9$ and to 48 decimals for $10 \le n \le 20$, the last two decimals again being treated as guard figures. The value of $\Gamma_{20}(20.5)$ was determined from the formula $\Gamma_{19}(20.5) = 1 - \frac{20}{20.5}\Gamma_{20}(20.5)$.

TABLE 3

n	$\Gamma_n(n+1)$									
4	0.52372	08704	07838	77766	75769	75593	86660	48727	78580	43665
5	0.51994	57538	64775	68821	34346	32507	84104	31536	81478	87906
6	0.51720	58142	19967	97031	31695	21499	10216	53040	55595	79910
7	0.51512	69527	70493	97856	08002	29281	45108	44467	87318	40198
8	0.51349	58469	29257	70751	62570	32861	16277	50607	14975	02227
9	0.51218	19943	76050	59331	41598	75483	65226	30795	54973	22154
10	0.51110	10921	86620	32355	85526	35981	16719	20453	32323	309
11	0.51019	62627	04216	50644	07137	94393	65938	18306	87300	628
12	0.50942	77437	63451	63838	67447	16936	64864	65048	01678	699
13	0.50876	69032	39266	60994	64208	95127	06776	53349	39915	836
14	0.50819	26005	56749	87188	07473	77755	64022	60300	56829	894
15	0.50768	88876	00355	53394	18119	96939	86807	43040	37618	825
16	0.50724	35071	53992	67161	89024	32507	65994	30463	35544	398
17	0.50684	68838	09181	78280	68775	99599	45014	76113	41866	320
18	0.50649	14287	06755	41776	15652	60279	22086	17472	45872	260
19	0.50617	10498	71381	60623	62865	49103	98633	92600	67424	949
20	0.50588	08005	53800	01150	70690	06386	77754	79946	62993	91

TABLE 4

n	$\Gamma_n\left(n+\frac{1}{2}\right)$									
4	0.49869	67817	44600	66346	81680	27784	51187	14425	53858	29470
5	0.49910	04970	51522	20594	67203	88220	83342	06805	40430	14520
6	0.49934	21135	60480	20657	09460	03094	52452	79395	75719	75088
7	0.49949	80535	26792	67444	28518	04849	04147	80106	16022	52110
8	0.49960	44966	71539	18881	76039	54113	35472	71422	90042	79728
9	0.49968	03657	10424	30671	67707	02113	48870	07480	23745	90208
10	0.49973	63354	46409	67524	26295	20584	00122	16192	85692	242
11	0.49977	88002	65633	21859	94295	27136	68655	01893	45922	921
12	0.49981	17784	95670	33213	45641	82375	63770	27704	49206	727
13	0.49983	78982	62394	36701	47637	57255	33360	92986	69220	639
14	0.49985	89370	11897	83306	32228	18634	18086	80862	56041	235
15	0.49987	61316	94446	67360	51546	57027	91771	27996	24864	949
16	0.49989	03646	05377	83256	78129	80596	36167	11802	05511	347
17	0.49990	22787	44567	49180	77571	65654	19960	48375	13273	662
18	0.49991	23516	64233	67482	09643	07122	62753	96535	66722	194
19	0.49992	09440	37290	98172	85944	00091	48814	37611	23830	019
20	0.49992	83323	72379	47462	34462	88502	21064	37411	21928	472

THE ASYMPTOTIC FORMULA FOR THE CONVERGING FACTOR $\Gamma_n(n+1)$

It is possible to derive asymptotic formulas for the converging factors $C_n(n+1)$ and $\Gamma_n(n+1)$, which furnish good approximations to these converging factors when n is reasonably large, say ≥ 10, the approximations improving with increasing values of n. We begin with $\Gamma_n(n+1)$, the converging factor when the argument of the exponential integral is a negative integer. Upon integrating by parts the integral $\int_0^\infty \frac{\exp(-u)}{1+\frac{u}{x}}\,du$ which furnishes $\Gamma(x)$, we obtain the relation

$$\Gamma(x) = 1 - \frac{1}{x}\int_0^\infty \frac{\exp(-u)}{\left(1+\frac{u}{x}\right)^2}\,du$$

and, on continuing to integrate by parts, we see that, n being any positive integer,

$$\Gamma(x) = 1 - \frac{1}{x} + \frac{2!}{x^2} - \ldots + (-1)^{n-1}\frac{(n-1)!}{x^{n-1}} + (-1)^n \frac{n!}{x^n}\int_0^\infty \frac{\exp(-u)}{\left(1+\frac{u}{x}\right)^{n+1}}\,du$$

Thus we have the relation

$$\Gamma_n(x) = \int_0^\infty \frac{\exp(-u)}{\left(1+\frac{u}{x}\right)^{n+1}}\,du, \quad n = 1, 2, \ldots .$$

Observe that the integration by parts which we have just indicated cannot be carried out when the argument of the exponential integral is positive, since the integral $\int_0^\infty \frac{\exp(-u)}{\left(1-\frac{u}{x}\right)^{n+1}}\,du$, which has a Cauchy principal value when $n=0$, does not exist when $n=1$. It follows from the relation just written that $\Gamma_n(x)$ is positive for all positive real numbers x, no matter what is the positive integer n, and this implies, by virtue of the relation $\Gamma_n(x) = 1 - \frac{n+1}{x}\Gamma_{n+1}(x)$, the bracketing inequalities

$$0 < \Gamma_n(x) < 1, \quad n = 1, 2, \ldots$$

To obtain a better estimate of $\Gamma_n(x)$, we make the substitution $u = \frac{n+1}{v} - x$ in the integral which furnishes $\Gamma_n(x)$, and obtain the relation

$$\Gamma_n(x) = \frac{x^{n+1}\exp x}{(n+1)^{n+1}} \int_0^{\frac{n+1}{x}} \exp\left(-\frac{n+1}{v}\right) v^{n+1} (n+1) \frac{dv}{v^2}$$

It follows on integration by parts, since the differential of $\exp\left(-\frac{n+1}{v}\right) v^{n+1}$ is $\exp\left(-\frac{n+1}{v}\right) v^{n+1} (n+1) \frac{v+1}{v^2}\, dv$, that

$$\Gamma_n(x) = \frac{1}{\frac{n+1}{x}+1} + \frac{x^{n+1}\exp x}{(n+1)^{n+1}} \int_0^{\frac{n+1}{x}} \exp\left(-\frac{n+1}{v}\right) v^{n+1} \frac{dv}{(v+1)^2} .$$

This relation tells us that not only is $\Gamma_n(x)$ positive, but that it is greater than $\frac{1}{\frac{n+1}{x}+1} = \frac{x}{x+n+1}$. In particular,

$$\Gamma_n(n+1) > \frac{1}{2},$$

which sharpens the inequality $\Gamma_n(n+1) > 0$. The inequality $\Gamma_n(n+1) < 1$ may be correspondingly sharpened. Since $\Gamma_{n+1}(x) > \frac{x}{x+n+2}$, we have

$$\Gamma_n(x) = 1 - \frac{n+1}{x}\Gamma_{n+1}(x) < \frac{1+x}{x+n+2} = \frac{1}{2} + \frac{x-n}{2(x+n+2)},$$

so that

$$\Gamma_n(n+1) < \frac{1}{2} + \frac{1}{2(2n+3)} .$$

Writing $\int_0^{\frac{n+1}{x}} \exp\left(-\frac{n+1}{v}\right) v^{n+1} \frac{dv}{(v+1)^2}$ in the form $\frac{1}{n+1}\int_{v=0}^{v=\frac{n+1}{x}} \frac{v^2}{(v+1)^3} \cdot d\left\{\exp\left(-\frac{n+1}{v}\right) v^{n+1}\right\}$, and integrating by parts, we see that

$$\Gamma_n(x) = \frac{x}{n+1+x} + \frac{(n+1)x}{(n+1+x)^3} + \frac{x^{n+1}\exp x}{(n+1)^{n+2}} \int_0^{\frac{n+1}{x}} \exp\left(-\frac{n+1}{v}\right) v^{n+1} \frac{v(v-2)}{(v+1)^4}\, dv.$$

If, then, $\frac{n+1}{x} \leq 2$, so that $v \leq 2$, we have the inequality

$$\Gamma_n(x) < \frac{x}{n+1+x} + \frac{(n+1)\,x}{(n+1+x)^3} \,,$$

and this implies, if $\frac{n+2}{x} \leq 2$, since $\Gamma_{n+1}(x) < \frac{x}{n+2+x} + \frac{(n+2)\,x}{(n+2+x)^3}$ and $\Gamma_n(x) = 1 - \frac{n+1}{x}\,\Gamma_{n+1}(x)$, the inequality

$$\Gamma_n(x) > \frac{1+x}{n+2+x} - \frac{(n+1)\,(n+2)}{(n+2+x)^3} \,.$$

In particular,

$$\frac{n+2}{2n+3} - \frac{(n+1)\,(n+2)}{(2n+3)^3} < \Gamma_n(n+1) < \frac{1}{2} + \frac{1}{8(n+1)} \,.$$

Setting $n=19$, we obtain the appraisal $0.50611 < \Gamma_{19}(20) < 0.50625$; the mean of the two bounds, 0.50618, is a good approximation to the value 0.50617 of $\Gamma_{19}(20)$, correct to 5 decimal places. Setting $x = n+1+h$, where we may take $|h| < \frac{1}{2}$, we obtain the appraisal

$$\frac{n+2+h}{2n+3+h} - \frac{(n+1)\,(n+2)}{(2n+3+h)^3} < \Gamma_n(n+1+h) < \frac{n+1+h}{2n+2+h} + \frac{(n+1)\,(n+1+h)}{(2n+2+h)^3}$$

of $\Gamma_n(n+1+h)$. In particular, when $h = -\frac{1}{2}$, we have the appraisal

$$\frac{2n+3}{4n+5} - \frac{8(n+1)\,(n+2)}{(4n+5)^3} < \Gamma_n\left(n+\frac{1}{2}\right) < \frac{2n+1}{4n+3} + \frac{4(n+1)\,(2n+1)}{(4n+3)^3}$$

of $\Gamma_n\left(n+\frac{1}{2}\right)$. Setting $n=19$, we obtain the inequalities

$$0.49984 < \Gamma_{19}(19.5) < 0.49999$$

The mean of the two bounds, 0.499915, is a good approximation to $\Gamma_{19}(19.5)$, whose value, to 6 decimal places, is 0.499921.

In order to secure closer estimates of $\Gamma_n(x)$, we continue to integrate by parts. Airey[16] was the first to treat the converging factor, and he was particularly interested in the case where x is an integer and $n = x-1$. To facilitate comparison of our results with those of

Airey, we set $(n+1) = (1-\beta)x$, so that $\beta=0$ when x is an integer and $n=x-1$. If $x=n+1+h$, $\beta = \frac{h}{x}$. The results of the two integrations by parts which we have already performed may be written as follows:

$$1.\quad \Gamma_n(x) = \frac{1}{2-\beta} + \frac{\exp x}{(1-\beta)^{n+1}} \int_0^{1-\beta} \exp\left(-\frac{n+1}{v}\right) v^{n+1} \frac{P_0(v)}{(v+1)^2}\, dv$$

$$2.\quad \Gamma_n(x) = \frac{1}{2-\beta} + \frac{1-\beta}{(2-\beta)^3 x} + \frac{\exp x}{(n+1)(1-\beta)^{n+1}} \int_0^{1-\beta} \exp\left(-\frac{n+1}{v}\right) v^{n+1} \frac{v P_1(v)}{(v+1)^4}\, dv$$

where $P_0(v)=1$, $P_1(v)=v-2$. The next integration by parts yields the relation

$$3.\quad \Gamma_n(x) = \frac{1}{2-\beta} + \frac{1-\beta}{(2-\beta)^3 x} - \frac{(1-\beta)(1+\beta)}{(2-\beta)^5 x^2} + \frac{\exp x}{(n+1)^2 (1-\beta)^{n+1}} \int_0^{1-\beta} \exp\left(-\frac{n+1}{v}\right) v^{n+1} \frac{v^2 P_2(v)}{(v+1)^6}\, dv,$$

where $P_2(v)$ is defined by the relation

$$-\frac{d}{dv}\left\{\frac{v^3 P_1(v)}{(v+1)^5}\right\} = \frac{v^2 P_2(v)}{(v+1)^6},$$

so that $P_2(v) = v^2 - 8v + 6$. Proceeding with the integration by parts, we construct in this way a sequence of polynomials, $P_j(v), j=0,1,2,\ldots$, where $P_{j+1}(v)$ is defined in terms of $P_j(v)$ by means of the relation

$$-\frac{d}{dv}\left\{\frac{v^{j+2} P_j(v)}{(v+1)^{2j+3}}\right\} = \frac{v^{j+1} P_{j+1}(v)}{(v+1)^{2j+4}}, \quad j=0,1,2,\ldots$$

$P_0(v)$ is 1 and $P_j(v)$ is a polynomial in v of degree j. $P_{j+1}(v)$ is determined, when $P_j(v)$ is known, by the formula

$$P_{j+1}(v) = \{(j+1)v - (j+2)\} P_j(v) - v(v+1) \frac{d}{dv} P_j(v)$$

Thus, if $P_j(v) = a_0 v^j + a_1 v^{j-1} + \ldots + a_j$ and $P_{j+1}(v) = b_0 v^{j+1} + b_1 v^j + \ldots + b_{j+1}$, we have $b_k = (k+1)\, a_k - (2j-k+3)\, a_{k-1}$, it being understood that a_{k-1} is zero if $k=0$, and that a_k is zero if $k=j+1$. Thus $b_0 = a_0$, which implies, since $P_0(v) = 1$, that the coefficient of the highest power of v in $P_j(v)$ is 1, for $j=1,2,\ldots$. Also $b_{j+1} = -(j+2)\, a_j$, which implies that the constant term in $P_j(v)$ is $(-1)^j\,(j+1)!$. The expressions furnishing $P_j(v)$, for $j=0,1,2,\ldots,9$, are given in Table 5.

TABLE 5

j	$P_j(v)$
0	1
1	$v-2$
2	v^2-8v+6
3	$v^3-22v^2+58v-24$
4	$v^4-52v^3+328v^2-444v+120$
5	$v^5-114v^4+1452v^3-4400v^2+3708v-720$
6	$v^6-240v^5+5610v^4-32120v^3+58140v^2-33984v+5040$
7	$v^7-494v^6+19950v^5-195800v^4+644020v^3-785304v^2+341136v-40320$
8	$v^8-1004v^7+67260v^6-1062500v^5+5765500v^4-12440064v^3+11026296v^2-3733920v+362880$
9	$v^9-2026v^8+218848v^7-5326160v^6+44765000v^5-155357384v^4+238904904v^3-162186912v^2+44339040v-3628800$

$P_2(v)$ has two real zeros, $4 \pm 10^{1/2}$, both of which are positive, and it follows from the relation

$$-\frac{d}{dv}\left\{\frac{v^4 P_2(v)}{(v+1)^7}\right\} = \frac{v^3 P_3(v)}{(v+1)^8}$$

that $P_3(v)$ has a real zero between these two zeros of $P_2(v)$. At either zero of $P_2(v)$, $P_3(v) = -\,v(v+1)\,\dfrac{d}{dv}\,P_2(v)$, so that $P_3(v)$ is positive at the smaller, and negative at the larger, of the two zeros of $P_2(v)$. Hence, $P_3(v)$ has three real zeros, all positive, the smallest being less than $4-10^{1/2}$ and the largest greater than $4+10^{1/2}$. We express this result by the statement that the zeros of $P_3(v)$ interlace those of $P_2(v)$. The same argument shows that $P_4(v)$ has four positive zeros, which interlace the three positive zeros of $P_3(v)$ and, generally, that $P_j(v)$ has j positive zeros, which interlace the $j-1$ positive zeros of $P_{j-1}(v)$, $j=2,3,\ldots$. In terms of the polynomials $P_j(v)$ we have the following expression giving $\Gamma_n(x)$

$$\Gamma_n(x) = a_0(\beta) + \frac{a_1(\beta)}{x} + \ldots + \frac{a_m(\beta)}{x^m} + \frac{\exp x}{(n+1)^m (1-\beta)^{n+1}} \int_0^{1-\beta} \exp\left(-\frac{n+1}{v}\right) v^{n+1}\, \frac{v_m P_m(v)}{(v+1)^{2m+2}}\, dv,$$

where $a_0(\beta) = \dfrac{1}{2-\beta}$ and

$$a_j(\beta) = \frac{(1-\beta)\,P_{j-1}(1-\beta)}{(2-\beta)^{2j+1}} \quad , \; j = 1, 2, \ldots, m,$$

and m is any positive integer. Denoting $a_j(0)$ simply by a_j, we have

$$a_0 = \frac{1}{2}, \quad a_j = \frac{P_{j-1}(1)}{2^{2j+1}} \quad , \; j = 1,2, \ldots$$

The values of a_j for $0 \le j \le 22$ were given by Airey[16] who did not, however, have the formula giving the remainder term in the expression furnishing $\Gamma_{x-1}(x)$. We have checked his results and have determined, in addition, a_{23}. The values of $P_j(1)$, $0 \le j \le 22$, are given in Table 6.

TABLE 6

j	$P_j(1)$
0	1
1	-1
2	-1
3	13
4	-47
5	-73
6	2447
7	-16811
8	-15551
9	172 6511
10	-189 94849
11	109 79677
12	29834 09137
13	-4 84211 03257
14	13 50023 66063
15	1012 53200 47141
16	-23203 31477 79359
17	1 30595 20092 04319
18	58 74028 26601 73759
19	-1862 05713 25553 80307
20	16905 21942 11969 07793
21	5 27257 18724 48118 05207
22	-230 95199 10538 07360 79793

Thus the first 10 terms of the expression which furnishes $\Gamma_{x-1}(x)$ when x is an integer are as follows

$$\frac{1}{2} + \frac{1}{2^3x} - \frac{1}{2^5x^2} - \frac{1}{2^7x^3} + \frac{13}{2^9x^4} - \frac{47}{2^{11}x^5}$$

$$- \frac{73}{2^{13}x^6} + \frac{2447}{2^{15}x^7} - \frac{16811}{2^{17}x^8} - \frac{15551}{2^{19}x^9}$$

If $x = 5$, the terms steadily decrease in numerical magnitude until we reach the 8th term, which is almost twice the numerical magnitude of the 7th term. On the other hand, when $x = 10$, the terms steadily decrease in numerical magnitude until we reach the 11th term, which is almost three times the numerical magnitude of the 10th term.

The appraisal of the error involved, when calculating $\Gamma_{x-1}(x)$, in stopping at any stated term of the series $\dfrac{1}{2} + \dfrac{1}{2^3x} - \dfrac{1}{2^5x^2} - \ldots$ is complicated by the fact that, when $j > 1$, $P_j(v)$ changes sign in the interval $0 \le v \le 1$. For example, when we take three terms of the series, we have

$$\Gamma_{x-1}(x) = \frac{1}{2} + \frac{1}{2^3 x} - \frac{1}{2^5 x^2} + \frac{\exp x}{x^2} \int_0^1 \exp\left(-\frac{n+1}{v}\right) v^{n+1} \frac{v^2 P_2(v)}{(v+1)^6}\, dv,$$

and $P_2(v)$ is positive over the part $0 \le v < 4 - 10^{1/2}$, and negative over the part $4 - 10^{1/2} < v \le 1$, of the interval of integration. Owing to the factor $\exp\left(-\frac{n+1}{v}\right) v^{n+3}$, the integrand of the integral on the right is very small when v is near zero, particularly when $n+1 = x$ is as large as 10, say, and the value of the integral is negative. However, an estimation of the remainder term is difficult, particularly when the number of terms of the series $\frac{1}{2} + \frac{1}{2^3 x} - \frac{1}{2^5 x^2} - \dots$, which we take is large. We avoid the necessity of this appraisal by constructing a continued fraction whose successive convergents bracket $\Gamma_{x-1}(x)$, and whose development as a power series in $\frac{1}{x}$ is precisely the series $\frac{1}{2} + \frac{1}{2^3 x} - \frac{1}{2^5 x^2} - \dots$. To do this we start with the relation

$$\int_0^\infty \frac{\{\exp(-u)\}u^j}{\left(1+\frac{u}{x}\right)^{n+1}}\, du = x^j \int_0^\infty \frac{\left\{\left(1+\frac{u}{x}\right)-1\right\}^j \exp(-u)}{\left(1+\frac{u}{x}\right)^{n+1}}\, du$$

$$= x^j \left\{\Gamma_{n-j}(x) - j\Gamma_{n-j+1}(x) + \frac{j(j-1)}{2}\Gamma_{n-j+2}(x) - \dots + (-1)^j \Gamma_n(x)\right\},$$

j being any positive integer not greater than n. Since the integral on the left is positive, it follows that

$$\Gamma_{n-j}(x) - j\Gamma_{n-j+1}(x) + \frac{j(j-1)}{2}\Gamma_{n-j+2}(x) - \dots + (-1)^j \Gamma_n(x) > 0,\ j = 1, 2, \dots, n.$$

On defining $\Gamma_n(x)$ for negative integral values of n by means of the relation $\Gamma_n(x) = 1 - \frac{n+1}{x}\Gamma_{n+1}(x)$, so that $\Gamma_{-1}(x) = 1$, $\Gamma_{-2}(x) = 1 + \frac{1}{x}$, $\Gamma_{-3}(x) = 1 + \frac{2}{x} + \frac{2!}{x^2}$, $\Gamma_{-4}(x) = 1 + \frac{3}{x} + \frac{3!}{x^2} + \frac{3!}{x^3}$ and so on; this relation remains true when $j > n$. Using the relations $\Gamma_{n-1}(x) = 1 - \frac{n}{x}\Gamma_n(x)$, $\Gamma_{n-2}(x) = 1 - \frac{n-1}{x} + \frac{n(n-1)}{x^2}\Gamma_n(x)$, $\Gamma_{n-3}(x) = 1 - \frac{n-2}{x} + \frac{(n-1)(n-2)}{x^2} - \frac{n(n-1)(n-2)}{x^3}\Gamma_n(x)$, and so on, we obtain a sequence of bounds, alternately upper and lower, for $\Gamma_n(x)$. Thus, when $j = 1$, $1 - \left(\frac{n}{x} + 1\right)\Gamma_n(x) > 0$, so that

1. $\Gamma_n(x) < \dfrac{x}{x+n}$.

When $j = 2$, $1 - \dfrac{n-1}{x} - 2 + \left(\dfrac{n(n-1)}{x^2} + \dfrac{2n}{x} + 1\right) \Gamma_n(x) > 0$, so that

2. $\Gamma_n(x) > \dfrac{x(x+n-1)}{x^2+2nx+n(n-1)}$.

When $j = 3$, $1 - \dfrac{n-2}{x} + \dfrac{(n-1)(n-2)}{x^2} + 3\,\dfrac{n-1}{x} - \left\{\dfrac{n(n-1)(n-2)}{x^3} + \dfrac{3n(n-1)}{x^2} + \dfrac{3n}{x} + 1\right\} \Gamma_n(x) > 0$,

so that

3. $\Gamma_n(x) < \dfrac{x\{x^2+(2n-1)x+(n-1)(n-2)\}}{x^3+3nx^2+3n(n-1)x+n(n-1)(n-2)}$,

and so on. We denote the factors which multiply x in the numerators of the fractions we encounter in this way by A_j, $j = 1, 2, \ldots$, so that $A_1 = 1$, $A_2 = x + n - 1$, $A_3 = x^2 + (2n - 1)x + (n - 1)(n - 2)$, and so on. We denote the denominators of these fractions by B_j, $j = 1, 2, \ldots$, so that $B_1 = x + n$, $B_2 = x^2 + 2nx + n(n - 1)$, $B_3 = x^3 + 3nx^2 + 3n(n-1)x + n(n-1)(n - 2)$. If $j > 2$, B_j is the following combination of B_{j-1} and B_{j-2}:

$$B_j = (x+n+1-j)\,B_{j-1} + (j-1)\,xB_{j-2},$$

and A_j is the same combination of A_{j-1} and A_{j-2}.

$$A_j = (x+n+1-j)\,A_{j-1} + (j-1)\,xA_{j-2} \;.$$

Thus the various fractions which furnish, alternately, upper and lower bounds for $\Gamma_n(x)$ are the successive convergents of the continued fraction

$$\cfrac{x}{x+n+\cfrac{x}{x+n-1+\cfrac{2x}{x+n-2+\cfrac{3x}{x+n-3+\ldots}}}}$$

On writing $n+1 = (1-\beta)\,x$, this continued fraction appears as

$$\cfrac{x}{(2-\beta)\,x-1+\cfrac{x}{(2-\beta)\,x-2+\cfrac{2x}{(2-\beta)\,x-3+\cfrac{3x}{(2-\beta)\,x-4+\dots}}}}$$

In particular, when x is an integer and we set $n = x-1$ so that $\beta = 0$, the successive convergents of the continued fraction

$$\cfrac{x}{2x-1+\cfrac{x}{2x-2+\cfrac{2x}{2x-3+\cfrac{3x}{2x-4+\dots}}}}$$

furnish, alternately, upper and lower bounds for the converging factor $\Gamma_{x-1}(x)$. On denoting by a_j, $j = 1, 2, \dots$, the numerators of the continued fraction whose convergents furnish, when multiplied by x, alternately upper and lower bounds for $\Gamma_n(x)$, so that $a_1 = 1$, $a_j = (j-1)\,x$, $j = 2, 3, \dots$, the theory of continued fractions tells us that

$$\frac{A_{j+1}}{B_{j+1}} - \frac{A_j}{B_j} = \frac{(-1)^j\, a_1 \dots a_{j+1}}{B_j\, B_{j+1}} = \frac{(-1)^j\, j!\, x^j}{B_j\, B_{j+1}}$$

Since the leading term of the product $B_j\, B_{j+1}$ is x^{2j+1}, it follows that the developments of $\frac{x\,A_j}{B_j}$ and $\frac{x\,A_{j+1}}{B_{j+1}}$ as power series in $\frac{1}{x}$ coincide up to and including the term involving $\frac{1}{x^{j-1}}$. For example

$$\frac{x\,A_1}{B_1} = \frac{x}{(2-\beta)\,x-1} = \frac{1}{2-\beta} + \frac{1}{(2-\beta)\,x} + \dots$$

$$\frac{x\,A_2}{B_2} = \frac{1}{2-\beta} + \frac{(1-\beta)}{(2-\beta)^3\,x} + \dots$$

The first j terms of the development of $\frac{x A_j}{B_j}$ as a power series in $\frac{1}{x}$ are the first j terms of the series which we obtained by integration by parts, and we obtain an upper bound for the numerical magnitude of the difference between the sum S_j of these j terms and $\Gamma_n(x)$ by taking the greater of the two numbers $\left| S_j - \frac{x A_j}{B_j} \right|$ and $\left| S_j - \frac{x A_{j+1}}{B_{j+1}} \right|$

The denominators, $b_j = x + n + 1 - j$, of the continued fraction whose convergents are $\frac{x A_j}{B_j}$, $j = 1, 2, \ldots$, are positive if $j < x+n+1$, but they become negative when $j > x+n+1$. It follows from the theory of continued fractions, which tells us that

$$\frac{x A_{j+1}}{B_{j+1}} - \frac{x A_{j-1}}{B_{j-1}} = (-1)^{j-1} \frac{a_1 a_2 \ldots a_j b_{j+1}}{B_{j+1} B_{j-1}} ,$$

that our upper bounds steadily decrease as j increases so long as $j < x+n+1$, but that they begin to increase when j becomes larger than $x+n+1$. Similarly, the lower bounds steadily increase as j increases so long as $j < x+n+1$, but they begin to increase when j becomes larger than $x+n+1$. Thus, the best we can do with the convergents of our continued fraction is to take j as the largest integer less than $x+n+1$; if j is odd, $\frac{x A_j}{B_j}$ is the best upper bound, and $\frac{x A_{j+1}}{B_{j+1}}$ is the best lower bound, to $\Gamma_n(x)$ which is furnished by the convergents of our continued fraction. On the other hand, if j is even, $\frac{x A_j}{B_j}$ is the best lower bound, and $\frac{x A_{j+1}}{B_{j+1}}$ is the best upper bound, to $\Gamma_n(x)$ which is furnished by these convergents. Correspondingly, the best number of terms of the series $\frac{1}{2-\beta} + \frac{1-\beta}{(2-\beta)^3 x} + \ldots$ to take to obtain an approximation to $\Gamma_n(x)$ is j, where j is the greatest integer less than $x+n+1 = (2-\beta)x$. In particular, when x is an integer and $n = x-1$, so that $\beta = 0$, the best number of terms of Airey's series $\frac{1}{2} + \frac{1}{8x} - \frac{1}{32x^2} - \ldots$ to take to obtain an approximation to $\Gamma_{x-1}(x)$ is $2x-1$. For example, when $x=5$, the sum of 9 terms of the series $\frac{1}{2} + \frac{1}{8.5} - \frac{1}{32.5^2} - \ldots$ is 0.52372 0838 . . . which is less than $\Gamma_4(5)$ by less than 4 units in the eighth decimal place. Similarly, when $x=10$, the sum of 19 terms of the series $\frac{1}{2} + \frac{1}{8.10} - \frac{1}{32.10^2} - \ldots$ is 0.51218 19943 76046 027 . . . , which is less than $\Gamma_9(10)$ by less than 5 units in the 15th decimal place. Use of Airey's series will furnish, then, $\Gamma_{x-1}(x)$ correct to seven decimal places if $x=5$ and to 14 decimal places if $x=10$, but it cannot be expected to do better than this.

The bounds furnished by the convergents $\frac{xA_j}{B_j}$ of our continued fraction can be improved by the following observation. These bounds were obtained from the fact that the integral $\int_0^\infty \frac{\exp(-u)\,u^j}{\left(1+\frac{u}{x}\right)^{n+1}}\,du$ is positive, but we know more than this. Since $\frac{u}{x}$ is positive, $1+\frac{u}{x} < \exp\frac{u}{x}$, so that $\left(1+\frac{u}{x}\right)^{n+1} < \exp\,(n+1)\frac{u}{x} = \exp(1-\beta)\,u$, where $(n+1) = (1-\beta)\,x$. Hence, $\int_0^\infty \frac{\exp(-u)\,u^j}{\left(1+\frac{u}{x}\right)^{n+1}}\,du > \int_0^\infty \{\exp-(2-\beta)\,u\}\,u^j\,du = \frac{j!}{(2-\beta)^{j+1}}$. It follows that, if j is odd, $\frac{\left\{xA_j - \frac{j!}{(2-\beta)^{j+1}}\right\}}{B_j}$ is an upper bound for $\Gamma_n(x)$, and that, if j is even, $\frac{\left\{xA_j + \frac{j!}{(2-\beta)^{j+1}}\right\}}{B_j}$ is a lower bound for $\Gamma_n(x)$. The developments of $\frac{\left\{xA_j - \frac{j!}{(2-\beta)^{j+1}}\right\}}{B_j}$, if j is odd, and of $\frac{\left\{xA_j + \frac{j!}{(2-\beta)^{j+1}}\right\}}{B_j}$, if j is even, as power series in $\frac{1}{x}$ coincide with the series $\frac{1}{2-\beta} + \frac{(1-\beta)}{(2-\beta)^3 x} + \dots$, which we obtained by integration by parts up to the term involving $\frac{1}{x^j}$, and not merely up to the term involving $\frac{1}{x^{j-1}}$, as did the development of $\frac{xA_j}{B_j}$. For example,

$$\frac{xA_1 - \frac{1}{(2-\beta)^2}}{B_1} = \frac{x - \frac{1}{(2-\beta)^2}}{(2-\beta)\,x-1} = \frac{1}{2-\beta} + \frac{(1-\beta)}{(2-\beta)^3 x} + \frac{(1-\beta)}{(2-\beta)^4 x^2} + \dots$$

$$\frac{xA_2 + \frac{2}{(2-\beta)^3}}{B_2} = \frac{(2-\beta)x^2 - 2x + \frac{2}{(2-\beta)^3}}{(2-\beta)^2x^2 - (5-3\beta)x + 2} = \frac{1}{(2-\beta)} + \frac{(1-\beta)}{(2-\beta)^3 x} - \frac{(1-\beta^2)}{(2-\beta)^5 x^2} + \dots$$

When $x=5$ and $n=4$, so that $\beta=0$, we have

$A_1=1$, $A_2=8$, $A_3=66$, $A_4=516$, $A_5=3900$, $A_6=28500$, $A_7=2\ 02500$, $A_8=14\ 02500$, $A_9=95\ 02500$; $B_1=9$, $B_2=77$; $B_3=629$, $B_4=4929$, $B_5=37225$, $B_6=2\ 72125$, $B_7=19\ 33125$, $B_8=133\ 90625$, $B_9=907\ 15625$. Thus

$$\frac{5A_9-\dfrac{9!}{2^{10}}}{B_9}=\frac{475\ 12145.625}{907\ 15625}=0.52374\ 820$$

is an upper bound for $\Gamma_4(5)$, while

$$\frac{5A_8+\dfrac{8!}{2^9}}{B_8}=\frac{70\ 12578.75}{133\ 90625}=0.52369\ 316$$

is a lower bound for $\Gamma_4(5)$. The mean of these two bounds, namely, 0.52372 068 is less than $\Gamma_4(5)$ by less than 2 units in the seventh decimal place. When $x=10$, the corresponding bounds, $\dfrac{10A_{19}-\dfrac{19!}{2^{20}}}{B_{19}}$ and $\dfrac{10A_{18}+\dfrac{18!}{2^{19}}}{B_{18}}$, for $\Gamma_9(10)$ are 0.51218 19944 79827 and 0.51218 19942 69253, respectively. Their mean, 0.51218 19943 74540, is less than $\Gamma_9(10)$ by less than 2 units in the twelfth decimal place.

THE ASYMPTOTIC FORMULA FOR THE CONVERGING FACTOR $C_n(n+1)$

The converging factor for the exponential integral $Ei(x)$ is furnished, when x is positive, by the formula

$$C_n(x)=\frac{1}{n!}\int_0^\infty \frac{\exp(-u)\,u^n}{1-\dfrac{u}{x}}\,du.$$

We transform this integral by means of a device due to Stieltjes.[20] Making the substitution $u=xv$, we obtain the relation

$$C_n(x) = \frac{x^{n+1}}{n!} \int_0^\infty \frac{\exp(-xv)v^n}{1-v}\, dv = \frac{x^{n+1}}{n!} \lim_{\delta \to 0} \left\{ \int_0^{1-\delta} \frac{\exp(-xv)v^n}{1-v}\, dv \right.$$

$$\left. + \int_{1+\delta}^\infty \frac{\exp(-xv)v^n}{1-v}\, dv \right\}, \; \delta > 0$$

We next make the substitution $v = 1 - t$ in the first, and the substitution $v = 1 + \tau$ in the second, of the two integrals on the right-hand side of this relation, and find that

$$C_n(x) = \frac{x^{n+1}\exp(-x)}{n!} \lim_{\delta \to 0} \left\{ \int_\delta^1 \frac{\exp(xt)(1-t)^n}{t}\, dt - \int_\delta^\infty \frac{\exp(-x\tau)(1+\tau)^n}{\tau}\, d\tau \right\}, \; \delta > 0$$

Writing $(\exp t)(1-t) = \exp(-\xi)$, we have $tt_\xi = 1 - t$ where t_ξ denotes the derivative of t with respect to ξ. Thus t_ξ is positive over the interval $\delta \leq t < 1$, and ξ increases monotonically from ϵ to ∞ as t increases from δ to 1, ϵ being the value of ξ when $t = \delta$, so that ϵ tends to zero with δ. Similarly, on writing $\{\exp(-\tau)\}(1+\tau) = \exp(-\xi)$, we have $\tau\tau_\xi = (1+\tau)$, so that τ_ξ is positive over the interval $\delta \leq \tau < \infty$. Thus ξ increases monotonically from ϵ' to ∞ as τ increases from δ to ∞, ϵ' being the value of ξ when $\tau = \delta$, so that ϵ' tends to zero with δ. We have, then, the relation

$$C_n(x) = \frac{x^{n+1}\exp(-x)}{n!} \lim_{\delta \to 0} \left\{ \int_\epsilon^\infty \frac{\{\exp(-x\xi)\}(1-t)^{n-x} t_\xi}{t}\, d\xi \right.$$

$$\left. - \int_{\epsilon'}^\infty \frac{\{\exp(-x\xi)\}(1+\tau)^{n-x}\, \tau_\xi}{\tau}\, d\xi \right\}$$

Since $\xi + t = -\log(1-t)$, ξ has, near $t = 0$, the development

$$\xi = \frac{t^2}{2} + \frac{t^3}{3} + \ldots$$

Setting $\xi = \frac{1}{2} z^2$, the relation $\frac{1}{2} z^2 = \frac{t^2}{2} + \frac{t^3}{3} + \ldots$ defines implicitly two functions of the complex variable z over a sufficiently small neighborhood of the origin. Since $t_\xi > 0$ when ξ is real and positive, these two functions are distinguishable one from the other by the fact that for one of them $t_z = zt_\xi$ is positive at $z=0$ and for the other t_z is negative at $z=0$. We select that one of the two functions for which t_z is positive at $z=0$, and write

$$t = b_1 z + b_2 z^2 + \dots \qquad , b_1 > 0$$

the constant term in the power series development being zero, since $t=0$ when $z=0$. Since $tt_z = z(1-t)$, we infer that $b_1^2 = 1$, which implies that $b_1 = 1$. The values of the various coefficients $b_2, b_3, \dots$, may be obtained from the relation $(b_1 + b_2 z + b_3 z^2 + \dots \quad)$ $(b_1 + 2b_2 z + 3b_3 z^2 + \dots \quad) = 1 - b_1 z - b_2 z^2 - \dots$ For example, $3b_1 b_2 = -b_1 = -1$, so that $b_2 = -\frac{1}{3}$; $4b_1 b_3 + 2b_2^2 = -b_2$, so that $b_3 = \frac{1}{2^2 \cdot 3^2}$, and so on. When j is even, we have

$$(j+1)(b_1 b_j + b_2 b_{j-1} + \dots \quad + b_{j/2} b_{j/2+1}) = -b_{j-1},$$

and when j is odd, we have

$$(j+1)(b_1 b_j + b_2 b_{j-1} + \dots \quad + b_{(j-1)/2}\, b_{(j+3)/2}) + \frac{1}{2}(j+1)\, b_{(j+1)/2}^2 = -b_{j-1}.$$

When j is large, say > 10, the determination of b_j from these relations is tedious. We shall later discuss methods for reducing the work involved, and shall give the values of b_j for $j \leq 42$. The function τ of the complex variable z satisfies the equation $\tau\tau_z = z(1+\tau)$, and it follows, on comparing this relation with the relation $tt_z = z(1-t)$, that $-\tau$ is the same function of $-z$ that t is of z, so that

$$\tau = b_1 z - b_2 z^2 + b_3 z^3 - \dots$$

Thus $t - \tau$ is a function of ξ:

$$\begin{aligned} t - \tau &= 2(b_2 z^2 + b_4 z^4 + \dots \quad) \\ &= 2^2 b_2 \xi + 2^3 b_4 \xi^2 + 2^4 b_6 \xi^3 + \dots \quad , \end{aligned}$$

a relation of which we shall shortly make essential use.

Both $\frac{t_z}{t}$ and $\frac{\tau_z}{\tau}$ have the same principal part, namely $\frac{1}{z}$, in their Laurent developments about $z=0$, and it follows that $(1-t)^{n-x} \frac{t_z}{t} - (1+\tau)^{n-x} \frac{\tau_z}{\tau}$ is regular near $z=0$. On writing our formula for $C_n(x)$ in the form

$$C_n(x) = \frac{x^{n+1}\exp(-x)}{n!} \lim_{\delta \to 0} \left\{ \int_{(2\epsilon)^{1/2}}^{\infty} \{\exp(-x\xi)\}(1-t)^{n-x} \frac{t_z}{t}\, dz \right.$$

$$\left. - \int_{(2\epsilon)^{1/2}}^{\infty} \exp(-x\xi)(1+\tau)^{n-x} \frac{\tau_z}{\tau}\, dz \right\}, \ \delta > 0,$$

we see that

$$C_n(x) = \frac{x^{n+1}\exp(-x)}{n!} \int_0^{\infty} \{\exp(-x\xi)\} \left\{ \frac{(1-t)^{n-x}\, t_z}{t} - \frac{(1+\tau)^{n-x}\, \tau_z}{\tau} \right\} dz$$

$$= \frac{x^{n+1}\exp(-x)}{n!} \int_0^{\infty} \exp(-x\xi) \left\{ \frac{(1-t)^{n-x}\, t_\xi}{t} - \frac{(1+\tau)^{n-x}\, \tau_\xi}{\tau} \right\} d\xi$$

Thus $C_n(x)$ is the product of the Laplace Transform of $\dfrac{(1-t)^{n-x}\, t_\xi}{t} - \dfrac{(1+\tau)^{n-x}\, \tau_\xi}{\tau}$ by $\dfrac{x^{n+1}\exp(-z)}{n!}$. In particular, when x is an integer and $n = x$, we have the following result:

$C_x(x)$ is the product of the Laplace Transform of $\dfrac{t_\xi}{t} - \dfrac{\tau_\xi}{\tau}$ by $\dfrac{x^{x+1}\exp(-x)}{x!}$.

Since $\dfrac{1}{t} = t_\xi + 1$ and $\dfrac{1}{\tau} = \tau_\xi - 1$, this result may be stated in the following equivalent form:

$C_x(x)$ is the product of the Laplace Transform of $t_\xi^2 - \tau_\xi^2 + t_\xi + \tau_\xi$ by $\dfrac{x^{x+1}\exp(-x)}{x!}$.

If we apply the Stieltjes substitution to the integral $\displaystyle\int_0^{\infty} \{\exp(-xv)\}\, v^n\, dv$, whose value is $\dfrac{n!}{x^{n+1}}$, in the way we have applied it to the integral $\displaystyle\int_0^{\infty} \frac{\{\exp(-xv)\}\, v^n}{1-v}\, dv$, we find that:

$\dfrac{n!}{x^{n+1}}$ is the product of the Laplace Transform of $(1-t)^{n-x}\, t_\xi + (1+\tau)^{n-x}\, \tau_\xi$ by $\exp(-x)$.

In particular, the product of the Laplace Transform of $t_\xi + \tau_\xi$ by $\dfrac{x^{x+1}\exp(-x)}{x!}$ is 1, and we have the following basic result:

$C_x(x)$ is 1 + the quotient of the Laplace Transform of $t_\xi^2 - \tau_\xi^2$ by the Laplace Transform of $t_\xi + \tau_\xi$.

Since $C_{x-1}(x) = 1 + C_x(x)$, this result may be stated in the following equivalent form:

$C_{x-1}(x)$ is 2 + the quotient of the Laplace Transform of $t_\xi^2 - \tau_\xi^2$ by the Laplace Transform of $t_\xi + \tau_\xi$.

When x is reasonably large, say ≥ 5, the dominant part of the Laplace Transform of any function of ξ arises from the values of the function near $\xi = 0$, this being due to the damping factor $\exp(-x\xi)$. For example, since $x!$ is the product of the Laplace Transform of $t_\xi + \tau_\xi$ by $x^{x+1}\exp(-x)$, and since $t_\xi + \tau_\xi$ has, near $\xi = 0$, the development

$$t_\xi + \tau_\xi = \frac{1}{z}(t_z + \tau_z) = 2^{\frac{1}{2}}\,\xi^{-\frac{1}{2}}\,\{b_1 + 3b_3(2\xi) + 5b_5(2\xi)^2 + \dots \quad \},$$

we have the following asymptotic formula for $x!$

$$x! \sim 2^{\frac{1}{2}}\,x^{x+1}\exp(-x)\left\{\frac{b_1\Gamma\left(\frac{1}{2}\right)}{x^{1/2}} + 3b_3\cdot 2\,\frac{\Gamma\left(\frac{3}{2}\right)}{x^{3/2}} + 5b_5\cdot 2^2\,\frac{\Gamma\left(\frac{5}{2}\right)}{x^{5/2}} + \dots\right\}$$

$$\sim (2\pi)^{\frac{1}{2}}\,x^{x+\frac{1}{2}}\exp(-x)\left\{b_1 + \frac{3b_3}{x} + \frac{3\cdot 5b_5}{x^2} + \frac{3\cdot 5\cdot 7b_7}{x^3} + \dots\right\}$$

Thus $b_1, 3b_3, 3\cdot 5b_5, \dots$ are the coefficients which appear in Stirling's asymptotic formula for $x!$, and this fact enables us to evaluate those of the b's which carry an odd subscript, when this subscript is large, in an easier way than that based on the relation previously given, which relation involves the b's with even subscripts as well as those with odd subscripts. Setting $L = (2\pi)^{-\frac{1}{2}}(\exp x)\,x^{-x}\,\Gamma(x)$, the asymptotic formula for $x! = x\Gamma(x)$ may be written in the form

$$L \sim \frac{b_1}{x^{1/2}} + \frac{3b_3}{x^{3/2}} + \frac{3\cdot 5b_5}{x^{5/2}} + \dots$$

On taking the logarithmic derivative of L, we find that $\frac{L_x}{L} = -\log x + \frac{\Gamma_x}{\Gamma}$, and since

$$\frac{\Gamma_x}{\Gamma} \sim \log x - \frac{1}{2x} - \frac{B_2}{2x^2} + \frac{B_4}{2 \cdot 2x^4} - \frac{B_6}{2 \cdot 3x^6} + \frac{B_8}{2 \cdot 4x^8} - \dots,$$

where the B's are the Bernoulli numbers $\left(B_2 = \frac{1}{6}, B_4 = \frac{1}{30}, B_6 = \frac{1}{42}, B_8 = \frac{1}{30}, \dots\right)$, we have the relation

$$\frac{L_x}{L} \sim -\frac{1}{2x}\left(1 + \frac{B_2}{x} - \frac{B_4}{2x^3} + \frac{B_6}{3x^5} - \frac{B_8}{4x^7} + \dots\right).$$

Hence,

$$L_x \sim -\frac{1}{2x^{3/2}}\left\{b_1 + \frac{3b_3 + b_1B_2}{x} + \frac{3 \cdot 5b_5 + 3b_3B_2}{x^2} + \frac{3 \cdot 5 \cdot 7b_7 + 3 \cdot 5b_5B_2 - \frac{1}{2}b_1B_4}{x^3} + \dots\right\},$$

and, on equating the expression on the right to

$$-\frac{1}{2x^{3/2}}\left\{b_1 + \frac{3^2b_3}{x} + \frac{3 \cdot 5^2b_5}{x^2} + \frac{3 \cdot 5 \cdot 7^2b_7}{x^3} + \dots\right\},$$

we obtain the following sequence of relations:

$$\begin{aligned} 3^2b_3 &= 3b_3 + b_1B_2, \\ 3 \cdot 5^2b_5 &= 3 \cdot 5b_5 + 3b_3B_2, \\ 3 \cdot 5 \cdot 7^2b_7 &= 3 \cdot 5 \cdot 7b_7 + 3 \cdot 5b_5B_2 - \frac{1}{2}b_1B_4, \end{aligned}$$

which serve to determine, one after the other, the numbers $b_3, b_5, b_7, \dots$, b_1 being known to be 1. We give the values of b_{2j+1}, $1 \le j \le 20$ in Table 7, listing also the coefficients $c_1 = 3b_3$, $c_2 = 3 \cdot 5b_5$, $c_3 = 3 \cdot 5 \cdot 7b_7, \dots$, which appear in Stirling's asymptotic formula for $x!$

The values to 45 decimal places of the coefficients c_j, $1 \le j \le 20$, are given in Table 8.

TABLE 7

j	b_{2j+1}	c_j
1	$\frac{1}{2^2\cdot 3^2}$	$\frac{1}{2^2\cdot 3}$
2	$\frac{1}{2^5\cdot 3^3\cdot 5}$	$\frac{1}{2^5\cdot 3^2}$
3	$-\frac{139}{2^7\cdot 3^5\cdot 5^2\cdot 7}$	$-\frac{139}{2^7\cdot 3^4\cdot 5}$
4	$-\frac{571}{2^{11}\cdot 3^8\cdot 5^2\cdot 7}$	$-\frac{571}{2^{11}\cdot 3^5\cdot 5}$
5	$\frac{163879}{2^{13}\cdot 3^9\cdot 5^2\cdot 7^2\cdot 11}$	$\frac{163879}{2^{13}\cdot 3^6\cdot 5\cdot 7}$
6	$\frac{52\ 46819}{2^{16}\cdot 3^{11}\cdot 5^3\cdot 7^2\cdot 11\cdot 13}$	$\frac{52\ 46819}{2^{16}\cdot 3^8\cdot 5^2\cdot 7}$
7	$-\frac{5347\ 03531}{2^{18}\cdot 3^{13}\cdot 5^4\cdot 7^2\cdot 11\cdot 13}$	$-\frac{5347\ 03531}{2^{18}\cdot 3^9\cdot 5^2\cdot 7}$
8	$-\frac{44831\ 31259}{2^{23}\cdot 3^{14}\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 17}$	$-\frac{44831\ 31259}{2^{23}\cdot 3^{10}\cdot 5^2\cdot 7}$
9	$\frac{43226\ 19216\ 12371}{2^{25}\cdot 3^{17}\cdot 5^5\cdot 7^2\cdot 11^2\cdot 13\cdot 17\cdot 19}$	$\frac{43226\ 19216\ 12371}{2^{25}\cdot 3^{13}\cdot 5^3\cdot 7\cdot 11}$
10	$\frac{6\ 23252\ 32025\ 21089}{2^{28}\cdot 3^{19}\cdot 5^5\cdot 7^4\cdot 11^2\cdot 13\cdot 17\cdot 19}$	$\frac{6\ 23252\ 32025\ 21089}{2^{28}\cdot 3^{14}\cdot 5^3\cdot 7^2\cdot 11}$
11	$-\frac{25834\ 62966\ 51342\ 04969}{2^{30}\cdot 3^{20}\cdot 5^5\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23}$	$-\frac{25834\ 62966\ 51342\ 04969}{2^{30}\cdot 3^{15}\cdot 5^3\cdot 7^2\cdot 11\cdot 13}$
12	$-\frac{15\ 79029\ 13885\ 49190\ 86429}{2^{34}\cdot 3^{22}\cdot 5^8\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23}$	$-\frac{15\ 79029\ 13885\ 49190\ 86429}{2^{34}\cdot 3^{17}\cdot 5^4\cdot 7^2\cdot 11\cdot 13}$
13	$\frac{7465\ 90869\ 96265\ 16022\ 03151}{2^{36}\cdot 3^{26}\cdot 5^8\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23}$	$\frac{7465\ 90869\ 96265\ 16022\ 03151}{2^{36}\cdot 3^{18}\cdot 5^4\cdot 7^2\cdot 11\cdot 13}$
14	$\frac{15115\ 13601\ 02809\ 79036\ 31961}{2^{39}\cdot 3^{27}\cdot 5^8\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23\cdot 29}$	$\frac{15115\ 13601\ 02809\ 79036\ 31961}{2^{39}\cdot 3^{19}\cdot 5^4\cdot 7^2\cdot 11\cdot 13}$
15	$-\frac{8\ 84927\ 22683\ 92873\ 14770\ 59871\ 90261}{2^{41}\cdot 3^{29}\cdot 5^{10}\cdot 7^5\cdot 11^2\cdot 13^2\cdot 17^2\cdot 19\cdot 23\cdot 29\cdot 31}$	$-\frac{8\ 84927\ 22683\ 92873\ 14770\ 59871\ 90261}{2^{41}\cdot 3^{21}\cdot 5^6\cdot 7^3\cdot 11\cdot 13\cdot 17}$
16	$-\frac{142\ 80171\ 24906\ 07530\ 60813\ 07010\ 97701}{2^{47}\cdot 3^{31}\cdot 5^{10}\cdot 7^5\cdot 11^3\cdot 13^2\cdot 17^2\cdot 19\cdot 23\cdot 29\cdot 31}$	$-\frac{142\ 80171\ 24906\ 07530\ 60813\ 07010\ 97701}{2^{47}\cdot 3^{22}\cdot 5^6\cdot 7^3\cdot 11\cdot 13\cdot 17}$
17	$\frac{23\ 55444\ 39310\ 99675\ 10921\ 43143\ 60000\ 87153}{2^{49}\cdot 3^{32}\cdot 5^{11}\cdot 7^6\cdot 11^3\cdot 13^2\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31}$	$\frac{23\ 55444\ 39310\ 99675\ 10921\ 43143\ 60000\ 87153}{2^{49}\cdot 3^{23}\cdot 5^6\cdot 7^3\cdot 11\cdot 13\cdot 17\cdot 19}$
18	$\frac{23466\ 08607\ 35190\ 37376\ 47919\ 57708\ 21151\ 21863}{2^{52}\cdot 3^{35}\cdot 5^{12}\cdot 7^6\cdot 11^4\cdot 13^2\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37}$	$\frac{23466\ 08607\ 35190\ 37376\ 47919\ 57708\ 21151\ 21863}{2^{52}\cdot 3^{26}\cdot 5^7\cdot 7^3\cdot 11^2\cdot 13\cdot 17\cdot 19}$
19	$\frac{260\ 30721\ 87220\ 37327\ 71509\ 99431\ 41656\ 23963\ 31667}{2^{54}\cdot 3^{37}\cdot 5^{12}\cdot 7^6\cdot 11^4\cdot 13^3\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37}$	$-\frac{260\ 30721\ 87220\ 37327\ 71509\ 99431\ 41656\ 23963\ 31667}{2^{54}\cdot 3^{27}\cdot 5^7\cdot 7^3\cdot 11^2\cdot 13\cdot 17\cdot 19}$
20	$\frac{7323\ 97274\ 26811\ 93597\ 69674\ 71475\ 43026\ 86956\ 30993}{2^{58}\cdot 3^{38}\cdot 5^{12}\cdot 7^7\cdot 11^4\cdot 13^3\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41}$	$-\frac{7323\ 97274\ 26811\ 93597\ 69674\ 71475\ 43026\ 86956\ 30993}{2^{58}\cdot 3^{28}\cdot 5^7\cdot 7^4\cdot 11^2\cdot 13\cdot 17\cdot 19}$

TABLE 8

j	c_j								
1	0.08333	33333	33333	33333	33333	33333	33333	33333	33333
2	0.00347	22222	22222	22222	22222	22222	22222	22222	22222
3	−0.00268	13271	60493	82716	04938	27160	49382	71604	93827
4	−0.00022	94720	93621	39917	69547	32510	28806	58436	21399
5	0.00078	40392	21720	06662	74740	34881	44228	88496	96257
6	0.00006	97281	37583	65857	77429	39882	85757	83308	29360
7	−0.00059	21664	37353	69388	28648	36225	60440	11873	91585
8	−0.00005	17179	09082	60592	19337	05784	30020	58822	81785
9	0.00083	94987	20672	08727	99933	57516	76498	34451	98182
10	0.00007	20489	54160	20010	55908	57193	02250	15052	06345
11	−0.00191	44384	98565	47752	65008	98858	32852	25448	76894
12	−0.00016	25162	62783	91581	68986	35123	98027	09981	05873
13	0.00640	33628	33808	06979	48236	38090	26579	58304	01894
14	0.00054	01647	67892	60451	51804	67508	57024	17355	47254
15	−0.02952	78809	45699	12050	54406	51054	69382	44465	65483
16	−0.00248	17436	00264	99773	09156	58368	74346	43239	75168
17	0.17954	01170	61234	85610	76994	07722	22633	05309	12823
18	0.01505	61130	40026	42441	23842	21877	13112	72602	59815
19	−1.39180	10932	65337	48139	91477	63542	27314	93580	45618
20	−0.11654	62765	99463	20085	07340	36907	14796	96789	37334

To determine the coefficients b_{2j}, $j = 1, 2, \dots$, we proceed as follows. On multiplying the two power series developments

$$t = b_1 z + b_2 z^2 + b_3 z^3 + \dots$$

$$\tau = b_1 z - b_2 z^2 + b_3 z^3 - \dots$$

we obtain the relation

$$t\tau = z^2 \{ b_1^2 + (2b_1 b_3 - b_2^2)\, z^2 + (2b_1 b_5 - 2b_2 b_4 + b_3^2)\, z^4 + \dots \quad \}$$

and, on using the relations $b_1 = 1$, $4b_1 b_3 + 2b_2^2 = -b_2$, $6b_1 b_5 + 6b_2 b_4 + 3b_3^2 = -b_4$, and so on, this relation may be written in the form

$$t\tau = z^2 \left\{ 1 - \left(\frac{1}{2}\, b_2 + 2b_2^2 \right) z^2 - \left(\frac{1}{3}\, b_4 + 4b_2 b_4 \right) z^4 - \left(\frac{1}{4}\, b_6 + 4b_2 b_6 + 2b_4^2 \right) z^6 - \dots \right\}$$

Since $tt_z = z(1-t)$, $\tau\tau_z = z(1+\tau)$, we have the relation

$$\frac{1}{t} - \frac{1}{\tau} = \frac{1}{z}\,(t_z - \tau_z) + 2 = 2 + 4b_2 + 2 \cdot 4b_4 z^2 + 2 \cdot 6b_6 z^4 + \dots \quad ,$$

and so the product of $2 + 4b_2 + 2\cdot 4b_4 z^2 + 2\cdot 6b_6 z^4 + \dots$ by

$$1-\left(\frac{1}{2}\,b_2+2b_2^2\right)z^2-\left(\frac{1}{3}\,b_4+4b_2b_4\right)z^4-\left(\frac{1}{4}\,b_6+4b_2b_6+2b_4^2\right)z^6-\dots$$

is $\dfrac{\tau - t}{z^2} = -2\{b_2 + b_4 z^2 + b_6 z^4 + \dots + b_8 z^6 +\}$. We obtain from this fact a sequence of relations which involve only the b's that carry even subscripts, and these relations enable us to determine, one after the other, the b_{2j}, $j = 1, 2, 3, \dots$. For example,

1. $1 + 2b_2 = - b_2$, so that $b_2 = -\dfrac{1}{3}$.

Using 1, we have $\dfrac{1}{2}\,b_2 + 2b_2^2 = \dfrac{1}{2\cdot 3^2}$, so that

2. $-\dfrac{1}{2\cdot 3^3} + 4b_4 = - b_4$; whence $b_4 = \dfrac{1}{2\cdot 3^3\cdot 5}$.

Continuing in this way, we obtain, one after the other, b_6, b_8, $b_{10}, \dots$. The values of b_{2j}, $1 \le j \le 21$, are given in Table 9.

We have seen that $C_{x-1}(x)$ is 2 plus the quotient of the Laplace Transform of $t_\xi^2 - \tau_\xi^2$ by the Laplace Transform of $t_\xi + \tau_\xi$. Since

$$t_\xi - \tau_\xi = 2\{2b_2 + 4b_4(2\xi) + 6b_6(2\xi)^2 + 8b_8(2\xi)^3 + \dots\},$$

$$t_\xi^2 - \tau_\xi^2 = 4b_2(t_\xi+\tau_\xi) + 2^2\cdot 4b_4\xi(t_\xi+\tau_\xi) + 2^3\cdot 6b_6\xi^2(t_\xi+\tau_\xi) + 2^4\cdot 8b_8\xi^3(t_\xi+\tau_\xi) + \dots$$

so that the Laplace Transform of $t_\xi^2 - \tau_\xi^2$ is the sum of the following expressions:

1. $4b_2$ times the Laplace Transform of $t_\xi + \tau_\xi$,
2. $-2^2\cdot 4b_4$ times the derivative of the Laplace Transform of $t_\xi + \tau_\xi$,
3. $2^3\cdot 6b_6$ times the second derivative of the Laplace Transform of $t_\xi + \tau_\xi$,
4. $-2^4\cdot 8b_8$ times the third derivative of the Laplace Transform of $t_\xi + \tau_\xi$,

and so on. On denoting the Laplace Transform of $(t_\xi + \tau_\xi)$ by $(2\pi)^{1/2} L$, we find that L has the asymptotic development

$$L \sim \frac{b_1}{x^{1/2}} + \frac{3b_3}{x^{3/2}} + \frac{3\cdot 5b_5}{x^{5/2}} + \frac{3\cdot 5\cdot 7b_7}{x^{7/2}} + \dots \quad .$$

TABLE 9

j	b_{2j}
1	$-\frac{1}{3}$
2	$\frac{1}{2\cdot 3^{3}\cdot 5}$
3	$-\frac{1}{2\cdot 3^{5}\cdot 5\cdot 7}$
4	$-\frac{1}{2^{3}\cdot 3^{6}\cdot 5\cdot 7}$
5	$\frac{281}{2^{3}\cdot 3^{9}\cdot 5^{3}\cdot 7\cdot 11}$
6	$\frac{5221}{2^{4}\cdot 3^{11}\cdot 5^{3}\cdot 7\cdot 11\cdot 13}$
7	$-\frac{5459}{2^{4}\cdot 3^{12}\cdot 5^{3}\cdot 7^{2}\cdot 11\cdot 13}$
8	$-\frac{912\;07079}{2^{7}\cdot 3^{14}\cdot 5^{5}\cdot 7^{3}\cdot 11\cdot 13\cdot 17}$
9	$\frac{26509\;86803}{2^{7}\cdot 3^{17}\cdot 5^{5}\cdot 7^{3}\cdot 11\cdot 13\cdot 17\cdot 19}$
10	$\frac{61718\;01683}{2^{8}\cdot 3^{18}\cdot 5^{6}\cdot 7^{3}\cdot 11\cdot 13\cdot 17\cdot 19}$
11	$-\frac{428\;39331\;45517}{2^{8}\cdot 3^{20}\cdot 5^{7}\cdot 7^{3}\cdot 11^{2}\cdot 13\cdot 17\cdot 19\cdot 23}$
12	$-\frac{1196\;39836\;48109}{2^{10}\cdot 3^{22}\cdot 5^{7}\cdot 7^{3}\cdot 11^{2}\cdot 13\cdot 17\cdot 19\cdot 23}$
13	$\frac{20869\;76249\;24077}{2^{10}\cdot 3^{23}\cdot 5^{7}\cdot 7^{4}\cdot 11^{2}\cdot 13^{2}\cdot 17\cdot 19\cdot 23}$
14	$\frac{29320\;11913\;05155\;66117}{2^{11}\cdot 3^{27}\cdot 5^{8}\cdot 7^{5}\cdot 11^{3}\cdot 13^{2}\cdot 17\cdot 19\cdot 23\cdot 29}$
15	$-\frac{27\;00231\;12146\;07564\;31181}{2^{11}\cdot 3^{29}\cdot 5^{9}\cdot 7^{5}\cdot 11^{3}\cdot 13^{2}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31}$
16	$-\frac{302\;52864\;76959\;66455\;59143}{2^{15}\cdot 3^{31}\cdot 5^{9}\cdot 7^{5}\cdot 11^{3}\cdot 13^{2}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31}$
17	$\frac{67822\;42429\;22326\;79335\;35073}{2^{15}\cdot 3^{32}\cdot 5^{10}\cdot 7^{5}\cdot 11^{3}\cdot 13^{3}\cdot 17^{2}\cdot 19\cdot 23\cdot 29\cdot 31}$
18	$\frac{5174\;85871\;06835\;35342\;63301\;48693}{2^{16}\cdot 3^{35}\cdot 5^{10}\cdot 7^{7}\cdot 11^{3}\cdot 13^{3}\cdot 17^{2}\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37}$
19	$-\frac{20321\;10359\;54227\;19515\;84056\;15907}{2^{16}\cdot 3^{36}\cdot 5^{10}\cdot 7^{7}\cdot 11^{3}\cdot 13^{3}\cdot 17^{2}\cdot 19^{2}\cdot 23\cdot 29\cdot 31\cdot 37}$
20	$-\frac{585\;30287\;26332\;92617\;24881\;45877\;26421}{2^{18}\cdot 3^{38}\cdot 5^{12}\cdot 7^{7}\cdot 11^{3}\cdot 13^{3}\cdot 17^{2}\cdot 19^{2}\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41}$
21	$\frac{1\;10855\;49579\;65750\;34381\;96928\;10335\;55329}{2^{18}\cdot 3^{40}\cdot 5^{12}\cdot 7^{8}\cdot 11^{3}\cdot 13^{3}\cdot 17^{2}\cdot 19^{2}\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43}$

TABLE 10

j	a_j
0	$\frac{2}{3}$
1	$\frac{2^{2}}{3^{3}\cdot 5}$
2	$\frac{2^{3}}{3^{4}\cdot 5\cdot 7}$
3	$-\frac{2^{4}}{3^{5}\cdot 5\cdot 7}$
4	$-\frac{2^{5}\cdot 281}{3^{8}\cdot 5^{2}\cdot 7\cdot 11}$
5	$\frac{2^{6}\cdot 5221}{3^{9}\cdot 5^{2}\cdot 7\cdot 11\cdot 13}$
6	$\frac{2^{7}\cdot 5459}{3^{10}\cdot 5^{2}\cdot 7\cdot 11\cdot 13}$
7	$-\frac{2^{8}\cdot 912\;07079}{3^{12}\cdot 5^{4}\cdot 7^{2}\cdot 11\cdot 13\cdot 17}$
8	$-\frac{2^{9}\cdot 26509\;86803}{3^{13}\cdot 5^{4}\cdot 7^{2}\cdot 11\cdot 13\cdot 17\cdot 19}$
9	$\frac{2^{10}\cdot 61718\;01683}{3^{14}\cdot 5^{4}\cdot 7^{2}\cdot 11\cdot 13\cdot 17\cdot 19}$
10	$\frac{2^{11}\cdot 428\;39331\;45517}{3^{16}\cdot 5^{5}\cdot 7^{2}\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23}$
11	$-\frac{2^{12}\cdot 1196\;39836\;48109}{3^{17}\cdot 5^{5}\cdot 7^{2}\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23}$
12	$-\frac{2^{13}\cdot 20869\;76249\;24077}{3^{18}\cdot 5^{5}\cdot 7^{3}\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23}$
13	$\frac{2^{14}\cdot 29320\;11913\;05155\;66117}{3^{22}\cdot 5^{6}\cdot 7^{3}\cdot 11^{2}\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29}$
14	$\frac{2^{15}\cdot 27\;00231\;12146\;07564\;31181}{3^{23}\cdot 5^{6}\cdot 7^{3}\cdot 11^{2}\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31}$
15	$-\frac{2^{16}\cdot 302\;52864\;76959\;66455\;59143}{3^{25}\cdot 5^{6}\cdot 7^{3}\cdot 11^{2}\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31}$
16	$-\frac{2^{17}\cdot 67822\;42429\;22326\;79335\;35073}{3^{26}\cdot 5^{7}\cdot 7^{3}\cdot 11^{2}\cdot 13^{2}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31}$
17	$\frac{2^{18}\cdot 5174\;85871\;06835\;35342\;63301\;48693}{3^{27}\cdot 5^{7}\cdot 7^{5}\cdot 11^{2}\cdot 13^{2}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37}$
18	$\frac{2^{19}\cdot 20321\;10359\;54227\;19515\;84056\;15907}{3^{28}\cdot 5^{7}\cdot 7^{5}\cdot 11^{2}\cdot 13^{2}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37}$
19	$-\frac{2^{20}\cdot 585\;30287\;26332\;92617\;24881\;45877\;26421}{3^{30}\cdot 5^{8}\cdot 7^{5}\cdot 11^{2}\cdot 13^{2}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41}$
20	$-\frac{2^{21}\cdot 1\;10855\;49579\;65750\;34381\;96928\;10335\;55329}{3^{31}\cdot 5^{8}\cdot 7^{5}\cdot 11^{2}\cdot 13^{2}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43}$

Hence,

$$-2L_x \sim \frac{b_1}{x^{3/2}} + \frac{3^2 b_3}{x^{5/2}} + \frac{3 \cdot 5^2 b_5}{x^{7/2}} + \dots ,$$

$$2^2 L_{xx} \sim \frac{3b_1}{x^{5/2}} + \frac{3^2 \cdot 5 b_3}{x^{7/2}} + \frac{3 \cdot 5^2 \cdot 7 b_5}{x^{9/2}} + \dots ,$$

$$-2^3 L_{xxx} \sim \frac{3 \cdot 5 b_1}{x^{7/2}} + \frac{3^2 \cdot 5 \cdot 7 b_3}{x^{9/2}} + \frac{3 \cdot 5^2 \cdot 7 \cdot 9}{x^{11/2}} + \dots ,$$

and so on. Writing

$$C_{x-1}(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots \quad ,$$

we have the following relation, which enables us to determine, one after the other, the coefficients $a_1, a_2, \dots$, a_0 being $2 + 4b_2 = \frac{2}{3}$:

$$\left(\frac{b_1}{x^{1/2}} + \frac{3b_3}{x^{3/2}} + \frac{3 \cdot 5 b_5}{x^{5/2}} + \dots .\right) \cdot \left(\frac{a_1}{x} + \frac{a_2}{x^2} + \dots .\right)$$

$$= 2 \cdot 4 b_4 \left(\frac{b_1}{x^{3/2}} + \frac{3^2 b_3}{x^{5/2}} + \frac{3 \cdot 5^2 b_5}{x^{7/2}} + \frac{3 \cdot 5 \cdot 7^2 b_7}{x^{9/2}} + \dots\right)$$

$$+ 2 \cdot 6 b_6 \left(\frac{3b_1}{x^{5/2}} + \frac{3^2 \cdot 5 b_3}{x^{7/2}} + \frac{3 \cdot 5^2 \cdot 7 b_5}{x^{9/2}} + \dots\right)$$

$$+ 2 \cdot 8 b_8 \left(\frac{3 \cdot 5 b_1}{x^{7/2}} + \frac{3^2 \cdot 5 \cdot 7 b_3}{x^{9/2}} + \dots\right)$$

$$+ \dots$$

Equating the coefficients of $\frac{1}{x^{3/2}}$, we find that $a_1 = 8b_4 = 2^2 \cdot 2! b_4$. Equating the coefficients of $\frac{1}{x^{5/2}}$, and using the value of a_1 that we have just determined, we find that

$a_2 = 48b_3b_4 + 36b_6 = \frac{4}{3}\ b_4 + 36b_6$. Now $\frac{4}{3}\ b_4 = -84b_6$, so that $a_2 = -48b_6 = -2^3 \cdot 3!b_6$. The equation which serves to determine a_3 is

$$a_3 + 3b_3a_2 + 3 \cdot 5b_5a_1 = 3 \cdot 5\ (2^3 \cdot 5b_4b_5 + 2^2 \cdot 3^2b_3b_6 + 2^4b_8),$$

and this yields the result

$$a_3 = \frac{-2^4}{3^5 \cdot 5 \cdot 7} = 2^4 \cdot 4!b_8.$$

Similarly, the equation which serves to determine a_4 is

$$a_4 + 3b_3a_3 + 3 \cdot 5b_5a_2 + 3 \cdot 5 \cdot 7b_7a_1 = 3 \cdot 5 \cdot 7\,(2^3 \cdot 7b_4b_7 + 2^2 \cdot 3 \cdot 5b_5b_6 + 2^4 \cdot 3b_3b_8 + 2^2 \cdot 5b_{10}),$$

and this yields the result

$$a_4 = -\frac{2^5 \cdot 281}{3^8 \cdot 5^2 \cdot 7 \cdot 11} = -2^5 \cdot 5!b_{10}.$$

Continuing in this way, we have verified the validity of the formula

$$a_j = (-1)^{j-1}\ 2^{j+1}\ (j+1)!\ b_{2j+2}$$

for $1 \leq j \leq 11$, but we have not been able to prove that this formula is valid for all positive integers j. Assuming that it is, the values of the coefficients a_j in the asymptotic formula for $C_{x-1}(x)$:

$$C_{x-1}(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots$$

for $0 \leq j \leq 20$ are given in Table 10.

The values, to 45 decimals, of the coefficients a_j, $0 \leq j \leq 20$, are given in Table 11.

To see how good an approximation to $C_{x-1}(x)$ is furnished by the asymptotic series $a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots$, we have calculated the sum of 10 and of 20 terms of this series when $x = 5$ and the sum of 20 terms when $x = 10$. The results are as follows:

TABLE 11

j	a_j								
0	0.66666	66666	66666	66666	66666	66666	66666	66666	66667
1	0.02962	96296	29629	62962	96296	29629	62962	96296	29630
2	0.00282	18694	88536	15520	28218	69488	53615	52028	21869
3	−0.00188	12463	25690	77013	52145	79659	02410	34685	47913
4	−0.00071	19598	88914	62142	49094	90753	11215	22891	07022
5	0.00067	83725	85094	12144	52874	26083	03650	82765	63146
6	0.00047	28641	94857	79339	46018	07953	43118	15976	19641
7	−0.00059	01363	95449	25981	15875	80677	23147	42977	47843
8	−0.00060	18476	34904	14883	51235	89341	79876	46299	24973
9	0.00093	41136	01724	33000	33378	19511	25879	86706	50297
10	0.00125	29105	43523	91124	40514	51244	20935	02778	36059
11	−0.00233	27163	37391	92602	80506	49803	38113	26243	79711
12	−0.00387	53803	04549	74631	00470	62290	30924	51771	25488
13	0.00842	84330	43714	92083	54506	43818	91054	14380	60794
14	0.01669	27963	87573	73765	89270	51579	43731	58500	45159
15	−0.04156	06326	53999	94960	39814	03377	46447	94307	01620
16	−0.09556	18035	42362	39396	76130	83996	97311	69279	75175
17	0.26811	45987	12206	69434	70947	35035	36395	26666	20372
18	0.70190	44500	19438	74033	84405	43349	91312	96923	58226
19	−2.19151	77207	85221	92950	89864	72162	83778	12408	21205
20	−6.43519	69808	13800	09084	29079	50653	40995	64243	94772

1. $a_0 + \frac{a_1}{5} + \ldots + \frac{a_9}{5^9} = 0.67268\ 95169\ 92$, which is less than $C_4(5)$ by less than 8 units in the eleventh decimal place;

2. $a_0 + \frac{a_1}{5} + \ldots + \frac{a_{19}}{5^{19}} = 0.67268\ 95170\ 6475$, which is greater than $C_4(5)$ by less than 2 units in the fourteenth decimal place;

3. $a_0 + \frac{a_1}{10} + \ldots + \frac{a_{19}}{10^{19}} = 0.66965\ 59030\ 74791\ 50278\ 5$, which is greater than $C_9(10)$ by less than 4 units in the twentieth decimal place.

Thus the sum of 20 terms of the asymptotic series furnishes $C_9(10)$ correct to 19 decimal places, and this implies, since $\frac{9!}{10^9} < 4 \cdot 10^{-4}$, that we may calculate, by means of this sum, $C(10) = 10\ \{ \exp(-10) \}\ Ei\ 10$ correct to 22 decimal places.

APPROXIMATIONS TO THE HIGHER COEFFICIENTS IN THE ASYMPTOTIC FORMULA FOR $C_{x-1}(x)$

The function t of the complex variable z which we have encountered in the preceding section is regular over a neighborhood of $z = 0$, its power series development about $z = 0$ being $b_1 z + b_2 z^2 + \dots$. This power series converges over the interior of the circle whose center is at $z = 0$ and whose radius is the distance from $z = 0$ to the singularity, or singularities, of t which are nearest $z = 0$. Since $tt_z = z(1-t)$, t_z is defined at any point of the complex z-plane at which $t \neq 0$, and the singularities of t in the complex z-plane are those points, other than $z = 0$, at which $t = 0$. Since $(1-t)\exp t = \exp\left(-\frac{1}{2} z^2\right)$ a necessary condition for t to be zero is the relation $z^2 = 4m\pi i$, where m is any integer. Consequently, the singularities of t lie among the points $z = \pm 2m^{1/2}\pi^{1/2}\exp\left(\pm\frac{i\pi}{4}\right)$, where, now, m is any positive integer. Taking m to be 1, we obtain four points $\pm\alpha$, $\pm\bar{\alpha}$, where $\alpha = 2\pi^{1/2}\exp\frac{i\pi}{4}$, whose common distance from the origin is $2\pi^{1/2}$, and there is no singularity of t whose distance from $z = 0$ is less than $2\pi^{1/2}$. These remarks are also applicable to the function $\tau(z)$, which is related to the function $t(z)$ by means of the formula $\tau(z) = -t(-z)$, and hence they are applicable to the function $t(z) - \tau(z)$, whose power series development about $z = 0$ is $2(b_2 z^2 + b_4 z^4 + \dots)$. The coefficients of the asymptotic formula for $C_{x-1}(x)$, namely

$$C_{x-1}(x) \sim \frac{2}{3} + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots,$$

are connected with the numbers $b_4, b_6, \dots$ by means of the formula

$$a_j = (-1)^{j-1}(j+1)!\,2^{j+1}\,b_{2j+2} \quad , j = 1, 2, \dots \quad ,$$

and so we direct our attention to the function $t-\tau$ rather than the function t. Since $t-\tau$ is a function of z^2 and not merely of z, we may confine ourselves to the points of the complex z-plane for which $-\frac{\pi}{2} < \arg z \leq \frac{\pi}{2}$, and in this right half of the complex z-plane the possible singularities of $t-\tau$ are the points $m^{1/2}\alpha$ and $m^{1/2}\bar{\alpha}$, where m is any positive integer. The point $m^{1/2}\alpha$ will be a singularity of t if, and only if, t is zero at α ; similarly, $m^{1/2}\alpha$ will be a singularity of τ if, and only if, τ is zero at α. Furthermore, since both t and τ are real when z is real, $m^{1/2}\bar{\alpha}$ will be a singularity of t, or of τ, if, and only if, $m^{1/2}\alpha$ is a singularity of t or of τ, respectively. If α were not a singularity of $t-\tau$, the radius of convergence of the power series development $b_2 z^2 + b_4 z^4 + \dots$ would be at least $2^{1/2}|\alpha| = 2^{3/2}\pi^{1/2}$. Now, the power series $b_2 z^2 + b_4 z^4 + \dots$ does not converge at z if $\left|\frac{b_{2k+2}}{b_{2k}}\right| z^2 > 1$ for k sufficiently large; from the values we have obtained for b_{2k},

$1 \le k \le 21$, we find that the ratios $\left|\dfrac{b_{38}}{b_{36}}\right|$, $\left|\dfrac{b_{40}}{b_{38}}\right|$ and $\left|\dfrac{b_{42}}{b_{40}}\right|$ are, approximately, $\dfrac{1}{16}$, $\dfrac{1}{13}$ and $\dfrac{1}{14}$, respectively, and, since the squared modulus of $2^{1/2}\alpha$, namely 8π, is about twice each of the reciprocals of these numbers, we infer that α and $\bar{\alpha}$ are singularities of $t-\tau$. If both t and τ were zero at $z=\alpha$, their sum $2(b_1 z + b_3 z^3 + \dots\ )$ would be zero at $z=\alpha$, so that $b_1 + b_3\alpha^2 + b_5\alpha^4 + \dots$ would be zero. The imaginary part of $b_1 + b_3\alpha^2 + b_5\alpha^4 + \dots$ is $2^2\pi b_3 - 2^6\pi^3 b_7 + 2^{10}\pi^5 b_{11} - \dots$, which is positive, since the numbers $b_3, b_7, b_{11}, \dots$ are alternately positive and negative. Hence one, but not both, of the two functions t and τ is zero at $z=\alpha$, and this implies that the points α and $\bar{\alpha}$ are singularities of the function $t-\tau$ of the complex variable z or, equivalently, that the points $2\pi i$ and $-2\pi i$ are singularities of the function $t-\tau$ of the complex variable $\xi = \dfrac{1}{2}z^2$. The fact that the imaginary parts of the first three terms $b_1\alpha$, $-b_2\alpha^2$, and $b_3\alpha^3$ of the power series which furnishes $t(\alpha)$ are positive indicates that it is t and not τ which is zero at α, and we shall proceed on this assumption. The nature of the singularities $\pm 2\pi i$ of the function $t-\tau$ of ξ is the same as that of the singularities α and $\bar{\alpha}$ of the function t of z, since τ is regular at α and $\bar{\alpha}$. Since $(1-t)\exp t = \exp(-\xi)$, $t+\xi = -\log(1-t)$, and if we take the determination of $\log(1-t)$ which is $-2\pi i$ at $t=0$, we obtain the relation $\xi - 2\pi i = \dfrac{t^2}{2} + \dfrac{t^3}{3} + \dots$, $|t|<1$. Writing $\xi - 2\pi i = \dfrac{1}{2}(z')^2$, it follows that

$$t = b_1 z' + b_2 (z')^2 + \dots$$

$$= b_1(2\xi - \alpha^2)^{1/2} + b_2(2\xi - \alpha^2) + b_3(2\xi - \alpha^2)^{3/2} + \dots, \text{ where } \alpha^2 = 4\pi i.$$

Thus the point $2\pi i$ is a branch point of the function $t-\tau$ of the complex variable ξ, the nonuniform part of the development of $t-\tau$ in the neighborhood of this branch point being

$$b_1(2\xi - \alpha^2)^{1/2} + b_3(2\xi - \alpha^2)^{3/2} + b_5(2\xi - \alpha^2)^{5/2} + \dots$$

(The development in question is valid over the circle of radius $2\pi^{1/2}$ whose center is $2\pi i$.) Applying the same argument to the point $-2\pi i$, we have the following result:

The points $\pm 2\pi i$ are branch points of the function $t-\tau$ of the complex variable ξ, and the nonuniform parts of the developments of $t-\tau$ in neighborhoods of these branch points are $b_1(2\xi - \alpha^2)^{1/2} + b_3(2\xi - \alpha^2)^{3/2} + \dots$ and $b_1(2\xi - \bar{\alpha}^2)^{1/2} + b_3(2\xi - \bar{\alpha}^2)^{3/2} + \dots$, respectively.

The function $t - \tau$ of the complex variable ξ furnishes, on setting $\xi = 2\pi \exp i\theta$, a function $f(\theta)$ of the real variable θ defined over the interval $-\pi \leq \theta \leq \pi$ and such that $f(-\pi) = f(\pi)$. At every point of the interval $-\pi \leq \theta \leq \pi$, except the points $\pm \frac{\pi}{2}$, $f(\theta)$ is indefinitely differentiable, the first derivative of $f(\theta)$ being the quotient of the derivative of $t - \tau$ with respect to ξ by 2π. At the points $\pm \frac{\pi}{2}$, however, $f(\theta)$, while continuous, fails to be differentiable owing to the terms $b_1(2\xi - \alpha^2)^{1/2}$ and $b_1(2\xi - \bar{\alpha}^2)^{1/2}$ in the developments of $t - \tau$ near its branch points $\pm 2\pi i$. If we subtract $b_1(2\xi - \alpha^2)^{1/2} + b_1(2\xi - \bar{\alpha}^2)^{1/2}$ from $t - \tau$, we obtain a new function of ξ having the points $\pm 2\pi i$ as branch points, the nonuniform parts of the developments of this function in neighborhoods of its two branch points being $b_3(2\xi - \alpha^2)^{3/2} + b_5(2\xi - \alpha^2)^{5/2} + \ldots$ and $b_3(2\xi - \bar{\alpha}^2)^{3/2} + b_5(2\xi - \bar{\alpha}^2)^{5/2} + \ldots$, respectively. This new function of ξ defines on the circle $\xi = 2\pi \exp i\theta$ a new function $f^*(\theta)$ of θ over the interval $-\pi \leq \theta \leq \pi$, which is not only continuous over this interval and such that $f^*(-\pi) = f^*(\pi)$, but is also continuously differentiable over the interval $-\pi \leq \theta \leq \pi$.

The function $f^*(\theta)$ possesses a continuous second derivative at all points of the interval $-\pi \leq \theta \leq \pi$, except for the points $\pm \frac{\pi}{2}$, at which points it fails to possess a second derivative owing to the terms $b_3(2\xi - \alpha^2)^{3/2}$ and $b_3(2\xi - \bar{\alpha}^2)^{3/2}$ in the developments, near its branch points, of the function of ξ which gave rise to $f^*(\theta)$. The fact that $f^*(\theta)$ is differentiable over the interval $-\pi \leq \theta \leq \pi$, while $f(\theta)$ fails to be differentiable at the points $\pm \frac{\pi}{2}$ of this interval, suggests that the Fourier coefficients of $f^*(\theta)$ tend more rapidly to zero, as their subscripts increase, than do those of $f(\theta)$ or, what is the same thing, that the coefficients in the power series development of $t - \tau - b_1(2\xi - \alpha^2)^{1/2} - b_1(2\xi - \bar{\alpha}^2)^{1/2}$ near $\xi = 0$ tend more rapidly to zero, as their subscripts increase, than do the coefficients in the power series development of $t - \tau$ near $\xi = 0$. We express this result by the statement that the coefficients in the power series development of $t - \tau$ near $\xi = 0$, whose subscripts are sufficiently large, are approximated by the corresponding coefficients in the power series development of $b_1(2\xi - \alpha^2)^{1/2} + b_1(2\xi - \bar{\alpha}^2)^{1/2}$. This method of approximating those coefficients in the power series development of a function of a complex variable whose subscripts are sufficiently large is due to Darboux[22], and we refer to it as the Darboux approximation. The coefficients of the various powers of ξ in the power series development, near $\xi = 0$, of $b_1(2\xi - \alpha^2)^{1/2} + b_1(2\xi - \bar{\alpha}^2)^{1/2}$ furnish the first term in the Darboux approximation to the coefficients of the corresponding powers of ξ in the power series development, near $\xi = 0$, of $t - \tau$; the coefficients of the various powers of ξ in the power series development, near $\xi = 0$, of $b_3(2\xi - \alpha^2)^{3/2} + b_3(2\xi - \bar{\alpha}^2)^{3/2}$ furnish the second term of this Darboux approximation, and so on. The process is not a convergent one, since the two series $b_1(2\xi - \alpha^2)^{1/2} + b_3(2\xi - \alpha^2)^{3/2} + \ldots$, $b_1(2\xi - \bar{\alpha}^2)^{3/2} + b_3(2\xi - \bar{\alpha}^2)^{3/2} + \ldots$ have no common point of convergence except for the point $\xi = 0$, their circles of convergence being the circles of radius 2π whose centers are $2\pi i$ and $-2\pi i$, respectively. We stop, accordingly, when the terms of the Darboux approximation begin to increase in absolute value.

The coefficient of ξ^k in the power series development of $t - \tau = 2(b_2 z^2 + b_4 z^4 + \ldots)$ $= 2(2b_2\xi + 2^2 b_4 \xi^2 + \ldots)$ near $\xi = 0$ is $2^{k+1} b_{2k}$, and the first term of the Darboux approximation to this is twice the real part of the coefficient of ξ^k in the power series development of $b_1(2\xi - \alpha^2)^{1/2} = (2\xi - \alpha^2)^{1/2}$ near $\xi = 0$. Thus the first term of the Darboux approximation to b_{2k} is the quotient of the real part of the coefficient of ξ^k in the power series development of $(2\xi - \alpha^2)^{1/2}$ near $\xi = 0$ by 2^k. Near $\xi = 0$ the argument of $2\xi - \alpha^2 = 2\xi - 2\pi i$ is $-\frac{\pi}{2}$, and so $(2\xi - \alpha^2)^{1/2} = -i\alpha\left(1 - \frac{2\xi}{\alpha^2}\right)^{1/2} = -i\alpha\left(1 - \frac{\xi}{\alpha^2} - \frac{1}{2\cdot 4}\frac{2^2\xi^2}{\alpha^4} - \frac{1\cdot 3}{2\cdot 4\cdot 6}\frac{2^3\xi^3}{\alpha^6} - \ldots\right)$ Hence, the coefficient of ξ^k in the power series development of $(2\xi - \alpha^2)^{1/2}$ near $\xi = 0$ is, if $k > 1$, $i\,\frac{1\cdot 3\ldots(2k-3)}{2\cdot 4\ldots\ 2k}\,\frac{2^k}{\alpha^{2k-1}}$, and the real part of this is $\frac{1\cdot 3\ldots(2k-3)}{2^{2k-1}\pi^{k-1/2}k!}\sin\left(k - \frac{1}{2}\right)\frac{\pi}{2}$, so that the first term of the Darboux approximation to b_{2k}, where $k > 1$, is $\frac{1\cdot 3\ldots(2k-3)}{2^{3k-1}\pi^{k-1/2}k!}$ $\sin\left(k - \frac{1}{2}\right)\frac{\pi}{2}$. The sign of this term, that is, of $\sin\left(k - \frac{1}{2}\right)\frac{\pi}{2}$, gives correctly the proper sign sequence, +, –, –, +, +, –, –, ... of the numbers b_4, b_6, b_8, ... Knowing the sign of b_{2k}, we may concentrate our attention upon its numerical magnitude, and we have the following result:

The first term of the Darboux approximation to $|b_{2k}|$ is $\frac{1\cdot 3\ldots(2k-3)}{2^{3k-1/2}\pi^{k-1/2}k!}$. An approximation to b_n similar to the one just given for b_{2k} has been published by G.N. Watson.[23] The coefficients a_j, $j = 1, 2, \ldots$, of the asymptotic formula $\frac{2}{3} + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots$ for $C_{x-1}(x)$ are connected with the numbers b_{2k} by means of the relation $a_j = (-1)^{j-1}(j+1)!\,2^{j+1}\,b_{2j+2}$, and so:

The first term of the Darboux approximation to $|a_j|$ is

$$\phi_j = \frac{1\cdot 3\ldots(2j-1)}{2^{2j+3/2}\pi^{j+1/2}}$$

The numbers ϕ_j satisfy the recurrence relation

$$\phi_{j-1} = \frac{4\pi}{2j-1}\,\phi_j \quad , \quad j = 2, 3, \ldots$$

If we calculate ϕ_j directly from the formula $\frac{1\cdot 3\ldots(2j-1)}{2^{2j+3/2}\pi^{j+1/2}}$ for a large value of j, say $j = 30$, the values of ϕ_j for smaller values of j may be determined, one after the other, from the recurrence formula just written, there being no loss of significant figures if $2j-1>4\pi$, that is, if $j>6$. We give, in Table 12, the values, to 24 significant figures, of ϕ_j for $19 \le j \le 30$.

TABLE 12

j	ϕ_j						
19		2.	13181	04399	21963	27559	227
20		6.	61611	93003	45524	30207	081
21		21.	58625	58600	81487	25646	51
22		73.	86452	54440	34768	16313	42
23		264.	50784	77299	92193	14249	0
24		989.	29668	91414	25672	71122	3
25		3857.	56072	74019	49795	91830	
26		15655.	72138	01012	86251	8684	
27		66029.	66430	07836	16995	1393	
28	2	88996.	05526	46246	63720	116	
29	13	10861.	79578	85719	67448	05	
30	61	54588.	96804	70619	43834	98	

The coefficient of ξ^k in the power series development of $(2\xi-\alpha^2)^{1/2}(2\xi-\alpha^2)$ $= i\alpha^3\left(1-\dfrac{2\xi}{\alpha^2}\right)^{3/2}$ near $\xi=0$ is, if $k>2$, $i\,\dfrac{3\cdot1\cdot3\ldots(2k-5)}{2\cdot4\ldots\quad 2k}\,\dfrac{2^k}{\alpha^{2k-3}}$, and the real part of this is $\dfrac{3\cdot1\cdot3\ldots(2k-5)}{2^{2k-3}\pi^{k-3/2}}\sin\left(k-\dfrac{3}{2}\right)\dfrac{\pi}{2}$. Thus, the second term of the Darboux approximation to b_{2k}, where $k>2$, is $b_3\,\dfrac{3\cdot1\cdot3\ldots(2k-5)}{2^{3k-3}\pi^{k-3/2}k!}\sin\left(k-\dfrac{3}{2}\right)\dfrac{\pi}{2}$. When k is even, $\sin\left(k-\dfrac{3}{2}\right)\dfrac{\pi}{2}=\sin\left(k-\dfrac{1}{2}\right)\dfrac{\pi}{2}$ and, when k is odd, $\sin\left(k-\dfrac{3}{2}\right)\dfrac{\pi}{2}=$ $-\sin\left(k-\dfrac{1}{2}\right)\dfrac{\pi}{2}$, and so the second term of the Darboux approximation to $|b_{2k}|$, where $k>2$, may be written in the form $(-1)^k\,b_3\,\dfrac{3\cdot1\cdot3\ldots(2k-5)}{2^{3k-3}\pi^{k-3/2}k!}\sin\left(k-\dfrac{1}{2}\right)\dfrac{\pi}{2}$. This implies that the second term of the Darboux approximation to $|a_j|$, where $j\ge2$, is $(-1)^{j+1}$ $\dfrac{3\cdot1\cdot3\ldots(2j-3)}{2^{2j-1/2}\pi^{j-1/2}}\,b_3$, which may be written in the form $(-1)^{j+1}\,\phi_j\,\dfrac{3\cdot2^2\pi}{2j-1}\,b_3$. Since $b_3=\dfrac{1}{2^2\cdot3^2}$, we see that the second term of the Darboux approximation to $|a_j|$ is

$$\phi_j \cdot \frac{\pi}{3(2j-1)} \quad , \quad j = 3, 5, 7, \ldots$$

$$-\phi_j \cdot \frac{\pi}{3(2j-1)} \quad ; \quad j = 2, 4, 6, \ldots$$

The coefficient of ξ^k in the power series development of $(2\xi - \alpha^2)^{1/2}\,(2\xi - \alpha^2)^2 = -i\alpha^5 \left(1 - \frac{2\xi}{\alpha^2}\right)^{5/2}$ near $\xi = 0$ is, if $k > 3$, $i\,\frac{5\cdot 3\cdot 1\cdot 3 \ldots (2k-7)}{2\cdot 4 \ldots \quad 2k}\,\frac{2^k}{\alpha^{2k-5}}$, and the real part of this is $\frac{5\cdot 3\cdot 1\cdot 3 \ldots (2k-7)}{2^{2k-5}\,\pi^{k-5/2}\,k!}\,\sin\left(k - \frac{5}{2}\right)\frac{\pi}{2}$. Thus, the third term of the Darboux approximation to b_{2k}, where $k > 3$, is $b_5\,\frac{5\cdot 3\cdot 1\cdot 3 \ldots (2k-7)}{2^{3k-5}\,\pi^{k-5/2}\,k!}\,\sin\left(k - \frac{5}{2}\right)\frac{\pi}{2}$.
Since $\sin\left(k - \frac{5}{2}\right)\frac{\pi}{2} = -\sin\left(k - \frac{1}{2}\right)\frac{\pi}{2}$, whether k is even or odd, the third term of the Darboux approximation to $|b_{2k}|$, where $k > 3$, is $-b_5\,\frac{5\cdot 3\cdot 1\cdot 3 \ldots (2k-7)}{2^{3k-9/2}\,\pi^{k-5/2}\,k!}$, and this implies that the third term of the Darboux approximation to $|a_j|$, where $j \geq 3$, is $-b_5\,\frac{5\cdot 3\cdot 1\cdot 3 \ldots (2j-5)}{2^{2j-5/2}\,\pi^{j-3/2}}$, and this may be written in the form

$$\phi_j \cdot -\frac{5\cdot 3}{(2j-1)\,(2j-3)}\;2^4\,\pi^2\,b_5$$

When dealing with the fourth term of the Darboux approximation to b_{2k} we encounter the factor $\sin\left(k - \frac{7}{2}\right)\frac{\pi}{2}$, which is $-\sin\left(k - \frac{1}{2}\right)\frac{\pi}{2}$, when k is even, and $\sin\left(k - \frac{1}{2}\right)\frac{\pi}{2}$, when k is odd. It follows, by the same argument as before, that the fourth term of the Darboux approximation to $|a_j|$, where $j \geq 4$, is

$$(-1)^j\,\phi_j \cdot \frac{7\cdot 5\cdot 3}{(2j-1)\,(2j-3)\,(2j-5)}\;2^6\,\pi^3\,b_7$$

Similarly, the fifth term of the Darboux approximation to $|a_j|$, where $j \geq 5$, is

$$\phi_j \cdot \frac{9\cdot 7\cdot 5\cdot 3}{(2j-1)\,(2j-3)\,(2j-5)\,(2j-7)}\;2^8\,\pi^4\,b_9,$$

and so on. Combining our results, we see that the Darboux approximation to $|a_j|$ may be written in the form

$$|a_j| \sim \phi_j \left\{ 1 + (-1)^{j-1} \frac{m_1}{(2j-1)} - \frac{m_2}{(2j-1)(2j-3)} \right.$$

$$\left. + (-1)^{j-1} \frac{m_3}{(2j-1)(2j-3)(2j-5)} - \frac{m_4}{(2j-1)\cdots(2j-7)} + \cdots \right\},$$

where it is understood that j is at least as great as the number of terms within the braces multiplying ϕ_j that we take, and

$$\begin{aligned} m_1 &= 3b_3 \cdot 2^2 \pi &&= c_1 \cdot 2^2 \pi, \\ m_2 &= 3 \cdot 5 b_5 \cdot 2^4 \pi^2 &&= c_2 \cdot 2^4 \pi^2, \\ m_3 &= -3 \cdot 5 \cdot 7 b_7 \cdot 2^6 \pi^3 &&= -c_3 \cdot 2^6 \pi^3, \\ m_4 &= -3 \cdot 5 \cdot 7 \cdot 9 b_9 \cdot 2^8 \pi^4 &&= -c_4 \cdot 2^8 \pi^4, \end{aligned}$$

and so on. Here $c_1, c_2, \ldots$ are the coefficients which appear in the Stirling asymptotic formula

$$\Gamma(x+1) \sim (2\pi)^{1/2} x^{x+1/2} \exp(-x) \left\{ 1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots \right\}$$

for the gamma function. Since $b_3, b_5, b_7, b_9, \ldots$ are alternately positive and negative in pairs, b_3 and b_5 being positive, b_7 and b_9 being negative, and so on, all the numbers m_j, $j = 1, 2, \ldots$ are positive. We give the values of m_j, for $1 \le j \le 9$, to 20 decimal places in Table 13. When j is even, the Darboux approximation to $|a_j|$ is

$$|a_j| \sim \phi_j \left\{ 1 - \frac{m_1}{2j-1} - \frac{m_2}{(2j-1)(2j-3)} - \frac{m_3}{(2j-1)(2j-3)(2j-5)} - \cdots \right\}$$

and, when j is odd, it is

$$|a_j| \sim \phi_j \left\{ 1 + \frac{m_1}{2j-1} - \frac{m_2}{(2j-1)(2j-3)} + \frac{m_3}{(2j-1)(2j-3)(2j-5)} - \cdots \right\},$$

it being understood that j is at least as great as the number of terms in the braces multiplying ϕ_j that we take.

TABLE 13

j	m_j				
1	1.	04719	75511	96597	74615
2	0.	54831	13556	16075	47882
3	5.	32083	01957	55154	32641
4	5.	72228	30226	76480	62366
5	245.	68978	99418	33917	69177
6	274.	57895	61285	07482	03641
7	29303.	05073	89922	77570	06838
8	32160.	28365	07278	92514	07956
9	65 60075.	15467	02926	63782	96686

The corresponding approximations to $|a_{19}|$ and $|a_{20}|$, obtained by stopping at the term involving m_8, are 2.19151 53 and 6.43519 70, respectively, so that the approximations furnish $|a_{19}|$ and $|a_{20}|$ correct to 5 and 7 decimal places, respectively. We give, in Table 14, to 8 significant figures, the approximations furnished in this way to the numbers $|a_j|$, where $21 \leq j \leq 30$.

TABLE 14

j	$\|a_j\|$
21	22.13216 4
22	72.03658 0
23	270.60613
24	966.93661
25	3939.2822
26	15330.078
27	67323.566
28	283428.43
29	1334757.4
30	6044163.1

APPROXIMATIONS TO THE HIGHER COEFFICIENTS IN STIRLING'S ASYMPTOTIC FORMULA FOR $\Gamma(x+1)$

The coefficients c_j, $j = 1, 2, \ldots$, in Stirling's asymptotic formula for $\Gamma(x+1)$:

$$\Gamma(x+1) \sim (2\pi)^{1/2} x^{x+1/2} \exp(-x) \left\{ 1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \ldots \right\},$$

are related to the b's which have odd subscripts greater than 1 by means of the formula

$$c_j = 1 \cdot 3 \ldots (2j+1)\, b_{2j+1} \quad , \; j = 1, 2, 3, \ldots .$$

The quotient of $t + \tau$ by z, namely $2\{b_1 + b_3 z^2 + b_5 z^4 + \ldots\}$, is a uniform function of $\xi = \frac{1}{2} z^2$, whose power series development near $\xi = 0$ is $2b_1 + 2^2 b_3 \xi + 2^3 b_5 \xi^2 + \ldots + 2^{k+1} b_{2k+1} \xi^k + \ldots$. Since both τ and

$$\frac{1}{z} = \{\alpha^2 + (z^2 - \alpha^2)\}^{-1/2} = \frac{1}{\alpha}\left\{1 + \frac{z^2 - \alpha^2}{\alpha^2}\right\}^{-1/2}$$

$$= \frac{1}{\alpha} - \frac{1}{2\alpha^3}(2\xi - \alpha^2) + \frac{1 \cdot 3}{2 \cdot 4 \alpha^5}(2\xi - \alpha^2)^2 + \ldots$$

are regular at $\xi = \frac{\alpha^2}{2} = 2\pi i$, the nonuniform part of $\frac{t+\tau}{z}$ near $\xi = 2\pi i$ is the product of $(2\xi - \alpha^2)^{1/2} + b_3(2\xi - \alpha^2)^{3/2} + \ldots$ by $\frac{1}{\alpha} - \frac{1}{2\alpha^3}(2\xi - \alpha^2) + \frac{1 \cdot 3}{2 \cdot 4 \alpha^5}(2\xi - \alpha^2)^2 - \ldots$, namely

$$\frac{1}{\alpha}(2\xi - \alpha^2)^{1/2} + \left(\frac{b_3}{\alpha} - \frac{1}{2\alpha^3}\right)(2\xi - \alpha^2)^{3/2} + \left(\frac{b_5}{\alpha} - \frac{b_3}{2\alpha^3} + \frac{1 \cdot 3}{2 \cdot 4 \alpha^5}\right)(2\xi - \alpha^2)^{5/2}$$

$$+ \left(\frac{b_7}{\alpha} - \frac{b_5}{2\alpha^3} + \frac{1 \cdot 3}{2 \cdot 4 \alpha^5} b_3 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \alpha^7}\right)(2\xi - \alpha^2)^{7/2}$$

$$+ \left(\frac{b_9}{\alpha} - \frac{b_7}{2\alpha^3}\, \frac{1 \cdot 3}{2 \cdot 4 \alpha^5} b_5 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \alpha^7} b_3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \alpha^9}\right)(2\xi - \alpha^2)^{9/2}$$

$$+\left(\frac{b_{11}}{\alpha}-\frac{b_9}{2\alpha^3}+\frac{1\cdot 3}{2\cdot 4\alpha^5}\,b_7-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\alpha^7}\,b_5+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\alpha^9}\,b_3\right.$$

$$\left.-\frac{1\cdot 3\ldots 9}{2\cdot 4\ldots 10\alpha^{11}}\right)(2\xi-\alpha^2)^{11/2}+\ldots\ .$$

The coefficient of ξ^k in the power series development of $\dfrac{t+\tau}{z}$ near $\xi = 0$ is $2^{k+1}\,b_{2k+1}$, and so the first term of the Darboux approximation to b_{2k+1} is the quotient of the real part of the coefficient of ξ^k in the power series development of $\dfrac{1}{\alpha}\,(2\xi-\alpha^2)^{1/2}$ near $\xi = 0$ by 2^k. Since the argument of $(2\xi-\alpha^2)$ at $\xi = 0$ is $-\dfrac{\pi}{2}$, $(2\xi-\alpha^2)^{1/2} = -i\alpha\left(1-\dfrac{2\xi}{\alpha^2}\right)^{1/2} = -i\alpha\left(1-\dfrac{\xi}{\alpha^2}-\dfrac{1\cdot 1}{2!}\,\dfrac{\xi^2}{\alpha^4}-\dfrac{1\cdot 1\cdot 3}{3!}\,\dfrac{\xi^3}{\alpha^6}-\ldots\right)$, and so the coefficient of ξ^k in the power series development of $\dfrac{1}{\alpha}\,(2\xi-\alpha^2)^{1/2}$ near $\xi = 0$ is, if $k > 1$, $i\,\dfrac{1\cdot 3\ldots(2k-3)}{k!\,\alpha^{2k}} = \dfrac{1\cdot 3\ldots(2k-3)}{2^{2k}\,\pi^k\,k!}\,i\,\exp\left(-i\,\dfrac{k\pi}{2}\right)$, and the real part of this is $\dfrac{1\cdot 3\ldots(2k-3)}{2^{2k}\,\pi^k\,k!}\,\sin\dfrac{k\pi}{2}$. Thus, the first term in the Darboux approximation to b_{2k+1}, where $k > 1$, is $\dfrac{1\cdot 3\ldots(2k-3)}{2^{3k}\,\pi^k\,k!}\,\sin k\,\dfrac{\pi}{2}$, and this implies that the first term in the Darboux approximation to c_j, where $j > 1$, is $\dfrac{\{1\cdot 3\ldots(2j-3)\}^2(2j-1)(2j+1)}{2^{3j}\,\pi^j\,j!}\,\sin j\,\dfrac{\pi}{2}$. When j is odd and $=2m+1$, say, this yields the approximation

$$c_j \sim (-1)^m\,\psi_j,\ \text{where}\ \psi_j = \frac{\{1\cdot 3\ldots(2j-3)\}^2\,(2j-1)\,(2j+1)}{2^{3j}\,\pi^j\,j!}\,,\ j = 3,\,5,\,7,\,\ldots\,,$$

and when j is even, it yields the approximation

$$c_j \sim 0,\ j = 2,\,4,\,6,\,\ldots\ .$$

From the values of c_j, $1 \le j \le 20$, given in Table 9, we see that c_2 is, roughly, $\dfrac{1}{25}$ of c_1 and that c_{2m+2} is, roughly, $\dfrac{1}{12}$ of c_{2m+1} for $1 \le m \le 9$.

The function ψ_j of j, $j = 2, 3, 4, \ldots$, satisfies the recurrence relation

$$\psi_{j-1} = \frac{8\pi j}{(2j-3)\,(2j+1)}\,\psi_j,\ j = 3,\,4,\,5,\,\ldots\ .$$

If $j \geq 8$, this relation enables us to readily determine, without loss of significant figures, ψ_{j-1} once ψ_j is known. We have calculated ψ_{30}, to 24 significant figures, from the formula

$$\psi_{30} = \frac{(1 \cdot 3 \ldots \ 57)^2 \ 59 \cdot 61}{2^{90} \pi^{30} \ 30!} \ ,$$

and have used the recurrence relation just given to determine ψ_j for $19 \leq j < 30$. The values, to 24 significant figures of ψ_j for $19 \leq j \leq 30$ are given in Table 15.

TABLE 15

j		ψ_j					
19		1.	44851	43088	66282	89175	531
20		4.	37158	08525	47949	93284	238
21		13.	89032	55111	13162	17990	29
22		46.	34962	80423	82061	99475	52
23		162.	04844	61686	23903	09323	5
24		592.	38269	53953	62203	49058	4
25		2259.	90680	11573	10402	57062	
26		8981.	50820	81353	05512	32576	
27		37126.	03013	72702	67965	4281	
28	1	59379.	09157	73567	15096	820	
29	7	09591.	15052	89175	39129	126	
30	32	72289.	86276	78062	66824	40	

The second term in the Darboux approximation to b_{2k+1} is the quotient by 2^k of the real part of the coefficient of ξ^k in the power series development of $\left(\frac{b_3}{\alpha} - \frac{1}{2\alpha^3}\right)(2\xi - \alpha^2)^{3/2}$ near $\xi = 0$; if $k > 2$, this coefficient is $i\left(\frac{b_3}{\alpha} - \frac{1}{2\alpha^3}\right) \frac{3 \cdot 1 \cdot 3 \ldots (2k-5)}{k! \, \alpha^{2k-3}}$, so that the second term the Darboux approximation to b_{2k-1} is

$$\frac{3 \cdot 1 \cdot 3 \ldots (2k-5)}{k!} \left\{ \frac{b_3 \sin (k-1) \frac{\pi}{2}}{2^{3k-2} \pi^{k-1}} - \frac{\frac{1}{2} \sin k \frac{\pi}{2}}{2^{3k} \pi^k} \right\}$$

This implies that the second term in the Darboux approximation to c_j, where $j > 2$, is

$$\frac{3\{1\cdot 3 \ldots (2j-5)\}^2 (2j-3)(2j-1)(2j+1)}{j!} \left\{ \frac{b_3 \sin (j-1) \frac{\pi}{2}}{2^{3j-2} \pi^{j-1}} - \frac{\frac{1}{2} \sin j \frac{\pi}{2}}{2^{3j} \pi^j} \right\},$$

which may be written in the form

$$\frac{3}{2j-3} \left\{ 2^2 \pi b_3 \sin (j-1) \frac{\pi}{2} - \frac{1}{2} \sin j \frac{\pi}{2} \right\} \psi_j .$$

When j is even, and equal to $2m+2$ say, $m = 1, 2, \ldots$, this reduces to $(-1)^m \frac{3}{4m+1} \cdot 2^2$ $\pi b_3 \psi_{2m+2} = (-1)^m \frac{\pi}{3(4m+1)} \psi_{2m+2} = (-1)^m \frac{\left(1 + \frac{5}{4m}\right)}{12\left(1 + \frac{1}{m}\right)} \psi_{2m+1}$, which indicates that, when m is reasonably large, c_{2m+2} is, roughly, $\frac{1}{12} c_{2m+1}$. Both c_{2m+2} and c_{2m+1} have the sign of $(-1)^m$, and so we concentrate our attention on the numerical value, $|c_j|$, of c_j. When j is odd, and $=2m+1$, say, the second term of the Darboux approximation to c_{2m+1} is $(-1)^{m+1} \frac{3}{2(4m-1)} \psi_{2m+1}$, and we express this by the statement that the second term of the Darboux approximation to $|c_{2m+1}|$ is $-\frac{3}{2(4m-1)} \psi_{2m+1}$; we mean by this that the second Darboux approximation to $|c_{2m+1}|$ is $\left\{ 1 - \frac{3}{2(4m-1)} \right\} \psi_{2m+1}$. Thus, we have the following results:

1. The second term in the Darboux approximation to $|c_{2m+2}|$ is

$$\frac{2^2 3 \pi}{4m+1} b_3 \psi_{2m+2} = \frac{\pi}{3(4m+1)} \psi_{2m+2} , \quad m = 1, 2, \ldots ;$$

2. The second term in the Darboux approximation to $|c_{2m+1}|$ is

$$-\frac{3}{2(4m-1)} \psi_{2m+1} , \quad m = 1, 2, \ldots .$$

Similarly, the third term in the Darboux approximation to b_{2k+1}, if $k > 3$, is

$$\frac{5 \cdot 3 \cdot 1 \ldots . (2k-7)}{k!} \left\{ \frac{b_5}{2^{3k-4} \pi^{k-2}} \sin (k-2) \frac{\pi}{2} - \frac{\frac{1}{2} b_3}{2^{3k-2} \pi^{k-1}} \sin (k-1) \frac{\pi}{2} + \frac{\frac{1 \cdot 3}{2 \cdot 4}}{2^{3k} \pi^k} \sin k \frac{\pi}{2} \right\},$$

and this implies that the third term of the Darboux approximation to c_j, where $j > 3$, is

$$\frac{5 \cdot 3}{(2j-3)(2j-5)} \left\{ \left(\frac{1 \cdot 3}{2 \cdot 4} - 2^4 \pi^2 b_5 \right) \sin j \frac{\pi}{2} - \frac{1}{2} \cdot 2^2 \pi b_3 \sin (j-1) \frac{\pi}{2} \right\} \psi_j \quad .$$

It follows that

1. The third term in the Darboux approximation to $|c_{2m+2}|$ is

$$\frac{-5 \cdot 3 \pi}{(4m+1)(4m-1)} \, 2 b_3 \psi_{2m+2} , \quad m = 1, 2, 3, \ldots .$$

1′. The third term in the Darboux approximation to $|c_{2m+1}|$ is

$$\frac{5 \cdot 3}{(4m-1)(4m-3)} \left\{ \frac{1 \cdot 3}{2 \cdot 4} - 2^4 \pi^2 b_5 \right\} \psi_{2m+1} , \; m = 2, 3, 4, \ldots .$$

Proceeding in this way, we obtain the following results:

2. The fourth term in the Darboux approximation to $|c_{2m+2}|$ is

$$\frac{7 \cdot 5 \cdot 3 \pi}{(4m+1)(4m-1)(4m-3)} \left\{ \frac{3}{2} b_3 - 2^6 \pi^2 b_7 \right\} \psi_{2m+2} , \; m = 2, 3, 4, \ldots .$$

2′. The fourth term in the Darboux approximation to $|c_{2m+1}|$ is

$$\frac{-7 \cdot 5 \cdot 3}{(4m-1)(4m-3)(4m-5)} \left\{ \frac{5}{2^4} - 2^3 \pi^2 b_5 \right\} \psi_{2m+1} , \quad m = 2, 3, 4, \ldots .$$

3. The fifth term in the Darboux approximation to $|c_{2m+2}|$ is

$$\frac{-9\cdot 7\cdot 5\cdot 3\pi}{(4m+1)\,(4m-1)\,(4m-3)\,(4m-5)}\left\{\frac{5}{2^2}\,b_3-2^5\pi^2 b_7\right\}\psi_{2m+2}\,,\ m=2,3,4,\ldots\,.$$

3′. The fifth term in the Darboux approximation to $|c_{2m+1}|$ is

$$\frac{9\cdot 7\cdot 5\cdot 3}{(4m-1)(4m-3)(4m-5)(4m-7)}\left\{\frac{35}{2^7}-2\cdot 3\pi^2 b_5+2^8\pi^4 b_9\right\}\psi_{2m+1}\,,\quad m=3,4,5,\ldots\,.$$

4. The sixth term in the Darboux approximation to $|c_{2m+2}|$ is

$$\frac{11\cdot 9\ \ldots\ 3\cdot\pi}{(4m+1)(4m-1)\ \ldots\ (4m-7)}\left\{\frac{35}{2^5}\,b_3-2^3\cdot 3\pi^2 b_7+2^{10}\pi^4 b_{11}\right\}\psi_{2m+2}\,,\quad m=3,4,5,\ldots\,.$$

4′. The sixth term in the Darboux approximation to $|c_{2m+1}|$ is

$$\frac{-11\cdot 9\ \ldots\ 3}{(4m-1)(4m-3)\ \ldots\ (4m-9)}\left\{\frac{63}{2^8}-5\pi^2 b_5+2^7\pi^4 b_9\right\}\psi_{2m+1},\quad m=3,4,5,\ldots\,.$$

5. The seventh term in the Darboux approximation to $|c_{2m+2}|$ is

$$\frac{-13\cdot 11\ \ldots\ 3\cdot\pi}{(4m+1)\ \ldots\ (4m-9)}\left\{\frac{63}{2^6}\,b_3-5\cdot 2^2\pi^2 b_7+2^9\pi^4 b_{11}\right\}\psi_{2m+2}\,,\quad m=3,4,5,\ldots\,.$$

5′. The seventh term in the Darboux approximation to $|c_{2m+1}|$ is

$$\frac{13\cdot 11\ \ldots\ 3}{(4m-1)\ \ldots\ (4m-11)}\left\{\frac{231}{2^{10}}-\frac{35}{2^3}\pi^2 b_5+2^5\cdot 3\pi^4 b_9-2^{12}\pi^6 b_{13}\right\}\psi_{2m+1}\,,\ m=4,5,6,\ldots\,.$$

6. The eight term in the Darboux approximation to $|c_{2m+2}|$ is

$$\frac{15\cdot 13\ \ldots\ 3\pi}{(4m+1)\ \ldots\ (4m-11)}\left\{\frac{231}{2^8}\,b_3-\frac{35}{2}\pi^2 b_7+2^7\cdot 3\cdot\pi^4 b_{11}-2^{14}\pi^6 b_{15}\right\}\psi_{2m+2}\,,$$

$$m=4,5,6,\ldots\,.$$

6′. The eighth term in the Darboux approximation to $|c_{2m+1}|$ is

$$\frac{-15\cdot 13 \,\ldots\, 3}{(4m-1)\ldots(4m-13)}\left\{\frac{429}{2^{11}} - \frac{63\,\pi^2}{2^4}\, b_5 + 5\cdot 2^4\,\pi^4\, b_9 - 2^{11}\,\pi^6\, b_{13}\right\}\psi_{2m+1}\,,$$

$$m = 4, 5, 6, \ldots\,.$$

7. The ninth term in the Darboux approximation to $|c_{2m+2}|$ is

$$\frac{-17\cdot 15 \ldots 3\,\pi}{(4m+1)\ldots(4m-13)}\left\{\frac{429}{2^9}\, b_3 - \frac{63\,\pi^2}{2^2}\, b_7 + 5\cdot 2^6\cdot\pi^4\, b_{11} - 2^{13}\,\pi^6\, b_{15}\right\}\psi_{2m+2}\,,$$

$$m = 4, 5, 6, \ldots\,.$$

7′. The ninth term in the Darboux approximation to $|c_{2m+1}|$ is

$$\frac{17\cdot 15 \ldots 3}{(4m-1)\ldots(4m-15)}\left\{\frac{6435}{2^{15}} - \frac{231}{2^6}\,\pi^2\, b_5 + 35\cdot 2\,\pi^4\, b_9 - 2^9\cdot 3\,\pi^6\, b_{13} + 2^{16}\,\pi^8\, b_{17}\right\}\psi_{2m+1}\,,\quad m = 5, 6, 7, \ldots\,.$$

Contenting ourselves with nine terms of the Darboux approximation to $|c_j|$, where $j > 9$, we collect these results as follows:

$$|c_{2m+2}| \sim \left\{\frac{q_1}{4m+1} - \frac{q_2}{(4m+1)(4m-1)} + \ldots - \frac{q_8}{(4m+1)(4m-1)\ldots(4m-13)}\right\}\pi\,\psi_{2m+2}\,,$$

$$m = 4, 5, \ldots$$

where $q_1 = 3\cdot 2^2\, b_3 = \dfrac{1}{3}$

$$q_2 = 5\cdot 3\cdot 2\, b_3 = \frac{5}{6}$$

$$q_3 = 7\cdot 5\cdot 3\left\{\frac{3}{2}\, b_3 - 2^6\cdot\pi^2\, b_7\right\}$$

$$q_4 = 9\cdot 7\cdot 5\cdot 3\left\{\frac{5}{2^2}\, b_3 - 2^5\,\pi^2\, b_7\right\}$$

$$q_5 = 11\cdot 9 \ldots 3\left\{\frac{35}{2^5}\, b_3 - 2^3\cdot 3\cdot\pi^2\, b_7 + 2^{10}\,\pi^4\, b_{11}\right\}$$

$$q_6 = 13\cdot 11 \,\ldots\, 3\left\{\frac{63}{2^6}\, b_3 - 2^2\cdot 5\cdot\pi^2\, b_7 + 2^9\,\pi^4\, b_{11}\right\}$$

$$q_7 = 15\cdot 13 \ldots 3 \left\{\frac{231}{2^8}\, b_3 - \frac{35}{2}\, \pi^2 b_7 + 2^7\cdot 3\,\pi^4 b_{11} - 2^{14}\pi^6 b_{15}\right\}$$

$$q_8 = 17\cdot 15 \ldots\ldots 3 \left\{\frac{429}{2^9}\, b_3 - \frac{63\,\pi^2}{2^2}\, b_7 + 5\cdot 2^6\,\pi^4 b_{11} - 2^{13}\pi^6 b_{15}\right\}$$

$$|c_{2m+1}| \sim \left\{1 - \frac{p_1}{4m-1} + \frac{p_2}{(4m-1)(4m-3)} - \cdots + \frac{p_8}{(4m-1)(4m-3)\ldots(4m-15)}\right\} \psi_{2m+1}\,,$$

$$m = 5, 6, 7, \ldots$$

where $p_1 = 1\cdot 5$

$$p_2 = 5\cdot 3 \left\{\frac{3}{8} - 2^4\pi^2 b_5\right\}$$

$$p_3 = 7\cdot 5\cdot 3 \left\{\frac{5}{16} - 2^3\pi^2 b_5\right\}$$

$$p_4 = 9\cdot 7\cdot 5\cdot 3 \left\{\frac{35}{128} - 2\cdot 3\,\pi^2 b_5 + 2^8\pi^4 b_9\right\}$$

$$p_5 = 11\cdot 9 \ldots 3 \left\{\frac{63}{256} - 5\,\pi^2 b_5 + 2^7\pi^4 b_9\right\}$$

$$p_6 = 13\cdot 11 \ldots 3 \left\{\frac{231}{2^{10}} - \frac{35}{2^3}\,\pi^2 b_5 + 2^5\cdot 3\,\pi^4 b_9 - 2^{12}\pi^6 b_{13}\right\}$$

$$p_7 = 15\cdot 13 \ldots 3 \left\{\frac{429}{2^{11}} - \frac{63}{16}\cdot\pi^2 b_5 + 2^4\cdot 5\,\pi^4 b_9 - 2^{11}\pi^6 b_{13}\right\}$$

$$p_8 = 17\cdot 15 \ldots 3 \left\{\frac{6435}{2^{15}} - \frac{231}{64}\,\pi^2 b_5 + 70\,\pi^4 b_9 - 2^9\cdot 3\,\pi^6 b_{13} + 2^{16}\pi^8 b_{17}\right\}$$

We give, in Table 16, the values, to 11 significant figures, of πq_j and p_j, for $1 \le j \le 8$.

TABLE 16

j	πq_j	p_j
1	1.04719 75512	1.50000 00000
2	2.61799 38780	5.07668 86444
3	19.06529 8055	30.89341 0255
4	127.02724 483	239.72229 870
5	1435.40438 46	2407.92829 67
6	7606.45224 60	28552.40994 4
7	234973.00138	400476.67498
8	3 452992.7755	6 367534.4062

Using these values and the values of ψ_{19} and ψ_{20} given in Table 15, we obtain the approximations 1.391812 and 0.116566 to $|c_{19}|$ and $|c_{20}|$, respectively. The correct values, to 6 decimal places, are 1.391801 and 0.116546.

By the same procedure approximations to $|c_j|$ for $21 \leqq j \leqq 30$ have been calculated to 6 or 7 significant figures. These results appear in the following table, which is similar to Table 14.

TABLE 17

j	$\|c_j\|$
21	13.3980
22	1.12093
23	156.802
24	13.1088
25	2192.56
26	183.199
27	36101.1
28	3015.17
29	691346.4
30	57722.53

GRAM'S METHOD FOR EVALUATING $Ei(x)$ WHEN $|x|$ IS LARGE

If x is any positive number and h is any positive number which is less than x, it follows from the definition of $Ei\,x$ that

$$Ei\,x - Ei(x-h) = \int_{x-h}^{x} \frac{\exp t}{t}\,dt = \frac{\exp x}{x}\int_0^h \frac{\exp(-v)}{1-\dfrac{v}{x}}\,dv, \quad t = x - v,$$

and

$$Ei(-x) - Ei\,\{-(x+h)\} = -\frac{\exp(-x)}{x}\int_0^h \frac{\exp(-v)}{1+\dfrac{v}{x}}\,dv.$$

On denoting the integral $\int_0^h \exp(-v)\,v^j\,dv$ by $x^j\,T_j(h)$, $j = 0, 1, 2, \ldots$, we have, then, the relations

$$Ei\,x = Ei(x-h) + \frac{\exp x}{x} \sum_{k=0}^{\infty} T_k(h)$$

$$-Ei\{-(x+h)\} = -Ei(-x) - \frac{\exp(-x)}{x} \sum_{k=0}^{\infty} (-1)^k\, T_k(h)$$

When $h=1$ we denote $T_j(h)$ simply by T_j, so that

$$T_j = \frac{R_j}{x^j}\ , \text{ where } R_j = \int_0^1 \exp(-v)\, v^j\, dv, \quad j = 0, 1, 2, \ldots\ .$$

Upon integration by parts, we obtain the recurrence relation

$$R_{j+1} = (j+1)\, R_j - \frac{1}{e}\ , \ j = 0, 1, 2, \ldots ,$$

which may be written in the equivalent form

$$R_j = \frac{R_{j+1} + \dfrac{1}{e}}{j+1}$$

Starting with the value $1 - \dfrac{1}{e}$ of R_0, we see that $R_1 = 1 - \dfrac{2}{e} = \dfrac{1}{e}\,(e-1-1)$, $R_2 = \dfrac{2}{e}\left(e - 1 - 1 - \dfrac{1}{2}\right)$, $R_3 = \dfrac{3!}{e}\left(e - 1 - 1 - \dfrac{1}{2!} - \dfrac{1}{3!}\right)$, and so on. Thus

$$R_j = \frac{j!}{e}\left(e - 1 - 1 - \frac{1}{2!} - \cdots - \frac{1}{j!}\right) = \frac{1}{(j+1)\,e}\left\{1 + \frac{1}{j+2} + \frac{1}{(j+2)(j+3)} + \cdots\right\}$$

For large values of j, say $j \geq 50$, the series $1 + \dfrac{1}{j+2} + \dfrac{1}{(j+2)(j+3)} + \ldots$ converges rapidly. We find that, to 45 decimal places,

$$R_{50} = 0.00735\ 47067\ 95800\ 18886\ 39367\ 19309\ 87336\ 72912\ 37121.$$

Starting with this result and using the recurrence formula, we obtain, without loss of significant figures, one after the other, R_{49}, R_{48}, ... , ending with R_0. The fact that $R_0 = 1 - \dfrac{1}{e}$ provides a useful check on the calculations. The values of R_j, $0 \leq j \leq 50$, to 45 decimal places are given in Table 18.

TABLE 18

j	R_j								
0	0.63212	05588	28557	67840	44762	29838	53913	25541	88869
1	0.26424	11176	57115	35680	89524	59677	07826	51083	77738
2	0.16060	27941	42788	39202	23811	49192	69566	27709	44345
3	0.11392	89412	56922	85447	16196	77416	62612	08670	21903
4	0.08783	63238	56249	09629	09549	39505	04361	60222	76483
5	0.07130	21781	09803	15985	92509	27363	75721	26655	71284
6	0.05993	36274	87376	63755	99817	94021	08240	85476	16571
7	0.05165	59512	40194	14132	43487	87986	11599	23875	04865
8	0.04536	81687	50110	80899	92665	33727	46707	16542	27787
9	0.04043	40775	79554	95939	78750	33385	74277	74422	38954
10	0.03646	13346	24107	27238	32265	63695	96690	69765	78408
11	0.03319	52396	93737	67461	99684	30494	17510	92965	51355
12	0.03046	34351	53409	77384	40973	95768	64044	41128	05135
13	0.02814	52158	22884	73837	77423	74830	86490	60206	55629
14	0.02615	35803	48944	01569	28694	77470	64781	68433	67727
15	0.02442	42640	62717	91379	75183	91898	25638	52047	04774
16	0.02290	87838	32044	29916	47705	00210	64129	58294	65251
17	0.02156	98839	73310	76420	55747	33419	44116	16550	98191
18	0.02037	84703	48151	43410	48214	31388	48004	23459	56306
19	0.01931	14954	43434	92639	60834	26219	65993	71273	58704
20	0.01835	04676	97256	20632	61447	54231	73787	51013	62974
21	0.01748	03804	70938	01125	35160	68705	03450	96828	11324
22	0.01668	89291	89193	92598	18297	41349	29834	55760	37992
23	0.01596	59301	80017	97598	65602	80872	40108	08030	62695
24	0.01530	28831	48989	10208	19229	70776	16507	18276	93554
25	0.01469	26375	53285	23045	25504	99242	66592	82465	27733
26	0.01412	91352	13973	67017	07892	10147	85326	69639	09932
27	0.01360	72096	05846	77301	57849	03830	57734	05797	57041
28	0.01312	24277	92267	32284	64535	37094	70466	87873	86012
29	0.01267	09648	04310	04095	16288	05584	97452	73883	83217
30	0.01224	95029	57858	90695	33403	97387	77495	42056	85414
31	0.01185	51505	22183	79395	80285	48859	56271	29304	36697
32	0.01148	53755	38439	08506	13897	93344	54594	63281	63181
33	0.01113	79515	97047	48543	03394	10208	55536	13835	73856
34	0.01081	09131	28172	18303	60161	76929	42141	95956	99989
35	0.01050	25183	14584	08466	50424	22368	28881	84036	88478
36	0.01021	12181	53584	72634	60034	35096	93659	50869	74086
37	0.00993	56305	11192	55320	66033	28425	19315	07722	30075
38	0.00967	45182	53874	70025	54027	09995	87886	18989	31735
39	0.00942	67707	29670	98836	51819	19677	81474	66125	26522
40	0.00919	13880	15397	21301	17530	16951	12899	70552	49780
41	0.00896	74674	59843	41188	63499	24834	82801	18194	29854
42	0.00875	41921	41980	97763	11730	72901	31562	89702	42724
43	0.00855	08209	33739	71654	49183	64595	11117	82746	26000
44	0.00835	66799	13105	20638	08842	72023	43097	66377	32847
45	0.00817	11549	18291	96554	42684	70892	93308	12521	67006
46	0.00799	36850	69988	09344	08258	90913	46087	01538	71156
47	0.00782	37571	17998	07012	32931	02771	20002	97861	33205
48	0.00766	09004	92465	04432	25451	62856	14056	22885	82696
49	0.00750	46829	59344	85020	91892	09789	42668	46947	40965
50	0.00735	47067	95800	18886	39367	19309	87336	72912	37121

The terms of the infinite series $T_0 \pm T_1 + T_2 \pm \cdots$, whose sums enable us to determine $Ei\,x$ and $-Ei\{-(x+1)\}$ once $Ei(x-1)$ and $-Ei(-x)$, respectively, are known, are furnished by the formula

$$T_j = \frac{R_j}{x^j} , \quad j = 0, 1, 2, \ldots .$$

When working to an accuracy of 45 decimal places j can assume high values, and the division of R_j by x^j is troublesome, requiring double and triple precision. Gram[9] introduced the idea of using a recurrence formula for the terms T_j of the series $\sum_{j=0}^{\infty} \frac{R_j}{x^j}$ rather than for the coefficients, R_j, of this power series. We use, to avoid loss of significant figures, this recurrence relation in the form

$$T_{j+1} = \frac{j+1}{x} T_j - \frac{1}{ex^{j+1}}$$

when $j + 1 \leq x$, and in the form

$$T_j = \frac{x}{j+1} \left\{ T_{j+1} + \frac{1}{ex^{j+1}} \right\}$$

when $x \leq j + 1$. To obtain a starting value when we use the recurrence relation in this latter form, we observe that $R_{j+1} < R_j$, since $v^{j+1} < v^j$ if $0 < v < 1$, and that $R_1 = 1 - \frac{2}{e} < \frac{1}{e}$. Hence, $R_j < \frac{1}{e}$ if $j > 0$, which implies, since $R_j = \left(R_{j+1} + \frac{1}{e}\right) \Big/ (j+1)$, that $R_j < \frac{2}{(j+1)e}$, $j = 0, 1, 2, \ldots$. When we are working with 45 decimal places, T_{j+1} may be set equal to zero when $\frac{2}{(j+2)\,ex^{j+1}} < 5 \cdot 10^{-46}$. If j is the largest integer for which $T_j > 5 \cdot 10^{-46}$, we have $T_j = \frac{1}{(j+1)\,ex^j}$, and we use this as our starting value when using the recurrence relation in the descending direction. For example, when $x = 5$ our starting value is $T_{61} = 0.0_{44}1$. A useful check on the calculations is furnished by the fact that the values given by the recurrence formula, used in both the ascending and descending directions, must check when they meet, that is, when j is the largest integer for which $j + 1 \leq x$. Denoting by $g(x)$ the integral $\int_0^1 \frac{\exp(-v)}{1 - \frac{v}{x}}\, dv$, the values, to 43 decimal places, of $g(x)$ as x assumes integral values from 6 to 20 inclusive are given in Table 19.

TABLE 19

x	$g(x)$								
6	0.68122	78279	40757	03656	16359	62245	36007	10235	592
7	0.67352	04591	20441	53743	90458	71333	71756	70577	239
8	0.66790	65113	46221	26358	28075	33319	95573	22226	682
9	0.66363	44345	81170	69399	04765	54396	74868	82766	133
10	0.66027	41897	28470	77258	32503	64544	49749	82444	341
11	0.65756	18497	16345	33880	84763	85101	98591	92050	073
12	0.65532	64243	57011	19801	29209	33646	11670	85046	353
13	0.65345	22483	66750	31035	74325	14110	72232	51891	458
14	0.65185	82731	89435	59818	27880	02717	49273	36619	220
15	0.65048	60147	75739	96076	59924	48944	78725	81029	827
16	0.64929	22100	80608	07076	11224	50103	91074	01314	207
17	0.64824	41667	78628	52750	48861	02595	03028	75370	370
18	0.64731	67200	10823	38473	80724	56584	41674	53519	673
19	0.64649	01834	29099	21568	33684	46957	33191	28643	655
20	0.64574	89350	52056	33230	93807	00127	01872	99699	520

The values, to 43 decimal places, of $g(-x)$ as x assumes integral values from 5 to 20, inclusive, are given in Table 20.

TABLE 20

x	$g(-x)$								
5	0.58490	60101	09348	06726	68048	84384	49214	87167	869
6	0.59207	38414	79257	77387	27049	35826	69746	81389	398
7	0.59735	00759	31870	08984	55078	13965	25746	22516	648
8	0.60139	67951	92274	30875	93051	25373	27045	53926	185
9	0.60459	91871	31697	78169	72988	85802	41762	41251	190
10	0.60719	66718	58243	67313	05185	47093	84565	32648	256
11	0.60934	59263	45769	76532	18931	99906	15192	39010	625
12	0.61115	37998	86363	82879	46013	73632	78000	19271	460
13	0.61269	56698	06235	89678	14053	79191	95204	70700	905
14	0.61402	62373	74046	39314	61461	69070	83102	42292	159
15	0.61518	61639	09603	96469	38319	93832	59076	78717	564
16	0.61620	63046	23461	20904	33314	82438	05591	50541	637
17	0.61711	04972	14345	49628	60165	37680	46773	27348	990
18	0.61791	74499	06612	47958	91931	83188	88114	95886	054
19	0.61864	20509	53324	18443	81831	61956	07133	73860	855
20	0.61929	62964	06298	55299	86484	25035	89317	94307	168

The value, to 50 significant figures, of $Ei\,5$, calculated from the formula

$$Ei\,5 = \gamma + \log 5 + \sum_{k=1}^{\infty} \frac{5^k}{k \cdot k!}\ , \text{ is}$$

$$Ei\,5 = 40.\ 18527\ 53558\ 03177\ 45509\ 14217\ 93795\ 86709\ 54190\ 87399|193\ .$$

From this result and the relation

$$Ei\,x = Ei(x-1) + \frac{\exp x}{x}\,g(x),$$

we obtain the values of $Ei\ x$ as x assumes integral values from 6 to 20, inclusive. These are given, to 44 significant figures, in Table 21.

TABLE 21

x	$Ei(x)$									
6		85.	98976	21424	39204	80358	34003	07990	69003	92967 85
7		191.	50474	33355	01395	95306	31482	72456	94743	32112 4
8		440.	37989	95348	38268	99742	45966	59393	39244	69328 4
9		1037.	87829	07170	89587	65757	32267	93622	21532	55708
10		2492.	22897	62418	77759	13844	01439	98524	84898	96471
11		6071.	40637	40986	11507	96488	72848	58515	51555	35365
12		14959.	53266	63975	28852	29246	18760	57532	80969	8844
13		37197.	68849	06890	35604	39164	52887	63479	02829	0814
14		93192.	51363	39653	71298	82452	83639	24416	71789	4978
15	2	34955.	85249	07683	03578	24574	58951	61172	62841	294
16	5	95560.	99867	08370	01850	16100	68484	62610	31861	022
17	15	16637.	89404	25168	84432	79743	28762	46260	38122	26
18	38	77904.	33059	74435	02996	46607	99507	98034	54755	10
19	99	50907.	25104	68447	60026	00253	82530	63332	36043	40
20	256	15652.	66405	65888	20481	12080	40980	71829	38269	8

Similarly, the value, to 46 decimal places, of $-Ei(-5)$, calculated from the formula

$$-Ei(-5) = \gamma + \log 5 + \left|\sum_{k=1}^{\infty} (-1)^k \frac{5^k}{k \cdot k!}\right. \text{ is}$$

$$-Ei(-5) = 0.\ 00114\ 82955\ 91275\ 32579\ 73305\ 61969\ 81972\ 20762\ 66095\ 5\ .$$

From this result, and the relation

$$-Ei\{-(x+1)\} = -\,Ei(-x) - \frac{\exp(-x)}{x}\,g(-x)$$

we obtain the values of $-Ei(-x)$ as x assumes integral values from 6 to 21, inclusive. These are given, to 45 decimal places, in Table 22.

TABLE 22

x	$-Ei\,(-x)$								
6	0. 00036	00824	52162	65865	92953	94115	77179	71888	27389
7	0. 00011	54817	31610	33821	64310	11456	04378	94989	08183
8	0. 00003	76656	22843	92490	17725	57995	95075	27208	90488
9	0. 00001	24473	54178	00627	21211	48565	23810	08025	67128
10	0. 00000	41569	68929	68532	42774	02859	81027	81803	84346
11	0. 00000	14003	00304	24744	17754	46564	23790	42857	83898
12	0. 00000	04751	08182	46724	93932	59461	26966	61441	83574
13	0. 00000	01621	86621	88014	32870	66907	43670	74383	03063
14	0. 00000	00556	56311	11145	18211	49899	20850	65129	55930
15	0. 00000	00191	86278	92147	86697	71043	09696	74133	88159
16	0. 00000	00066	40487	24944	10427	85696	12935	96427	39491
17	0. 00000	00023	06431	98982	16544	94245	00440	70416	13170
18	0. 00000	00008	03609	03448	28677	65720	72130	12471	78184
19	0. 00000	00002	80782	90970	60795	26732	41525	64574	52825
20	0. 00000	00000	98355	25290	64988	16903	96987	10889	47761
21	0. 00000	00000	34532	01267	14675	62666	67883	78457	75270

APPENDIX A

$n = 5$

$C_5(6) = 0.67167\ 41597\ 23592\ 90621\ 95365\ 26431\ 71574\ 70413\ 39625$

d_2	−0.05597	28466	43632	74218	49613	77202	64297	89201	11635
d_3	−0.04933	63503	72929	69531	27820	69199	70633	56755	43152
d_4	$+.0_{2}849$	92457	40100	90950	51378	16033	40674	94144	47545
d_5	$+.0_{3}51$	13122	47084	20190	97410	26835	53597	79339	25099
d_6	$-.0_{3}30$	71057	49320	05330	70511	93061	41907	33078	90918
d_7	$+.0_{4}3$	16981	48781	81233	17277	65036	49948	71884	14771
d_8	$+.0_{5}$	17753	89746	82014	55199	34120	95672	21456	03906
d_9		$-.0_{6}8500$	23458	45136	10349	48851	92876	41546	89741
d_{10}		$+.0_{7}979$	13689	65403	50632	43425	77336	92541	10063
d_{11}		$-.0_{8}19$	56264	21346	95393	55839	48189	28543	39559
d_{12}		$-.0_{8}10$	61038	65702	59740	32349	88156	31646	71874
d_{13}		$+.0_{9}1$	88317	02689	46311	24923	56488	01388	51308
d_{14}		$-.0_{10}$	16512	88919	76694	11805	43406	99838	14228
d_{15}			$+.0_{12}476$	26024	30304	51559	47235	66626	06088
d_{16}			$+.0_{13}97$	59359	95022	14983	47238	25004	65864
d_{17}			$-.0_{13}19$	97801	79908	42071	51990	66340	20195
d_{18}			$+.0_{14}2$	24104	35772	41668	81820	39840	54421
d_{19}			$-.0_{15}$	17860	32140	30771	64217	33778	85609
d_{20}				$+.0_{17}960$	34790	77858	26985	90849	64768
d_{21}				$-.0_{18}10$	68759	32749	15678	57803	28649
d_{22}				$-.0_{19}5$	57506	40076	71194	96795	30781
d_{23}				$+.0_{20}$	96622	46481	42622	95560	14535
d_{24}				$-.0_{20}$	11561	18256	91730	09035	33358
d_{25}					$+.0_{21}1205$	63944	56392	66141	91907
d_{26}					$-.0_{22}119$	10130	49474	91310	97848
d_{27}					$+.0_{23}11$	67280	54505	89555	20692
d_{28}					$-.0_{24}1$	16705	02513	00158	80719
d_{29}					$+.0_{25}$	12071	61010	67901	67468
d_{30}						$-.0_{26}1296$	50926	64827	72088
d_{31}						$+.0_{27}144$	21326	82134	03200
d_{32}						$-.0_{28}16$	53178	15006	91287
d_{33}						$+.0_{29}1$	94345	32313	57162
d_{34}						$-.0_{30}$	23334	39957	55368
d_{35}							$+.0_{31}2852$	49648	77740
d_{36}							$-.0_{32}354$	18045	13729
d_{37}							$+.0_{33}44$	58558	45119
d_{38}							$-.0_{34}5$	68195	69104
d_{39}							$+.0_{35}$	73217	42653
d_{40}								$-.0_{36}9530$	34976
d_{41}								$+.0_{36}1252$	01855
d_{42}								$-.0_{37}165$	88258
d_{43}								$+.0_{38}22$	15136
d_{44}								$-.0_{39}2$	97964
d_{45}								$+.0_{40}$	40353
d_{46}									$-.0_{41}5500$
d_{47}									$+.0_{42}754$
d_{48}									$-.0_{42}104$
d_{49}									$+.0_{43}14$
d_{50}									$-.0_{44}2$

$C_4(5) = 0.67268\ 95170\ 64739\ 00786\ 17640\ 74824\ 64036\ 16676\ 57502$

$n = 6$

$C_6(7) = 0.67095\ 13224\ 50164\ 67304\ 61715\ 04486\ 00544\ 04599\ 27971$

d_2	−0.04792	50944	60726	04807	47265	36034	71467	43185	66284
d_3	−0.04305	47529	08502	28113	57403	29901	45574	53029	93687
d_4	$+.0_2632$	46197	56651	17469	57838	40204	96721	10795	55262
d_5	$+.0_350$	73221	10911	35949	57887	13402	33105	43138	50647
d_6	$-.0_321$	09816	74076	38028	98744	62076	58624	95868	76393
d_7	$+.0_41$	54809	78235	65147	43562	8699[illegible]	53441	72083	14443
d_8	$+.0_5$	18324	07614	76464	23103	65243	56866	65915	83487
d_9		$-.0_64784$	16494	50426	36958	60458	22355	15863	64799
d_{10}		$+.0_7353$	33440	51105	32993	19698	29204	71097	95710
d_{11}		$+.0_816$	24442	71939	91000	34590	58835	16946	54645
d_{12}		$-.0_96$	33360	83838	62309	48811	84242	75684	64248
d_{13}		$+.0_{10}$	65669	09166	19304	54408	25847	01002	95784
d_{14}			$-.0_{11}2248$	34033	89680	09780	56650	69768	91642
d_{15}			$-.0_{12}325$	64120	87274	12547	43168	92611	79169
d_{16}			$+.0_{13}63$	68712	91953	49982	07180	21954	87314
d_{17}			$-.0_{14}5$	82649	46083	88127	44300	12257	69898
d_{18}			$+.0_{15}$	28066	09456	13080	82904	14177	98420
d_{19}				$+.0_{17}582$	40420	10170	47007	71105	66904
d_{20}				$-.0_{17}279$	51267	41473	71257	50465	61211
d_{21}				$+.0_{18}34$	06700	19178	93728	85974	19438
d_{22}				$-.0_{19}2$	83048	28969	37643	17240	21084
d_{23}				$+.0_{20}$	17517	77752	46797	64554	72326
d_{24}					$-.0_{22}713$	45591	29397	03973	32395
d_{25}					$-.0_{24}2$	25620	35092	96395	39970
d_{26}					$+.0_{24}4$	23000	54981	98427	79295
d_{27}					$-.0_{25}$	56996	79335	65411	25512
d_{28}						$+.0_{26}5693$	43301	43202	42906
d_{29}						$-.0_{27}504$	52872	43567	76728
d_{30}						$+.0_{28}42$	56142	85334	58487
d_{31}						$-.0_{29}3$	55905	12979	12340
d_{32}						$+.0_{30}$	30254	09004	54572
d_{33}							$-.0_{31}2650$	32792	92788
d_{34}							$+.0_{32}240$	36441	85745
d_{35}							$-.0_{33}22$	53902	98051
d_{36}							$+.0_{34}2$	17659	37484
d_{37}							$-.0_{35}$	21551	48461
d_{38}								$+.0_{36}2179$	49457
d_{39}								$-.0_{37}224$	42970
d_{40}								$+.0_{38}23$	47594
d_{41}								$-.0_{39}2$	48992
d_{42}								$+.0_{40}$	26738
d_{43}									$-.0_{41}2904$
d_{44}									$+.0_{42}319$
d_{45}									$-.0_{43}35$
d_{46}									$+.0_{44}4$

$C_5(6) = 0.67167\ 41597\ 23592\ 90621\ 95365\ 26431\ 71574\ 70413\ 39625$

$n = 7$

$C_7(8) = 0.67041\ 06361\ 50233\ 20732\ 48806\ 52522\ 58909\ 47249\ 09018$

d_{2}	−0.04190	06647	59389	57545	78050	40782	66181	84203	06814
d_{3}	−0.03817	49446	03384	20204	51829	13268	11151	51316	41099
d_{4}	$+.0_{2}488$	82968	30298	19317	47923	05643	34363	63692	25941
d_{5}	$+.0_{3}46$	55439	32054	78573	36503	42267	36842	42413	68433
d_{6}	$-.0_{3}15$	03336	76886	91920	50634	17020	42053	66161	68085
d_{7}	$+.0_{5}$	77938	95165	47731	24416	09997	41347	84902	79287
d_{8}	$+.0_{5}$	14965	06417	63403	15308	14797	47697	16747	53329
d_{9}		$-.0_{6}2745$	27034	81318	83984	46227	46151	73873	37582
d_{10}		$+.0_{7}121$	77961	19605	83006	90015	62095	85601	41061
d_{11}		$+.0_{8}17$	35766	16877	96067	22114	44604	46793	85534
d_{12}		$-.0_{9}3$	25743	63596	49372	35763	27632	76045	14395
d_{13}		$+.0_{10}$	20895	74483	46119	24106	20067	19862	96031
d_{14}			$+.0_{12}483$	02636	70980	55556	98810	34980	59518
d_{15}			$-.0_{12}230$	48428	31165	39182	53361	76746	59411
d_{16}			$+.0_{13}23$	23623	34347	65876	41418	87626	70560
d_{17}			$-.0_{14}1$	03893	71458	80256	17715	73715	29923
d_{18}				$-.0_{16}3871$	04302	41677	23196	18989	35152
d_{19}				$+.0_{16}1141$	92426	98884	51547	67865	28702
d_{20}				$-.0_{17}111$	40948	81482	05351	31065	31939
d_{21}				$+.0_{19}6$	46586	60302	12234	99127	62560
d_{22}				$-.0_{20}$	13848	69232	17338	54264	85692
d_{23}					$-.0_{21}1858$	23571	70819	49485	33029
d_{24}					$+.0_{22}294$	72975	94511	39023	05965
d_{25}					$-.0_{23}26$	07639	25487	26935	34051
d_{26}					$+.0_{24}1$	71721	17993	60742	11756
d_{27}						$-.0_{26}8597$	73807	15983	14683
d_{28}						$+.0_{27}269$	72208	92726	79842
d_{29}						$+.0_{29}4$	50654	98791	52065
d_{30}						$-.0_{29}1$	66838	34823	67041
d_{31}						$+.0_{30}$	18364	90104	51045
d_{32}							$-.0_{31}1572$	16243	81310
d_{33}							$+.0_{32}121$	00112	49056
d_{34}							$-.0_{34}8$	90027	45726
d_{35}							$+.0_{35}$	64860	07520
d_{36}								$-.0_{36}4791$	92769
d_{37}								$+.0_{37}363$	68014
d_{38}								$-.0_{38}28$	50078
d_{39}								$+.0_{39}2$	30561
d_{40}								$-.0_{40}$	19193
d_{41}									$+.0_{41}1638$
d_{42}									$-.0_{42}143$
d_{43}									$+.0_{43}13$
d_{44}									$-.0_{44}1$

$C_6(7) = 0.67095\ 13224\ 50164\ 67304\ 61715\ 04486\ 00544\ 04599\ 27971$

$n = 8$

$C_8(9) = 0.66999\ 10087\ 76206\ 46897\ 98473\ 55147\ 74517\ 86251\ 13002$

d_{2}	−0.03722	17227	09789	24827	66581	86397	09695	43680	61833
d_{3}	−0.03427	98723	91867	46309	06179	12118	73355	89356	99123
d_{4}	$+.0_{2}389$	05927	74594	21215	41253	31187	59160	08659	76644
d_{5}	$+.0_{3}41$	59444	73188	68032	16470	35274	85260	34549	28723
d_{6}	$-.0_{3}11$	05613	91491	43729	18955	64954	84915	95951	96671
d_{7}	$+.0_{5}$	39273	63107	30068	93607	35784	98797	38748	61132
d_{8}	$+.0_{5}$	11537	47913	06017	31440	33534	16574	08676	55121
d_{9}		$-.0_{6}1624$	36375	45409	96853	45729	11498	64298	28421
d_{10}		$+.0_{8}34$	24216	28918	58224	89755	86521	26333	42230
d_{11}		$+.0_{8}12$	94891	03598	22369	74223	94406	92939	03092
d_{12}		$-.0_{9}1$	63592	75634	25039	74131	65972	19895	02559
d_{13}			$+.0_{11}5711$	29948	82718	86456	03395	37998	30151
d_{14}			$+.0_{12}709$	09415	07537	25795	26363	74808	86592
d_{15}			$-.0_{12}115$	84161	18431	41093	25722	13224	61055
d_{16}			$+.0_{14}7$	14256	96312	04101	41385	19608	05759
d_{17}				$+.0_{16}1019$	93316	59316	80153	98035	91954
d_{18}				$-.0_{16}4516$	02362	30941	27802	48680	36228
d_{19}				$+.0_{17}469$	40638	62559	21756	08644	50644
d_{20}				$-.0_{18}24$	45943	17542	68808	68419	80700
d_{21}				$+.0_{20}$	10466	79832	24626	54813	50071
d_{22}				$+.0_{20}$	11243	13338	71472	98572	40551
d_{23}					$-.0_{21}1245$	48663	20758	60982	63972
d_{24}					$+.0_{23}80$	56971	82898	03259	39031
d_{25}					$-.0_{24}3$	05860	71416	86298	85657
d_{26}						$-.0_{26}1754$	07681	52332	42725
d_{27}						$+.0_{26}1446$	36506	73444	20562
d_{28}						$-.0_{27}148$	00706	35042	30605
d_{29}						$+.0_{28}10$	33652	37960	69105
d_{30}						$-.0_{30}$	56204	49132	65828
d_{31}							$+.0_{31}2338$	64645	22960
d_{32}							$-.0_{33}56$	57482	18562
d_{33}							$-.0_{34}1$	77862	67774
d_{34}							$+.0_{35}$	37669	77304
d_{35}								$-.0_{36}3501$	30034
d_{36}								$+.0_{37}261$	96216
d_{37}								$-.0_{38}17$	80582
d_{38}								$+.0_{39}1$	16039
d_{39}									$-.0_{41}7490$
d_{40}									$+.0_{42}489$
d_{41}									$-.0_{43}33$
d_{42}									$+.0_{44}2$

$C_7(8) = 0.67041\ 06361\ 50233\ 20732\ 48806\ 52522\ 58909\ 47249\ 09018$

$n = 9$									
$C_9(10) = 0.66965\ 59030\ 74791\ 50274\ 69509\ 06002\ 39100\ 44134\ 63914$									
d_{2}	−0.03348	27951	53739	57513	73475	45300	11955	02206	73196
d_{3}	−0.03110	11469	89750	69499	08434	96979	99202	99852	88454
d_{4}	$+.0_{2}316$	96559	03074	79150	27469	50906	00239	10044	13464
d_{5}	$+.0_{3}36$	84504	67549	03057	95971	13867	11964	93193	52692
d_{6}	$-.0_{4}8$	35318	04013	66574	00122	08670	69334	39600	19615
d_{7}	$+.0_{5}$	18962	90807	61376	94353	73402	24343	44920	10929
d_{8}		$+.0_{6}8782$	22104	50461	69245	57435	68736	62814	49289
d_{9}		$-.0_{7}991$	34084	92945	22759	09254	30824	84971	51169
d_{10}		$+.0_{9}1$	39846	59860	45355	86258	53086	87019	29112
d_{11}		$+.0_{9}8$	88505	62675	82447	27696	99090	51043	44182
d_{12}		$-.0_{10}$	82611	73744	12102	29924	37859	02070	80959
d_{13}			$+.0_{12}791$	04017	33559	84856	88886	29029	27903
d_{14}			$+.0_{12}516$	63010	84827	31619	89159	55176	35844
d_{15}			$-.0_{13}53$	49241	12807	61516	90247	46676	65532
d_{16}			$+.0_{14}1$	78597	53795	54319	58528	45281	08420
d_{17}			$+.0_{15}$	14656	94423	99367	07963	48365	76064
d_{18}				$-.0_{16}2376$	47550	01908	66632	82041	66120
d_{19}				$+.0_{17}147$	99797	24394	67849	61728	33758
d_{20}				$-.0_{19}2$	17742	98807	95112	54953	98376
d_{21}				$-.0_{20}$	49737	79753	50312	37441	18411
d_{22}					$+.0_{21}5737$	43971	05234	87460	08568
d_{23}					$-.0_{22}332$	54728	73760	23855	13348
d_{24}					$+.0_{24}7$	96311	62463	54421	69994
d_{25}					$+.0_{25}$	56572	99898	54070	93734
d_{26}						$-.0_{26}8502$	44845	87331	51974
d_{27}						$+.0_{27}609$	22467	01542	77250
d_{28}						$-.0_{28}28$	38077	72694	01192
d_{29}						$+.0_{30}$	63943	82530	65366
d_{30}							$+.0_{31}3279$	02264	35019
d_{31}							$-.0_{32}523$	59517	61664
d_{32}							$+.0_{33}40$	47633	69302
d_{33}							$-.0_{34}2$	33832	60776
d_{34}							$+.0_{35}$	10790	71283
d_{35}								$-.0_{37}380$	14751
d_{36}								$+.0_{39}6$	98458
d_{37}								$+.0_{40}$	34784
d_{38}									$-.0_{41}5225$
d_{39}									$+.0_{42}420$
d_{40}									$-.0_{43}28$
d_{41}									$+.0_{44}2$
$C_8(9) = 0.66999\ 10087\ 76206\ 46897\ 98473\ 55147\ 74517\ 86251\ 13002$									

$n = 10$

$C_{10}(11) = 0.66938\ 21330\ 06108\ 66931\ 32580\ 83438\ 23344\ 00650\ 58853$

d_2	−0.03042	64605	91186	75769	60571	85610	82879	27302	29948
d_3	−0.02845	90023	88412	92377	59965	34205	40431	55921	07276
d_4	$+.0_2263$	18969	94464	21656	87283	36096	06913	04433	30722
d_5	$+.0_332$	60257	16555	56468	18378	76178	56595	98876	14262
d_6	$-.0_46$	45761	45109	72787	84532	98742	25604	43971	42455
d_7		$+.0_67978$	07196	14029	33621	02808	76584	81359	12214
d_8		$+.0_66703$	57894	73574	80104	38398	64653	53118	26784
d_9		$-.0_7622$	28993	47501	29034	91010	08018	31376	82086
d_{10}		$-.0_810$	02699	57691	48352	63811	89022	62461	15346
d_{11}		$+.0_95$	97156	93590	21591	41563	04944	16991	63930
d_{12}		$-.0_{10}$	42166	86907	58251	15858	95192	14579	14302
d_{13}			$-.0_{12}637$	44410	48409	63295	49389	07986	30715
d_{14}			$+.0_{12}327$	62105	47971	27524	02839	28690	91647
d_{15}			$-.0_{13}23$	93485	24989	01951	76305	21779	91832
d_{16}			$+.0_{15}$	17841	89357	18191	77509	88056	86283
d_{17}			$+.0_{15}$	11272	80723	39095	63294	90443	15568
d_{18}				$-.0_{16}1057$	97786	13549	58351	12755	50762
d_{19}				$+.0_{18}37$	18083	38300	46349	40378	73675
d_{20}				$+.0_{19}1$	59791	82662	91262	32934	36141
d_{21}				$-.0_{20}$	29930	38924	49660	58870	41543
d_{22}					$+.0_{21}1936$	96837	81618	22327	86927
d_{23}					$-.0_{23}50$	13009	91090	67321	51268
d_{24}					$-.0_{24}2$	96960	64342	92704	29196
d_{25}					$+.0_{25}$	44145	69219	34880	81644
d_{26}						$-.0_{26}2820$	56524	96780	82559
d_{27}						$+.0_{28}98$	27947	57513	20084
d_{28}						$+.0_{30}$	54227	07934	81949
d_{29}						$-.0_{30}$	35568	35673	00359
d_{30}							$+.0_{31}2961$	37959	34026
d_{31}							$-.0_{32}156$	22589	75720
d_{32}							$+.0_{34}5$	34552	05435
d_{33}								$-.0_{36}4085$	60877
d_{34}								$-.0_{36}1068$	78868
d_{35}								$+.0_{37}104$	99850
d_{36}								$-.0_{39}6$	58121
d_{37}								$+.0_{40}$	32414
d_{38}									$-.0_{41}1295$
d_{39}									$+.0_{43}39$
d_{40}									$-.0_{44}1$

$C_9(10) = 0.66965\ 59030\ 74791\ 50274\ 69509\ 06002\ 39100\ 44134\ 63910$

$n = 11$

$C_{11}(12) = 0.66915\ 42786\ 33243\ 76931\ 23142\ 24278\ 12072\ 08024\ 08638$

d_2	-0.02788	14282	76385	15705	46797	59344	92169	67001	00360
d_3	-0.02622	88095	40200	82460	80733	46703	05990	57388	83313
d_4	$+.0_{2}222$	01636	85353	90897	66437	45821	96044	61232	65631
d_5	$+.0_{3}28$	91359	13313	08647	83583	06056	92030	20207	63680
d_6	$-.0_{4}5$	09144	89609	99085	23393	78834	81336	05031	53945
d_7		$+.0_{6}1946$	55051	74831	70914	04368	47571	26309	30476
d_8		$+.0_{6}5161$	65669	24742	32572	86960	99461	84592	35840
d_9		$-.0_{7}400$	36855	60858	98439	78704	22622	83361	55715
d_{10}		$-.0_{8}12$	98616	40641	76221	78988	52465	46952	81953
d_{11}		$+.0_{9}4$	01689	54297	55004	98246	88994	52673	40722
d_{12}		$-.0_{10}$	21666	44837	71728	00706	43933	84947	60180
d_{13}			$-.0_{12}908$	28309	26245	31344	67703	77136	55247
d_{14}			$+.0_{12}199$	25076	53648	31596	35262	39807	87371
d_{15}			$-.0_{13}10$	45126	45693	50616	69033	16556	52044
d_{16}			$-.0_{15}$	22125	93584	61314	30207	10893	02118
d_{17}				$+.0_{16}6858$	53642	38880	61531	11229	63116
d_{18}				$-.0_{17}437$	35733	03702	11184	36018	56814
d_{19}				$+.0_{19}4$	44691	01876	12893	80425	89735
d_{20}				$+.0_{19}1$	47027	51533	56525	84199	69383
d_{21}				$-.0_{20}$	13433	49732	33108	77239	76101
d_{22}					$+.0_{22}511$	65124	41434	69082	70942
d_{23}					$+.0_{24}7$	88829	69280	96222	53679
d_{24}					$-.0_{24}2$	40653	49773	94729	86478
d_{25}					$+.0_{25}$	16622	84750	97924	31406
d_{26}						$-.0_{27}560$	63362	18119	80124
d_{27}						$-.0_{29}6$	31596	71070	39964
d_{28}						$+.0_{29}2$	17608	55169	70414
d_{29}						$-.0_{30}$	15693	80096	78539
d_{30}							$+.0_{32}659$	75465	65853
d_{31}							$-.0_{33}11$	01838	36820
d_{32}							$-.0_{35}$	82860	61522
d_{33}								$+.0_{36}9478$	22740
d_{34}								$-.0_{37}563$	53159
d_{35}								$+.0_{38}23$	05202
d_{36}								$-.0_{40}$	56317
d_{37}									$-.0_{42}626$
d_{38}									$+.0_{42}174$
d_{39}									$-.0_{43}13$
d_{40}									$+.0_{44}1$

$C_{10}(11) = 0.66938\ 21330\ 06108\ 66931\ 32580\ 83438\ 23344\ 00650\ 58853$

$n = 12$

$C_{12}(13) = 0.66896\ 16871\ 06122\ 82925\ 20774\ 85568\ 66422\ 65282\ 79016$

d_2	−0.02572	92956	57927	80112	50799	03291	10247	02510	87654
d_3	−0.02432	15745	81644	72814	74317	99831	22551	43460	98069
d_4	$+.0_{2}189$	79619	11593	49972	24495	25053	55344	25632	57343
d_5	$+.0_{3}25$	73804	14388	78045	01174	41532	56941	14475	85672
d_6	$-.0_{4}4$	08315	89532	53079	45261	11957	90257	05102	71612
d_7		$-.0_{6}1361$	63485	64390	86039	64503	13356	03119	33582
d_8		$+.0_{6}4017$	76287	80805	91803	25610	38305	28143	63526
d_9		$-.0_{7}263$	08092	45145	78191	33678	46094	75401	96364
d_{10}		$-.0_{8}12$	69257	35188	41446	77865	41718	86534	81510
d_{11}		$+.0_{9}2$	72731	92986	22326	28757	57085	89795	45535
d_{12}		$-.0_{10}$	11094	83254	23103	47682	48629	65392	40509
d_{13}			$-.0_{12}825$	99964	11509	37612	82683	60976	84375
d_{14}			$+.0_{12}119$	96059	27322	66684	88590	75044	45810
d_{15}			$-.0_{14}4$	37665	97800	04081	92603	00972	54138
d_{16}			$-.0_{15}$	26110	91155	39449	30767	04617	57855
d_{17}				$+.0_{16}3870$	77177	76630	18483	60067	21176
d_{18}				$-.0_{17}169$	62482	33603	69160	06053	52573
d_{19}				$-.0_{19}3$	30981	76403	90176	52838	47671
d_{20}				$+.0_{20}$	89427	44558	76086	59269	17147
d_{21}					$-.0_{21}5339$	07380	11470	89862	80202
d_{22}					$+.0_{23}79$	34651	83093	71426	04780
d_{23}					$+.0_{23}12$	01822	87569	93038	42726
d_{24}					$-.0_{24}1$	14027	49350	00677	27524
d_{25}						$+.0_{26}4722$	57528	70840	66516
d_{26}						$-.0_{28}11$	94345	76243	60756
d_{27}						$-.0_{28}12$	56992	98827	65490
d_{28}						$+.0_{30}$	96519	66056	56673
d_{29}							$-.0_{31}3834$	37004	65839
d_{30}							$+.0_{33}37$	63351	48340
d_{31}							$+.0_{34}6$	71306	35275
d_{32}							$-.0_{35}$	59071	75101
d_{33}								$+.0_{36}2841$	46788
d_{34}								$-.0_{38}78$	49929
d_{35}								$-.0_{40}$	37910
d_{36}								$+.0_{40}$	19609
d_{37}									$-.0_{41}1389$
d_{38}									$+.0_{43}64$
d_{39}									$-.0_{44}2$

$C_{11}(12) = 0.66915\ 42786\ 33243\ 76931\ 23142\ 24278\ 12072\ 08024\ 08638$

$n = 13$

$C_{13}(14) = 0.66879\ 67632\ 76254\ 16516\ 75869\ 82985\ 99144\ 33040\ 86388$

d_2	−0.02388	55986	88437	64875	59852	49392	35683	72608	60228
d_3	−0.02267	21143	48169	63577	35245	11933	69729	34637	68561
d_4	$+.0_2164$	11061	02374	04564	42242	64021	31158	42437	88677
d_5	$+.0_323$	01098	56266	76361	70946	77940	69215	65212	65912
d_6	$-.0_43$	32339	92663	18885	39249	72068	15205	19863	10931
d_7		$-.0_63133$	25513	13908	66688	24954	40591	67816	67350
d_8		$+.0_63163$	14921	92109	33982	74524	54458	45487	31986
d_9		$-.0_7175$	96776	68436	23850	58501	92024	41238	74512
d_{10}		$-.0_811$	28170	94115	59423	76771	48044	56245	27581
d_{11}		$+.0_91$	87522	63698	65052	52118	29042	01322	67210
d_{12}			$-.0_{11}5562$	96467	67596	15455	53401	29513	71498
d_{13}			$-.0_{12}663$	55527	94801	64102	48286	95863	50600
d_{14}			$+.0_{13}72$	39379	23979	68106	06026	18243	56782
d_{15}			$-.0_{14}1$	66646	57813	87568	48743	23616	88783
d_{16}			$-.0_{15}$	21159	28824	87297	22655	70374	95648
d_{17}				$+.0_{16}2122$	66886	60438	33660	64914	35375
d_{18}				$-.0_{18}59$	23048	60079	97815	77512	57562
d_{19}				$-.0_{19}3$	97188	01424	80725	47630	40599
d_{20}				$+.0_{20}$	48105	78882	68327	14223	17961
d_{21}					$-.0_{21}1921$	52300	09815	70533	44621
d_{22}					$-.0_{23}25$	17469	41760	38094	22341
d_{23}					$+.0_{24}7$	68747	28970	63380	76510
d_{24}					$-.0_{25}$	45130	11382	63153	76004
d_{25}						$+.0_{27}898$	21554	89292	31279
d_{26}						$+.0_{28}62$	29320	08326	49984
d_{27}						$-.0_{29}6$	66094	91285	13574
d_{28}						$+.0_{30}$	29987	86368	29634
d_{29}							$-.0_{32}427$	50066	56936
d_{30}							$-.0_{33}41$	88177	23282
d_{31}							$+.0_{34}3$	88007	79621
d_{32}							$-.0_{35}$	17500	14386
d_{33}								$+.0_{37}372$	28746
d_{34}								$+.0_{38}10$	95516
d_{35}								$-.0_{39}1$	51992
d_{36}									$+.0_{41}8381$
d_{37}									$-.0_{42}289$
d_{38}									$+.0_{44}4$

$C_{12}(13) = 0.66896\ 16871\ 06122\ 32925\ 20774\ 85568\ 66422\ 65282\ 79016$

$n = 14$

$C_{14}(15) = 0.66865\ 39464\ 12270\ 33310\ 40940\ 04574\ 12613\ 90972\ 13591$

d_2	−0.02228	84648	80409	01110	34698	00152	47087	13032	40453
d_3	−0.02123	16237	83092	93283	98457	86659	89018	34976	33758
d_4	$+.0_{2}143$	30556	03828	13016	03834	52668	86902	36966	02362
d_5	$+.0_{3}20$	66586	84903	73882	93108	26346	45885	45161	49657
d_6	$-.0_{4}2$	74038	78092	74249	22993	06493	34625	88475	26118
d_7		$-.0_{6}4022$	42060	45032	26239	52260	82267	90612	47552
d_8		$+.0_{6}2518$	29770	96578	95888	91435	99254	17689	68825
d_9		$-.0_{7}119$	43674	86871	10673	12438	71939	00377	07430
d_{10}		$-.0_{9}9$	62244	64764	92665	53863	25012	02095	30680
d_{11}		$+.0_{9}1$	30703	76572	85074	71824	67406	41947	45541
d_{12}			$-.0_{11}2641$	64875	20239	75756	71202	54774	03724
d_{13}			$-.0_{12}507$	71272	15601	01347	40579	36331	58466
d_{14}			$+.0_{13}44$	00911	49157	39491	77613	02033	73637
d_{15}			$-.0_{15}$	48184	39433	78895	72204	45742	84766
d_{16}			$-.0_{15}$	15325	60656	87733	56643	94232	87926
d_{17}				$+.0_{16}1150$	56509	58363	26582	38248	89771
d_{18}				$-.0_{18}15$	68148	17053	48871	32059	25327
d_{19}				$-.0_{19}3$	04666	11408	42269	81969	06084
d_{20}				$+.0_{20}$	24522	68112	71173	32631	57136
d_{21}					$-.0_{22}589$	80161	41527	60859	24561
d_{22}					$-.0_{23}36$	77832	49391	19256	32550
d_{23}					$+.0_{24}4$	05485	43849	66331	87944
d_{24}					$-.0_{25}$	15689	81275	41962	15806
d_{25}						$-.0_{28}77$	14648	63891	30690
d_{26}						$+.0_{28}45$	17557	67023	70469
d_{27}						$-.0_{29}2$	70967	53280	80251
d_{28}							$+.0_{31}6663$	25170	37610
d_{29}							$+.0_{32}194$	01490	82821
d_{30}							$-.0_{33}27$	31040	89865
d_{31}							$+.0_{34}1$	34472	55082
d_{32}								$-.0_{36}2995$	01703
d_{33}								$-.0_{38}78$	04446
d_{34}								$+.0_{38}10$	92252
d_{35}								$-.0_{40}$	55871
d_{36}									$+.0_{41}1599$
d_{37}									$-.0_{44}3$
d_{38}									$-.0_{44}3$

$C_{13}(14) = 0.66879\ 67632\ 76254\ 16516\ 75869\ 82985\ 99144\ 33040\ 86388$

$n = 15$

$C_{15}(16) = 0.66852\ 90719\ 66716\ 05797\ 90869\ 97889\ 23880\ 02248\ 96284$

d_2	-0.02089	15334	98959	87681	18464	68684	03871	25070	28009
d_3	-0.01996	28527	70876	67179	95063	97138	16505	36455	40500
d_4	$+.0_2126$	21889	34556	09206	57869	63439	03959	17725	57023
d_5	$+.0_318$	64262	12908	15379	42044	81792	27508	35819	41405
d_6	$-.0_42$	28574	99990	59230	24667	64295	83765	63508	56917
d_7		$-.0_64400$	10830	04107	12446	77643	01115	30846	14285
d_8		$+.0_62026$	37310	94437	34467	14904	66340	41245	55913
d_9		$-.0_882$	01997	62162	44189	51677	73816	72216	09952
d_{10}		$-.0_98$	05120	82718	59604	75936	28137	43695	62915
d_{11}		$+.0_{10}$	92347	76075	84319	52903	63949	94938	47949
d_{12}			$-.0_{11}1097$	41948	51916	19822	93621	41669	93565
d_{13}			$-.0_{12}380$	66695	64237	14014	55963	90840	86179
d_{14}			$+.0_{13}26$	99147	28513	38848	26880	14386	61223
d_{15}				$+.0_{16}1160$	97354	37375	57813	84122	61788
d_{16}				$-.0_{15}10611$	56987	61326	10134	21000	88235
d_{17}				$+.0_{17}619$	94170	76264	12515	86587	83640
d_{18}					$+.0_{20}25194$	73836	23601	96003	49883
d_{19}					$-.0_{19}2\ 05419$	98701	69895	70771	87768
d_{20}					$+.0_{20}12118$	07817	17511	66402	06930
d_{21}					$-.0_{22}109$	94516	79108	25468	06401
d_{22}					$-.0_{23}27$	86712	96750	67703	33160
d_{23}					$+.0_{24}1$	96473	37520	71507	99282
d_{24}						$-.0_{26}4510$	87151	62960	88673
d_{25}						$-.0_{27}220$	53114	70401	11678
d_{26}						$+.0_{28}24$	09651	48853	33848
d_{27}						$-.0_{30}$	93976	44443	94834
d_{28}							$+.0_{32}285$	07256	99390
d_{29}							$+.0_{32}185$	33278	55198
d_{30}							$-.0_{33}11$	79109	03125
d_{31}							$+.0_{35}$	33951	59755
d_{32}								$+.0_{37}247$	28419
d_{33}								$-.0_{38}79$	28919
d_{34}								$+.0_{39}4$	35526
d_{35}								$-.0_{40}$	12284
d_{36}									$-.0_{43}10$
d_{37}									$+.0_{43}21$
d_{38}									$-.0_{44}1$

$C_{14}(15) = 0.66865\ 39464\ 12270\ 33310\ 40940\ 04574\ 12613\ 90972\ 13591$

$n = 16$

$C_{16}(17) = 0.66841\ 89593\ 70525\ 45706\ 35900\ 03013\ 07512\ 75110\ 46446$

d_2	−0.01965	93811	57956	63109	01055	88323	91397	43385	60189
d_3	−0.01883	68870	13413	46544	74468	39673	57200	10063	30973
d_4	$+.0_{2}112$	01476	79385	25040	34183	25108	00926	43729	05193
d_5	$+.0_{3}16$	88976	03480	85251	56914	53402	84158	75707	61296
d_6	$-.0_{4}1$	92611	34282	25012	72732	90118	84526	66884	97173
d_7		$-.0_{6}4481$	57964	59959	45550	56409	15773	09868	88893
d_8		$+.0_{6}1646$	92941	42975	94791	07977	81580	42911	52349
d_9		$-.0_{8}56$	82258	60678	74593	32074	59718	10675	96927
d_{10}		$-.0_{9}6$	67956	55275	68961	47760	62612	45569	57529
d_{11}		$+.0_{10}$	66106	01141	36706	99517	00993	81103	59210
d_{12}			$-.0_{12}290$	24300	38896	15426	11021	17385	14675
d_{13}			$-.0_{12}283$	36242	24750	91920	37867	60056	47887
d_{14}			$+.0_{13}16$	69728	77985	95841	97913	02513	10661
d_{15}			$+.0_{15}$	19451	13462	53922	08890	53038	71759
d_{16}				$-.0_{16}7211$	38161	48546	59306	14331	22747
d_{17}				$+.0_{17}331$	94107	68591	08442	93343	46340
d_{18}				$+.0_{19}5$	12543	56478	75214	95514	68088
d_{19}				$-.0_{19}1$	31330	93452	65827	59172	16012
d_{20}					$+.0_{21}5831$	60056	24045	61555	16577
d_{21}					$+.0_{23}41$	17345	45616	00918	96035
d_{22}					$-.0_{23}17$	90439	33374	28291	05169
d_{23}					$+.0_{25}$	90210	53679	33047	27411
d_{24}						$-.0_{27}697$	06620	71156	36336
d_{25}						$-.0_{27}172$	89634	78177	16365
d_{26}						$+.0_{28}11$	35627	80600	87207
d_{27}						$-.0_{30}$	26659	45136	04686
d_{28}							$-.0_{32}873$	57693	12522
d_{29}							$+.0_{32}103$	69088	32364
d_{30}							$-.0_{34}4$	18325	23188
d_{31}								$+.0_{36}4137$	89114
d_{32}								$+.0_{37}533$	18126
d_{33}								$-.0_{38}37$	78911
d_{34}								$+.0_{39}1$	23505
d_{35}									$-.0_{42}706$
d_{36}									$-.0_{42}161$
d_{37}									$+.0_{43}10$

$C_{15}(16) = 0.66852\ 90719\ 66716\ 05797\ 90869\ 97889\ 23880\ 02248\ 96284$

$n = 17$

$C_{17}(18) = 0.66832\ 11376\ 92053\ 06079\ 09122\ 85663\ 76541\ 04682\ 94792$

d_2	−0.01856	44760	47001	47391	08586	74601	77126	14018	97078
d_3	−0.01783	09453	31592	53800	33015	30570	30476	80962	26034
d_4	$+.0_{2}100$	07960	00580	26511	00106	00921	00952	17457	02433
d_5	$+.0_{3}15$	36417	92547	46086	18139	90298	73629	64568	15737
d_6	$-.0_{4}1$	63796	75586	27379	09174	12522	35825	00373	12788
d_7		$-.0_{6}4393$	94754	20490	56849	96152	09989	52081	97929
d_8		$+.0_{6}1351$	07214	34519	53521	69344	35456	60867	68738
d_9		$-.0_{8}39$	59647	90467	07230	39250	63417	67499	13284
d_{10}		$-.0_{9}5$	52613	24001	75491	37867	15914	98629	86384
d_{11}		$+.0_{10}$	47907	98133	76071	43524	85972	56332	31198
d_{12}			$+.0_{12}118$	63632	08512	52588	39676	92958	21496
d_{13}			$-.0_{12}210$	81887	68710	34934	12658	52871	07219
d_{14}			$+.0_{13}10$	40479	79304	45244	26686	04739	54652
d_{15}			$+.0_{15}$	24130	26142	40005	60885	38468	58304
d_{16}				$-.0_{16}4869$	56150	83490	72360	99589	47324
d_{17}				$+.0_{17}175$	76053	17306	68617	28937	78754
d_{18}				$+.0_{19}5$	80750	76201	47240	33048	29347
d_{19}				$-.0_{20}$	81957	79792	20230	82818	14933
d_{20}					$+.0_{21}2712$	35388	47325	40390	26818
d_{21}					$+.0_{23}73$	30878	36702	97023	84335
d_{22}					$-.0_{23}10$	73696	55096	18125	98732
d_{23}					$+.0_{25}$	39348	90278	77540	45381
d_{24}						$+.0_{27}390$	44395	10265	49896
d_{25}						$-.0_{27}108$	26568	35830	91651
d_{26}						$+.0_{29}4$	94914	13216	89618
d_{27}							$-.0_{31}4199$	99810	30532
d_{28}							$-.0_{32}756$	97258	60844
d_{29}							$+.0_{33}48$	64986	68839
d_{30}							$-.0_{34}1$	21087	69510
d_{31}								$-.0_{36}2208$	52300
d_{32}								$+.0_{37}329$	08317
d_{33}								$-.0_{38}14$	01033
d_{34}								$+.0_{40}$	21774
d_{35}									$+.0_{41}1049$
d_{36}									$-.0_{43}90$
d_{37}									$+.0_{44}3$

$C_{16}(17) = 0.66841\ 89593\ 70525\ 45706\ 35900\ 03013\ 07512\ 75110\ 46446$

$n = 18$

$C_{18}(19) = 0.66823\ 36582\ 16749\ 55546\ 74074\ 39908\ 71627\ 59446\ 40700$

d_2	-0.01758	50962	68861	83040	70370	37892	33463	88406	48439
d_3	-0.01692	68387	27408	35682	78232	61828	33913	54792	75493
d_4	$+.0_3 89$	95475	32251	14343	27698	26623	38621	11220	85196
d_5	$+.0_3 14$	03015	64193	72403	25973	04792	99783	46420	09839
d_6	$-.0_4 1$	40443	45203	68213	68048	80268	31908	23187	02933
d_7		$-.0_6 4213$	19496	02730	23441	20475	06077	69979	68363
d_8		$+.0_6 1117$	99879	44653	45606	16602	59042	44822	66325
d_9		$-.0_8 27$	66547	01488	28897	12352	89861	18061	99779
d_{10}		$-.0_9 4$	57373	45427	67818	58975	92896	27275	07728
d_{11}		$+.0_{10}$	35120	96439	06732	45464	65161	83688	09938
d_{12}			$+.0_{12} 311$	59142	97341	05959	93316	29853	97361
d_{13}			$-.0_{12} 157$	32818	44027	63226	65526	14339	82098
d_{14}			$+.0_{14} 6$	51757	50657	96300	70144	82543	47255
d_{15}			$+.0_{15}$	23186	71350	23210	58538	59153	44283
d_{16}				$-.0_{16} 3288$	02042	47185	72198	10427	12209
d_{17}				$+.0_{18} 91$	08858	60469	84819	29125	94585
d_{18}				$+.0_{19} 5$	08629	95962	56813	65541	77205
d_{19}				$-.0_{20}$	50593	34592	23289	37697	72256
d_{20}					$+.0_{21} 1191$	16740	23646	53817	67094
d_{21}					$+.0_{23} 67$	09272	65038	49276	55214
d_{22}					$-.0_{24} 6$	22036	99974	77245	51499
d_{23}					$+.0_{25}$	15962	33717	17734	83931
d_{24}						$+.0_{27} 558$	99834	38529	26362
d_{25}						$-.0_{28} 61$	84904	72089	34140
d_{26}						$+.0_{29} 1$	99843	69157	29610
d_{27}							$+.0_{31} 1927$	81430	79852
d_{28}							$-.0_{32} 473$	48623	66326
d_{29}							$+.0_{33} 20$	56225	10304
d_{30}							$-.0_{35}$	21547	20057
d_{31}								$-.0_{36} 2393$	56381
d_{32}								$+.0_{37} 157$	47973
d_{33}								$-.0_{39} 4$	21976
d_{34}									$-.0_{41} 2822$
d_{35}									$+.0_{42} 779$
d_{36}									$-.0_{43} 36$
d_{37}									$+.0_{44} 1$

$C_{17}(18) = 0.66832\ 11376\ 92053\ 06079\ 09122\ 85663\ 76541\ 04682\ 94792$

$n = 19$

$C_{19}(20) = 0.66815\ 49634\ 35226\ 78819\ 83897\ 04143\ 02536\ 96920\ 53972$

d_{2}	−0.01670	38740	85880	66970	49597	42603	57563	42423	01349
d_{3}	−0.01610	98708	63803	97767	65013	41913	21414	55252	56622
d_{4}	$+.0_{3}81$	29185	83466	15753	41807	97104	29022	58852	25890
d_{5}	$+.0_{3}12$	85819	65299	39347	53977	81534	96053	24198	43531
d_{6}	$-.0_{4}1$	21319	03416	35937	42597	47539	82577	40665	37030
d_{7}		$-.0_{6}3985$	03891	43740	87845	66402	11432	77715	75867
d_{8}		$+.0_{7}932$	58941	60263	27234	48202	21641	29279	22301
d_{9}		$-.0_{8}19$	30931	34102	02944	61217	86453	87536	21125
d_{10}		$-.0_{9}3$	79402	79766	72503	66486	20717	78207	26661
d_{11}		$+.0_{10}$	26022	54235	31490	82391	27243	78043	67672
d_{12}			$+.0_{12}388$	14513	24275	43584	09254	31748	86176
d_{13}			$-.0_{12}118$	00109	20856	90790	00147	29073	19238
d_{14}			$+.0_{14}4$	09238	95167	37273	87830	93022	15228
d_{15}			$+.0_{15}$	20235	87961	71229	85235	04755	87687
d_{16}				$-.0_{16}2227$	42858	10330	38019	86451	12595
d_{17}				$+.0_{18}45$	30287	55276	63597	30230	77100
d_{18}				$+.0_{19}4$	04799	91585	07657	96034	80005
d_{19}				$-.0_{20}$	31096	54220	54740	63312	78203
d_{20}					$+.0_{22}465$	08596	51331	03517	27015
d_{21}					$+.0_{23}51$	89243	54828	85697	54138
d_{22}					$-.0_{24}3$	53369	79778	94779	92191
d_{23}						$+.0_{26}5619$	33044	14781	43639
d_{24}						$+.0_{27}466$	92749	50739	18101
d_{25}						$-.0_{28}33$	65118	06465	04356
d_{26}						$+.0_{30}$	71990	77328	62867
d_{27}							$+.0_{31}2765$	47770	22352
d_{28}							$-.0_{32}261$	89048	43690
d_{29}							$+.0_{34}7$	87492	38967
d_{30}								$+.0_{36}5586$	28189
d_{31}								$-.0_{36}1540$	45298
d_{32}								$+.0_{38}65$	88713
d_{33}								$-.0_{40}$	86051
d_{34}									$-.0_{41}5513$
d_{35}									$+.0_{42}391$
d_{36}									$-.0_{43}11$

$C_{20}(21) = 0.66808\ 37935\ 81999\ 53426\ 39404\ 80412\ 44492\ 86773\ 05041$

$C_{18}(19) = 0.66823\ 36582\ 16749\ 55546\ 74074\ 39908\ 71627\ 59446\ 40700$

APPENDIX B

$n = 4$

$\Gamma_4(5) = 0.52372\ 08704\ 07838\ 77766\ 75769\ 75593\ 86660\ 48727\ 78580\ 43665$

δ_1	+0.04744	17408	15677	55533	51539	51187	73320	97455	57160	87330
δ_2	$-.0_{2}439$	03462	25106	32243	16037	46371	65345	07417	20697	17037
δ_3	$+.0_{3}53$	32647	34321	79839	21549	32014	30037	35874	52772	23402
δ_4	$-.0_{4}5$	98746	54242	68668	43259	61113	57754	17814	77564	57661
δ_5	$+.0_{5}$	69606	72354	62713	14479	66613	31340	49159	43495	39097
δ_6		$-.0_{6}8357$	09748	98503	42362	04268	23368	39067	25336	25406
δ_7		$+.0_{6}1033$	66667	39105	69858	04272	58224	76939	72632	86785
δ_8		$-.0_{7}131$	40243	67029	65819	69786	26217	35206	20185	94126
δ_9		$+.0_{8}17$	13026	22334	36404	85882	22350	89033	94050	24412
δ_{10}		$-.0_{9}2$	28544	34893	90588	29678	08077	32923	44522	71394
δ_{11}		$+.0_{10}$	31145	93133	35207	36106	94951	83436	98073	64080
δ_{12}			$-.0_{11}4328$	17133	78763	26096	41717	15272	67376	60591
δ_{13}			$+.0_{12}612$	34267	70657	44435	38129	01753	57428	10542
δ_{14}			$-.0_{13}88$	07427	67010	50848	60801	48864	76280	87032
δ_{15}			$+.0_{13}12$	86186	37849	32637	73084	46629	50167	35449
δ_{16}			$-.0_{14}1$	90479	49453	21425	46577	79775	15338	97053
δ_{17}			$+.0_{15}$	28577	21583	16014	00618	25003	29908	24915
δ_{18}				$-.0_{16}4339$	11117	05928	03898	95053	31385	51905
δ_{19}				$+.0_{17}666$	21163	36457	24545	36688	73484	13054
δ_{20}				$-.0_{17}103$	35015	87340	43248	07252	51927	42694
δ_{21}				$+.0_{18}16$	18774	49617	72924	05802	03740	55619
δ_{22}				$-.0_{19}2$	55832	13937	59503	75191	59027	94132
δ_{23}				$+.0_{20}$	40771	82755	38147	55722	61844	13784
δ_{24}					$-.0_{21}6548$	88247	97961	84996	54691	68111
δ_{25}					$+.0_{21}1059$	64945	81676	90765	39420	22139
δ_{26}					$-.0_{22}172$	64326	42485	50357	51892	26925
δ_{27}					$+.0_{23}28$	31067	91566	25899	89405	15948
δ_{28}					$-.0_{24}4$	67089	14993	65647	54084	14272
δ_{29}					$+.0_{25}$	77508	08699	66948	65951	17054
δ_{30}					$-.0_{25}$	12931	61868	58251	14714	37589
δ_{31}						$+.0_{26}2168$	64813	36335	30066	05605
δ_{32}						$-.0_{27}365$	45768	43258	07788	13471
δ_{33}						$+.0_{28}61$	87101	32654	62432	75769
δ_{34}						$-.0_{28}10$	52053	52319	49669	06801
δ_{35}						$+.0_{29}1$	79636	49076	12882	80223
δ_{36}						$-.0_{30}$	30794	25440	12620	77291
δ_{37}							$+.0_{31}5298$	84921	72654	17783
δ_{38}							$-.0_{32}915$	06938	56180	43986
δ_{39}							$+.0_{32}158$	56816	41772	85381
δ_{40}							$-.0_{33}27$	56773	07337	96600
δ_{41}							$+.0_{34}4$	80780	52984	93577
δ_{42}							$-.0_{35}$	84099	85475	57607
δ_{43}							$+.0_{35}$	14753	37619	55056
δ_{44}								$-.0_{36}2595$	27849	63975
δ_{45}								$+.0_{37}457$	74597	81023
δ_{46}								$-.0_{38}80$	94081	62173
δ_{47}								$+.0_{38}14$	34729	94125
δ_{48}								$-.0_{39}2$	54912	87270
δ_{49}								$+.0_{40}$	45393	54736
δ_{50}									$-.0_{41}8101$	04488
δ_{51}									$+.0_{41}1448$	76605
δ_{52}									$-.0_{42}259$	61713
δ_{53}									$+.0_{43}46$	61391
δ_{54}									$-.0_{44}8$	38524
δ_{55}									$+.0_{44}1$	51114
δ_{56}									$-.0_{45}$	27281
δ_{57}										$+.0_{46}4933$
δ_{58}										$-.0_{47}894$
δ_{59}										$+.0_{47}162$
δ_{60}										$-.0_{48}29$
δ_{61}										$+.0_{49}5$
δ_{62}										$-.0_{49}1$

$\Gamma_3(4) = 0.52925\ 22708\ 88284\ 45431\ 27142\ 87920\ 73349\ 78796\ 46897\ 15465$

$n = 5$

$\Gamma_5(6) = 0.51994\ 57538\ 64775\ 68821\ 34346\ 32507\ 84104\ 31536\ 81478\ 87906$

δ_1	$+0.03989$	15077	29551	37642	68692	65015	68208	63073	62957	75812
δ_2	$-.0_2343$	73050	92513	26425	75836	21026	63800	06221	10498	81514
δ_3	$+.0_330$	65809	33578	81854	72796	14152	73900	44492	36553	86704
δ_4	$-.0_42$	82531	95429	32905	55027	95568	83279	00241	24229	75049
δ_5	$+.0_5$	26851	79004	80620	34419	08320	06922	28085	41423	86210
δ_6		$-.0_62626$	92844	32460	08724	84370	23189	52878	98173	96433
δ_7		$+.0_7264$	05284	25853	80525	47669	01613	93114	56009	04943
δ_8		$-.0_827$	22217	14649	81377	03042	19064	99735	54544	34827
δ_9		$+.0_92$	87341	03097	30648	46953	70839	88780	96996	88253
δ_{10}		$-.0_{10}$	31003	23422	63157	19369	68442	42223	21059	22834
δ_{11}			$+.0_{11}3414$	16003	81883	84972	94453	86429	31134	52160
δ_{12}			$-.0_{12}383$	18158	59462	12977	73249	84108	24860	06537
δ_{13}			$+.0_{13}43$	77128	25408	76730	42236	58800	37578	64771
δ_{14}			$-.0_{14}5$	08277	22438	93925	09708	17177	48362	36563
δ_{15}			$+.0_{15}$	59929	80780	96273	11796	14368	39270	03754
δ_{16}				$-.0_{16}7167$	36091	47736	92136	42294	61105	96332
δ_{17}				$+.0_{17}868$	62011	24384	51768	05721	04742	09697
δ_{18}				$-.0_{17}106$	57834	70089	43990	52508	33192	74489
δ_{19}				$+.0_{18}13$	22886	13551	93997	46673	43016	65409
δ_{20}				$-.0_{19}1$	65983	64707	94183	10660	18619	24436
δ_{21}				$+.0_{20}$	21037	74057	26567	16126	62634	68737
δ_{22}					$-.0_{21}2691$	84327	44949	14816	66532	81387
δ_{23}					$+.0_{22}347$	50850	23014	91770	24115	67265
δ_{24}					$-.0_{23}45$	23914	44431	34196	45359	75842
δ_{25}					$+.0_{24}5$	93585	49041	27347	51789	55182
δ_{26}					$-.0_{25}$	78464	90910	05062	27042	46110
δ_{27}					$+.0_{25}$	10445	02603	59248	26730	76548
δ_{28}						$-.0_{26}1399$	64464	07105	87071	45204
δ_{29}						$+.0_{27}188$	73184	07315	75976	28735
δ_{30}						$-.0_{28}25$	60047	74063	74325	93521
δ_{31}						$+.0_{29}3$	49215	28712	16741	09205
δ_{32}						$-.0_{30}$	47891	34477	05689	61815
δ_{33}							$+.0_{31}6601$	22314	41063	30028
δ_{34}							$-.0_{32}914$	29917	05725	58296
δ_{35}							$+.0_{32}127$	21811	85557	26726
δ_{36}							$-.0_{33}17$	77924	02655	29042
δ_{37}							$+.0_{34}2$	49513	46167	94702
δ_{38}							$-.0_{35}$	35156	84459	88581
δ_{39}								$+.0_{36}4972$	61291	13238
δ_{40}								$-.0_{37}705$	90580	50192
δ_{41}								$+.0_{37}100$	56087	58206
δ_{42}								$-.0_{38}14$	37369	52532
δ_{43}								$+.0_{39}2$	06113	07497
δ_{44}								$-.0_{40}$	29647	25322
δ_{45}									$+.0_{41}4277$	13029
δ_{46}									$-.0_{42}618$	81360
δ_{47}									$+.0_{43}89$	77586
δ_{48}									$-.0_{43}13$	05892
δ_{49}									$+.0_{44}1$	90441
δ_{50}									$-.0_{45}$	27841
δ_{51}										$+.0_{46}4080$
δ_{52}										$-.0_{47}599$
δ_{53}										$+.0_{48}88$
δ_{54}										$-.0_{48}13$
δ_{55}										$+.0_{49}2$

$\Gamma_4(5) = 0.52372\ 08704\ 07838\ 77766\ 75769\ 75593\ 86660\ 48727\ 78580\ 43665$

$n = 6$

$\Gamma_6(7) = 0.51720\ 58142\ 19967\ 97031\ 31695\ 21499\ 10216\ 53040\ 55595\ 79910$

δ_1	$+0.03441$	16284	39935	94062	63390	42998	20433	06081	11191	59820
δ_2	$-.0_{2}253$	16440	04347	48582	46016	37108	87439	54850	35636	67316
δ_3	$+.0_{3}19$	19952	56560	29098	71961	61794	84340	87994	13502	45335
δ_4	$-.0_{4}1$	49891	49363	72446	30515	66334	48560	35246	95968	20308
δ_5	$+.0_{5}$	12029	64654	94418	16194	42812	85678	21014	98680	58350
δ_6		$-.0_{7}991$	06368	31492	44875	37643	30415	63002	66853	40361
δ_7		$+.0_{8}83$	69667	51887	31779	41666	66170	47285	58445	98683
δ_8		$-.0_{9}7$	23548	13719	30757	49035	29771	82714	34959	49100
δ_9		$+.0_{10}$	63942	51897	95987	84880	23611	73825	38868	08001
δ_{10}			$-.0_{11}5769$	07917	56440	26066	20167	33051	24865	98701
δ_{11}			$+.0_{12}530$	72989	97015	93254	74583	66774	29083	17055
δ_{12}			$-.0_{13}49$	72487	47207	05313	11862	09913	43304	95804
δ_{13}			$+.0_{14}4$	73934	23105	51457	45613	84032	38928	27752
δ_{14}			$-.0_{15}$	45903	60450	01569	95573	96182	45963	02735
δ_{15}				$+.0_{16}4513$	65934	33823	40434	41752	68942	17407
δ_{16}				$-.0_{17}450$	15414	14601	72642	93195	84954	51073
δ_{17}				$+.0_{18}45$	49552	62714	51140	50656	67553	37139
δ_{18}				$-.0_{19}4$	65587	87323	37508	44802	90377	89385
δ_{19}				$+.0_{20}$	48209	80541	41362	21277	20519	28531
δ_{20}					$-.0_{21}5047$	40643	07459	42508	49235	53086
δ_{21}					$+.0_{22}533$	97444	89701	48818	55890	69708
δ_{22}					$-.0_{23}57$	04693	22956	94702	84678	37929
δ_{23}					$+.0_{24}6$	15124	16604	70226	34368	43311
δ_{24}					$-.0_{25}$	66909	58764	28135	35678	53736
δ_{25}						$+.0_{26}7338$	40024	27151	88520	87890
δ_{26}						$-.0_{27}811$	16478	19378	05546	19893
δ_{27}						$+.0_{28}90$	33004	03490	41561	24479
δ_{28}						$-.0_{28}10$	12987	40126	29315	52235
δ_{29}						$+.0_{29}1$	14358	75690	92866	88945
δ_{30}						$-.0_{30}$	12992	23216	62011	04221
δ_{31}							$+.0_{31}1484$	95148	18825	08555
δ_{32}							$-.0_{32}170$	69824	71348	38168
δ_{33}							$+.0_{33}19$	72952	35078	33747
δ_{34}							$-.0_{34}2$	29226	55200	99493
δ_{35}							$+.0_{35}$	26765	23833	05402
δ_{36}								$-.0_{36}3140$	06570	21877
δ_{37}								$+.0_{37}370$	06441	61339
δ_{38}								$-.0_{38}43$	80280	92980
δ_{39}								$+.0_{39}5$	20634	37336
δ_{40}								$-.0_{40}$	62129	07237
δ_{41}									$+.0_{41}7442$	47476
δ_{42}									$-.0_{42}894$	81596
δ_{43}									$+.0_{42}107$	96452
δ_{44}									$-.0_{43}13$	07074
δ_{45}									$+.0_{44}1$	58758
δ_{46}									$-.0_{45}$	19343
δ_{47}										$+.0_{46}2364$
δ_{48}										$-.0_{47}290$
δ_{49}										$+.0_{48}36$
δ_{50}										$-.0_{49}4$

$\Gamma_5(6) = 0.51994\ 57538\ 64775\ 68821\ 34346\ 32507\ 84104\ 31536\ 81478\ 87906$

$n = 7$

$\Gamma_7(8) = 0.51512\ 69527\ 70493\ 97856\ 08002\ 29281\ 45108\ 44467\ 87318\ 40198$

δ_1	+0.03025	39055	40987	95712	16004	58562	90216	88935	74636	80396
δ_2	$-.0_2194$	15290	07167	91653	84495	55767	18852	38843	49570	59616
δ_3	$+.0_312$	80208	10026	54689	93044	44909	26095	14380	28360	35241
δ_4	$-.0_5$	86643	27400	71271	39841	17873	33737	98496	86902	68796
δ_5		$+.0_66012$	22030	44985	82873	75760	73030	98310	44638	20242
δ_6		$-.0_7427$	26772	20342	23504	78843	86049	94105	87414	21794
δ_7		$+.0_831$	06326	93599	34782	60487	89509	49236	63758	85755
δ_8		$-.0_92$	30778	59030	43913	45850	82507	25796	50212	25781
δ_9		$+.0_{10}$	17501	36407	71992	70745	57353	49206	45306	39993
δ_{10}			$-.0_{11}1353$	36302	20499	55632	89762	91016	89163	34323
δ_{11}			$+.0_{12}106$	60438	57374	94738	04758	81853	46708	25387
δ_{12}			$-.0_{14}8$	54521	97225	25853	56937	17518	22454	39660
δ_{13}			$+.0_{15}$	69637	98735	32608	88240	48190	19777	79488
δ_{14}				$-.0_{16}5764$	35723	38643	09751	92615	60385	00904
δ_{15}				$+.0_{17}484$	24394	07127	68906	13857	99158	39814
δ_{16}				$-.0_{18}41$	25088	51027	46412	85771	54384	58290
δ_{17}				$+.0_{19}3$	56061	72111	23300	78043	07346	75293
δ_{18}				$-.0_{20}$	31119	09882	90761	90026	49039	80094
δ_{19}					$+.0_{21}2751$	97314	98058	05645	36877	80497
δ_{20}					$-.0_{22}246$	09386	42405	85043	51622	95760
δ_{21}					$+.0_{23}22$	24017	02783	81820	35377	20021
δ_{22}					$-.0_{24}2$	03008	36115	48262	18911	98272
δ_{23}					$+.0_{25}$	18706	88692	80822	79069	83205
δ_{24}						$-.0_{26}1739$	35713	36010	52981	25420
δ_{25}						$+.0_{27}163$	10871	99844	53514	59933
δ_{26}						$-.0_{28}15$	41988	25750	24685	63773
δ_{27}						$+.0_{29}1$	46901	64238	64353	56934
δ_{28}						$-.0_{30}$	14097	79616	94341	85223
δ_{29}							$+.0_{31}1362$	39308	80157	13705
δ_{30}							$-.0_{32}132$	53710	96401	60264
δ_{31}							$+.0_{33}12$	97545	41249	11119
δ_{32}							$-.0_{34}1$	27800	35996	63387
δ_{33}							$+.0_{35}$	12660	42110	58808
δ_{34}								$-.0_{36}1261$	13058	36997
δ_{35}								$+.0_{37}126$	28847	00446
δ_{36}								$-.0_{38}12$	71045	66477
δ_{37}								$+.0_{39}1$	28546	48750
δ_{38}								$-.0_{40}$	13060	92731
δ_{39}									$+.0_{41}1332$	97080
δ_{40}									$-.0_{42}136$	62267
δ_{41}									$+.0_{43}14$	06072
δ_{42}									$-.0_{44}1$	45280
δ_{43}									$+.0_{45}$	15068
δ_{44}										$-.0_{46}1569$
δ_{45}										$+.0_{47}164$
δ_{46}										$-.0_{48}17$
δ_{47}										$+.0_{49}2$

$\Gamma_6(7) = 0.51720\ 58142\ 19967\ 97031\ 31695\ 21499\ 10216\ 53040\ 55595\ 79910$

$n = 8$

$\Gamma_8(9) = 0.51349\ 58469\ 29257\ 70751\ 62570\ 32861\ 16277\ 50607\ 14975\ 02227$

δ_1	$+0.02699$	16938	58515	41503	25140	65722	32555	01214	29950	04454
δ_2	$-.0_2153$	58531	93110	01316	28335	47214	40571	51597	20881	90114
δ_3	$+.0_48$	95571	39583	52608	98954	55936	73459	65765	14660	72690
δ_4	$-.0_5$	53471	13871	03116	15111	58560	09407	68475	55582	52771
δ_5		$+.0_63266$	12119	95755	17497	60802	12038	93491	27477	89642
δ_6		$-.0_7203$	91783	59598	12734	12372	82461	13946	09229	07175
δ_7		$+.0_813$	00170	10485	35947	43052	82896	90891	08202	04818
δ_8		$-.0_{10}$	84582	11725	82184	89427	66258	26585	33597	31308
δ_9			$+.0_{11}5609$	24607	74248	12947	85435	97839	97188	01133
δ_{10}			$-.0_{12}378$	87669	51266	13032	18859	27178	06210	05790
δ_{11}			$+.0_{13}26$	04275	26910	29380	71056	17983	99469	77321
δ_{12}			$-.0_{14}1$	82016	13545	31549	69643	20382	31573	34857
δ_{13}			$+.0_{15}$	12924	60219	13163	09856	38424	70171	19386
δ_{14}				$-.0_{17}931$	69146	42585	19209	21652	84767	59825
δ_{15}				$+.0_{18}68$	13212	09946	83207	55198	61711	85778
δ_{16}				$-.0_{19}5$	05066	04255	17149	90667	06247	44462
δ_{17}				$+.0_{20}$	37928	62754	58685	67149	44112	52921
δ_{18}					$-.0_{21}2883$	56429	01657	18663	68902	06738
δ_{19}					$+.0_{22}221$	80483	94436	75831	28328	14345
δ_{20}					$-.0_{23}17$	25205	07200	52191	63873	50117
δ_{21}					$+.0_{24}1$	35613	19623	48043	46328	43992
δ_{22}					$-.0_{25}$	10767	90232	68163	24256	86273
δ_{23}						$+.0_{27}863$	21162	09761	81851	96083
δ_{24}						$-.0_{28}69$	83315	01467	46451	46605
δ_{25}						$+.0_{29}5$	69871	34304	74046	93670
δ_{26}						$-.0_{30}$	46890	65989	74635	81206
δ_{27}							$+.0_{31}3888$	87498	85889	43800
δ_{28}							$-.0_{32}324$	96243	96617	62204
δ_{29}							$+.0_{33}27$	35057	23571	13279
δ_{30}							$-.0_{34}2$	31784	71688	51882
δ_{31}							$+.0_{35}$	19772	30766	42781
δ_{32}								$-.0_{36}1697$	30804	34750
δ_{33}								$+.0_{37}146$	58121	30402
δ_{34}								$-.0_{38}12$	73211	18924
δ_{35}								$+.0_{39}1$	11204	76296
δ_{36}									$-.0_{41}9764$	48197
δ_{37}									$+.0_{42}861$	75807
δ_{38}									$-.0_{43}76$	42657
δ_{39}									$+.0_{44}6$	80994
δ_{40}									$-.0_{45}$	60954
δ_{41}										$+.0_{46}5480$
δ_{42}										$-.0_{47}495$
δ_{43}										$+.0_{48}45$
δ_{44}										$-.0_{49}4$

$\Gamma_7(8) = 0.51512\ 69527\ 70493\ 97856\ 08002\ 29281\ 45108\ 44467\ 87318\ 40198$

$n = 9$

$\Gamma_9(10) = 0.51218\ 19943\ 76050\ 59331\ 41598\ 75483\ 65226\ 30795\ 54973\ 22154$

δ_1	$+0.02436$	39887	52101	18662	83197	50967	30452	61591	09946	44308
δ_2	$-.0_{2}124$	51109	66701	34303	73882	42806	87808	69948	67802	21800
δ_3	$+.0_{4}6$	50663	78382	56706	51777	12681	44996	53417	16316	88397
δ_4	$-.0_{5}$	34745	63354	94257	32341	78180	55571	69046	42260	37976
δ_5		$+.0_{6}1894$	67294	06971	78686	17235	85116	98973	48803	01616
δ_6		$-.0_{7}105$	42565	73161	34200	81994	04646	94740	73503	58562
δ_7		$+.0_{9}5$	98162	48324	47141	06704	56000	85322	90282	18311
δ_8		$-.0_{10}$	34580	66811	79017	08685	43457	94819	28747	94006
δ_9			$+.0_{11}2035$	49406	47765	95583	10383	39416	57170	07669
δ_{10}			$-.0_{12}121$	90233	40535	91572	71292	40612	36998	77096
δ_{11}			$+.0_{14}7$	42246	11294	60725	96340	53939	02610	74879
δ_{12}			$-.0_{15}$	45916	81990	70375	32535	22959	67612	51693
δ_{13}				$+.0_{16}2883$	93502	83828	64123	52848	16628	54318
δ_{14}				$-.0_{17}183$	78053	36311	24883	36093	01794	37653
δ_{15}				$+.0_{18}11$	87501	21773	07432	15575	26705	74856
δ_{16}				$-.0_{20}$	77753	42046	53673	26613	85426	66021
δ_{17}					$+.0_{21}5155$	80903	45251	40641	88147	05357
δ_{18}					$-.0_{22}346$	03329	64543	99470	49005	47500
δ_{19}					$+.0_{23}23$	49338	12716	64956	32053	34791
δ_{20}					$-.0_{24}1$	61269	95759	13672	57084	76064
δ_{21}					$+.0_{25}$	11187	32441	50785	50628	82547
δ_{22}						$-.0_{27}783$	89673	63929	35489	60721
δ_{23}						$+.0_{28}55$	45703	42950	62702	64365
δ_{24}						$-.0_{29}3$	95944	93303	25514	98974
δ_{25}						$+.0_{30}$	28517	93264	65459	05041
δ_{26}							$-.0_{31}2071$	28690	87126	19324
δ_{27}							$+.0_{32}151$	65057	07363	76374
δ_{28}							$-.0_{33}11$	18871	75138	11621
δ_{29}							$+.0_{35}$	83158	72787	82308
δ_{30}								$-.0_{36}6224$	33434	09508
δ_{31}								$+.0_{37}469$	03893	96379
δ_{32}								$-.0_{38}35$	57425	83655
δ_{33}								$+.0_{39}2$	71493	95152
δ_{34}								$-.0_{40}$	20843	66825
δ_{35}									$+.0_{41}1609$	44373
δ_{36}									$-.0_{42}124$	95923
δ_{37}									$+.0_{44}9$	75349
δ_{38}									$-.0_{45}$	76518
δ_{39}										$+.0_{46}6032$
δ_{40}										$-.0_{47}478$
δ_{41}										$+.0_{48}38$
δ_{42}										$-.0_{49}3$

$\Gamma_8(9) = 0.51349\ 58469\ 29257\ 70751\ 62570\ 32861\ 16277\ 50607\ 14975\ 02227$

$n = 10$

$\Gamma_{10}(11) = 0.51110\ 10921\ 86620\ 32355\ 85526\ 35981\ 16719\ 20453\ 32323\ 309$

δ_1	$+0.02220$	21843	73240	64711	71052	71962	33438	40906	64646	618
δ_2	$-.0_{2}102$	96834	53423	91304	46471	20582	26412	46386	68640	805
δ_3	$+.0_{4}4$	87428	88023	10261	28534	20009	60763	30702	20964	561
δ_4	$-.0_{5}$	23538	31386	02189	54643	66827	26634	30978	28870	776
δ_5		$+.0_{6}1158$	89510	44924	53544	51220	34206	28601	69114	374
δ_6		$-.0_{8}58$	13783	46037	46127	07516	38562	53708	32635	249
δ_7		$+.0_{9}2$	96999	67640	58643	00402	05418	25574	91518	836
δ_8		$-.0_{10}$	15440	77629	87119	11380	51787	37160	05225	599
δ_9			$+.0_{12}816$	45260	83221	97384	59397	92983	17337	176
δ_{10}			$-.0_{13}43$	88083	99138	48594	37087	40257	98905	294
δ_{11}			$+.0_{14}2$	39572	33831	07555	80283	87685	01904	906
δ_{12}			$-.0_{15}$	13278	69901	48980	91317	91482	74832	965
δ_{13}				$+.0_{17}746$	75068	64459	76693	04005	99675	351
δ_{14}				$-.0_{18}42$	58404	43953	52604	41918	36868	538
δ_{15}				$+.0_{19}2$	46108	07956	55490	04658	53968	043
δ_{16}				$-.0_{20}$	14407	09024	19114	62552	88972	115
δ_{17}					$+.0_{22}853$	82640	70271	67001	82460	617
δ_{18}					$-.0_{23}51$	20180	91251	29432	03922	571
δ_{19}					$+.0_{24}3$	10535	48829	02628	10367	322
δ_{20}					$-.0_{25}$	19038	97476	20097	94421	912
δ_{21}						$+.0_{26}1179$	46986	47889	31694	907
δ_{22}						$-.0_{28}73$	79960	70133	09350	111
δ_{23}						$+.0_{29}4$	66193	62244	62180	612
δ_{24}						$-.0_{30}$	29720	28152	94589	137
δ_{25}							$+.0_{31}1911$	39703	82004	941
δ_{26}							$-.0_{32}123$	96668	75666	447
δ_{27}							$+.0_{34}8$	10526	52810	339
δ_{28}							$-.0_{35}$	53406	82271	812
δ_{29}								$+.0_{36}3545$	35255	302
δ_{30}								$-.0_{37}237$	04330	482
δ_{31}								$+.0_{38}15$	95806	156
δ_{32}								$-.0_{39}1$	08143	710
δ_{33}									$+.0_{41}7375$	326
δ_{34}									$-.0_{42}506$	076
δ_{35}									$+.0_{43}34$	930
δ_{36}									$-.0_{44}2$	425
δ_{37}									$+.0_{45}$	169
δ_{38}										$-.0_{46}12$
δ_{39}										$+.0_{47}1$

$\Gamma_{9}(10) = 0.51218\ 19943\ 76050\ 59331\ 41598\ 75483\ 65226\ 30795\ 54973\ 214$

$n = 11$

$\Gamma_{11}(12) = 0.51019\ 62627\ 04216\ 50644\ 07137\ 94393\ 65938\ 18306\ 87300\ 627$

δ_1	$+0.02039$	25254	08433	01288	14275	88787	31876	36613	74601	256
δ_2	$-.0_3 86$	56522	04242	67488	69354	85979	08371	05815	70702	937
δ_3	$+.0_4 3$	74493	58752	61570	46901	91590	20769	80796	33864	907
δ_4	$-.0_5$	16503	26467	45302	26758	63803	84837	60606	09573	748
δ_5		$+.0_7 740$	47156	72592	08528	81925	22066	96144	57373	166
δ_6		$-.0_8 33$	80979	02306	28676	54267	00910	62970	20659	494
δ_7		$+.0_9 1$	57018	26560	46337	51251	41734	23365	23541	694
δ_8			$-.0_{11} 7413$	21362	27509	77947	84285	71518	34650	528
δ_9			$+.0_{12} 355$	61895	96353	52815	61047	80361	77491	975
δ_{10}			$-.0_{13} 17$	32441	02351	72380	94738	07217	43102	258
δ_{11}			$+.0_{15}$	85664	55783	55602	13899	35449	37682	275
δ_{12}				$-.0_{16} 4297$	23452	53816	34125	32266	49675	227
δ_{13}				$+.0_{17} 218$	57527	90447	47464	07360	58599	869
δ_{14}				$-.0_{18} 11$	26730	03326	75071	55067	26220	694
δ_{15}				$+.0_{20}$	58834	59762	11093	04759	37757	738
δ_{16}					$-.0_{21} 3110$	51382	64454	34438	71283	339
δ_{17}					$+.0_{22} 166$	42395	59291	46221	81213	191
δ_{18}					$-.0_{24} 9$	00715	80321	36161	50938	847
δ_{19}					$+.0_{25}$	49289	91567	38286	19629	737
δ_{20}						$-.0_{26} 2726$	10927	01853	04386	626
δ_{21}						$+.0_{27} 152$	32332	77503	46833	664
δ_{22}						$-.0_{29} 8$	59522	45717	20696	537
δ_{23}						$+.0_{30}$	48961	18790	10526	959
δ_{24}							$-.0_{31} 2814$	44885	16354	756
δ_{25}							$+.0_{32} 163$	20395	96701	757
δ_{26}							$-.0_{34} 9$	54375	90105	950
δ_{27}							$+.0_{35}$	56262	80175	659
δ_{28}								$-.0_{36} 3342$	75091	169
δ_{29}								$+.0_{37} 200$	09714	196
δ_{30}								$-.0_{38} 12$	06454	617
δ_{31}								$+.0_{40}$	73248	500
δ_{32}									$-.0_{41} 4477$	068
δ_{33}									$+.0_{42} 275$	417
δ_{34}									$-.0_{43} 17$	049
δ_{35}									$+.0_{44} 1$	062
δ_{36}										$-.0_{46} 67$
δ_{37}										$+.0_{47} 4$

$\Gamma_{10}(11) = 0.51110\ 10921\ 86620\ 32355\ 85526\ 35981\ 16719\ 20453\ 32323\ 309$

$n = 12$

$\Gamma_{12}(13) = 0.50942\ 77437\ 63451\ 63838\ 67447\ 16936\ 64864\ 65048\ 01678\ 699$

δ_1	$+0.01885$	54875	26903	27677	34894	33873	29729	30096	03357	398
δ_2	$-.0_3 73$	78872	33229	47854	90776	70624	26611	64713	50553	321
δ_3	$+.0_4 2$	93895	87933	22542	55288	54843	35667	94281	33078	915
δ_4	$-.0_5$	11908	98283	94795	69598	84792	82812	48004	67110	351
δ_5		$+.0_7 490$	74241	32877	49601	75221	55596	82125	82410	018
δ_6		$-.0_8 20$	55630	97491	90102	07758	20708	70812	33801	282
δ_7		$+.0_{10}$	87490	34978	45577	58462	16938	71053	61608	620
δ_8			$-.0_{11} 3781$	86854	81962	76894	00950	70392	24646	514
δ_9			$+.0_{12} 165$	95483	68976	47644	18801	93196	52290	354
δ_{10}			$-.0_{14} 7$	38951	01610	48207	25240	90661	93351	619
δ_{11}			$+.0_{15}$	33372	49952	50883	41363	26871	36773	877
δ_{12}				$-.0_{16} 1527$	97130	27467	66697	38381	99626	561
δ_{13}				$+.0_{18} 70$	89290	70215	00636	68577	06165	692
δ_{14}				$-.0_{19} 3$	33166	76454	24496	81576	26480	618
δ_{15}				$+.0_{20}$	15852	76680	84136	79434	16350	760
δ_{16}					$-.0_{22} 763$	39581	56245	15422	11762	607
δ_{17}					$+.0_{23} 37$	18917	94215	77037	16283	822
δ_{18}					$-.0_{24} 1$	83202	22257	70607	21216	839
δ_{19}						$+.0_{26} 9122$	67271	87903	56058	839
δ_{20}						$-.0_{27} 459$	01351	36366	47033	873
δ_{21}						$+.0_{28} 23$	32817	44943	97340	130
δ_{22}						$-.0_{29} 1$	19710	71369	39371	795
δ_{23}							$+.0_{31} 6200$	58392	86420	913
δ_{24}							$-.0_{32} 324$	06718	56025	991
δ_{25}							$+.0_{33} 17$	08446	01767	289
δ_{26}							$-.0_{35}$	90823	29154	612
δ_{27}								$+.0_{36} 4867$	36757	172
δ_{28}								$-.0_{37} 262$	88642	615
δ_{29}								$+.0_{38} 14$	30541	227
δ_{30}								$-.0_{40}$	78410	939
δ_{31}									$+.0_{41} 4328$	002
δ_{32}									$-.0_{42} 240$	507
δ_{33}									$+.0_{43} 13$	452
δ_{34}									$-.0_{45}$	757
δ_{35}										$+.0_{46} 43$
δ_{36}										$-.0_{47} 2$

$\Gamma_{11}(12) = 0.51019\ 62627\ 04216\ 50644\ 07137\ 94393\ 65938\ 18306\ 87300\ 627$

$n = 13$

$\Gamma_{13}(14) = 0.50876\ 69032\ 39266\ 60994\ 64208\ 95127\ 06776\ 53349\ 39915\ 836$

δ_1	+0.01753	38064	78533	21989	28417	90254	13553	06698	79831	672
δ_2	$-.0_{3}63$	64400	65726	30189	09589	56000	40260	38063	68022	465
δ_3	$+.0_{4}2$	34848	75467	84216	01883	07958	18256	74215	31220	181
δ_4		$-.0_{6}8806$	81768	39728	36839	58161	53282	88976	44419	963
δ_5		$+.0_{7}335$	50186	09010	50256	41601	16278	10554	00930	587
δ_6		$-.0_{8}12$	97946	28958	17630	15492	08081	98169	45512	101
δ_7		$+.0_{10}$	50973	13999	29208	09191	58555	86232	91833	310
δ_8			$-.0_{11}2031$	34240	83073	75200	61414	36493	55509	041
δ_9			$+.0_{13}82$	11342	71966	13533	16906	89336	20489	305
δ_{10}			$-.0_{14}3$	36562	35112	26557	64572	73850	75472	945
δ_{11}			$+.0_{15}$	13981	95064	58022	69835	04935	21181	664
δ_{12}				$-.0_{17}588$	50708	41905	78436	76725	90567	766
δ_{13}				$+.0_{18}25$	08701	81250	16565	09216	04791	744
δ_{14}				$-.0_{19}1$	08266	23077	31275	30808	59789	243
δ_{15}					$+.0_{21}4728$	45038	89422	43228	07465	440
δ_{16}					$-.0_{22}208$	91239	15932	07003	76619	368
δ_{17}					$+.0_{24}9$	33404	07135	45206	65075	769
δ_{18}					$-.0_{25}$	42157	91418	42102	10558	674
δ_{19}						$+.0_{26}1924$	15387	03474	38193	568
δ_{20}						$-.0_{28}88$	71617	62538	68810	059
δ_{21}						$+.0_{29}4$	13069	54432	53359	568
δ_{22}						$-.0_{30}$	19416	00913	99660	042
δ_{23}							$+.0_{32}921$	03568	16321	116
δ_{24}							$-.0_{33}44$	07985	33684	686
δ_{25}							$+.0_{34}2$	12776	07661	664
δ_{26}							$-.0_{35}$	10356	20084	340
δ_{27}								$+.0_{37}508$	10496	013
δ_{28}								$-.0_{38}25$	12269	358
δ_{29}								$+.0_{39}1$	25149	005
δ_{30}									$-.0_{41}6279$	568
δ_{31}									$+.0_{42}317$	300
δ_{32}									$-.0_{43}16$	142
δ_{33}									$+.0_{45}$	827
δ_{34}										$-.0_{46}43$
δ_{35}										$+.0_{47}2$

$\Gamma_{12}(13) = 0.50942\ 77437\ 63451\ 63838\ 67447\ 16936\ 64864\ 65048\ 01678\ 699$

$n = 14$

$\Gamma_{14}(15) = 0.50819\ 26005\ 56749\ 87188\ 07473\ 77755\ 64022\ 60300\ 56829\ 894$

δ_1	$+0.01638$	52011	13499	74376	14947	55511	28045	20601	13659	788
δ_2	$-.0_{3}55$	45522	38391	91863	45301	57080	57422	21408	88234	542
δ_3	$+.0_{4}1$	90608	54189	46715	54366	74605	67116	07136	72068	725
δ_4		$-.0_{6}6651$	52921	27175	72956	65712	12421	47145	29039	649
δ_5		$+.0_{7}235$	58376	48535	28766	58214	53922	10418	12227	171
δ_6		$-.0_{9}8$	46594	54597	70597	69003	87381	87629	91370	671
δ_7		$+.0_{10}$	30858	16554	19375	44782	11016	73402	85898	391
δ_8			$-.0_{11}1140$	47282	09374	68658	46116	64161	36797	564
δ_9			$+.0_{13}42$	72417	39356	53587	37751	48606	31676	681
δ_{10}			$-.0_{14}1$	62176	77885	91622	15688	90289	52477	248
δ_{11}				$+.0_{16}6235$	64737	20128	14690	74804	94437	162
δ_{12}				$-.0_{17}242$	77488	21717	70758	69272	19767	618
δ_{13}				$+.0_{19}9$	56768	97072	86877	09886	69428	821
δ_{14}				$-.0_{20}$	38154	36054	66161	18100	94473	703
δ_{15}					$+.0_{21}1539$	10756	43681	32534	54034	887
δ_{16}					$-.0_{23}62$	78227	95045	58875	34516	460
δ_{17}					$+.0_{24}2$	58884	56913	89444	23414	919
δ_{18}					$-.0_{25}$	10787	88352	46281	85296	750
δ_{19}						$+.0_{27}454$	14023	45382	67367	908
δ_{20}						$-.0_{28}19$	30780	31490	24147	499
δ_{21}						$+.0_{30}$	82876	88985	65161	565
δ_{22}							$-.0_{31}3590$	57062	48296	041
δ_{23}							$+.0_{32}156$	96326	04576	212
δ_{24}							$-.0_{34}6$	92174	38934	063
δ_{25}							$+.0_{35}$	30782	07922	592
δ_{26}								$-.0_{36}1380$	16408	516
δ_{27}								$+.0_{38}62$	37388	367
δ_{28}								$-.0_{39}2$	84057	722
δ_{29}								$+.0_{40}$	13032	811
δ_{30}									$-.0_{42}602$	278
δ_{31}									$+.0_{43}28$	028
δ_{32}									$-.0_{44}1$	313
δ_{33}										$+.0_{46}62$
δ_{34}										$-.0_{47}3$

$\Gamma_{13}(14) = 0.50876\ 69032\ 39266\ 60994\ 64208\ 95127\ 06776\ 53349\ 39915\ 836$

$n = 15$

$\Gamma_{15}(16) = 0.50768\ 88876\ 00355\ 53394\ 18119\ 96939\ 86807\ 43040\ 37618\ 825$

δ_1	+0.01537	77752	00711	06788	36239	93879	73614	86080	75237	650
δ_2	$-.0_{3}48$	75025	36800	04255	20576	31024	63472	87139	25937	938
δ_3	$+.0_{4}1$	56812	31181	45398	58728	13815	43113	09831	31189	573
δ_4		$-.0_{6}5116$	69258	40432	75272	81724	64268	64094	23772	505
δ_5		$+.0_{7}169$	31149	32666	01888	61569	06794	88939	90819	493
δ_6		$-.0_{9}5$	68002	36025	52336	25201	66466	73569	96579	648
δ_7		$+.0_{10}$	19313	28535	73581	66127	90880	89072	50712	041
δ_8			$-.0_{12}665$	39239	31349	95875	03081	59740	29131	083
δ_9			$+.0_{13}23$	22130	50147	10174	49423	07328	51052	542
δ_{10}			$-.0_{15}$	82063	98612	29136	63539	69319	90343	266
δ_{11}				$+.0_{16}2935$	92503	84483	91201	98763	01497	163
δ_{12}				$-.0_{17}106$	29979	33098	82543	21850	50306	786
δ_{13}				$+.0_{19}3$	89389	02512	05482	39239	19689	238
δ_{14}				$-.0_{20}$	14426	73148	97224	90072	79358	086
δ_{15}					$+.0_{22}540$	44940	96064	30908	03713	515
δ_{16}					$-.0_{23}20$	46520	12750	51424	36000	892
δ_{17}					$+.0_{25}$	78311	09897	26500	43484	188
δ_{18}						$-.0_{26}3027$	26959	34562	21471	313
δ_{19}						$+.0_{27}118$	18856	79745	49154	230
δ_{20}						$-.0_{29}4$	65880	69055	84632	707
δ_{21}						$+.0_{30}$	18536	57294	86707	029
δ_{22}							$-.0_{32}744$	25678	44405	839
δ_{23}							$+.0_{33}30$	14675	30007	197
δ_{24}							$-.0_{34}1$	23160	41860	263
δ_{25}								$+.0_{36}5073$	47987	813
δ_{26}								$-.0_{37}210$	68764	292
δ_{27}								$+.0_{39}8$	81794	912
δ_{28}								$-.0_{40}$	37187	031
δ_{29}									$+.0_{41}1579$	842
δ_{30}									$-.0_{43}67$	599
δ_{31}									$+.0_{44}2$	913
δ_{32}									$-.0_{45}$	126
δ_{33}										$+.0_{47}6$

$\Gamma_{14}(15) = 0.50819\ 26005\ 56749\ 87188\ 07473\ 77755\ 64022\ 60300\ 56829\ 894$

$n = 16$

$\Gamma_{16}(17) = 0.50724\ 35071\ 53992\ 67161\ 89024\ 32507\ 65994\ 30463\ 35544\ 398$

δ_1	$+0.01448$	70143	07985	34323	78048	65015	31988	60926	71088	796
δ_2	$-.0_{3}43$	19123	73014	44122	15746	18293	72893	57616	32897	804
δ_3	$+.0_{4}1$	30552	62186	72988	52434	72149	33321	45435	37614	884
δ_4		$-.0_{6}3999$	88900	37962	91033	37965	65293	06604	70100	535
δ_5		$+.0_{7}124$	18766	76871	77899	21566	82053	28791	69818	810
δ_6		$-.0_{9}3$	90633	96163	54234	86397	33193	60506	32797	599
δ_7		$+.0_{10}$	12445	51128	50918	68087	74391	86845	50012	488
δ_8			$-.0_{12}401$	50850	68966	40059	03122	15556	45459	268
δ_9			$+.0_{13}13$	11300	72273	15206	22831	47597	23974	324
δ_{10}			$-.0_{15}$	43343	13306	69175	90190	20738	97330	007
δ_{11}				$+.0_{16}1449$	54828	40935	74643	13635	62502	963
δ_{12}				$-.0_{18}49$	03687	51606	14575	48074	11567	460
δ_{13}				$+.0_{19}1$	67754	31247	06126	61791	87819	090
δ_{14}					$-.0_{21}5801$	87795	87335	78363	21291	456
δ_{15}					$+.0_{22}202$	81079	72389	84684	42203	882
δ_{16}					$-.0_{24}7$	16350	29823	26725	58527	271
δ_{17}					$+.0_{25}$	25559	77285	71292	81621	844
δ_{18}						$-.0_{27}921$	02666	55603	75068	483
δ_{19}						$+.0_{28}33$	50881	17899	79196	675
δ_{20}						$-.0_{29}1$	23057	20256	19667	995
δ_{21}							$+.0_{31}4560$	44914	04212	450
δ_{22}							$-.0_{32}170$	51166	77446	273
δ_{23}							$+.0_{34}6$	43045	81265	619
δ_{24}							$-.0_{35}$	24455	05596	874
δ_{25}								$+.0_{37}937$	63588	934
δ_{26}								$-.0_{38}36$	23604	743
δ_{27}								$+.0_{39}1$	41121	462
δ_{28}									$-.0_{41}5537$	299
δ_{29}									$+.0_{42}218$	859
δ_{30}									$-.0_{44}8$	712
δ_{31}									$+.0_{45}$	349
δ_{32}										$-.0_{46}14$
δ_{33}										$+.0_{47}1$

$\Gamma_{15}(16) = 0.50768\ 88876\ 00355\ 53394\ 18119\ 96939\ 86807\ 43040\ 37618\ 825$

$n = 17$

$\Gamma_{17}(18) = 0.50684\ 68838\ 09181\ 78280\ 68775\ 99599\ 45014\ 76113\ 41866\ 320$

δ_1	+0.01369	37676	18363	56561	37551	99198	90029	52226	83732	640
δ_2	$-.0_{3}38$	53124	87447	03946	42136	23012	19554	22109	64652	536
δ_3	$+.0_{4}1$	09841	30466	00414	50091	11421	93059	00009	23806	415
δ_4		$-.0_{6}3171$	69195	40142	60960	13140	11786	21136	17764	456
δ_5		$+.0_{8}92$	74624	59065	01104	07677	09065	55818	35059	376
δ_6		$-.0_{9}2$	74591	04714	14136	42362	50284	75840	43804	850
δ_7			$+.0_{11}8229$	31092	38706	43942	87305	73655	59689	793
δ_8			$-.0_{12}249$	59048	85358	67833	46655	68332	13977	784
δ_9			$+.0_{14}7$	65911	87954	71262	99882	38644	17088	345
δ_{10}			$-.0_{15}$	23774	60058	78576	29165	72916	33069	958
δ_{11}				$+.0_{17}746$	32456	69839	79502	89610	19986	209
δ_{12}				$-.0_{18}23$	68743	70984	17322	19132	69136	179
δ_{13}				$+.0_{20}$	75994	04983	84460	55685	57951	786
δ_{14}					$-.0_{21}2463$	81176	01487	02136	36651	766
δ_{15}					$+.0_{23}80$	70441	15376	83735	87228	196
δ_{16}					$-.0_{24}2$	67020	52703	37998	89947	429
δ_{17}						$+.0_{26}8921$	66867	02365	22183	920
δ_{18}						$-.0_{27}300$	95315	52447	71507	571
δ_{19}						$+.0_{28}10$	24711	07480	42792	537
δ_{20}						$-.0_{30}$	35208	96459	11233	429
δ_{21}							$+.0_{31}1220$	54931	57309	676
δ_{22}							$-.0_{33}42$	67859	81190	880
δ_{23}							$+.0_{34}1$	50494	91354	197
δ_{24}								$-.0_{36}5350$	52299	968
δ_{25}								$+.0_{37}191$	75252	788
δ_{26}								$-.0_{39}6$	92573	759
δ_{27}								$+.0_{40}$	25204	764
δ_{28}									$-.0_{42}924$	069
δ_{29}									$+.0_{43}34$	123
δ_{30}									$-.0_{44}1$	269
δ_{31}										$+.0_{46}48$
δ_{32}										$-.0_{47}2$

$\Gamma_{16}(17) = 0.50724\ 35071\ 53992\ 67161\ 89024\ 32507\ 65994\ 30463\ 35544\ 398$

$n = 18$

$\Gamma_{18}(19) = 0.50649\ 14287\ 06755\ 41776\ 15652\ 60279\ 22086\ 17472\ 45872\ 260$

δ_1	$+0.01298$	28574	13510	83552	31305	20558	44172	34944	91744	520
δ_2	$-.0_{3}34$	58643	94561	67547	05948	81027	85356	23409	62094	224
δ_3	$+.0_{5}$	93287	58057	72839	61704	35150	10023	64775	41339	517
δ_4		$-.0_{6}2547$	08717	64712	63767	05931	24138	53371	97568	568
δ_5		$+.0_{8}70$	38543	76553	78880	25615	66203	30001	34989	560
δ_6		$-.0_{9}1$	96813	80161	73339	63689	59994	99678	31078	185
δ_7			$+.0_{11}5567$	68506	60391	06673	29671	90528	53687	877
δ_8			$-.0_{12}159$	31292	48034	31952	74606	35482	19518	118
δ_9			$+.0_{14}4$	60992	58592	75719	82991	11848	31977	452
δ_{10}			$-.0_{15}$	13486	88151	81242	51409	81009	89958	800
δ_{11}				$+.0_{17}398$	85121	25363	29873	28438	13938	426
δ_{12}				$-.0_{18}11$	92060	86826	46249	25935	00059	743
δ_{13}				$+.0_{20}$	35998	13230	26426	69045	05313	300
δ_{14}					$-.0_{21}1098$	14872	44306	69924	09275	290
δ_{15}					$+.0_{23}33$	83355	40923	03827	60290	198
δ_{16}					$-.0_{24}1$	05255	58562	75269	37245	397
δ_{17}						$+.0_{26}3305$	67345	33275	23656	011
δ_{18}						$-.0_{27}104$	78492	13673	94562	775
δ_{19}						$+.0_{29}3$	35173	13739	04798	893
δ_{20}						$-.0_{30}$	10816	32243	76851	010
δ_{21}							$+.0_{32}352$	07853	01056	343
δ_{22}							$-.0_{33}11$	55738	61863	381
δ_{23}							$+.0_{35}$	38251	79659	593
δ_{24}								$-.0_{36}1276$	23173	179
δ_{25}								$+.0_{38}42$	91484	705
δ_{26}								$-.0_{39}1$	45412	696
δ_{27}									$+.0_{41}4964$	001
δ_{28}									$-.0_{42}170$	693
δ_{29}									$+.0_{44}5$	911
δ_{30}									$-.0_{45}$	206
δ_{31}										$+.0_{47}7$

$\Gamma_{17}(18) = 0.50684\ 68838\ 09181\ 78280\ 68775\ 99599\ 45014\ 76113\ 41866\ 320$

$n = 19$

$\Gamma_{19}(20) = 0.50617\ 10498\ 71381\ 60623\ 62865\ 49103\ 98633\ 92600\ 67424\ 949$

δ_1	+0.01234	20997	42763	21247	25730	98207	97267	85201	34849	898
δ_2	$-.0_3 31$	21765	04021	32768	33340	65519	62697	99613	66835	726
δ_3	$+.0_5$	79898	76499	21267	50979	76807	70245	73331	36584	872
δ_4		$-.0_6 2068$	88419	38073	38088	61545	43295	07329	41414	943
δ_5		$+.0_8 54$	18934	01506	25797	89611	72116	23094	72456	469
δ_6		$-.0_9 1$	43547	74044	61959	68542	79326	89158	45045	321
δ_7			$+.0_{11} 3845$	07742	77994	06136	83392	87083	62435	111
δ_8			$-.0_{12} 104$	12615	83050	97287	67046	01346	24279	292
δ_9			$+.0_{14} 2$	85022	42335	34960	72988	44688	91030	543
δ_{10}				$-.0_{16} 7884$	60353	27567	52522	02079	95011	662
δ_{11}				$+.0_{17} 220$	38326	07763	34078	76283	13821	276
δ_{12}				$-.0_{19} 6$	22287	07093	46100	99124	45392	478
δ_{13}				$+.0_{20}$	17747	26189	02474	04233	84010	892
δ_{14}					$-.0_{22} 511$	11071	39211	79231	46703	923
δ_{15}					$+.0_{23} 14$	86127	77609	89147	38565	696
δ_{16}					$-.0_{25}$	43618	36559	26407	95883	005
δ_{17}						$+.0_{26} 1292$	02059	37515	75404	040
δ_{18}						$-.0_{28} 38$	61636	64898	73782	195
δ_{19}						$+.0_{29} 1$	16436	98183	53558	410
δ_{20}							$-.0_{31} 3541$	15007	61122	639
δ_{21}							$+.0_{32} 108$	60471	50312	156
δ_{22}							$-.0_{34} 3$	35831	92966	345
δ_{23}							$+.0_{35}$	10468	47123	735
δ_{24}								$-.0_{37} 328$	89149	714
δ_{25}								$+.0_{38} 10$	41241	457
δ_{26}								$-.0_{40}$	33212	554
δ_{27}									$+.0_{41} 1067$	159
δ_{28}									$-.0_{43} 34$	535
δ_{29}									$+.0_{44} 1$	125
δ_{30}										$-.0_{46} 37$
δ_{31}										$+.0_{47} 1$

$\Gamma_{18}(19) = 0.50649\ 14287\ 06755\ 41776\ 15652\ 60279\ 22086\ 17472\ 45872\ 260$

$\Gamma_{20}(21) = 0.50588\ 08005\ 53800\ 01150\ 70690\ 06386\ 77754\ 79946\ 62993\ 91$

REFERENCES

1. von Soldner, J., "Théorie et Tables d'une Nouvelle Fonction Transcendante," Munich (1809).

2. Bretschneider, C.A., "Berechnung der Grundzahl der natürlichen Logarithmen so wie mehrerer anderer mit ihr zusammenhängender Zahlwerthe," Archiv der Mathematik und Physik, Vol. 1, pp. 96–101 (1841).

3. Chebyshev, P.L., "Sur la fonction qui détermine la totalité des nombres premiers inférieurs à une limite donnée," Mémoires présentés à l'Académie Impériale des Sciences de St. Pétersbourg, Vol. 6, pp. 141–157 (1851).

4. Hadamard, J., "Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques," Bulletin de la Société Mathématique de France, Vol. 24, pp. 199–220 (1896).

5. de la Vallée Poussin, C., "La fonction $\zeta(s)$ de Riemann et les nombres premiers en général," Annales de la Société Scientifiques de Bruxelles, Vol. 20, pp. 183–256 (1896).

6. Ingham, A.E., "The Distribution of Prime Numbers," Cambridge Tracts in Mathematics, No. 30, Cambridge University Press, Cambridge (1932).

7. Bretschneider, C.A., "Ueber die Berechnung des Integrallogarithmen und einiger mit ihm zusammenhängenden anderen Functionen," Zeitschrift für Mathematik und Physik, Vol. 6, pp. 127–139 (1861).

8. Glaisher, J.W.L., "Tables of the Numerical Values of the Sine-Integral, Cosine-Integral, and Exponential-Integral," Philosophical Transactions of the Royal Society of London, Vol. 160, pp. 367–388 (1870).

9. Gram, J.P., "Om beregning af funktionen $Li\ (e^x)$," Det Koneglige Danske Videnskabernes Selskabs Skrifter, Vol. 2, pp. 263–272 (1884).

10. British Association for the Advancement of Science, "Circular and Hyperbolic Functions, Exponential, Sine and Cosine Integrals, Factorial (Gamma) and Derived Functions, Integrals of Probability Integral," Mathematical Tables, Vol. 1, Cambridge University Press, Cambridge (1931). Second edition 1946, third edition 1951.

11. New York Works Progress Administration Mathematical Tables Project, "Tables of Sine, Cosine and Exponential Integrals," Vols. I and II (1940).

12. Coulson, C.A. and Duncanson, W.E., "Some New Values for the Exponential Integral," Philosophical Magazine, Series 7, Vol. 33, pp. 754–761 (1942).

13. Kotani, M., Amemiya, A., Ishiguro, E., and Kimura, T., "Table of Molecular Integrals," Maruzen Co., Tokyo (1955).

14. Harris, F.E., "Tables of the Exponential Integral $Ei\ (x)$," Mathematical Tables and other Aids to Computation, Vol. 11, pp. 9–16 (1957).

15. Miller, J. and Hurst, R.P., "Simplified Calculation of the Exponential Integral," Mathematical Tables and other Aids to Computation, Vol. 12, pp. 187–193 (1958).

16. Airey, J.R., "The 'Converging Factor' in Asymptotic Series and the Calculation of Bessel, Laguerre and Other Functions," Philosophical Magazine, Series 7, Vol. 24, pp. 521–552 (1937).

17. Rosser, J.B., "Transformations to Speed the Convergence of Series," Journal of Research of the National Bureau of Standards, Vol. 46, pp. 56–64 (1951).

18. Miller, J.C.P., "A Method for the Determination of Converging Factors, Applied to the Asymptotic Expansions for the Parabolic Cylinder Functions," Proceedings of the Cambridge Philosophical Society, Vol. 48, pp. 243–254 (1952).

19. Dingle, R.B., "Asymptotic Expansions and Converging Factors I. General Theory and Basic Converging Factors," Proceedings of the Royal Society of London, Vol. 244, pp. 456–475 (1958). See also ibid., pp. 476–490.

20. Stieltjes, T.J., "Recherches sur quelques séries semiconvergentes," Annales Scientifiques de l'École Normale Supérieure, Série 3, Vol. 3, pp. 201–258 (1886).

21. Davis, H.T., "Tables of the Higher Mathematical Functions," Vol. 1, Principia Press, Bloomington, Indiana, p. 180 (1933).

22. Darboux, G., "Mémoire sur l'approximation des fonctions de très-grands nombres, et sur une classe étendue de devéloppements en série," Journal de Mathématique, Sér. 3, Vol. 4, pp. 5–36, 377–416 (1878).

23. Watson, G.N., "Theorems stated by Ramanujan (V): Approximations connected with e^x," Proceedings of the London Mathematical Society, Second Series, Vol. 29, pp. 293–308 (1929).

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