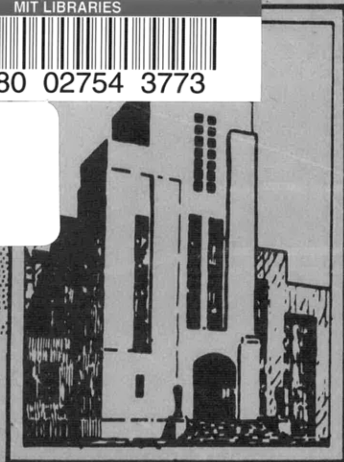
Report 1497



DEPARTMENT OF THE NAVY

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HYDROMECHANICS

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AXISYMMETRIC ELASTIC DEFORMATIONS AND STRESSES IN A RING-STIFFENED, PERFECTLY CIRCULAR CYLINDRICAL SHELL UNDER EXTERNAL HYDROSTATIC PRESSURE

by

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STRUCTURAL MECHANICS LABORATORY

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PREFACE

The analysis presented in this report was developed by the authors during the course of research conducted at Brooklyn Polytechnic Institute under Contract N6 ONR-26303 sponsored jointly by the Office of Naval Research and the Bureau of Ships. The original report, prepared in June 1951, had a limited distribution and restricted contents because of classification.

Familiarity of the authors with the application of the analysis to actual problems within the U. S. Navy and with the need for more rational design criteria by naval architects, prompted them to revise, extend, and finally rewrite the original theory in a more concise and usable form. It was also felt that publication of the more important results in the form of a Model Basin report would facilitate their dissemination among cognizant codes within the Bureau of Ships and others working in the field of pressure vessel analysis.

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NOTATION

A_f	Actual cross-sectional area of frame in sq in.
A_{eff}	Effective cross-sectional area of frame in sq in.
A, B, C, F	Arbitrary constants
b	Faying width of circular ring frames in contact with shell in in.
C_1, C_2, C_3, C_4	Arbitrary constants
d	Depth of circular ring frame in in.
D	Flexural rigidity of shell $= \dfrac{Eh^3}{12(1-\upsilon^2)}$
E	Young's modulus of elasticity in psi
F_1, F_2, F_3, F_4, F_5	Functions of load and geometry of shell
h	Thickness of shell plating in in.
$L = L_f - b$	Distance between adjacent faying flanges (unsupported length of shell)
L_f	Center-to-center distance between adjacent frames
M_x, M_ϕ	Longitudinal and circumferential bending moments per unit circumferential length in in-lb/in.
N_x, N_ϕ	Longitudinal and circumferential normal forces per unit circumferential length in lb/in.
p	Hydrostatic pressure in psi
P_r	Radial pressure in psi
Q_o	Transverse shear force per unit circumferential length transmitted by the shell to a ring frame in lb/in.
Q_x	Transverse shear force per unit circumferential length in lb/in.

Q^*	**Total load on circular ring frame per unit circumferential length in lb/in.**
R	**Radius to median surface of shell**
u, v, w	**Longitudinal, circumferential, and radial displacements**
x, θ, r	**Longitudinal, circumferential, and radial coordinates**
$\alpha = \dfrac{A_{eff}}{L_f h}$	Ratio of effective frame area to shell area
$\beta = \dfrac{b}{L_f}$	Ratio of faying width to frame spacing
$\gamma \equiv \dfrac{p}{p^*} = \dfrac{p\sqrt{3(1-\upsilon^2)}}{2E(h/R)^2}$	Measure of beam-column effect ($\gamma = 0$ **implies** no such effect)
$\varepsilon_x, \varepsilon_\phi$	**Longitudinal and circumferential strains**
$\eta_1 = \frac{1}{2}\sqrt{1-\gamma}$	
$\eta_2 = \frac{1}{2}\sqrt{1+\gamma}$	
$\theta = \sqrt[4]{3(1-\upsilon^2)}\ \dfrac{L}{\sqrt{Rh}}$	Shell flexibility parameter
$\lambda_1, \lambda_2, \lambda_3, \lambda_4$	**Roots of characteristic equation**
υ	**Poisson's ratio**
$\sigma_{\phi M}$	**Circumferential membrane stress in psi**
σ_{Φ}	**Total circumferential stress in psi**

NOTATION (Cont'd)

σ_{xb}	Longitudinal bending stress in psi
σ_x	Total longitudinal stress in psi

$$(\ )_{,x} \equiv \frac{d(\ )}{dx}\ ;\ (\ )_{,xx} \equiv \frac{d^2(\ )}{dx^2}\ ;\ \text{etc.}$$

ABSTRACT

This investigation of the axisymmetric elastic deformations and stresses in a ring-stiffened circular cylindrical shell under hydrostatic pressure was undertaken to include properly the effect of the axial load and the load boundary condition of Viterbo at the rings. Neither of the previous solutions to this problem, notably those of Von Sanden and Günther and of Viterbo, considered the nonlinear "beam-column" effect produced by the combination of longitudinal compression with longitudinal bending. Numerical calculations show that this nonlinearity can become significant and a graphical presentation of the refined analysis included herein clearly indicates its importance in certain ranges of the geometric parameters of interest to pressure-vessel designers.

INTRODUCTION

The ability of a circular cylindrical shell reinforced by uniformly spaced ring frames to support a pressure load is of interest to engineers in general and to naval architects in particular. The problem of determining the amount of external pressure that can be taken by such a structure before failure by axisymmetric yielding occurs in the shell plating has been considered by many investigators, among them Von Sanden and Günther [1] and Viterbo.[2] However, the criteria developed by these authors for the limiting maximum bending and circumferential stresses in the shell plating and the peripheral load supported by the frames do not reflect properly the "beam-column effect" of the hydrostatic pressure on the structure. For this reason the earlier theories were modified to include the complete effect of the load.

The modified theory [3] is based on the assumption that all deformations are small. The results obtained are therefore not likely to be reliable when the radial displacement of any shell element is greater than the thickness of the shell. The analysis is also based on the assumption that the shell structure is perfectly circular and initially stress-free.

The present report contains the derivation of all the equations necessary for the calculation of the maximum longitudinal and circumferential stresses in the shell plating and the shear transmitted from the shell to the reinforcing rings. Since the theory proved cumbersome, a graphical solution of the equations was developed by M. A. Krenzke and R. D. Short at the David Taylor Model Basin. The results were first presented in Reference 4.

[1]References are listed on page 49 .

BASIC EQUATIONS

The differential equation governing the axisymmetric elastic deformations of a circular cylindrical shell of finite length and under the action of external hydrostatic pressure is derived as Equation [A12] in Appendix A, i.e.,

$$Dw,_{xxxx} + \frac{pR}{2} w,_{xx} + \frac{Eh}{R^2} w = - p (1-\upsilon/2) \qquad [1]$$

This equation is also given in Reference 5. The term $\frac{pR}{2} w,_{xx}$ which renders the solution of Equation [1] to be a nonlinear function of the pressure, was not considered in the analyses of References 1 and 2. Equation [1] is an ordinary differential equation with constant coefficients and may be solved by the usual method of assuming a complementary solution of the form:

$$w_c \sim e^{\lambda x} \qquad [2]$$

The four roots of the auxiliary equation are

$$\lambda_1;\lambda_2;\lambda_3;\lambda_4 = \pm \left\{ - \frac{pR}{4D} \pm \left[\left(\frac{pR}{4D}\right)^2 - \frac{Eh}{DR^2} \right]^{1/2} \right\}^{1/2} \qquad [3]$$

The complementary solution is consequently given by

$$w_c = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + c_3 e^{\lambda_3 x} + c_4 e^{\lambda_4 x} \qquad [4]$$

where c_1 to c_4 are arbitrary constants of integration to be determined from boundary conditions arising from the enforcement of structural continuity and force equilibrium at the supporting ring frames.

A particular integral of Equation [1] is

$$w_p = - \frac{pR^2}{Eh} (1-\upsilon/2) \qquad [5]$$

and may be checked by substitution into Equation [1]. The expression for w_p represents the axisymmetric radial displacement of a long unstiffened cylindrical shell under hydrostatic pressure.

Since from Equations [3] $\lambda_1 = - \lambda_2$ and $\lambda_3 = -\lambda_4$ and because of the relations

$$e^{\lambda x} = \cosh \lambda x + \sinh \lambda x$$
$$e^{-\lambda x} = \cosh \lambda x - \sinh \lambda x \qquad [6]$$

the complete solution for the equation of bending and stretching of the curved shell element can be written as

$$w = A \sinh \lambda_1 x + B \cosh \lambda_1 x + C \sinh \lambda_3 x + F \cosh \lambda_3 x - \frac{pR^2}{Eh} (1-\upsilon/2) \qquad [7]$$

where A, B, C, and F are new arbitrary constants of integration.

The plane midway between a pair of frames is chosen as the origin of the axial coordinate x as shown in Figure 1 to take advantage of symmetry. This requires that the solution for the radial displacement w be an even function of x, and, consequently, the arbitrary constants A and C of Equation [7] must be set equal to zero, so that

$$w = B \cosh \lambda_1 x + F \cosh \lambda_3 x - \frac{pR^2}{Eh} (1-\upsilon/2) \qquad [8]$$

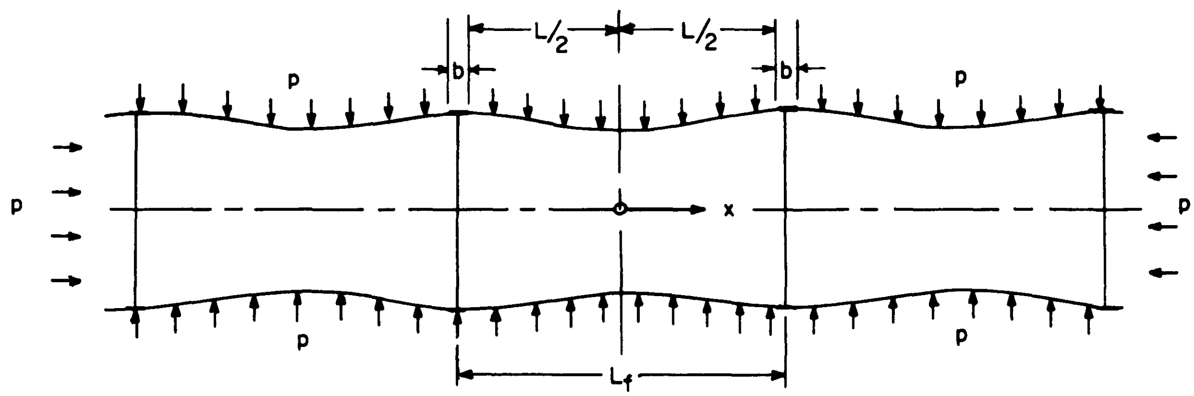

Figure 1 - Symmetrically Loaded Cylindrical Shell with Equally Spaced Reinforcing Ring Frames

The characteristic roots λ_1 and λ_3, which govern the nature of Solution [8], can be rewritten in the form

$$\lambda_1;\lambda_3 = \sqrt{2}\,\frac{\theta}{L}\left\{-\left(\frac{p}{p^*}\right) \pm \left[\left(\frac{p}{p^*}\right)^2 - 1\right]^{1/2}\right\}^{1/2} \qquad [9]$$

where

$$\theta = \sqrt[4]{3(1-\upsilon^2)}\;\frac{L}{\sqrt{Rh}} \qquad [10]$$

and

$$p^* = \frac{2E(h/R)^2}{\sqrt{3(1-\upsilon^2)}}$$

is the critical load for the axisymmetric elastic buckling of an unstiffened cylindrical shell under the action of uniform axial pressure; see Reference 5.

From Equations [9] it is seen that Solution [8] depends upon whether

$$\frac{p}{p^*} \lesseqgtr 1.0 \text{ or } \frac{p}{p^*} = 0 \qquad [11]$$

The Conditions [11] can therefore be termed a measure of the "beam-column effect" arising from the axial portion of the hydrostatic pressure loading. In the investigation of the present problem by Von Sanden and Günther,[1] in which the term $\frac{pR}{2} w,_{xx}$ appearing in Equation [1] was not considered, the conditions stipulated by [11] do not arise since for this approximation the ratio $\frac{p}{p^*}$ does not appear in the expression for the roots λ_1 and λ_3.

The special case in which $\frac{p}{p^*} = 0$ corresponds to that of zero axial pressure. This essentially is the solution considered in Reference 1 where the effect of the axial pressure on the bending deformations was neglected. Only the effect of the axial pressure on the membrane deformations was considered therein.

For the special case of equal roots, i.e., $\lambda_1 = \lambda_3 \equiv \bar{\lambda}_1$, which arises when $\frac{p}{p^*} = 1.0$, the solution for the displacement w given by [8] no longer is valid and must be modified to

$$w = \overline{B} \cosh \overline{\lambda}_1 x + \overline{F}\, \overline{\lambda}_1 x \sinh \overline{\lambda}_1 x - \frac{pR^2}{Eh} (1 - \upsilon/2) \qquad [12]$$

The four possible sets of characteristic roots λ_1 and λ_3 arising from Conditions [11], are considered in Appendix B.

BOUNDARY CONDITIONS

In order to evaluate the two remaining constants B and F in Solution [8] or $\overline{B}$ and $\overline{F}$ in Solution [12], conditions of continuity on the displacements and rotations at the junction of the shell and frame must be considered; see Figure 2.

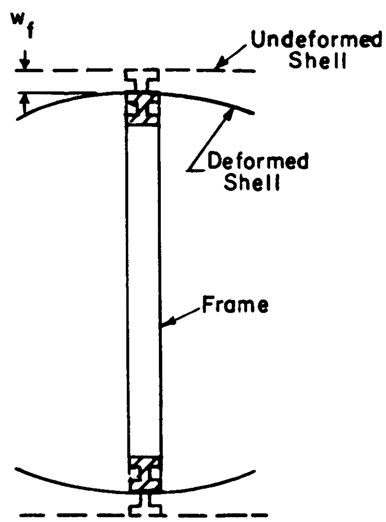

Figure 2 - Elastic Deformations at Junction of Shell and Frame.

The mode of deformation assumed to occur is such that all bays of shell plating between adjacent ring frames deflect radially inward with no axial rotation at the stiffener location so that the following condition must be satisfied:

$$\frac{dw}{dx} = 0 \quad \text{at} \quad x = \frac{L}{2} \qquad [13]$$

If the faying width b of the frame is a sizable percentage of the length of shell between frame centers, then the total load supported by a ring frame per unit length of the circumference must be

$$|Q^*| = 2Q_o + pb$$

where Q_o is the transverse shear transmitted by the shell to the frame and p is the hydrostatic pressure. Consequently, the hoop stress in the circular ring frame loaded as shown in Figure 3 is

$$\sigma_\phi = - \frac{(2Q_o + pb)R}{A_f + bh} \qquad [14]$$

The corresponding axisymmetric strain is (see Equation [A10])

$$\varepsilon_\phi = \frac{w_f}{R} = \frac{\sigma_\phi}{E} \qquad [15]$$

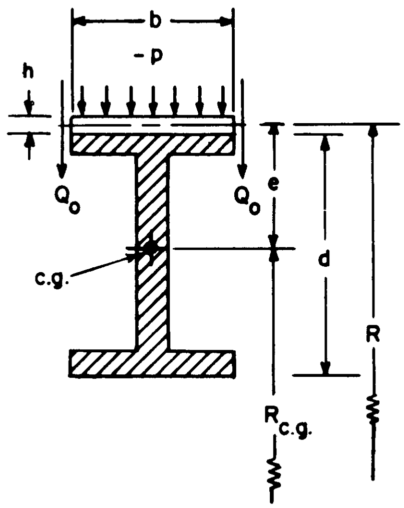

Figure 3 - Cross Section of Loaded Ring Frame

so that the radial displacement of the frame is

$$w_f = -\frac{(2Q_o + pb)R^2}{E(A_f + bh)} \qquad [16]$$

Von Sanden and Günther used relation [16] in their solution of the present problem. However, Viterbo pointed out in Reference 2 that the use of Equation [16] leads to the inconsistent result that the total load Q* per unit circumferential length carried by a strip of the shell of width b is less than pb when the area A_f of the supporting frame is set equal to zero. This inconsistency may be removed by considering the axial load per unit circumferential length, $-\frac{pR}{2}$, caused by hydrostatic pressure in addition to the radial pressure -p as shown in Figure 4. In such a case the total load supported by a ring frame per unit circumferential length is

$$|Q^*| = 2Q_o + pb(1-\upsilon/2) \qquad [17]$$

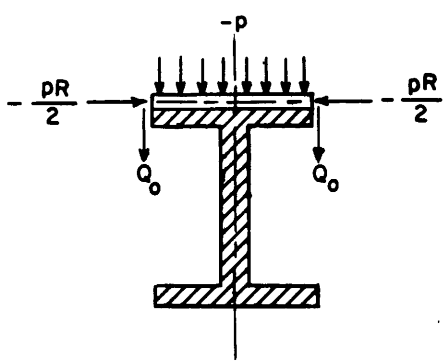

Figure 4 - Viterbo Modification to Frame Loading

Viterbo arrived at the same result by considering the radial pressure -p to be resolved into the two components, $-\upsilon p/2$ and $-p + \upsilon p/2$. The first component together with the longitudinal pressure $-pR/2$ produces no radial deformation of the frames, so that only the effect of the transverse load $-p(1-\upsilon/2)$ need be considered.

Equation [16] was also based on the assumption that the whole area of the frame is effectively concentrated at the radius of the median surface of the shell plating. This does not permit a distinction between internal and external frames. A correction to Equation [16] can very easily be made to allow for the fact that the frame cross-sectional area is concentrated at its centroid and the shell area bh is concentrated at its median surface. This can be seen if Equation [16] is rewritten as

$$\frac{E}{R^2}(A_f + bh)w_f = -2Q_o - pb \qquad [16a]$$

so that now it becomes

$$E\left(\frac{A_f}{R_{cg}R} + \frac{bh}{R^2}\right) w_f \equiv Kw_f = -2Q_o - pb(1-\upsilon/2) \qquad [18]$$

where R is the radius to the median surface of the shell and R_{cg} is the radius to the centroid of the frame cross section; see Figure 3. Equation [18] includes both the so-called Viterbo effect and the frame effect just discussed.

Wilson in Reference 6 has considered the question of the frame rigidity constant K in greater detail. He has derived the following expression taking into account the shape of cross section of the frame:

$$K = E\left(\frac{A_f}{R_{cg}^2} + \frac{bh}{R^2}\right) + 3E\left(\frac{I_f}{R_{cg}^4} + \frac{bh^3}{12R^4}\right) \qquad [19]$$

For practical purpose, the second term of [19] is of no consequence so that

$$K \approx E\left(\frac{A_f}{R_{cg}^2} + \frac{bh}{R^2}\right) \qquad [20]$$

It should be noted that the frame rigidity constant K of Equations [18] and [20] do not agree exactly.

A further refinement was introduced by Wilson in which all points of the frame and associated plating are assumed not to deflect radially the same amount. This may be necessary only for very deep frames with thin webs. In such a case the frame rigidity constant is given by

$$K = \frac{Eb(t_f + h)}{R^2} + \frac{R_1}{R} \frac{E}{\left[(1+\upsilon)\frac{k_1}{R_1} - (1-\upsilon)k_2R_1\right]} \qquad [21]$$

where

$$k_1 = \mp \frac{R_1^2 R_2^2 [h_w \pm (1-\upsilon)\, R_2\lambda_f/E]}{-h_w R_2^2 [h_w \mp (1+\upsilon)\, R_2\lambda_f/E]\; h_w R_2^2\, [h_w \pm (1-\upsilon)\, R_2\lambda_f/E]}$$

[22]

$$k_2 = \pm \frac{R_1^2 [h_w \mp (1-\upsilon)\, R_2\lambda_f/E]}{h_w R_1^2 [h_w \mp (1-\upsilon)\, R_2 L_f/E] - h_w R_2^2\, [h_w \pm (1-\upsilon)\, R_2\lambda_f/E]}$$

and

$$\lambda_f = \frac{E h_f f}{R_f^2}$$

is the flange rigidity constant. The upper and lower signs in the expressions for the k's refer to external and internal frames, respectively, and the dimensions of these frames are given in Figures 5a and 5b. The value given for K by Equation [21] will not differ significantly from that given by Equation [20] except for deep frames with thin webs in which the ratio of depth-of-frame to radius-of-shell is > 0.2.

It is convenient to introduce the idea of an effective frame area A_{eff} so that the boundary condition [18] can be written in the form

$$\frac{E}{R^2}\,(A_{eff} + bh)\, w_f = -\,2Q_o - pb(1-\upsilon/2) \qquad [23]$$

where from Equation [18]

$$A_{eff} = A_f\,(\frac{R}{R_{cg}}) \qquad [24a]$$

or from Equation [20]

$$A_{eff} = A_f \left(\frac{R}{R_{cg}}\right)^2 \qquad [24b]$$

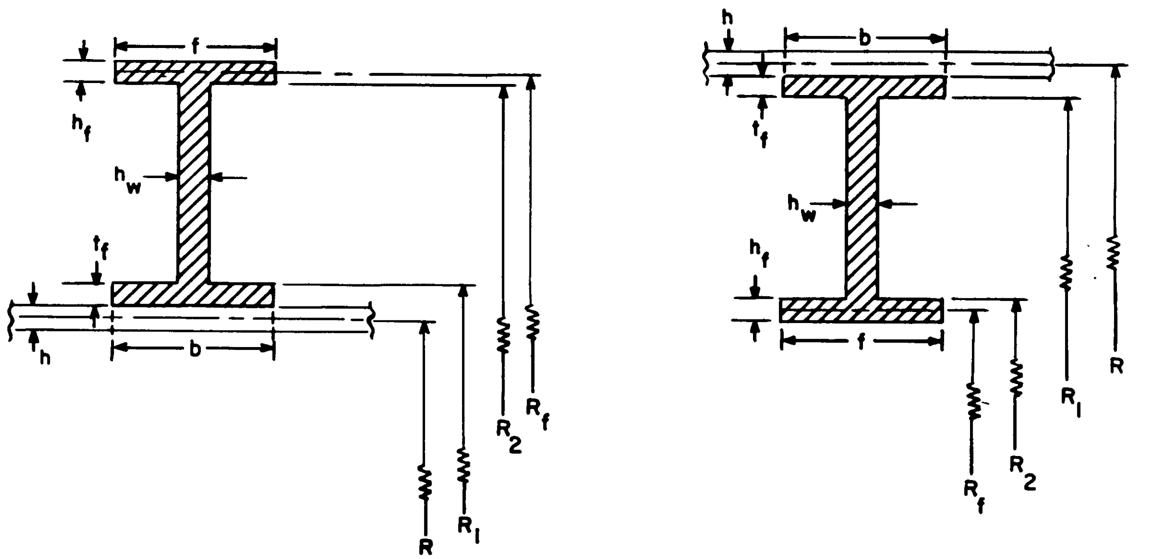

Figure 5a — External Deep Frame   Figure 5b — Internal Deep Frame

Figure 5 - Dimensions of Deep Frames

If the rigidity of a deep frame with a thin web as defined by Equation [21] is to be used, then

$$A_{eff} = bt_f + \frac{RR_1}{\left[(1+\nu)\frac{k_1}{R_1} - (1-\nu)\, k_2 R_1\right]} \qquad [25]$$

Since the transverse shear Q_x in the shell is related to the radial displacement w by Equations [A3] and [A4], i.e.,

$$Q_x = \frac{dM_x}{dx} - N_x \frac{dw}{dx} = - D \frac{d^3 w}{dx^3} + \frac{pR}{2} \frac{dw}{dx} \qquad [26]$$

for hydrostatic pressure, the boundary condition which requires continuity of radial displacement at the junction of the shell edge and frame in view of Equation [23] is

$$\frac{E}{R^2}(A_{eff} + bh)\, w_f = 2D\frac{d^3w}{dx^3} - pR\frac{dw}{dx} - pb\,(1-\upsilon/2) \text{ at } x = \frac{L}{2} \qquad [27]$$

where the effective frame area A_{eff} is that given by either [24] or [25] depending upon its geometry.

When Solution [8] is substituted into the boundary conditions [13] and [27], and the resulting equations are solved for the constants B and F, the following obtains:

$$B = \frac{-p(1-\upsilon/2)\frac{A_{eff}}{h}(\lambda_3 \sinh \lambda_3 \frac{L}{2})}{(\lambda_1 \sinh \lambda_1 \frac{L}{2})(K' \cosh \lambda_3 \frac{L}{2} - 2D\lambda_3^3 \sinh \lambda_3 \frac{L}{2}) - (\lambda_3 \sinh \lambda_3 \frac{L}{2})(K' \cosh \lambda_1 \frac{L}{2} - 2D\lambda_1^3 \sinh \lambda_1 \frac{L}{2})}$$

$$F = \frac{p(1-\upsilon/2)\frac{A_{eff}}{h}(\lambda_1 \sinh \lambda_1 \frac{L}{2})}{(\lambda_1 \sinh \lambda_1 \frac{L}{2})(K' \cosh \lambda_3 \frac{L}{2} - 2D\lambda_3^3 \sinh \lambda_3 \frac{L}{2}) - (\lambda_3 \sinh \lambda_3 \frac{L}{2})(K' \cosh \lambda_1 \frac{L}{2} - 2D\lambda_1^3 \sinh \lambda_1 \frac{L}{2})} \qquad [28]$$

When Solution [12] is substituted into the same boundary conditions and the resulting equations are solved for the constants $\overline{B}$ and $\overline{F}$, the following obtains:

$$\overline{B} = \frac{p(1-\upsilon/2)\frac{A_{eff}}{h}(\sinh \overline{\lambda}_1 \frac{L}{2} + \overline{\lambda}_1 \frac{L}{2} \cosh \overline{\lambda}_1 \frac{L}{2})}{K'(\overline{\lambda}_1 \frac{L}{2} + \sinh \overline{\lambda}_1 \frac{L}{2} \cosh \overline{\lambda}_1 \frac{L}{2}) + 4D\overline{\lambda}_1^3 \sinh^2 \overline{\lambda}_1 \frac{L}{2}} \qquad [29]$$

$$\bar{F} = \frac{-p(1-\upsilon/2)\,\frac{A_{eff}}{h}\,(\sinh \bar{\lambda}_1 \frac{L}{2})}{K'(\bar{\lambda}_1 \frac{L}{2} + \sinh \bar{\lambda}_1 \frac{L}{2} \cosh \bar{\lambda}_1 \frac{L}{2}) + 4D\bar{\lambda}_1^3 \sinh^2 \lambda_1 \frac{L}{2}} \qquad [29]$$

where
$$K' = \frac{E}{R^2}(A_{eff} + bh)$$

in both Equations [28] and [29].

DETERMINATION OF LONGITUDINAL AND CIRCUMFERENTIAL STRESSES IN SHELL

The expression for bending stress σ_b in the shell plating is derived on the same assumption used in simple beam theory that the bending stresses vary linearly across the thickness, i.e., $\sigma_y = \frac{My}{I}$. Thus the maximum stress corresponding to the longitudinal bending moment M_x is given by

$$\sigma_{xb} = \frac{M_x h/2}{I} = \pm \frac{Dh}{2I} \frac{d^2w}{dx^2} \qquad [30]$$

where the plus sign is for the outer fiber and the minus sign for the inner fiber of the shell plating, and

$$D = \frac{Eh^3}{12(1-\upsilon^2)} \quad ; \quad I = \frac{h^3}{12}$$

so that,

$$\sigma_{xb} = \pm \frac{Eh}{2(1-\upsilon^2)} \frac{d^2w}{dx^2} \qquad [31]$$

The longitudinal curvatures or second derivatives of the radial displacement w(x) from Equations [8] and [12], respectively, are

$$\frac{d^2w}{dx^2} = B\lambda_1^2 \cosh \lambda_1 x + F\lambda_3^2 \cosh \lambda_3 x \qquad [32]$$

and

$$\frac{d^2w}{dx^2} = \bar{B}\bar{\lambda}_1^2 \cosh \bar{\lambda}_1 x + \bar{F}\bar{\lambda}_1^3 x \sinh \bar{\lambda}_1 x + 2\bar{F}\bar{\lambda}_1^2 \cosh \bar{\lambda}_1 x \qquad [33]$$

where the constants B, F and $\bar{B}$, $\bar{F}$ are given by Equations [28] and [29], respectively. Thus the bending stresses [31] become:

$$\sigma_{xb} = \pm \frac{Eh}{2(1-\upsilon^2)} [B\lambda_1^2 \cosh \lambda_1 x + F\lambda_3^2 \cosh \lambda_3 x] \qquad [34]$$

and

$$\sigma_{xb} = \pm \frac{Eh}{2(1-\upsilon^2)} [\bar{B}\bar{\lambda}_1^2 \cosh \bar{\lambda}_1 x + \bar{F}\bar{\lambda}_1^3 x \sinh \bar{\lambda}_1 x + 2\bar{F}\bar{\lambda}_1^2 \cosh \bar{\lambda}_1 x] \qquad [35]$$

The total longitudinal stress σ_X in the shell is then given as the sum of the longitudinal bending component σ_{xb} plus the longitudinal membrane component, i.e.,

$$\sigma_X = \sigma_{xb} - \frac{pR}{2h} \qquad [36]$$

so that

$$\sigma_{X_i^o} = -\frac{pR}{2h} \pm \frac{Eh}{2(1-\upsilon^2)} [B\lambda_1^2 \cosh \lambda_1 x + F\lambda_3^2 \cosh \lambda_3 x] \qquad [37]$$

and

$$\sigma_{X_i^o} = -\frac{pR}{2h} \pm \frac{Eh}{2(1-\upsilon^2)} [\bar{B}\bar{\lambda}_1^2 \cosh \lambda_1 x + \bar{F}\bar{\lambda}_1^3 x \sinh \bar{\lambda}_1 x + 2\bar{F}\bar{\lambda}_1^2 \cosh \bar{\lambda}_1 x] \qquad [38]$$

where the subscripts o and i refer to outer and inner fibers of the shell plating in conjunction with the plus and minus signs, respectively.

From Equations [A9] and [A10], the circumferential membrane stress σ_{ϕ_M} in the shell for the case of hydrostatic pressure is given by

$$\sigma_{\phi M} = \frac{N_\phi}{h} = E\,\frac{w}{r} - \upsilon\,\frac{pR}{2h} \qquad [39]$$

When the appropriate expression for the radial displacement w is substituted into [39], the circumferential membrane stress becomes

$$\sigma_{\phi M} = -\frac{pR}{h} + \frac{E}{R}\,[B \cosh \lambda_1 x + F \cosh \lambda_3 x] \qquad [40]$$

and

$$\sigma_{\phi M} = -\frac{pR}{h} + \frac{E}{R}\,[\bar{B} \cosh \bar{\lambda}_1 x + \bar{F}\,\bar{\lambda}_1 x \sinh \bar{\lambda}_1 x] \qquad [41]$$

Equations [40] and [41] give the axisymmetric hoop stress which is assumed to be uniform through the thickness of the shell plating.

For the case of axial symmetry, the bending moment M_ϕ in the circumferential direction is

$$M_\phi = \upsilon M_x = \pm\,\upsilon D\,\frac{d^2 w}{dx^2} \qquad [42]$$

Thus the circumferential bending stress corresponding to the circumferential bending moment M_ϕ given above is seen to be

$$\sigma_{\phi b} = \upsilon \sigma_{xb} \qquad [43]$$

The total circumferential stress σ_Φ in the shell is then given as the sum of the circumferential membrane component $\sigma_{\phi M}$ plus the circumferential bending component $\sigma_{\phi b}$, i.e.,

$$\sigma_\Phi = \sigma_{\phi M} + \upsilon \sigma_{xb} \qquad [44]$$

Hence with the appropriate expressions, Equations [40] or [41] for $\sigma_{\phi M}$ and Equations [34] or [35] for σ_{xb}, the following expressions are obtained from Equation [44]

$$\sigma_{\Phi_i^o} = -\frac{pR}{h} + \left[\frac{E}{R} \pm \frac{\upsilon E h \lambda_1^2}{2(1-\upsilon^2)}\right] B \cosh \lambda_1 x + \left[\frac{E}{R} \pm \frac{\upsilon E h \lambda_3^2}{2(1-\upsilon^2)}\right] F \cosh \lambda_3 x \qquad [45]$$

and

$$\sigma_{\Phi_i^o} = -\frac{pR}{h} + \left[\frac{E}{R} \pm \frac{\upsilon Eh\bar{\lambda}_1^2}{2(1-\upsilon^2)}\right] [\bar{B} \cosh \bar{\lambda}_1 x + \bar{F}\, \bar{\lambda}_1 x \sinh \bar{\lambda}_1 x] \pm \frac{\upsilon Eh\bar{\lambda}_1^2}{(1-\upsilon^2)} [\bar{F} \cosh \bar{\lambda}_1 x] \qquad [46]$$

Equations [37], [38] and [45], [46] give the distribution of total longitudinal stress and total circumferential stress in the shell plating between adjacent ring frames.

The points of critical stress are at midbay (x=o) and at a frame ($x=\frac{L}{2}$). At these locations the total longitudinal and total circumferential stresses $\sigma_{X_i^o}$ and $\sigma_{\Phi_i^o}$, respectively, become

At midbay

$$\sigma_{X_i^o m} = -\frac{pR}{2h} \pm \frac{Eh}{2(1-\upsilon^2)} [B\lambda_1^2 + F\lambda_3^2] \qquad [47]$$

$$\rightarrow \sigma_{X_i^o m} = -\frac{pR}{2h} \pm \frac{Eh\bar{\lambda}_1^2}{2(1-\upsilon^2)} [\bar{B} + 2\bar{F}] \qquad [48]$$

$$\sigma_{\Phi_i^o m} = -\frac{pR}{h} + \frac{E}{R} [B + F] \pm \frac{\upsilon Eh}{2(1-\upsilon^2)} [B\lambda_1^2 + F\lambda_3^2] \qquad [49]$$

$$\rightarrow \sigma_{\Phi_i^o m} = -\frac{pR}{h} + \frac{E}{R} [\bar{B}] \pm \frac{\upsilon Eh\bar{\lambda}_1^2}{2(1-\upsilon^2)} [\bar{B} + 2\bar{F}] \qquad [50]$$

At a frame

$$\sigma_{X_i^o f} = -\frac{pR}{2h} \pm \frac{Eh}{2(1-\upsilon^2)} [B\lambda_1^2 \cosh \lambda_1 \frac{L}{2} + F\lambda_3^2 \cosh \lambda_3 \frac{L}{2}] \qquad [51]$$

$$\sigma_{x_1^o f} = -\frac{pR}{2h} \pm \frac{Eh\bar{\lambda}_1^2}{2(1-\upsilon^2)}\left[\bar{B}\cosh\bar{\lambda}_1\frac{L}{2} + \bar{F}\bar{\lambda}_1\frac{L}{2}\sinh\bar{\lambda}_1\frac{L}{2} + 2\bar{F}\cosh\bar{\lambda}_1\frac{L}{2}\right] \quad [52]$$

$$\sigma_{\Phi_1^o f} = -\frac{pR}{h} + \frac{E}{R}\left[B\cosh\lambda_1\frac{L}{2} + F\cosh\lambda_3\frac{L}{2}\right] \pm \frac{\upsilon Eh}{2(1-\upsilon^2)}\left[B\lambda_1^2\cosh\lambda_1\frac{L}{2} + F\lambda_3^2\cosh\lambda_3\frac{L}{2}\right] \quad [53]$$

$$\sigma_{\Phi_1^o f} = -\frac{pR}{h} + \left[\frac{E}{R} \pm \frac{\upsilon Eh\bar{\lambda}_1^2}{2(1-\upsilon^2)}\right]\left[\bar{B}\cosh\bar{\lambda}_1\frac{L}{2} + \bar{F}\bar{\lambda}_1\frac{L}{2}\sinh\bar{\lambda}_1\frac{L}{2}\right] \pm \frac{\upsilon Eh\bar{\lambda}_1^2}{(1-\upsilon^2)}\left[\bar{F}\cosh\bar{\lambda}_1\frac{L}{2}\right] \quad [54]$$

where the constants B and F are given by Equations [28] and $\bar{B}$ and $\bar{F}$ are given by Equations [29].

DETERMINATION OF TOTAL RADIAL LOAD ON A RING FRAME

The total radial load supported by a ring frame per unit circumferential length is given by Equation [17] as

$$|Q^*| = 2Q_o + pb(1-\upsilon/2)$$

and, since the transverse shears Q_o are given by Equation [26], evaluated at the shell edge where the shell meets the frame, i.e., at $x=\frac{L}{2}$, then the total compressive load on the ring becomes

$$Q^* = 2D\left[\frac{d^3w}{dx^3}\right]_{x=\frac{L}{2}} - pR\left[\frac{dw}{dx}\right]_{x=\frac{L}{2}} - pb(1-\upsilon/2) \quad [55]$$

Differentiating Equation [8] and then substituting the result into Equation [55] and introducing the condition prescribed by Equation [13] leads to the following expression for the total ring load:

$$Q^* = \frac{Eh^3}{6(1-\upsilon^2)} [B\lambda_1^3 \sinh \lambda_1 \frac{L}{2} + F\lambda_3^3 \sinh \lambda_3 \frac{L}{2}] - pb(1-\upsilon/2) \qquad [56]$$

where the constants B and F are given by Equations [28] .

For the special case of equal roots, i.e., $\lambda_1 = \lambda_3 \equiv \bar{\lambda}_1$, the deflection w (x) given by Equation [12] is used in conjunction with Equation [55] to yield the following expression for the total ring load:

$$Q^* = \frac{Eh^3\bar{\lambda}_1^3}{6(1-\upsilon^2)} [\bar{B} \sinh \bar{\lambda}_1 \frac{L}{2} + \bar{F}\,\bar{\lambda}_1 \frac{L}{2} \cosh \bar{\lambda}_1 \frac{L}{2} + 3\bar{F} \sinh \bar{\lambda}_1 \frac{L}{2}] - pb(1-\upsilon/2) \qquad [57]$$

where the constants $\bar{B}$ and $\bar{F}$ are given by Equations [29] .

Once the total ring load Q* is determined from either Equation [56] or [57] , as the case may be, the circumferential stress at the frame flange can be found from the following equation:

$$\sigma_{\phi f} = \frac{Q^* R^2}{(A_{eff} + bh)(R + d + \frac{h}{2})} \qquad [58]$$

Equation [58] results from a substitution of [23] for w_f into [15] and replacement of R by $(R + d + \frac{h}{2})$ in the latter equation. The derivation of equation [58] is based on the assumption that the circumferential stress σ_ϕ varies linearly across the depth of the frame, from the point where it meets the shell to the flange. Therefore, Equation [58] for the flange stress is valid only for frames that are not too deep according to the same criterion which governs the use of either Equation [20] or [21] for the frame stiffness K. Otherwise the circumferential stress varies according to a hyperbolic distribution across the depth of the frame, and in that case the Lamé solution for a thick-walled cylinder can be used in place of [58] .

SUMMARY OF FORMULAS FOR CASE (1) WHERE $\frac{p}{p^*} < 1.0$

The most important case of a ring-stiffened cylindrical shell under the action of hydrostatic pressure loading encountered in actual practice is that corresponding to $\frac{p}{p^*} < 1.0$. This is the case where the applied pressure is less than that which would cause axisymmetric elastic buckling of an unstiffened cylindrical shell of the same dimensions. In this instance, the characteristic roots λ_1 and λ_3 which govern the nature of the solution for the axisymmetric deformations and stresses are given by Equations [B6]. Substituting these values of λ_1 and λ_3 into Equations [28] gives the following expressions for the integration constants B and F:

$$B = \frac{-p(1-\nu/2)\frac{A_{eff}}{h}\left(\eta_1 - i\eta_2\right)[\sinh \eta_1\theta \cos \eta_2\theta - i \cosh \eta_1\theta \sin \eta_2\theta]}{2i\eta_1\eta_2K'\left[\frac{\sinh \eta_1\theta \cosh \eta_1\theta}{\eta_1} + \frac{\sin \eta_2\theta \cos \eta_2\theta}{\eta_2}\right] + 8i\eta_1\eta_2\left(\eta_1^2 + \eta_2^2\right)\left(\frac{2\theta}{L}\right)^3 D[\cosh^2 \eta_1\theta - \cos^2 \eta_2\theta]}$$

F = [59]

$$F = \frac{p(1-\nu/2)\frac{A_{eff}}{h}\left(\eta_1 + i\eta_2\right)[\sinh \eta_1\theta \cos \eta_2\theta + i \cosh \eta_1\theta \sin \eta_2\theta]}{2i\eta_1\eta_2K'\left[\frac{\sinh \eta_1\theta \cosh \eta_1\theta}{\eta_1} + \frac{\sin \eta_2\theta \cos \eta_2\theta}{\eta_2}\right] + 8i\eta_1\eta_2\left(\eta_1^2 + \eta_2^2\right)\left(\frac{2\theta}{L}\right)^3 D[\cosh^2 \eta_1\theta - \cos^2 \eta_2\theta]}$$

where the following nondimensional parameters have been introduced:

$$\eta_1 = \frac{1}{2}\sqrt{1-\gamma}$$

$$\eta_2 = \frac{1}{2}\sqrt{1+\gamma} \qquad [60]$$

$$\gamma \equiv \frac{p}{p^*} = \frac{p}{2E}\sqrt{3(1-\upsilon^2)}\left(\frac{R}{h}\right)^2$$

$$\theta = \sqrt[4]{3(1-\upsilon^2)}\ \frac{L}{\sqrt{Rh}} \qquad [60]$$

It should be noted that the constants B and F as given above are complex quantities. Even so, when they are substituted into Equation [8], Equations [47], [49], [51], [53], and [56], the deflections, stresses, and ring load, come out to be real quantities, as they should.

Upon substitution of Equations [59] for the constants B and F into Equation [8], the following expression for the axisymmetric radial displacement w as a function of the axial coordinate x is obtained:

$$w(x) = \qquad [61]$$

$$-\frac{pR^2}{Eh}(1-\upsilon/2)\left\{1-\left[\frac{\alpha F_2}{\alpha+\beta+(1-\beta)F_1}\left(\cosh 2\theta\eta_1\frac{x}{L}\cos 2\theta\eta_2\frac{x}{L}+\frac{\left(\sqrt{\frac{0.91}{3}}\frac{F_4}{F_2}+\gamma\right)}{4\eta_1\eta_2}\sinh 2\theta\eta_1\frac{x}{L}\cdot\right.\right.\right.$$

$$\left.\left.\left.\cdot\sin 2\theta\eta_2\frac{x}{L}\right)\right]\right\}$$

where

$$\alpha = \frac{A_{eff}}{L_f h} \quad ; \quad \beta = \frac{b}{L_f}$$

and the functions F_1, F_2, and F_4 are defined by Equations [72], [73], and [75], respectively. A distribution of w(x) for a given shell geometry can be conveniently determined from Equation [61] with the aid of the curves given in Figures 6 and 7. Of particular interest are the radial displacements at midbay, i.e. at x=o, and at the reinforcing frames, i.e. at $x=\frac{L}{2}$. Substituting these values of x into Equation [61], respectively, the following expressions are obtained:

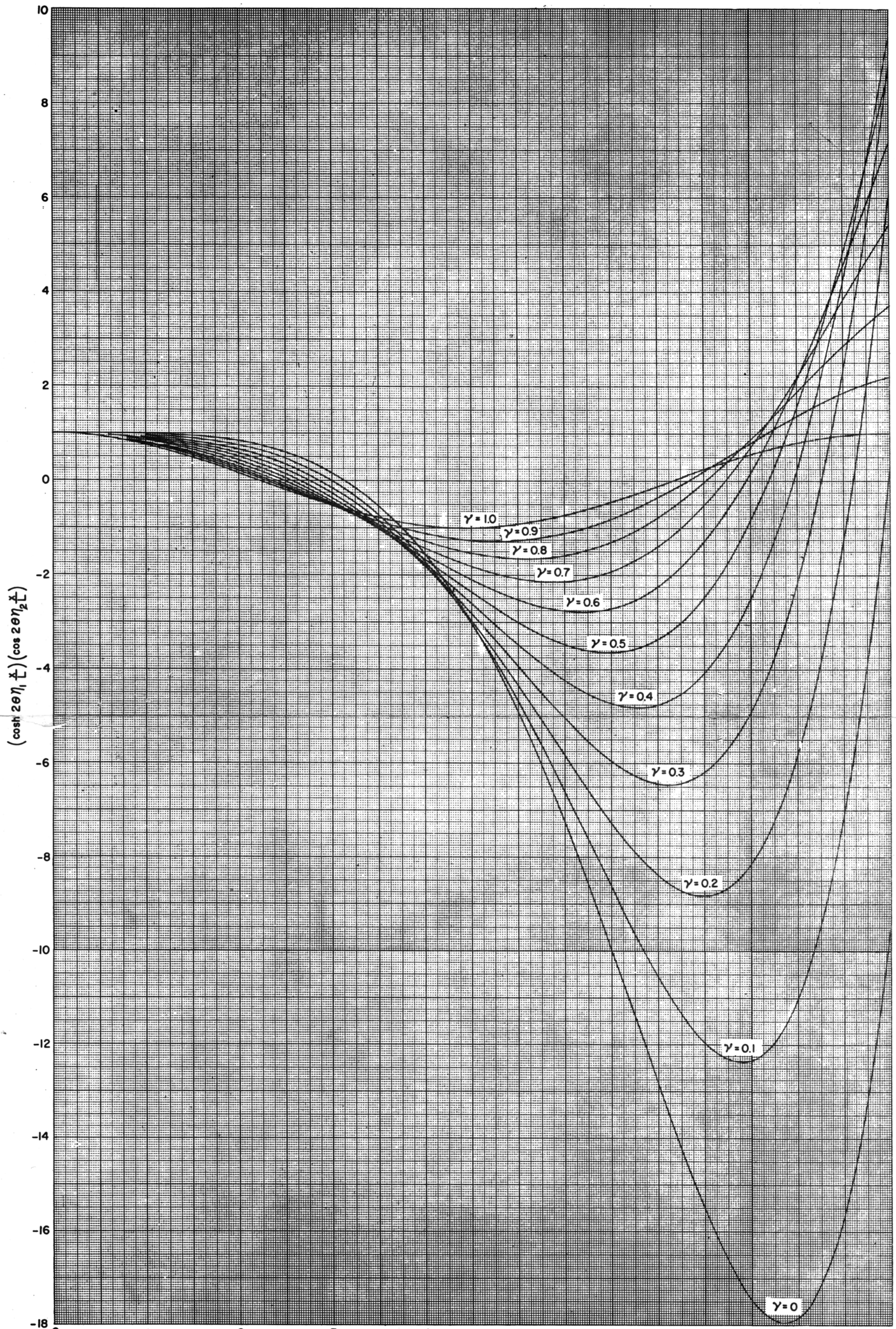

Figure 6 - Plot of $(\cosh 2\theta\eta_1 \frac{X}{L})$ $(\cos 2\theta\eta_2 \frac{X}{L})$

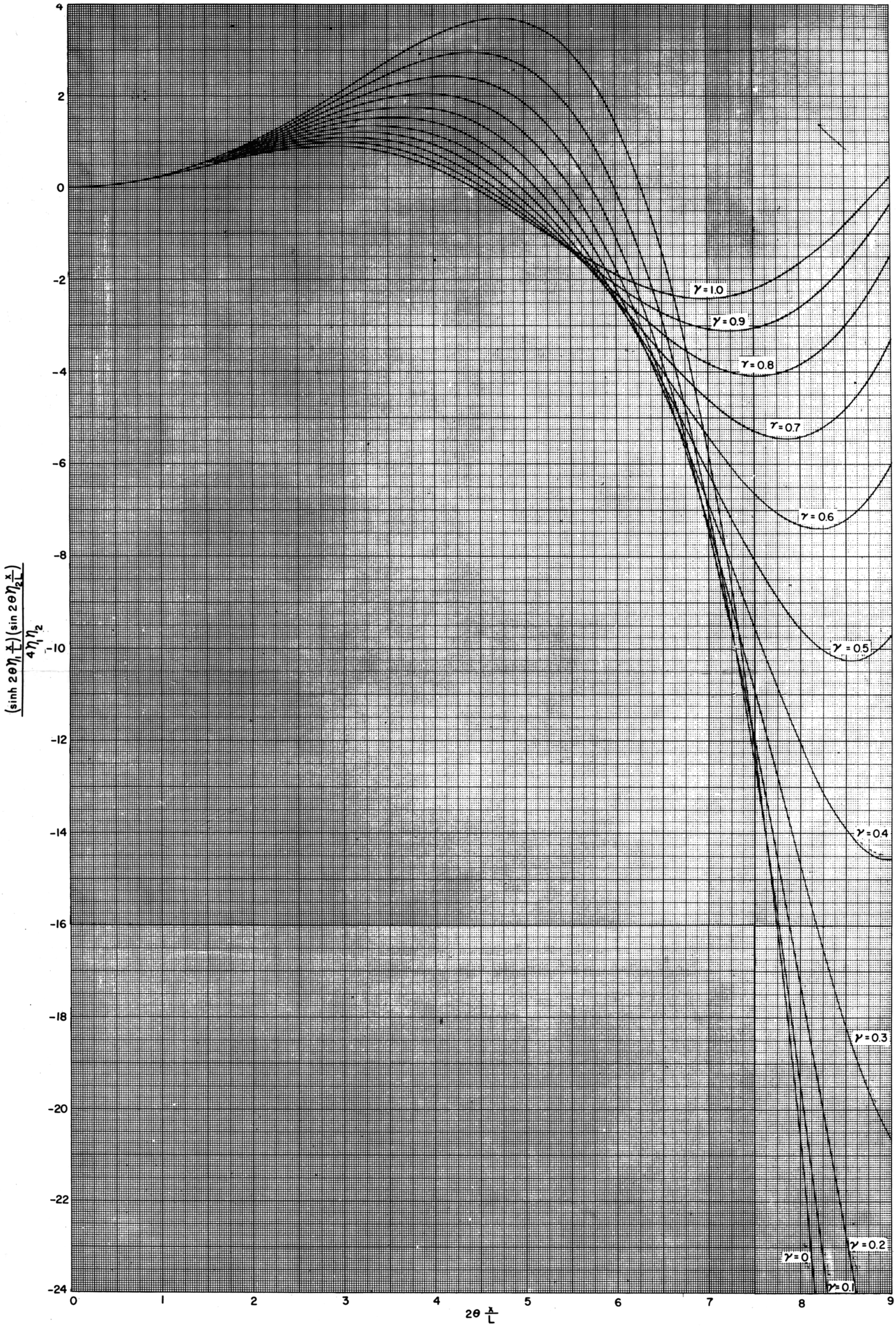

Figure 7 - Plot of $(\sinh 2\theta\eta_1 \frac{x}{L})$ $(\sin 2\theta\eta_2 \frac{x}{L}) \div 4\eta_1\eta_2$

At midbay

$$w_m = -\frac{pR^2}{Eh}(1-\upsilon/2)\left\{\frac{\alpha F_2}{\alpha+\beta+(1-\beta)F_1}\right\} \qquad [62]$$

At a frame

$$w_f =$$

$$-\frac{pR^2}{Eh}(1-\upsilon/2)\left\{1-\frac{\alpha F_2}{\alpha+\beta+(1-\beta)F}\left(\cosh\eta_1\theta\cos\eta_2\theta+\frac{\sqrt{\frac{0.91}{3}}\,\frac{F_4}{F_2}+\gamma}{4\eta_1\eta_2}\sinh\eta_1\theta\sin\eta_2\theta\right)\right\}$$

[63]

Upon substitution of Equations [59] for the constants B and F into Equations [47], [49], [51], and [53], respectively, the following expressions for the critical stresses are obtained:

$$\frac{\sigma_{X_i^o m}}{\sigma_u} = \frac{1}{2} \pm \frac{\sigma_{xbm}}{\sigma_u} \qquad [64]$$

$$\frac{\sigma_{\Phi_i^o m}}{\sigma_u} = 1-\left(1-\frac{\sigma_{\phi Mf}}{\sigma_u}\right)F_2 \pm \upsilon\frac{\sigma_{xbm}}{\sigma_u} \qquad [65]$$

$$\frac{\sigma_{X_i^o f}}{\sigma_u} = \frac{1}{2} \pm \left(1-\frac{\sigma_{\phi Mf}}{\sigma_u}\right)\sqrt{\frac{0.91}{1-\upsilon^2}}\,F_3 \qquad [66]$$

$$\frac{\sigma_{\Phi_i^o f}}{\sigma_u} = 1-\left(1-\frac{\sigma_{\phi Mf}}{\sigma_u}\right) \pm \upsilon\underbrace{\left(1-\frac{\sigma_{\phi Mf}}{\sigma_u}\right)}_{\text{"a"}}\sqrt{\frac{0.91}{1-\upsilon^2}}\,F_3 \qquad [67]$$

pict their behavior with variations in the geometric parameters of the shell:

$$F_1 = \left(\frac{4}{\theta}\right)\left[\frac{\cosh^2 \eta_1\theta - \cos^2 \eta_2\theta}{\dfrac{\cosh \eta_1\theta \sinh \eta_1\theta}{\eta_1} + \dfrac{\cos \eta_2\theta \sin \eta_2\theta}{\eta_2}}\right] \quad [72]$$

$$F_2 = \left[\frac{\dfrac{\cosh \eta_1\theta \sin \eta_2\theta}{\eta_2} + \dfrac{\sinh \eta_1\theta \cos \eta_2\theta}{\eta_1}}{\dfrac{\cosh \eta_1\theta \sinh \eta_1\theta}{\eta_1} + \dfrac{\cos \eta_2\theta \sin \eta_2\theta}{\eta_2}}\right] \quad [73]$$

$$F_3 = \sqrt{\frac{3}{0.91}}\left[\frac{-\dfrac{\cosh \eta_1\theta \sinh \eta_1\theta}{\eta_1} + \dfrac{\cos \eta_2\theta \sin \eta_2\theta}{\eta_2}}{\dfrac{\cosh \eta_1\theta \sinh \eta_1\theta}{\eta_1} + \dfrac{\cos \eta_2\theta \sin \eta_2\theta}{\eta_2}}\right] \quad [74]$$

$$F_4 = \sqrt{\frac{3}{0.91}}\left[\frac{\dfrac{\cosh \eta_1\theta \sin \eta_2\theta}{\eta_2} - \dfrac{\sinh \eta_1\theta \cos \eta_2\theta}{\eta_1}}{\dfrac{\cosh \eta_1\theta \sinh \eta_1\theta}{\eta_1} + \dfrac{\cos \eta_2\theta \sin \eta_2\theta}{\eta_2}}\right] \quad [75]$$

The form of the expressions for the stresses, Equations [64] through [69], and the functions F_1, F_2, F_3, and F_4, Equations [72] through [75], were developed by M. A. Krenzke and R. D. Short in Reference 4. The curves of Figures 8 through 11 in this report were taken from Reference 4.

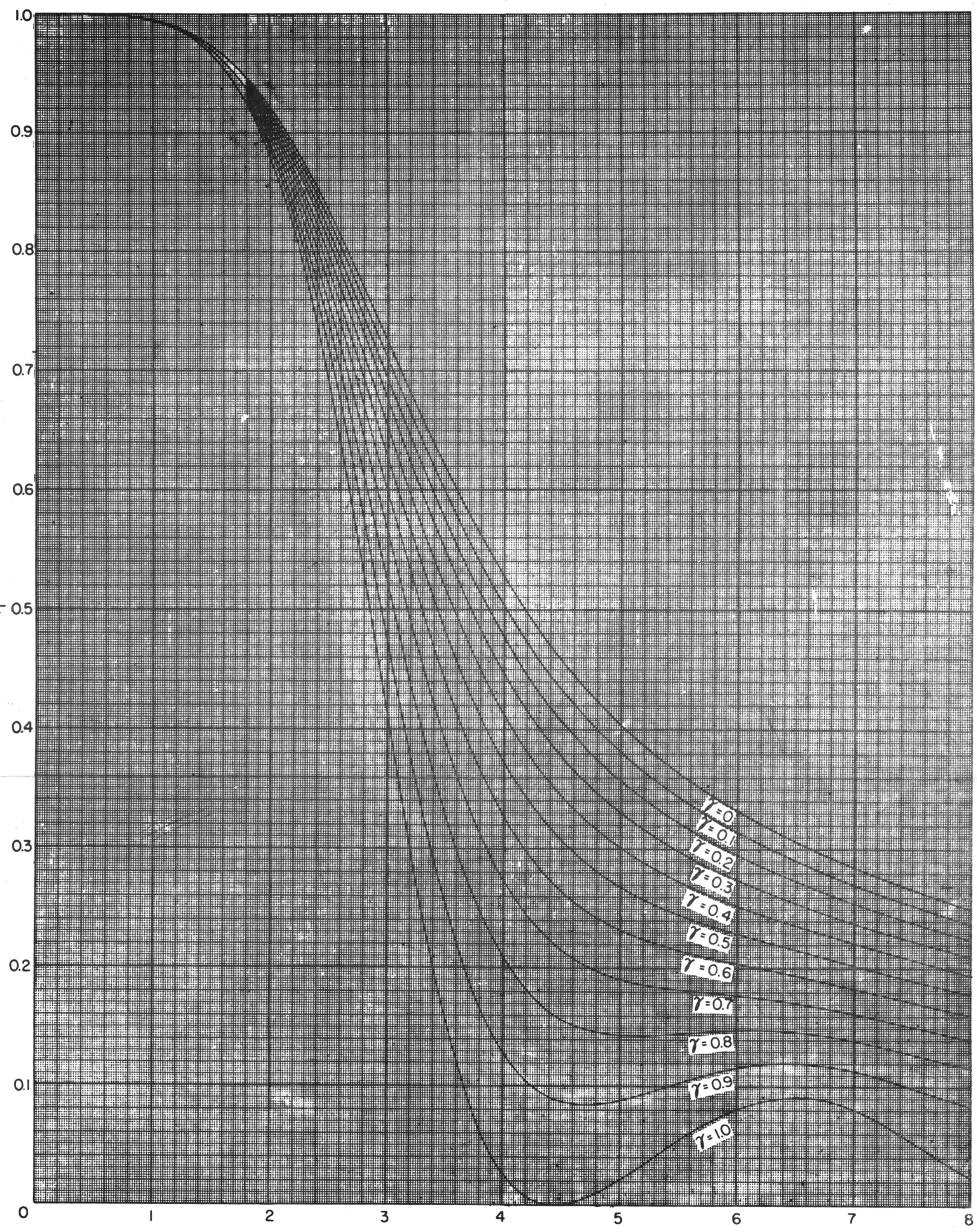

Figure 8 - Plot of the Function $F_1(\theta, \gamma)$

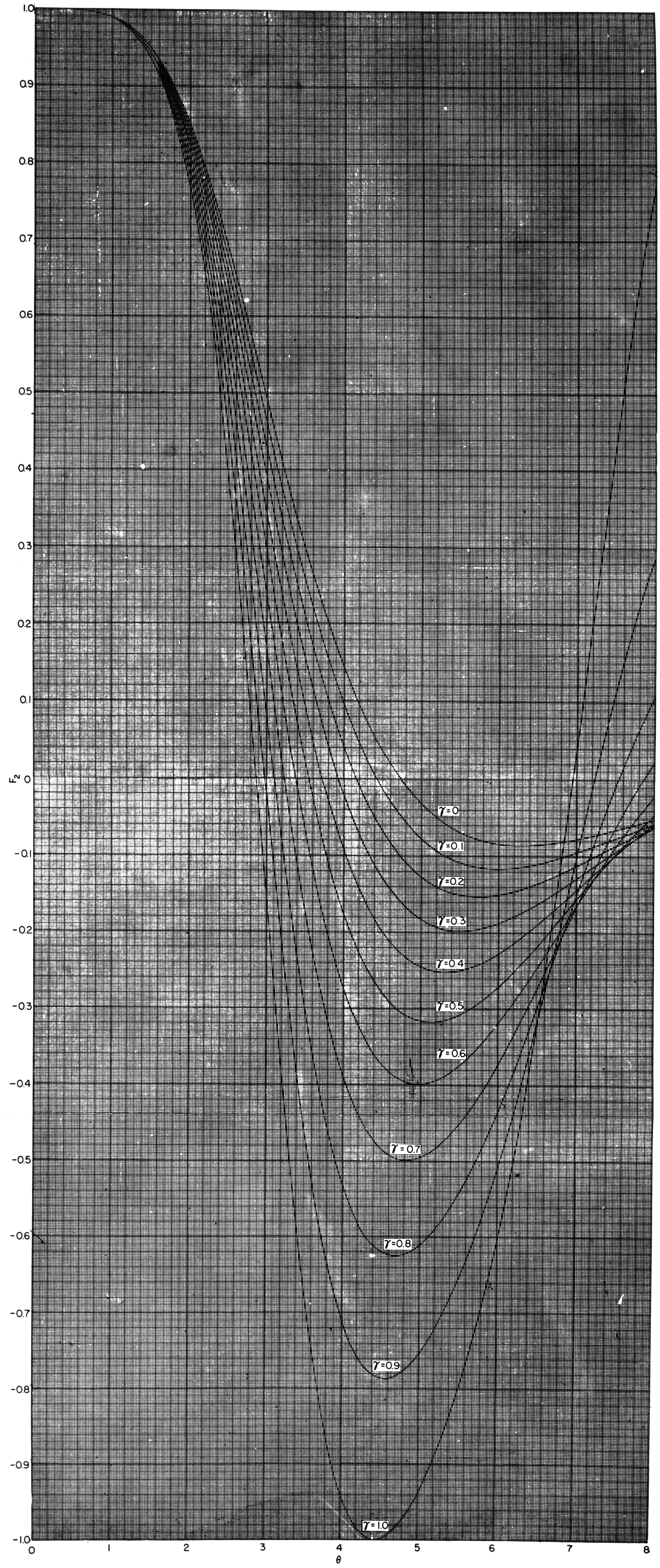

Figure 9 - Plot of the Function $F_2(\theta, \gamma)$

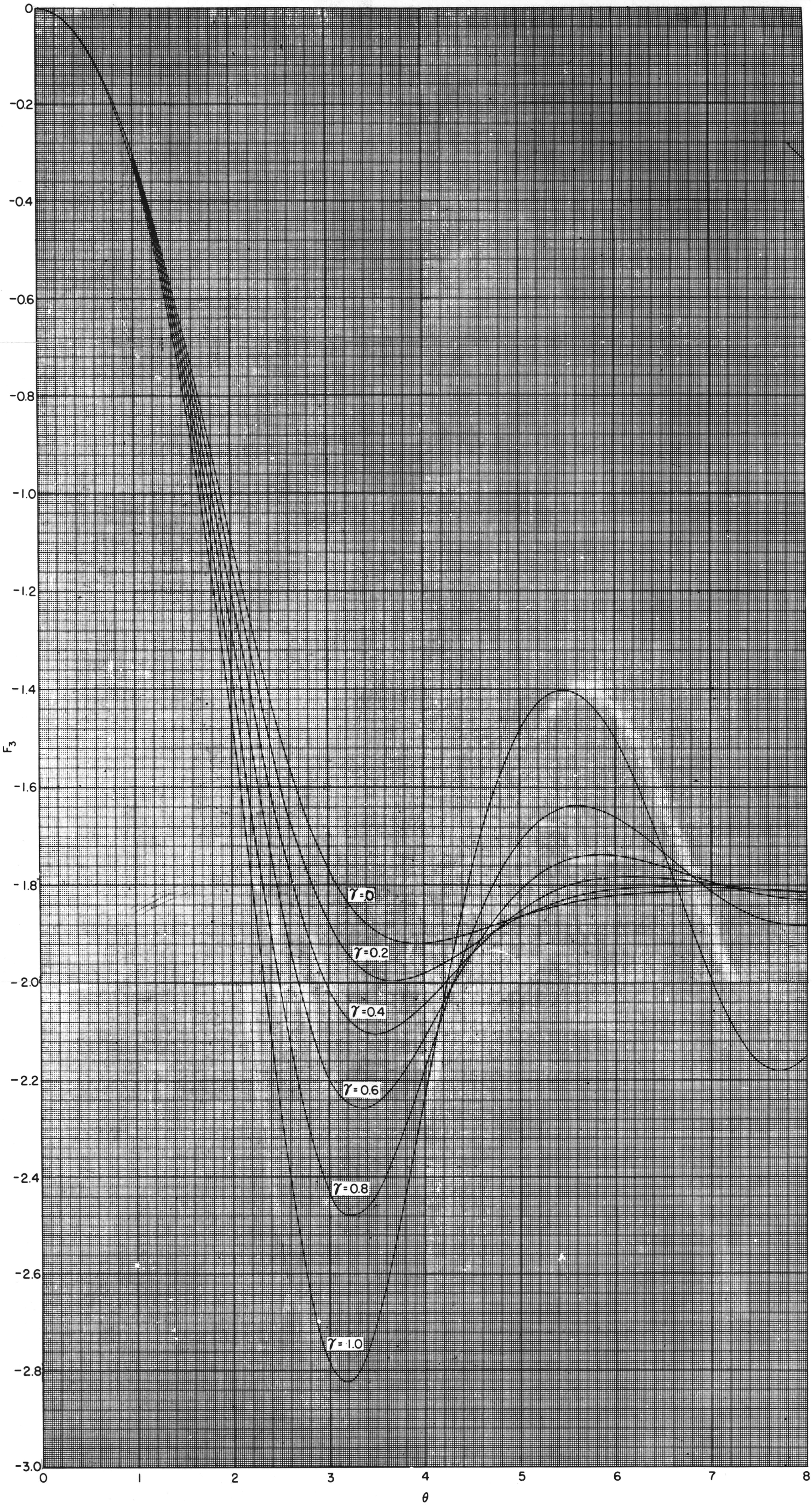

Figure 10 - Plot of the Function F_3 (θ, γ)

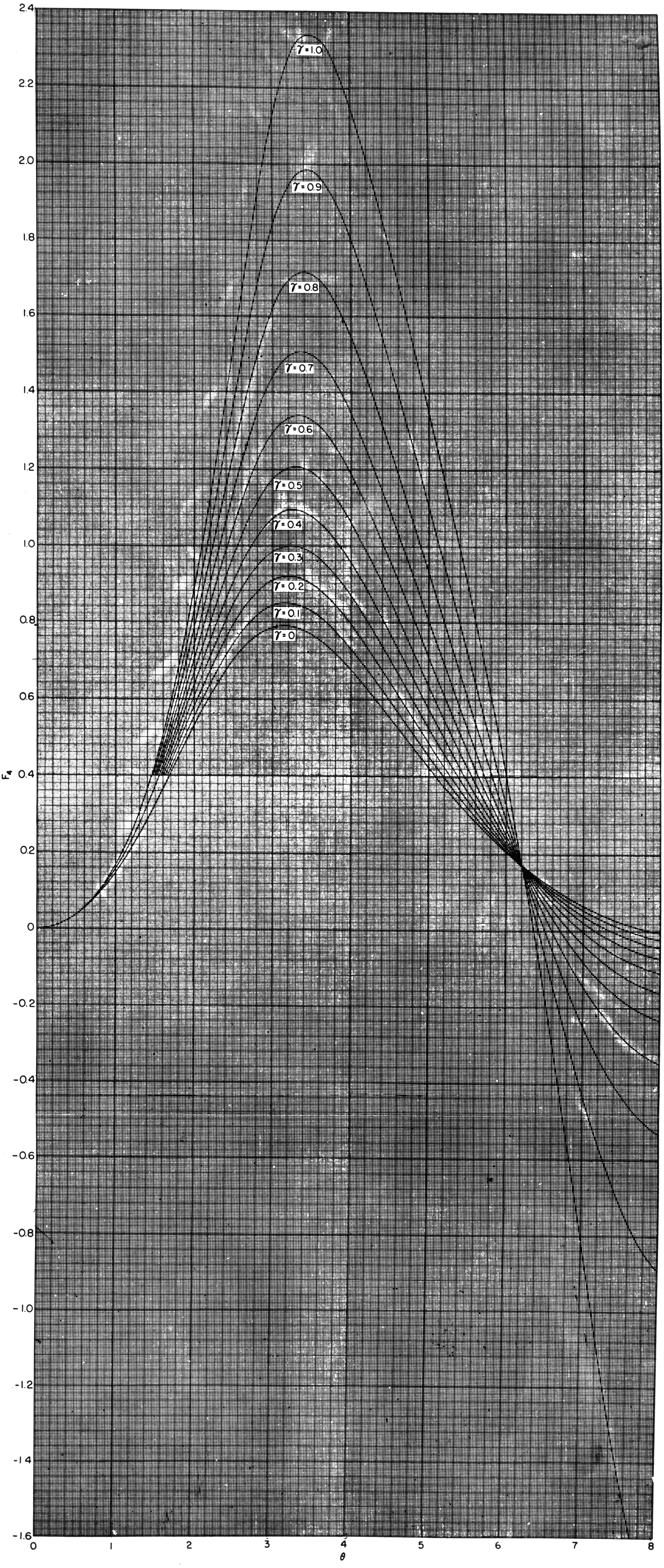

Figure 11 - Plot of the Function $F_4(\theta, \gamma)$

It should be noted that, for the special case where $\gamma = 0$, i.e., no "beam-column effect," the curves of Figures 8 through 11 correspond to the solution developed by Von Sanden and Günther,[1] modified to include the "Viterbo effect" at the frames. From this graphical representation, the influence of the "beam-column effect" on the axisymmetric deflections and stresses in a ring-stiffened cylinder for different geometries can be immediately assessed.

CRITERION FOR AXISYMMETRIC ELASTIC BUCKLING

The special case of equal roots, Case (2) in Appendix B where $\lambda_1 = \lambda_3$, corresponds to the solution in which the applied hydrostatic pressure p is equal in magnitude to the axial pressure p* for axisymmetric elastic buckling of an unstiffened cylindrical shell, that is, $\frac{p}{p^*} = 1.0$. Hence,

$$p_{cr} = p^* = \frac{2E(h/R)^2}{\sqrt{3(1-\upsilon^2)}} \qquad [76]$$

and the corresponding buckling stress is

$$\sigma_{cr} = \frac{E(h/R)}{\sqrt{3(1-\upsilon^2)}} \qquad [77]$$

The present formulation of the axisymmetric deformation problem requires that the edges of the shell where they meet the frames remain horizontal so that the buckling stress must be larger than that given by Equation [77] . Mathematically, this means that the axial stress resultant N_x required to produce buckling of a cylindrical shell with edges restrained against axial rotation must be

$$\left(\frac{N_x}{2D}\right)^2 > \frac{Eh}{DR^2} \qquad [78]$$

This corresponds to Case (3) in which $\frac{p}{p^*} > 1.0$, and the characteristic roots given by Equation [B10] in Appendix B are

$$\lambda_1 = i\,\frac{2\theta}{L}\,(\eta_1 + \eta_2) \qquad [79]$$

$$\lambda_3 = i\,\frac{2\theta}{L}\,(\eta_1 - \eta_2) \qquad [79]$$

In small-displacement theory, the condition for instability of a structure may be obtained by requiring that the displacement become infinitely large at buckling. This criterion would not apply to the Von Sanden and Günther formulation of the problem because the "beam-column" term necessary for buckling has not been included. However, with the present formulation a criterion for axisymmetric elastic buckling can be extracted by setting the denominator of the constants B and F appearing in the solution, Equation [8], equal to zero. Therefore, upon substitution of Equations [79] into Equations [28] and thence setting the denominators of B and F, which are identical, equal to zero, the following obtains:

$$\alpha + \beta + (1-\beta)\,F_5 = 0 \qquad [80]$$

where

$$F_5 = \left(\frac{4}{\theta}\right)\left[\frac{\cos^2 \eta_1'\theta - \cos^2 \eta_2\theta}{\dfrac{\cos \eta_1'\theta \sin \eta_1'\theta}{\eta_1'} + \dfrac{\cos \eta_2\theta \sin \eta_2\theta}{\eta_2}}\right] \qquad [81]$$

and

$$\eta_1' = \frac{1}{2}\sqrt{\gamma - 1}$$

It should be noted that the function F_5 can also be derived directly by simply replacing η_1 by $i\eta_1'$ in the function F_1 given by Equation [72].

The buckling criterion given by Equation [80] represents an exact solution to the axisymmetric elastic buckling problem of a thin cylindrical shell reinforced by ring frames of finite rigidity and loaded by hydrostatic pressure. Equation [80] is a transcendental equation to be solved for the critical pressure p_{cr} for a given shell and frame geometry defined by the nondimensional parameters α, β, and θ.

In practice, it usually turns out that β is on the order of 0.04 and less, whereas α is on the order of 0.4. Thus, if β is neglected in Equation [80], the buckling criterion simplifies to

$$\alpha + F_5 = 0 \qquad [82]$$

Hence, a good approximation to the buckling pressure can be found from a numerical solution of Equation [82]. A more exact solution of Equation [80], in which the effect of β is not neglected, can be found by using the numerical result obtained above as a first approximation to p_{cr} and then carrying out at least two more calculations bracketing this value of pressure so as to satisfy Equation [80]. For this purpose, a convenient graphical representation of the function F_5 is given as Figure 12.

For the special case in which the frame area becomes very large, i.e., $\alpha \rightarrow \infty$, corresponding to very rigid bulkheads, the buckling criterion [80] simplifies to

$$\eta_1' \cos \eta_2\theta \sin \eta_2\theta + \eta_2 \cos \eta_1'\theta \sin \eta_1'\theta = 0 \qquad [83]$$

Physical reasoning leads one to believe that, for a given shell geometry, the buckling load given by [80] should be somewhat smaller than that resulting from [83] since lighter frames tend to reduce the buckling strength of a ring-stiffened cylinder.

For engineering structures where the frames are closely spaced, buckling will occur with circumferential "ripples" so that the criteria given above do not apply to such cases. If the frame spacing is large, then both axisymmetric and asymmetric[7] (lobar) elastic buckling must be considered. This latter problem has received further analytic treatment by Reynolds; see Reference 8. Axisymmetric and asymmetric buckling of ring-stiffened cylindrical shells in the inelastic range has been treated by Lunchick[9] and Reynolds,[10] respectively.

A more comprehensive study of the axisymmetric elastic buckling of a cylindrical shell with restrained edges is given in Reference 11.

CRITERION FOR DETERMINATION OF INFLECTION POINTS IN DISPLACEMENT SHAPE

In the fabrication of ring-stiffened cylindrical shells from flat plating, i.e., rolled and welded structures, it is important to locate circumferential seams at points where the longitudinal bending is a minimum, or zero. It is necessary to avoid putting welded joints at points of high stress so as to minimize the fatigue of such joints. For this purpose, the designer would

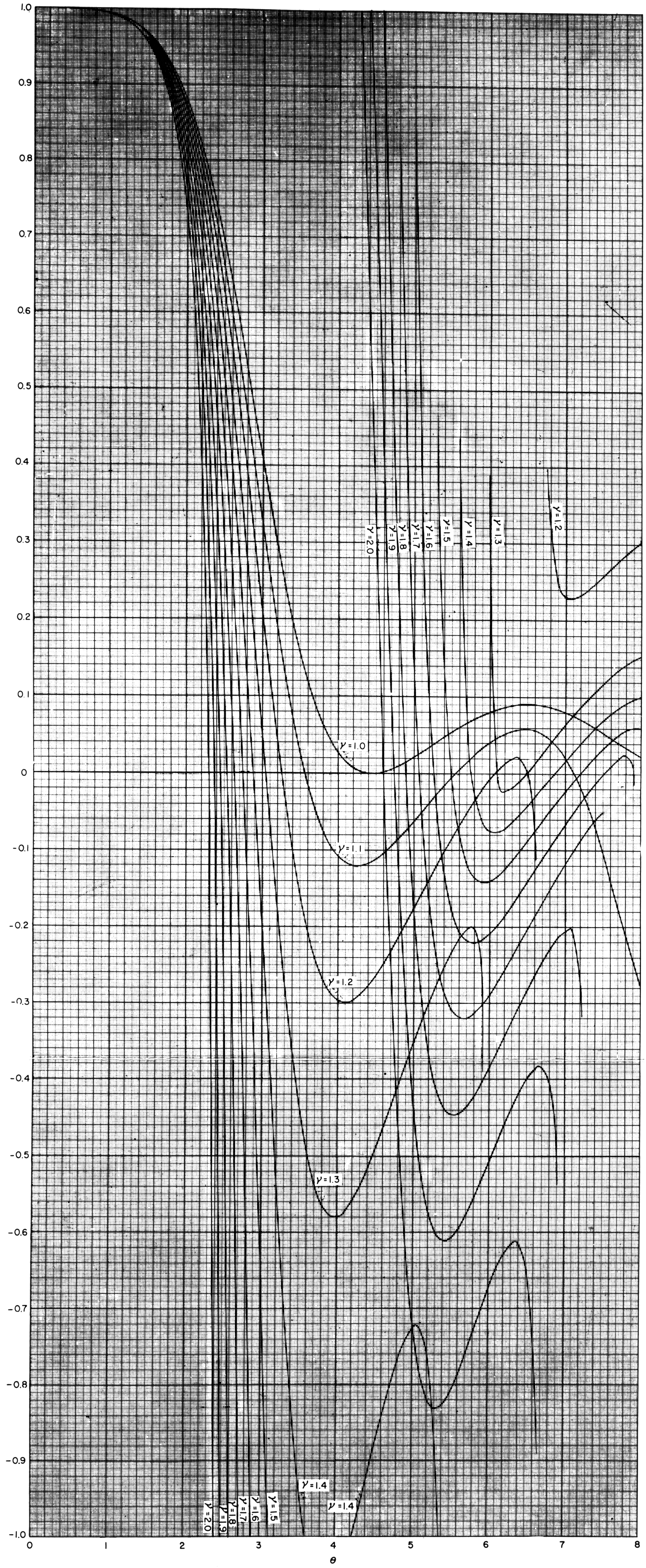

Figure 12 - Plot of the Function $F_5(\theta,\gamma)$

like to know where such locations of zero bending exist.

The condition for the existence of an inflection point in the displacement shape of the cylindrical shell is that the longitudinal curvature, or second derivative of the displacement function w(x), vanishes at that location. Thus, differentiating Equation [8] twice with respect to x and then setting the result equal to zero leads to the following criterion:

$$B\lambda_1^2 \cosh \lambda_1 x + F\lambda_3^2 \cosh \lambda_3 x = 0 \qquad [84]$$

Upon substitution of the values for the constants B and F given by Equations [28] into the above, the following obtains:

$$-\lambda_1 \sinh \lambda_3 \frac{L}{2} \cosh \lambda_1 x + \lambda_3 \sinh \lambda_1 \frac{L}{2} \cosh \lambda_3 x = 0 \qquad [85]$$

Case (1), where $\frac{p}{p*} < 1.0$, is the one of greatest practical importance, so that substituting the values of λ_1 and λ_3 given by Equations [B6] into Condition [85] leads to the following relation for determining the value of x where the inflection points occur:

$$\tanh 2\theta \eta_1 \frac{x}{L} \tan 2\theta \eta_2 \frac{x}{L} = \frac{4\eta_1 \eta_2}{\gamma + \sqrt{\frac{3}{0.91}} \frac{F_2}{F_4}} \qquad [86]$$

For a given shell geometry, a numerical solution of Equation [86] can be easily found with the aid of the curves given in Figures 6 and 7.

DERIVATION OF EFFECTIVE-WIDTH FORMULA

In the case of a ring-stiffened cylinder under some loading condition, such as hydrostatic pressure, a portion of the shell will act effectively with the ring frame to resist direct stress and bending moment caused by the interaction between shell and frames. A knowledge of this effective width is of particular interest in a study of the buckling strength of the ring itself or in an elastic general-instability analysis of the entire cylinder. It is also important in calculating the stresses in the frame flanges of imperfectly circular cylindrical shell structures.

A number of formulas for determining effective width in problems related to ring-stiffened cylindrical shells are presently available. For the case of axisymmetric deformations, H. Bleich,[12] P. P. Biljaard,[13] and B. Thürlimann[14] have developed such formulas. The more complex case of asymmetrical deformations was considered by C. B. Biezeno and J. J. Koch;

where

$$\overline{N}'_{\phi} = N'_{\phi}\left(\frac{L}{2}\right) \qquad [89]$$

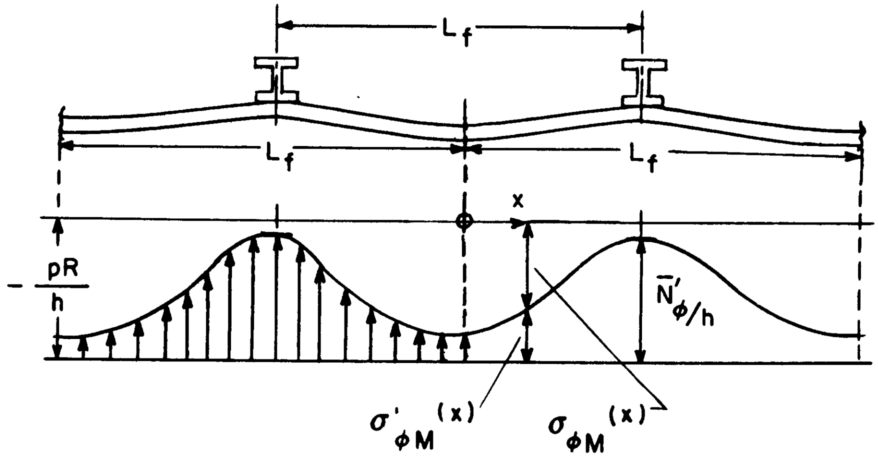

Figure 13a

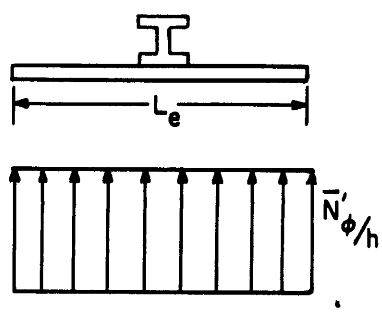

Figure 13b

Figure 13 - Frame-Shell Cross Sections and Associated Circumferential Stress Distribution

Substituting Equations [87] and [88] into [89] gives the following expression for the effective width L_e:

their results are given in the form of tables and three-dimensional plots in Reference 15.

Consideration of the "beam-column" effect can influence the axisymmetric elastic deformations and stresses in a ring-stiffened cylinder under hydrostatic pressure; this is evident from the plots of the F_1, F_2, F_3, F_4, and F_5 functions given herein. For this reason it was deemed necessary to derive a new formula for effective width based on the present formulation of the overall problem. In deriving this formula, some equivalence-of-structure criterion must be established between the actual structure shown in Figure 13a and a "fictitious" one comprised of the ring frame and an effective width L_e of shell plating acting as another flange as is shown in Figure 13b. This "equivalent" ring-shell combination must be such that the flange and web elements do not deform in themselves, i.e., the cross section shown in Figure 13b does not change shape.

If the shell possesses large bending rigididity in the x-direction so that it does not deform between adjacent ring frames, then the entire shell plating is effective, i.e., $L_e = L$. The effective width L_e may be looked upon as a measure of the longitudinal flexibility of the shell. The determination of L_e is based on the condition that the resultant hoop force in the two structures, shown in Figures 13a and 13b, is the same. The circumferential membrane stress $\sigma_{\phi M}(x)$ in the shell is given by Equation [40]; it may be rewritten as follows:

$$\sigma'_{\phi M}(x) \equiv \sigma_{\phi M}(x) + \frac{pR}{h} = \frac{E}{R}\left[B \cosh \lambda_1 x + F \cosh \lambda_3 x\right] \qquad [87]$$

The term on the right side of Equation [87] represents the bending action of the shell between adjacent elastic ring frames. The total hoop force is then given by

$$S'_\phi = \int_{-\frac{L}{2}}^{+\frac{L}{2}} N'_\phi(x)\,dx = 2\int_{o}^{\frac{L}{2}} \sigma'_{\phi M}(x)\; h dx \qquad [88]$$

The equivalence criterion is defined to be

$$S'_\phi = L_e \overline{N'}_\phi \qquad [89]$$

$$L_e = \frac{2\left[\frac{B}{\lambda_1}\sinh\lambda_1\frac{L}{2} + \frac{F}{\lambda_3}\sinh\lambda_3\frac{L}{2}\right]}{B\cosh\lambda_1\frac{L}{2} + F\cosh\lambda_3\frac{L}{2}} \qquad [90]$$

For Case (1) where $\frac{p}{p^*} < 1.0$, the integration constants B and F are given by Equations [59] and the characteristic roots λ_1 and λ_3 are given by Equations [B6]. Substituting these values into Equation [90] and simplifying the resulting expression yields

$$L_e = LF_1 \qquad [91]$$

where the function F_1 is defined by Equation [72] and is plotted on Figure 8.

The effective-width formula given by Equation [91] may be specialized for the case of zero "beam-column" effect, i.e., $\frac{p}{p^*} = 0$, corresponding to Case (4) in Appendix B. Setting $\eta_1\theta = \eta_2\theta = \frac{\theta}{2}$ in Equation [72] for the function F_1 and then substituting this value of F_1 into Equation [91] gives

$$L_e = \frac{2\sqrt{Rh}}{\sqrt[4]{3(1-\upsilon^2)}}\left[\frac{\cosh\theta - \cos\theta}{\sinh\theta + \sin\theta}\right] \qquad [92]$$

This same result, Equation [92], is obtained when the values of the characteristic roots λ_1 and λ_3 given by Equations [B13] are substituted into Equations [28] for the constants B and F, and thence into Equation [90].

For $\upsilon = 0.3$, Equation [92] becomes

$$L_e = 1.56\sqrt{Rh}\left[\frac{\cosh\theta - \cos\theta}{\sinh\theta + \sin\theta}\right] \qquad [93]$$

An expression similar to Equation [93] was derived by B. Thürlimann in Reference 14.

FAILURE CRITERIA FOR AXISYMMETRIC COLLAPSE

Equations have been developed which give the axisymmetric radial deflections and the principal stresses in the longitudinal and circumferential directions at any point in the shell plating between adjacent ring frames of a stiffened thin-walled cylinder under uniform hydrostatic pressure. From these general equations, formulas for the maximum stresses at the critically stressed locations of the cylinder have been derived. These formulas permit the determination of the pressure at which the shell and frame stresses reach the yield strength of the material used for these elements. However, a number of failure criteria may be introduced to predict the pressure at which axisymmetric collapse of the structure can occur. In other words, the question of how the biaxial stresses in the cylinder plating combine and interact to produce axisymmetric collapse due to yielding of the material will now be examined.

The simplest and most obvious criterion of failure is derived from the maximum principal stress theory of Rankine.[16] The maximum stresses occur in the circumferential direction on the outside surface of the shell plating midbay between frames and in the longitudinal direction on the inside surface of the shell plating at a frame; these stresses are determined from Equations [65] and [66], respectively. Which of the two stresses is higher depends upon the geometry of the cylindrical shell and reinforcing rings, but in most cases of interest $\sigma_{x_i f} > \sigma_{\phi o_m}$. However, extensive Model Basin experimental data has shown that the stress $\sigma_{\phi o_m}$ is determinative in causing axisymmetric collapse. Therefore, the criterion of failure based on this stress together with Rankine's theory is expressed as

$$\sigma_{\Phi o_m} \equiv \sigma_y \qquad [94]$$

where $\sigma_{\Phi o_m}$ can be determined from Equation [65] and σ_y is the yield strength of the shell material.

A more realistic criterion of failure considers the biaxial state of stress in the shell and is derived from the energy of distortion theory,[16] which grew out of the analytical work of Huber, Von Mises, and Hencky. Since the octahedral-shearing-stress theory gives the same results as the energy-of-distortion theory and permits the use of a more familiar quantity such as stress, the former theory will be used to derive other criteria of failure. For a biaxial state of stress at midbay, defined by the principal stresses σ_{xm} and $\sigma_{\phi m}$, the octahedral shearing stress is given by

$$\tau_G = \frac{1}{3}\left[(\sigma_{xm} - \sigma_{\Phi m})^2 + \sigma_{xm}^2 + \sigma_{\Phi m}^2\right]^{\frac{1}{2}} \qquad [95]$$

However, since according to this theory inelastic action at any point in a body under any combination of stresses begins only when the octahedral shearing stress τ_G becomes equal to $(\sqrt{2}/3)\,\sigma_y$, then Equation [95] leads to the following general criterion:

$$\left[\sigma^2_{Xm} + \sigma^2_{\Phi m} - \sigma_{Xm}\,\sigma_{\Phi m}\right]^{\frac{1}{2}} \equiv \sigma_y \qquad [96]$$

Essentially, two distinct criteria for predicting axisymmetric collapse can be derived from Equation [96] depending upon whether the outer-fiber stresses or mid-fiber (membrane) stresses at midbay are used. For yielding on the outer surface of the shell plating, the following criterion results:

$$\left[\sigma^2_{X^o m} + \sigma^2_{\Phi^o m} - \sigma_{X^o m}\,\sigma_{\Phi^o m}\right]^{\frac{1}{2}} \equiv \sigma_y \qquad [97]$$

where the stresses $\sigma_{X^o m}$ and $\sigma_{\Phi^o m}$ are given by Equations [64] and [65], respectively.

If it is assumed that collapse does not occur until yielding has penetrated through half the thickness of the shell plating, then the following criterion is applicable:

$$\left[\sigma^2_{XM} + \sigma^2_{\Phi M} - \sigma_{XM}\,\sigma_{\Phi M}\right]^{\frac{1}{2}} \equiv \sigma_y \qquad [98]$$

where the membrane stresses σ_{XM} and $\sigma_{\Phi M}$ can be found from Equations [64] and [65] and with [69] to be

$$\frac{\sigma_{XM}}{\sigma_u} = \frac{1}{2}$$

$$[99]$$

$$\frac{\sigma_{\Phi M}}{\sigma_u} = 1 - \frac{(1-\nu/2)\,\alpha F_2}{\alpha + \beta + (1-\beta) F_1}$$

For the general case in which the "beam-column" effect is considered, i.e. $\gamma \neq 0$, it is readily seen that the criteria [94], [97], and [98] lead to transcendental equations for the collapse pressure since the stresses are nonlinear functions of the pressure p. However, a good first approximation can be found by assuming that $\gamma = 0$ so that the stresses then become linear in the pressure p permitting explicit, closed-form solutions for collapse pressure p_c. In such a case, Equations [94], [97], and [98], respectively, lead to

$$|p_c| = \frac{\sigma_y h/R}{1-A[F_2-\upsilon F_4\sqrt{\frac{0.91}{1-\upsilon^2}}]} \qquad [100]$$

$$|p_c| = \frac{\sigma_y h/R}{\left\{\frac{3}{4}+A^2[F_2^2+F_2F_4(1-2\upsilon)\sqrt{\frac{0.91}{1-\upsilon^2}}+F_4^2(1-\upsilon+\upsilon^2)(\frac{0.91}{1-\upsilon^2})]-\frac{3}{2}A[F_2-\upsilon F_4\sqrt{\frac{0.91}{1-\upsilon^2}}]\right\}^{1/2}} \qquad [101]$$

$$|p_c| = \frac{\sigma_y h/R}{\left\{\frac{3}{4}+A^2F_2^2-\frac{3}{2}AF_2\right\}^{1/2}} \qquad [102]$$

where

$$A \equiv \frac{(1-\upsilon/2)\alpha}{\alpha+\beta+(1-\beta)F_1}$$

in Equations [100], [101], and [102]. One significant result to be pointed out is that the collapse pressure p_c given by Equation [101] becomes identical to that given by Equation [102] for the special case when $F_4=0$. This is as it should be because, from Equation [68], it can be seen that $F_4=0$ implies that the longitudinal bending stress at midbay is zero, i.e. $\sigma_{xbm} = 0$. Hence the stresses [64] and [65] become the membrane stresses given by Equations [99].

For the general case in which $\gamma \neq 0$, the stresses are nonlinear functions of the pressure, and Equations [100], [101], and [102] are then transcendental in the pressure by virtue of the complicated fashion in which the load parameter γ appears in the F_1, F_2, F_3, and F_4 functions; see Equations [72] through [75]. However, a numerical iteration procedure can be used in which the collapse pressures p_c are first calculated for $\gamma = 0$. These values of pressure are then used as a first approximation to determine a new value of γ from Equation [60]. Then, with this value of γ, the stress functions can be found from the curves of Figures 8 through 11, and new values of p_c can be computed from the respective formulas. This process can be repeated again, but probably only one or at most two iterations are required to obtain satisfactory convergence.

For the case of zero frame area, i.e. $\alpha = 0$, Equation [100] degenerates to

$$|p_c| = \frac{\sigma_y h}{R} \qquad [103]$$

whereas Equations [101] and [102] degenerate to

$$|p_c| = \frac{2}{\sqrt{3}} \frac{\sigma_y h}{R} \qquad [104]$$

Equations [103] and [104] are well-known results for the case of the unstiffened cylinder under hydrostatic pressure.

Lunchick[17] derived another criterion of failure for axisymmetric collapse based on the plastic-hinge concept. He made use of the Hencky-Huber-Von Mises criterion of yielding, Equation [96], and allowed for the plastic reserve strength after the initiation of yielding in the shell plating at midbay. A complete discussion of this plastic-hinge theory and the criteria given by Equations [94], [97], and [98] together with experimental results found at the Model Basin is given in Reference 17.

CONCLUSIONS

1. The well-known formulas of Von Sanden and Günther[1] for the ring load and critical shell stresses in a ring-stiffened cylindrical shell under hydrostatic pressure have been replaced by Equations [70], [66], and [65].

2. The equations of the present analysis indicate that cylindrical shell structures may collapse by axisymmetric buckling. However, for shells with closely spaced rings, buckling may occur with an asymmetric (lobar) pattern in the circumferential direction, and both cases of instability must

be investigated to determine the lower value of critical pressure.

3. The significant results of the present analysis have been plotted in convenient graphical form as Figures 6 through 12 for ease of numerical calculation and visual representation of the variation in frame and shell deformations and stresses with geometry. These curves indicate that, for certain ranges of geometry, variations in the load parameter γ, which is a measure of the "beam-column" effect, can lead to large differences in prediction of deformation and stress between the theory of Von Sanden and Günther and that of the present report.

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APPENDIX A

DERIVATION OF GOVERNING DIFFERENTIAL EQUATION

For purposes of analysis an element cut from a thin cylindrical shell of mean radius R and uniform thickness h is shown in Figure 14. This element supports a radial pressure P_r (in psi) which is considered positive when it is directed away from the axis of the cylinder. The stress resultants N_x and N_ϕ (in lb/in.) are positive when directed so as to cause tension. The axial, circumferential, and radial coordinates are x, ϕ, and r, respectively, and the corresponding displacements are u, v, and w.

The equilibrium of forces acting in the radial direction requires that

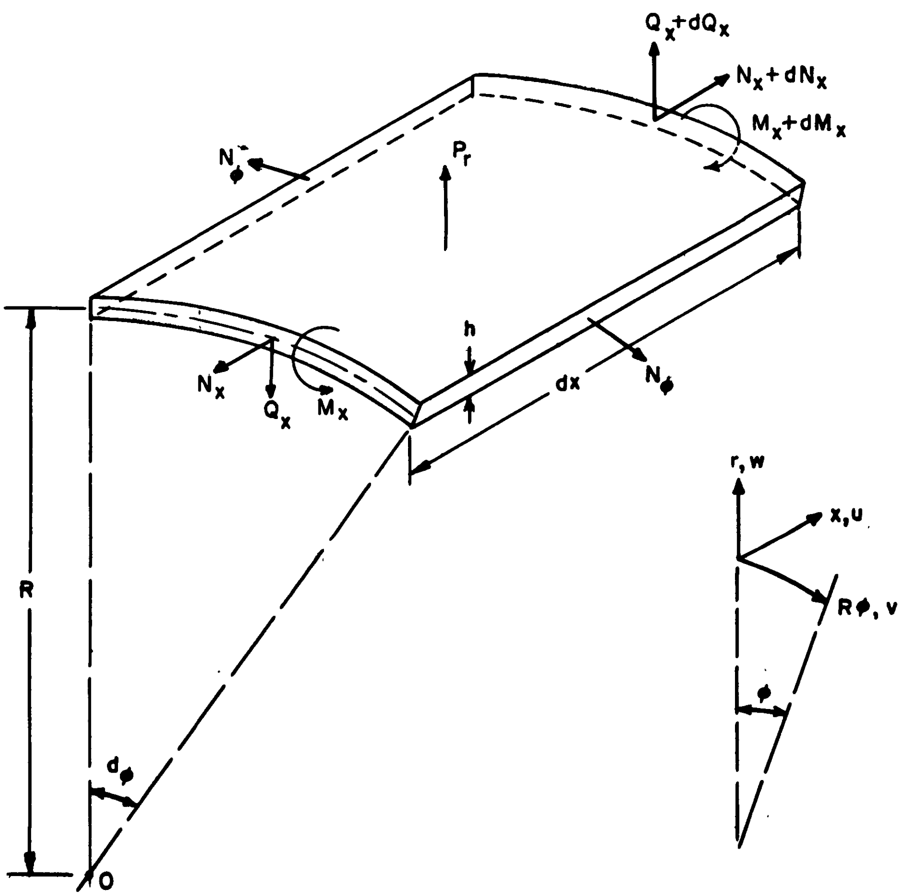

Figure 14 - Element of a Cylindrical Shell

$$\frac{dQ_x}{dx} R d\phi dx + N_x \frac{d^2w}{dx^2} R d\phi dx - N_\phi d\phi dx + P_r R d\phi dx = 0 \qquad [A1]$$

The second term of this equation represents an effective radial pressure per unit length of cylinder caused by the axially applied stress resultant N_x as shown in Figure 15. It is noted that neither Von Sanden and Günther nor Viterbo considered this term.

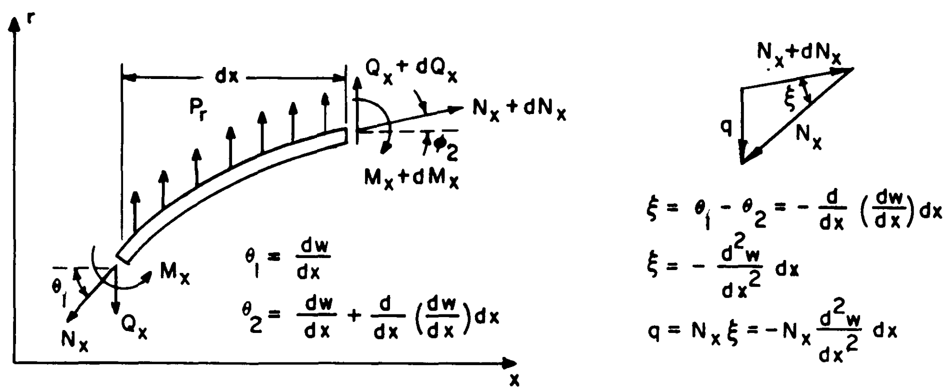

Figure 15 - Effect of Axial Pressure

Division of each term of Equation [A1] by the element of area $R d\phi dx$ leads to the differential equation

$$\frac{dQ_x}{dx} + N_x \frac{d^2w}{dx^2} - \frac{N_\phi}{R} = - P_r \qquad [A2]$$

Moment equilibrium on the element of Figure 14 requires that

$$Q_x = \frac{dM_x}{dx} - N_x \frac{dw}{dx} \qquad [A3]$$

The well-known moment-curvature relationship

$$M_x = - D \frac{d^2w}{dx^2} \qquad [A4]$$

is introduced into Equation [A3] ; and the result, into Equation [A2] to obtain

$$D \frac{d^4 w}{dx^4} - N_x \frac{d^2 w}{dx^2} + \frac{N_\phi}{R} = P_r \qquad [A5]$$

where D is the bending rigidity of the shell plating defined as

$$D = \frac{Eh^3}{12(1-\upsilon^2)} \qquad [A6]$$

The two-dimensional Hooke's law for the shell plate may be given as

$$\varepsilon_x = \frac{1}{Eh} [N_x - \upsilon N_\phi] \qquad [A7]$$

$$\varepsilon_\phi = \frac{1}{Eh} [N_\phi - \upsilon N_x] \qquad [A8]$$

Equation [A8] when solved for N_ϕ yields

$$N_\phi = Eh\varepsilon_\phi + \upsilon N_x \qquad [A9]$$

By axial symmetry and with the sign convention adopted, the circumferential strain ε_ϕ in the shell is

$$\varepsilon_\phi = \frac{w}{R-r} \approx \frac{w}{R} \qquad [A10]$$

where r is the distance from the median surface of the shell to some point across the thickness; its order of magnitude is O(h/2) and less.

If Equation [A10] is substituted into [A9] and the result into Equation [A5], then

$$D \frac{d^4 w}{dx^4} - N_x \frac{d^2 w}{dx^2} + \frac{Eh}{R^2} w = P_r - \frac{\upsilon}{R} N_x \qquad [A11]$$

For the case of external hydrostatic pressure

$$P_r = - p \text{ psi}$$

and

$$N_x = - \frac{pR}{2} \text{ lbs/in.}$$

so that Equation [A11] becomes

$$D \frac{d^4 w}{dx^4} + \frac{pR}{2} \frac{d^2 w}{dx^2} + \frac{Eh}{R^2} w = - p(1-\upsilon/2) \qquad [A12]$$

It can be shown that, if the circumferential strain ε_ϕ in the shell is approximated by

$$\varepsilon_\phi = \frac{w}{R-r} \approx \frac{w}{R}\left(1+\frac{r}{R}\right) \qquad [A13]$$

instead of [A10], then an additional term, $\frac{\upsilon D}{R^2} \cdot \frac{d^2 w}{dx^2}$ appears in the differential equation [A12] . Order of magnitude considerations show that this term is of little significance. Therefore, for simplicity, it is omitted from the differential equation.

APPENDIX B

THE FOUR POSSIBLE CASES: $\frac{p}{p*} \lesseqgtr 1.0$ AND $\frac{p}{p*} = 0$

<u>Case (1):</u> $\frac{p}{p*} < 1.0$

For Case (1), the value of

$$\left[\left(\frac{p}{p^*}\right)^2 - 1\right]^{1/2} \qquad [B1]$$

is a pure imaginary number. Consequently, the roots λ_1 and λ_3 are unequal and assume the complex-number form

$$\begin{aligned} \lambda_1 &= c + ig \\ \lambda_3 &= c - ig \end{aligned} \qquad [B2]$$

where $i = \sqrt{-1}$.

From Equations [9] and [B2]

$$c + ig = \sqrt{2}\,\frac{\theta}{L}\left\{-\left(\frac{p}{p^*}\right) + \left[\left(\frac{p}{p^*}\right)^2 - 1\right]^{1/2}\right\}^{1/2} \qquad [B3]$$

Squaring both sides of [B3] and equating the real and imaginary parts gives

$$\begin{aligned} c^2 - g^2 &= -2\left(\frac{\theta}{L}\right)^2\left(\frac{p}{p^*}\right) \\ cg &= \left(\frac{\theta}{L}\right)^2\left[1 - \left(\frac{p}{p^*}\right)^2\right]^{1/2} \end{aligned} \qquad [B4]$$

Simultaneous solution of Equations [B4] yields

$$c = \frac{\theta}{L}\left[1 - \left(\frac{p}{p^*}\right)\right]^{1/2}$$

[B5]

$$g' = \frac{\theta}{L}\left[1 + \left(\frac{p}{p^*}\right)\right]^{1/2}$$

since the quantities c and g must be real. Thus the characteristic roots of [B2] become

$$\lambda_1 = \frac{2\theta}{L}\,(\eta_1 + i\eta_2)$$

[B6]

$$\lambda_3 = \frac{2\theta}{L}\,(\eta_1 - i\eta_2)$$

where

$$\eta_1 = \frac{1}{2}\sqrt{1 - \gamma}$$

$$\eta_2 = \frac{1}{2}\sqrt{1 + \gamma}$$

[B7]

$$\gamma = \frac{p}{p^*}$$

<u>Case (2):</u> $\frac{p}{p^*} = 1.0$

For Case (2), the characteristic roots λ_1 and λ_3 of Equation [9] become equal and pure imaginary, and are given by

$$\bar{\lambda}_1 = \bar{\lambda}_3 = i\sqrt{2}\,\frac{\theta}{L} \qquad \text{[B8]}$$

For equal roots, the solution for the displacement w(x) given by Equation [12] must be used.

This solution occurs when the applied hydrostatic pressure p is equal in magnitude to the axial pressure p* for axisymmetric elastic buckling of an unstiffened cylindrical shell.

Case (3): $\frac{p}{p^*} > 1.0$

For Case (3), the value of the quantity under the second radical in Equation [9] is a real number. Hence, the roots are unequal and pure imaginary, and assume the form

$$\begin{aligned} \lambda_1 &= i\,(c + g) \\ \lambda_3 &= i\,(c - g) \end{aligned} \qquad [B9]$$

where c and g are again given by [B5]. The characteristic roots of [B9] can then be written as

$$\begin{aligned} \lambda_1 &= i\,\frac{2\theta}{L}\,(\eta_1 + \eta_2) \\ \lambda_3 &= i\,\frac{2\theta}{L}\,(\eta_1 - \eta_2) \end{aligned} \qquad [B10]$$

where the quantities η_1 and η_2 are given by [B7].

Case (4): $\frac{p}{p^*} = 0$ (No axial pressure)

For Case (4), λ_1 and λ_3 become

$$\begin{aligned} \lambda_1 &= \sqrt{2}\,\frac{\theta}{L}\,(+\,i)^{1/2} \\ \lambda_3 &= \sqrt{2}\,\frac{\theta}{L}\,(-\,i)^{1/2} \end{aligned} \qquad [B11]$$

By de Moivre's theorem

$$\begin{aligned} (+\,i)^{1/2} &= \pm\frac{1}{\sqrt{2}}\,(1 + i) \\ (-\,i)^{1/2} &= \pm\frac{1}{\sqrt{2}}\,(1 - i) \end{aligned} \qquad [B12]$$

so that, for the positive sign of the above quantities, the characteristic roots become

$$\lambda_1 = \frac{\theta}{L}(1 + i)$$

$$\lambda_3 = \frac{\theta}{L}(1 - i) \qquad [B13]$$

The same solution would result if the minus sign in [B12] was considered.

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