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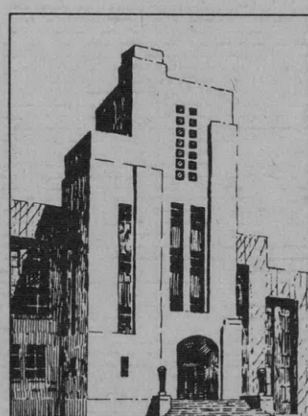
A SYSTEMATIC EVALUATION OF MICHELL'S INTEGRAL

by

Georg P. Weinblum



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NOTATION

Symbol	Definition
A	Area
$A(x)$	Sectional area curve
$A_0 = \beta BH$	Midship section area
a	Coefficients of polynomials
$a(\xi)$	Area curve of dimensionless section
$a^*(\xi)$	Dimensionless sectional area curve with unit ordinate at the midship section
B	Beam
b	Coefficients of polynomials
$b = \frac{B}{2}$	Half beam radius
C	A constant, coefficients
D	Diameter for bodies of revolution
E	Resistance function
e	Eccentricity
$F = \frac{U}{\sqrt{gL}}$	Froude number
H	Draft
$I(\gamma)$	Resistance function
$J(\gamma)$	Resistance function
$K = 2\frac{H}{L}$	Dimensionless curvature at the midship section
L	Length
$\frac{L}{D}$	Length-diameter ratio of body
$l = \frac{L}{2}$	Half-length
M	Resistance function
$\mathfrak{M}$	General functions of the type $\mathfrak{M}_{ij}[$or; K; $\gamma_0]$ where o and r are the indices of the E function

m, n	Integers, exponents
R	Resistance
R^*	Dimensionless resistance
S_y	Static moment
$S\eta$	Dimensionless static moment
$t = -\dfrac{\partial X(1)}{\partial \xi}$	Taylor's tangent value
U	Velocity in the x direction
$\mathit{\nabla}$	Volume displacement
$v(\xi)$	Fining function
$W(z) = 2\int_{-l}^{+l} y(x,z)\,dx$	Waterline area
W_0	Load waterline area
w_0	Dimensionless waterplane area
$w(\zeta)$	Dimensionless waterline area
$w^*(\zeta)$	Dimensionless waterline area reduced to unity at the load waterline
X	Axis
$X_s(\xi)$	Symmetric parts of the dimensionless waterline equation
$X_a(\xi)$	Asymmetric parts of the dimensionless waterline equation
x	Coordinate
x_0	Longitudinal coordinate of a centroid
Y	Axis
y	Coordinate
$y = \pm\, y(x,z)$	Equation of hull
Z	Axis
z	Coordinate
$\alpha = C_w$	Area coefficient of load waterline
$\beta = C_x$	Midship area coefficient
γ	Variable of integration

$\gamma_0 = \frac{1}{2F^2}$	
$\delta = C_b$	Block coefficient
$\zeta = \frac{z}{H}$	Dimensionless coordinate
$\zeta = K(\xi)$	Equation of longitudinal midsection
$\eta = \frac{y}{b}$	Dimensionless coordinate
$\eta = \pm\, \eta(\xi, \zeta)$	Dimensionless equation of hull
$\eta = X(\xi)$	Dimensionless equation of load waterline
$\eta = Z(\zeta)$	Dimensionless equation of midship section
η_s	Symmetric } parts of η
η_a	Asymmetric } parts of η
$\xi = \frac{x}{l}$	Dimensionless coordinate
ξ_0	Dimensionless longitudinal coordinate of a centroid
$\phi = C_p$	Prismatic coefficient
$\psi = \frac{L}{\nabla^{1/3}}$	Ratio of slenderness

INTRODUCTION

For nearly thirty years attempts have been made to evaluate Michell's wave resistance formula[1] in such a way that useful deductions for the profession can be immediately obtained.

Havelock succeeded in explaining the most characteristic features of the wave resistance of ships, Wigley and the author compared results of theoretical computations with experimental data and the author tried to develop methods for finding ship forms of low wave resistance. Although much interesting information has been accumulated, the results remain rather sporadic. The bibliography of the subject can be found in TMB Report 710 (Reference 2).

So far, notwithstanding various efforts, no better solution to the wave resistance of normal surface ships has been found than Michell's integral. For this reason and another to be mentioned later it was decided that a more comprehensive attempt should be made to evaluate this integral. On request of the Taylor Model Basin the author submitted a research program to the Mathematics Department of the Office of Naval Research and was fortunate to find kind interest and strong support, for which he feels especially indebted to Dr. Mina Rees, Dr. John Wehausen and Dr. E. Bromberg. A contract was granted to the Bureau of Standards which at present has completed the computing work connected with the first stage of the program. The author wishes to express his thanks to Dr. Alt, Dr. Levin, Dr. Abramowitz, Mr. Blum, and Mr. Hirschberger, who have contributed decisively to the success of the work. The extensive calculations were started with the full understanding and with the hope that Michell's analysis will be superseded by better "theories," but it was thought that even in this case the simple linearized solution would not lose its significance.

Before beginning the computations careful consideration was given to related attempts made by Sretensky.[3] This well-known author came to rather disappointing conclusions concerning the practical use of Michell's integral. It has been shown, however, that Sretensky's approach is not quite consistent and his final negative statement is not conclusive.[2]

The author wishes to acknowledge that besides ONR, the Model Basin and the Wave Panel of the Society of Naval Architects gave full encouragement to the work. The Model Basin initiated a similar project on the wave resistance of submerged bodies of revolution which has been successfully completed.[4]

The program of the present work has been already discussed in the author's review on wave resistance.[2] The most interesting problem in dealing with the wave resistance of normal ocean-going hull forms consists of finding the appropriate longitudinal displacement distribution, i.e., the sectional area curve. The proper vertical distribution of the displacement though of comparable basic importance can be treated in a more summary way. Clearly the separation of longitudinal and vertical distributions is an artifice, which in the later stage of the present

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[1]References are listed on page 59.

work will be eliminated; besides, the dependence of the wave resistance upon two fundamental quantities – the draft and the midship area coefficient – can be judged already from the results so far obtained.

The tables can be applied essentially in two ways. First, if suitable restrictions are introduced, hull forms of least wave resistance may be calculated directly for a sufficient number of Froude speed parameters $F = U/\sqrt{gL}$. This approach is useful and necessary but experience has shown that it does not cover all practical needs.

Secondly, wave-resistance curves may be calculated for a large number of systematically varied forms. From these curves trends in resistance change due to systematic form variations can be established and the influence of various form parameters can be studied. At present, emphasis is laid on the second procedure but some forms of least resistance also have been investigated. A complete survey of the field requires, clearly, both methods of computation.

Part I of the present report deals with some basic geometrical properties of hulls and in Part II it is shown how Michell's integral can be evaluated for simplified ship forms.

The piece de resistance is the collection of tables in Appendix II.

PART I

GEOMETRY OF THE SHIP

DESCRIPTION OF THE HULL FORM

GENERAL REMARKS ON ALGEBRAICALLY DEFINED SHIP LINES

Any treatise on theoretical naval architecture should include a chapter on the geometric properties of ship forms and their analytical representation which may be called "Geometry of Ships." This terminology agrees with the corresponding one used by Mr. Owen in "The Principles of Naval Architecture" although the notation "Geometry of Ships" has been used in a narrower and not quite adequate sense for special problems in the field of static stability.

Much ingenuity has been displayed in describing the geometric properties of hulls by form parameters and coefficients and in developing graphical procedures for the design of these hulls. A characteristic feature is the wider use of integral relations, especially of integral curves, amongst which the sectional area curve is the most important. Differential relations, although well known, are much less popular.

The present day's graphical method of hull design is efficient from a restricted practical viewpoint. Its flexibility and power should not be underestimated, but it does not furnish a satisfactory foundation for scientific work. It is thought that the lack of a general and rigorous method of representing ship forms is responsible to a considerable extent for the back-wardness in some branches of theoretical naval architecture.

Quite a few attempts have been made to base the design procedure on mathematical equations. The ideas underlying these attempts were sometimes rather mystic insofar as unproven superior resistance qualities were claimed for analytically defined lines. D.W. Taylor approached the problem in a much more realistic way. According to his statements he developed "mathematical formulae not with the idea that they give lines of least resistance but simply to obtain lines possessing desired shape."[5]

He was quite successful in representing sectional area curves and waterlines by fifth degree polynomials.

Our present aim is somewhat more general than Taylor's: we wish to develop equations which enable us to represent lines possessing desired shapes and which at the same time are suitable for finding criteria for this desired shape from the point of view of fundamental mechanical properties like resistance, stability, seaworthiness, etc. That means that the expressions for the ship surface must be sufficiently general and that their application in various theories dealing with mechanical properties of ships must lead to a reasonable amount of mathematical work.[2]

There exists another purely practical viewpoint from which it is desirable to derive equations for the hull: the reduction of work in the mold loft. Although this requirement is basic, we shall not consider it as a primary one within the scope of the present report.

Even from the point of view of mechanics alone the problem cannot be handled in an exhaustive way since the dependence of important hydrodynamic effects upon the geometric properties of lines and surfaces is almost completely unknown. For instance, we do not know what limits of slopes and curvatures must be established to avoid unfavorable pressure gradients which may lead to separation or high tangential resistance. In this respect we must be satisfied by Taylor's criterion to obtain lines possessing a well defined shape. Even so, the possibility of making form variations in a systematic and rigorous way is a necessary and valuable condition for experimental research.

GENERAL PROPERTIES OF SHIP HULLS AND SHIP LINES

Axes of Reference; General Expressions for the Hull and the Main Ship Lines in Dimensional and Dimensionless Coordinates

Let us assume a system of axes as shown in Figure 1. The XY-plane coincides with the design load waterline, the XZ-plane is the plane of symmetry, and YZ-plane is the plane of the midship section. The positive direction of Z is downward.

This system of reference differs from that usually accepted in buoyancy and stability calculations where the XY-plane contains or intersects the keel and Z-axis points upwards.

Some differences in notation and definitions arise because of this discrepancy which, however, are of minor consequence.

As usual the principal dimensions of the ship are denoted by L, B, and H.

For a summary description of hulls the following definitions are proposed:

1. A *fine* ship is a ship with a low prismatic coefficient ϕ. Consequently the block coefficient δ must also be small, while the magnitude of the midship area coefficient β is not decisive.

2. A *slender* ship is a ship with a high value of the length displacement ratio $L/\nabla^{1/3} = \psi = \textcircled{M}$ (Froude), or low value ∇/L^3 (Taylor).

3. A *narrow* ship is a ship with a low B/L ratio.

4. A *thin* ship is a narrow ship with a low B/H ratio. In extreme cases it can be described as a body with wedgelike waterlines and sections. This concept is important in connection with Michell's theory of wave resistance ("Michell's ship").

5. Bodies of revolution with a large L/D ratio are called very elongated bodies of revolution.

Throughout this report broad use will be made of dimensionless coordinates.

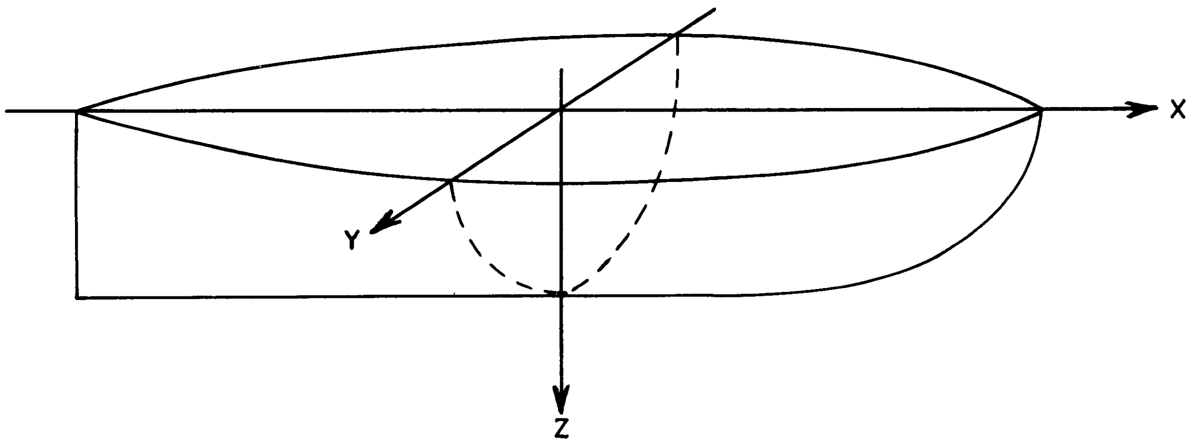

Figure 1 - Axes of Reference

Dimensionless offsets of hulls, ship lines, and integral curves have been familiar in naval architecture for a long time as an indispensable means for systematizing actual ship forms.

It is fairly obvious that the use of dimensionless coordinates is advantageous when studying geometrical properties of hulls; for instance, simple connections between the equations of the hull and the well-known form coefficients are immediately established.

In an earlier report[2] the writer has tried to carry out a strict division between the principal dimensions and the "pure-shape" of a hull when applied to investigations on wave resistance. The procedure appears to be legitimate within certain limitations. The same applies to some extent to investigations on seaworthiness. Although the results may be different when dealing with viscous phenomena, it is hoped that the consistent use of dimensionless representation will contribute appreciably to increase our knowledge of the hydrodynamical and mechanical properties of hulls.

From the present point of view, the use of such parameters as $L/\nabla^{1/3}$ cannot be recommended for a detailed analysis, since here the pure form constant, $\delta = C_b$* and the proportions of principal dimensions are mixed together. Our purpose is to approximate the ship form by as many characteristic values as possible, not to merge several known parameters into a single one. Therefore, the use of the separate ratios L/B, B/H, and δ instead of $L/\nabla^{1/3}$ is preferable. The latter ratio is suitable only as a first orientation.

The equation of the hull may be written as

$$y = \pm\, y(x,\, z) \qquad [1]$$

The double sign appears because the hull consists of two essentially symmetric halves.

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*Throughout this report Greek letters are used for the form coefficients: $C_b = \delta$, $C_x = \beta$, $C_w = \alpha$, $C_p = \phi$.

In most cases it is sufficient to consider

$$y = + y(x, z) \qquad [1a]$$

In what follows, dimensionless coordinates are preferably used, defined by

$$\xi = \frac{x}{l} \qquad \eta = \frac{y}{b} \qquad \zeta = \frac{z}{H} \qquad [2]$$

with $l = L/2$ and $b = B/2$.

Thus the following equation corresponds to [1a].

$$\eta = \eta\,(\xi, \zeta) \qquad [2a]$$

Basic relations will be given in a dimensional as well as in a dimensionless form.

The main purpose of the following synopsis is to work out a consistent system of symbols and notation.

The symbol y and in dimensionless representation η, will be used not only for the equation of the surface, but also for equations of ship lines when no confusion can be caused.

In later applications it will be assumed that the thickness of the stem, the sternpost, and an eventual keel is zero; otherwise expressed, the hull form is faired down to the centerplane at these locations.

Thus, Equation [3] given below for the load waterline complies with the conditions $X(\pm 1) = 0$, and Equation [4] for the midship section complies with $Z(1) = 0$. These assumptions will be always tacitly made unless the contrary has been stated. It is easy to derive equations of ship lines with finite ordinates at their ends; the same applies to the equation of the hull when the thickness of the keel, stem, and sternpost are constant and equal, but complications arise when variable "intercepts" must be considered.

PRINCIPAL SHIP LINES AND INTEGRAL CURVES

The following notations are proposed:

1. The hull equation

$$y = y(x, z) = b\eta = b\eta(\xi, \zeta)$$

2. The load waterline

$$y(x,0) = b\eta\,(\xi, 0)$$
$$\eta\,(\xi,0) = X(\xi) \qquad [3]$$

3. The midship section

$$y(0,z) = b\eta\,(0,\zeta)$$
$$\eta\,(0,\zeta) = Z(\zeta) \qquad [4]$$

4. **Longitudinal midsection (centerplane contour)**

$$y(x, z) = 0$$

$$\eta(\xi, \zeta) = 0 \quad \zeta = K(\xi) \qquad [5]$$

5. **Sectional area curve**

$$A(x) = 2\, b H \int_0^{K(\xi)} \eta(\xi, \zeta)\, d\zeta = b H \; a(\xi) \qquad [6]$$

where

$$a(\xi) = 2 \int_0^{K(\xi)} \eta(\xi, \zeta)\, d\zeta \qquad [6a]$$

The midship section area $A(0)$ is denoted by $A_0 = \beta B H$. At the midship section $\xi = 0$, $a(0) = 2\beta$

When the centerplane contour is a rectangle

$$A(x) = 2 \int_0^{H} y(x,z)\, dz$$

$$a(\xi) = 2 \int_0^{1} \eta(\xi, \zeta)\, d\zeta \qquad [7]$$

To obtain a dimensionless sectional area curve with a unit ordinate at the midship section we define

$$A(x) = \beta\, BH\, a^*(\xi)$$

$$a^*(\xi) = \frac{1}{2\beta}\, a(\xi) = \frac{1}{\beta} \int_0^{K(\xi)} \eta(\xi, \zeta)\, d\zeta \qquad [8]$$

The prismatic coefficient may then be defined

$$\phi = \frac{1}{2} \int_{-1}^{+1} a^*(\xi)\, d\xi \qquad [8a]$$

6. **Waterline area curve**

$$W(z) = 2 \int_{-l}^{+l} y(x, z)\, dx = 2\, bl \int_{-1}^{+1} \eta(\xi, \zeta)\, d\xi \qquad [9]$$

$$= bl\, w(\zeta) = \alpha\, BL\, w^*(\zeta)$$

where

$$w(\zeta) = 2\int_{-1}^{+1} \eta(\xi,\zeta)\, d\xi \qquad [9a]$$

$$\alpha = \frac{1}{2}\int_{-1}^{+1} X(\xi)\, d\xi = \frac{W_0}{BL} \qquad [9b]$$

α is the area coefficient of the load waterline and the load waterline area $W(0)$ is denoted by W_0.

a. $w(\xi) = W(z)/bl$ is the area curve of the dimensionless waterlines $\eta(\xi,\zeta)$ at the depth ζ_0. With $w(0) = w_0$, $w_0 = 4\alpha$.

b. $w^*(\zeta)$ is the dimensionless waterline area curve reduced to unity at the load waterline $w^*(0) = 1$.

c. We note further that $\alpha(\zeta) = W(z)/BL$ is the curve of area coefficients of waterlines at a depth z, referred to LB.

SUMMARY OF EQUATIONS FOR MATHEMATICAL SHIP LINES

$y = y(x, z)$	Equation of hull
$\xi = \frac{x}{l},\ \eta = \frac{y}{b},\ \zeta = \frac{z}{H}$	Dimensionless coordinates, where $l = \frac{L}{2}$, $b = \frac{B}{2}$
$\eta = \eta(\xi,\zeta)$	Dimensionless equation of hull
$\eta = \eta(\xi,0) = X(\xi)$	Dimensionless equation of waterplane.
$\eta = \eta(0,\zeta) = Z(\zeta)$	Dimensionless equation of midship section
$0 = \eta(\xi,\zeta);\ \ \zeta = K(\xi)$	Dimensionless equation of centerplane
$a(\xi) = 2\int_0^{K(\xi)} \eta(\xi,\zeta)\, d\zeta$	Dimensionless area of section
$a^*(\xi) = \frac{1}{\beta}\int_0^{K(\xi)} \eta(\xi,\zeta)\, d\zeta$	Dimensionless sectional area curve with unit ordinate at the midship section
$\frac{\nabla}{lbH} = \int_{-1}^{+1} a(\xi)\, d\xi = 4\delta$	Dimensionless volume
$a(0) = a_0 = 2\int_0^{1} Z(\zeta)\, d\zeta$	Dimensionless midship section area

$$w(\zeta) = 2\int_{-1}^{+1} \eta(\xi,\zeta)\, d\xi$$ Dimensionless waterline area

$$w^*(\zeta) = \frac{1}{2\alpha}\int_{-1}^{+1} \eta(\xi,\zeta)\, d\xi$$ Dimensionless waterline area reduced to unity at the load waterline

$$w(0) = w_0 = 2\int_{-1}^{+1} X(\xi)\, d\xi$$ Dimensionless waterplane area

$$C_w = \alpha = \frac{W(0)}{LB} = \frac{blw_0}{LB} = \frac{w_0}{4} = \frac{1}{2}\int_{-1}^{+1} X(\xi)\, d\xi$$ Load waterline coefficient

$$C_x = \beta = \frac{A(0)}{BH} = \frac{bHa_0}{BH} = \frac{a_0}{2} = \int_0^1 Z(\zeta)\, d\zeta$$ Midship area coefficient

$$C_B = \delta = \frac{\forall}{LBH} = \frac{1}{4}\int_{-1}^{+1} a(\xi)\, d\xi$$ Block coefficient

$$C_p = \phi = \frac{\forall}{LA_0} = \frac{\forall}{LBH} \times \frac{BH}{A_0} = \frac{\delta}{\beta} = \frac{1}{2}\int_{-1}^{+1} a^*(\xi)\, d\xi$$ Prismatic coefficient

SYMMETRY AND ANTISYMMETRY WITH RESPECT TO THE MIDSHIP SECTION

Application to Calculations

The description of the ship form by suitable coefficients can be highly improved by treating separately the forebody and the afterbody. Astonishingly, this rather trivial and well known procedure has only recently found a broader application.

Taking, for example, the equation of the load waterline $X(\xi)$ and denoting the pertinent parameters for the forebody and afterbody by the subscripts F and A we obtain a consistent set of coefficients by calculating moments of various orders

$$\int_0^1 X(\xi)\, d\xi = \alpha_F \qquad \int_0^1 X(\xi)\, \xi^2\, d\xi = i_F, \text{ etc.}$$

$$\int_0^1 X(\xi)\, \xi d\xi = \sigma_F \qquad \int_{-1}^0 X(\xi)\, d\xi = \alpha_A$$

$$\int_{-1}^0 X(\xi)\, \xi d\xi = \sigma_A$$

In the same way suitable parameters can be established for the entrance and run. To my knowledge Tulin (TINA, 1924) was the first to propose the ratios $\bar{\xi}_F = \sigma_F/\alpha_F$ and $\bar{\xi}_A = \sigma_A/\alpha_A$ as form parameters.

The difference $\alpha_F - \alpha_A$ can be used as an independent characteristic value for the description of the asymmetry beside the most popular distance of the centroid Equation [14].

For our present purpose, however, we do not need to dwell upon this matter and may confine ourselves to some remarks which are important for the resistance calculation.

A basic procedure is to split up the surface equation into a main part symmetric with respect to the midsection (y an even function with respect to x)

$$y_s(x,z) = b\,\eta_s(\xi,\zeta)$$

and an asymmetric (skew) deviation (y an odd function with respect to x)

$$y_a(x,z) = b\,\eta_a(\xi,\zeta)$$

$$y(x,z) = y_s(x,z) + y_a(x,z) \qquad [10]$$

$$\eta(\xi,\zeta) = \eta_s(\xi,\zeta) + \eta_a(\xi,\zeta) \qquad [10a]$$

The same applies to any curve dependent upon x (or ξ), such as

$$A(x), \quad a^*(\xi), \quad X(\xi), \quad \text{etc}$$

For instance

$$X(\xi) = X_s(\xi) + X_a(\xi) \qquad [10b]$$

Integrating over the total length we obtain

$$\int_{-1}^{+1} X(\xi)\, d\xi = 2\int_0^1 X_s(\xi)\, d\xi \qquad [11]$$

since the integral over an odd function with equal and opposite limits disappears,

$$\int_{-1}^{+1} X_a(\xi)\, d\xi = 0$$

This elementary remark is very useful in the whole field of theoretical naval architecture. Thus from Equation [11] it follows, for instance, that the area W and the area coefficient α of the load waterline depend only upon the symmetrical part of $X_s(\xi)$, while the odd terms $X_a(\xi)$ only yield a contribution to the static moment S_y or S_η with respect to the transverse axis y or η.

$$w_0 = 4\int_0^1 X_s(\xi)\, d\xi = 4\alpha \qquad [12]$$

$$S_\eta = 4\int_0^1 X_a(\xi)\ \xi\, d\xi = 4\, S_\eta^* \qquad [13]$$

Let x_0 be the longitudinal coordinate of the centroid. Then with $\xi_0 = x_0/l$ we obtain

$$\xi_0 = \frac{S_\eta^*}{\alpha} = \frac{\int_0^1 X_a(\xi)\ \xi\, d\xi}{\int_0^1 X_s(\xi)\, d\xi} \qquad [14]$$

In shipbuilding practice the ratio $e_0 = x_0/L = \xi_0/2$ is commonly used.

When a curve is given analytically or graphically by $X = X(\xi)$, $-1 \le \xi \le 1$, then

$$X_s = \frac{1}{2}[X(\xi) + X(-\xi)], \quad -1 \le \xi \le 1 \qquad [15]$$

$$X_a = \frac{1}{2}[X(\xi) - X(-\xi)], \quad -1 \le \xi \le 1 \qquad [15a]$$

It is clear that X_s is symmetric, X_a asymmetric, and that $X = X_s + X_a$, $-1 \le \xi \le 1$.

These trivial considerations can save labor when performing routine computations in shipbuilding practice.

REPRESENTATION OF SHIP HULLS BY POLYNOMIALS

GENERAL CONSIDERATIONS

Any function $y = y(x,z)$ which is continuous in a given domain can be approximated within any degree of accuracy desired by a complete set of orthogonal functions. As such one could, for instance, choose the Fourier series or the Legendre polynomials.[6] Since, however, a technically satisfactory solution must be restricted to a small number of terms, the mentioned functions do not appear to be practical in our case. Using a modest number of Fourier series terms, the approximating function generally will not be fair, i.e., exhibit a larger number of points of inflection.

An interesting example showing that the orthodox approach is not always the simplest may be quoted from the field of aerodynamics: in a study of lift distribution over wings, Fuchs[7] has demonstrated that by selecting properly the coefficients and the terms of a set of trigonometric functions a better approximation can be found than by the Fourier expansion with the same (small) number of terms.

The successful application of spline curves* to ship design suggests that an analytical representation of ship forms by polynomials should be rather simple.

This way has been tried with good results so far as waterlines are concerned. Its mathematical justification follows from Weierstrass' theorem[1]: a continuous function $y(x,z)$ within prescribed boundaries can be approximated with any desired degree of accuracy by a polynomial in x,z.

Thus

$$y = \sum\sum A_{mn} x^n z^m$$

$$\eta = \sum\sum a_{mn} \xi^n \zeta^m \qquad [16]$$

can be assumed as general expressions for the ship hull.

The general equation [16] does not lend itself easily to a discussion. Besides the boundary conditions, Equation [16] must fulfill the basic inequality $\eta(\xi,\zeta) \geq 0$.

For design purposes the block coefficient δ the main area coefficients α and β, and various other integral and differential relations can be prescribed. The most familiar and powerful approach is to assume the form of the sectional area curve

$$a^*(\xi) = \frac{1}{\beta}\int_0^{K(\xi)} \eta(\xi,\zeta)\, d\zeta \qquad [8]$$

It is difficult to comply with conditions of fairness since these have not yet been properly formulated. However, assuming a reasonable number of arbitrary parameters one is, at least in principle, enabled to derive ship forms from general mechanical considerations like minimum wave resistance, considerations on seaworthiness, etc.

Actually, so far, Equation [16] has not been systematically discussed. Instead of the general approach, some intuitive procedures of constructing the hull equation have been proposed by the author. These procedures follow to some extent the graphical method of design and are largely based on the equations of the load waterline and the midship section

$$X(\xi) = 1 - \sum a_n \xi^n \qquad [17]$$

$$Z(\zeta) = 1 - \sum b_m \zeta^m \qquad [18]$$

*The equations of the simplest spline curves are polynomials.

These are investigated separately and then are connected in such a way that the boundary condition on the contour line, which will generally be assumed in the form $\eta(\xi, \zeta) = 0$ (Equation [5]), and other conditions are easily fulfilled. In what follows it will be assumed that the thickness of the stern and the keel is zero. This restriction is by no means necessary, but it simplifies the work considerably. Then from Equations [17] and [18] we obtain immediately

$$\sum a_n = 1 \qquad \textbf{[17a]}$$

and

$$\sum b_m = 1 \qquad [18a]$$

since

$$X(1) = Z(1) = 0$$

By introducing additional functions the flexibility of forms can be appreciably increased. Thus the problem may be split into two parts:

1. The study of appropriate lines (waterlines and sections) to which some attention has already been given and which will be investigated more thoroughly in the section on Equations of Waterlines and Sectional Area Curves.

2. The construction of the hull from these elements. Simple examples will be discussed in a later section.

REMARKS ON THE PROPERTIES OF THE BINOMIALS

$$\eta = 1 - \xi^n \quad \text{or} \quad \eta = 1 - \zeta^m \qquad [19]$$

with n,m positive integers are equations of general parabolas. Obviously the parabola $\eta_1 = 1 - \eta = \xi^n$ has the following important properties within the region

$$0 \leq \xi \leq 1$$

1. $\eta_1 = 0$ and $\eta_1 = 1$ for all n at the points $\xi = 0$ and $\xi = 1$; respectively.
2. With increasing n, the parabolas approach the axes $\eta_1 = 0$ and $\xi = 1$
3. By folding the curves around the line $\eta_1 = \xi$, we obtain the curves

$$\eta_1 = \xi^{1/n} \qquad [20]$$

Introducing again our usual axis of reference the curves

$$\eta = 1 - \eta_1 = 1 - \xi^n$$

where n is no longer restricted to integral values, are called Chapman's parabolas. They have been frequently recommended as ship lines (see Figure 2).

Although almost useless for actual design work, these simple curves can be applied with success for various theoretical estimates.

EQUATIONS OF WATERLINES AND SECTIONAL AREA CURVES

GENERAL CONSIDERATIONS

D.W. Taylor's investigation on the properties of these lines dependent upon three parameters, ϕ or α, t and K, the curvature at the midship section, has been performed in a truly classical style. Unfortunately, Taylor's work had not found the proper response, and only lately Benson[8] and Sparks[9] have applied his results to various problems of design.

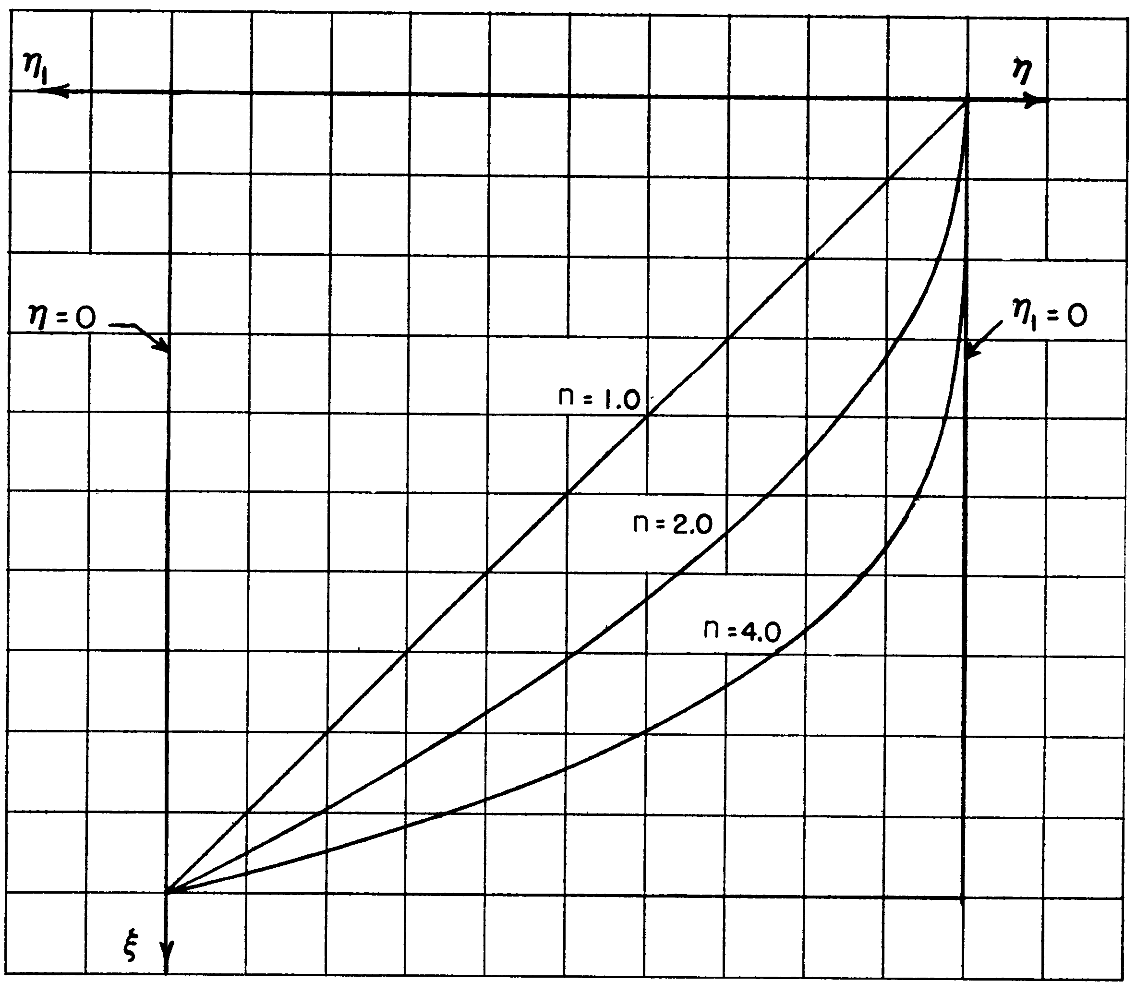

Figure 2 - The Binomial $\eta = 1 - \xi^n$

Combining Taylor's curves with a parallel middle body for higher prismatics, it is thought that a close approximation to a large number of empirical waterlines and sectional area curves of "normal" shape can be obtained.

There are, however, objections to his system. Taylor's lines must be applied separately to the forebody and the afterbody; they are not suitable for representing simultaneously the whole range of a line. Besides, generally our final purpose is to find the equation of the *surface*, not of a single *line*. In this case it may be preferable to obtain a practically cylindrical part by using higher powers of the variable ξ instead of inserting a rigorously parallel middle body. Therefore we shall use the axes of reference introduced in Figure 1 and base the representation on polynomials which admit, if necessary, a greater variety of powers than in Taylor's system.

When deriving equations of the lines involved it is advantageous to make use of the fact that hulls are frequently roughly symmetric with respect to the midship section, in so far as normal ocean-going ships are considered.

SYMMETRIC FORMS

In this section the lines discussed are those which are symmetric with respect to the midship section, i.e., the corresponding polynomials are even functions in ξ. Normally such polynomials consist only of terms with even powers of ξ. However, by introducing the absolute value of ξ, $|\xi|$, even terms of the type $|\xi|^{2n+1}$ with odd exponents can be obtained. This artifice is widely used for the third power $|\xi|^3$, since the geometric properties of the two functions ξ^2 and ξ^4 with the lowest even integer exponents are so widely different that it is desirable to have an intermediate element. With higher exponents the difference in character between consecutive even powers ξ^{2n} and ξ^{2n+2} gradually disappears so that there is less advantage in inserting terms of the type $|\xi|^{2n+1}$.

In principle the aforementioned trick is unnecessary since, from the completeness property of the power functions (Weierstrass' theorem), it follows that symmetric functions can be approximated by even powers only.[6] But appreciable simplification in actual work seems to be possible by this simple procedure.

Let us start with families of curves which depend upon two arbitrary parameters which are called "basic families."

The general equation of these curves is given by

$$\eta(\xi) = \langle n_1 \; n_2 \; n_3 \rangle = 1 - a_{n_1} \xi^{n_1} - a_{n_2} \xi^{n_2} - a_{n_3} \xi^{n_3} \qquad [21]$$

Following an earlier assumption, the ordinate η at the midship section ($\xi = 0$) is equal to unity, and at the stern and stem ($\xi = \pm 1$) is equal to zero.

From Equation [17a] one relation between the coefficients is obtained,

$$a_{n_1} + a_{n_2} + a_{n_3} = 1 \qquad [21a]$$

so that only two arbitrary parameters are left in Equation [21]. This can be explicitly expressed by rewriting Equation [21] as

$$\eta(\xi) = 1 - \xi^{n_3} - a_{n_1}\left(\xi^{n_1} - \xi^{n_3}\right) - a_{n_2}\left(\xi^{n_2} - \xi^{n_3}\right) \qquad [22]$$

The parameters a_{n_1}, a_{n_2} are easily expressed in terms of the coefficients α (or ϕ) and t using the two conditions

$$\int_0^1 X(\xi)\, d\xi = \alpha \qquad \frac{\partial X(1)}{\partial \xi} = -t$$

The symbolic expression for the lines $< n_1, n_2, n_3 >$ may be rewritten in a more explicit way as

$$< n_1 \quad n_2 \quad n_3;\ \alpha\ ;\ t > \qquad [23]$$

Let us take an example

$$< 2\ 4\ 6; \alpha; t > = 1 - a_2\,\xi^2 - a_4\,\xi^4 - a_6\,\xi^6 \qquad [24]$$

The area condition gives immediately

$$\alpha = \frac{6}{7} - \frac{4}{21}\,a_2 - \frac{2}{35}\,a_4 \qquad [25]$$

the tangent condition

$$t = 6 - 4a_2 - 2a_4 \qquad [26]$$

resulting in

$$\begin{aligned} a_2 &= 9 - \frac{105}{8}\,\alpha + \frac{3}{8}\,t \\ a_4 &= -15 + \frac{105}{4}\,\alpha - \frac{5}{4}\,t \\ a_6 &= 7 - \frac{105}{8}\,\alpha + \frac{7}{8}\,t \end{aligned} \qquad [27]$$

Assuming for example $\alpha = 2/3$, $t = 2$, one obtains

$$a_2 = 1 \quad a_4 = a_6 = 0 \quad \eta = 1 - \xi^2$$

i.e., the common parabola.

Two sets of curves belonging to this family are shown in Figures 15 and 16 of TMB Report 710 (Reference 2).

For the general expression $\eta(\xi) = < n_1\ n_2\ n_3; \alpha\ ;\ t >$ the following relations are obtained

$$a_{n_1} = \frac{n_1 + 1}{(n_2 - n_1)(n_3 - n_1)} \left[n_2 n_3 - \alpha (n_2 + 1)(n_3 + 1) + t \right]$$

$$a_{n_2} = - \frac{n_2 + 1}{(n_2 - n_1)(n_3 - n_2)} \left[n_1 n_3 - \alpha (n_1 + 1)(n_3 + 1) + t \right] \qquad [28]$$

$$a_{n_3} = \frac{n_3 + 1}{(n_3 - n_1)(n_3 - n_2)} \left[n_1 n_2 - \alpha (n_1 + 1)(n_2 + 1) + t \right]$$

Introducing these expressions into the general equation, one obtains

$$\eta(\xi) = \eta_0(\xi) + \alpha \eta_1(\xi) + t \eta_2(\xi) \qquad [29]$$

where (a) $\eta_0(\xi)$ complies with

$$\eta_0(0) = 1 \quad \eta_0(1) = 0 \quad \int_0^1 \eta_0(\xi)\, d\xi = 0 \quad \frac{\partial \eta_0(1)}{\partial \xi} = 0 \qquad [30]$$

(b) $\eta_1(\xi)$ satisfies

$$\eta_1(0) = \eta_1(1) = 0 \quad \int_0^1 \eta_1 \, d\xi = 1 \quad \frac{\partial \eta_1(1)}{\partial \xi} = 0 \qquad [31]$$

(c) $\eta_2(\xi)$ satisfies

$$\eta_2(0) = \eta_2(1) = 0 \quad \int_0^1 \eta_2 \, d\xi = 0 \quad \frac{\partial \eta_2(1)}{\partial \xi} = - t \qquad [32]$$

It is easily verified that

$$\eta_0(\xi) = 1 - \frac{(n_1+1) n_2 n_3}{(n_2 - n_1)(n_3 - n_1)} \xi^{n_1} + \frac{(n_2+1) n_1 n_3}{(n_2 - n_1)(n_3 - n_2)} \xi^{n_2} - \frac{(n_3 - 1) n_1 n_2}{(n_3 - n_1)(n_3 - n_2)} \xi^{n_3}$$

$$\eta_1(\xi) = \frac{(n_1+1)(n_2+1)(n_3+1)}{(n_2 - n_1)(n_3 - n_1)(n_3 - n_2)} \left[(n_3 - n_2) \xi^{n_1} - (n_2 - n_1) \xi^{n_2} + (n_2 - n_1) \xi^{n_3} \right] \qquad [33]$$

$$\eta_2(\xi) = - \frac{1}{(n_2 - n_1)(n_3 - n_1)(n_3 - n_2)} \left[(n_1 + 1)(n_3 - n_2) \xi^{n_1} - (n_2 + 1)(n_3 - n_1) \xi^{n_2} + (n_2 + 1)(n_2 - n_1) \xi^{n_3} \right]$$

A basic family is a linear function of the form parameters α and t. This leads to a simple representation of sets of curves and admits of linear interpolation.

Althought any basic family contains only two arbitrary parameters, it is easy to obtain a wide variety of forms by mixing two or more sets. Thus $< n_1 n_2 n_3 n_4; \alpha ; t >$ can be immediately obtained from $B < n_1 n_2 n_3; \alpha ; t > + (1-B) < n_2 n_3 n_4; \alpha ; t >$ with B an arbitrary parameter.

The same result may be obtained from a function

$$\underset{\infty}{\Delta}(\xi) = < n_1 n_2 n_3 n_4; 0; 0 > = \sum_i C_{n_i} \xi^{n_i} \qquad [34]$$

where C_{n_1} can be put equal to one.

$\underset{\infty}{\Delta}(\xi)$ complies with the conditions

$$\underset{\infty}{\Delta}(0) = \underset{\infty}{\Delta}(1) = 0 \qquad \frac{\partial \underset{\infty}{\Delta}(0)}{\partial \xi} = \frac{\partial \underset{\infty}{\Delta}(1)}{\partial \xi} = 0 \qquad \int_0^1 \underset{\infty}{\Delta}(\xi)\, d\xi = 0 \qquad [35]$$

The coefficients are therefore connected by

$$1 + C_{n_2} + C_{n_3} + C_{n_4} = 0 \qquad [36]$$

with

$$C_{n_2} = -\frac{(n_4-n_1)(n_3-n_1)(n_2+1)}{(n_4-n_2)(n_3-n_2)(n_1+1)} \qquad C_{n_3} = \frac{(n_4-n_1)(n_2-n_1)(n_3+1)}{(n_4-n_3)(n_3-n_2)(n_1+1)} \qquad [37]$$

for instance

$$< 2\ 4\ 6\ 8; 0; 0 > = \xi^2 - 5\xi^4 + 7\xi^6 - 3\xi^8 \qquad [38]$$

Thus we can write

$$< n_1 n_2 n_3 n_4; \alpha ; t > = < n_1 n_2 n_3; \alpha ; t > + B < n_1 n_2 n_3 n_4; 0; 0> \qquad [39]$$

with B an arbitrary parameter.

This apparently clumsy procedure presents in fact advantages. Dependent upon the character of the polynomials, geometrical interpretations for the parameter B can be found. For instance, when the lowest power in Equation [34] is ξ^2 as in Equation [38], B is equal to $K/2 + a_2$ where K is the dimensionless curvature at $\xi = 0$.

Where the lowest exponent is three or more, other suitable geometric interpretations of parameters may be found. Resistance theory should be helpful in this respect.

ANTISYMMETRIC TERMS

In a similar way expressions for asymmetric (skew) terms may be obtained.

In TMB Report 758 (Reference 4) the function

$$X_a = a_1 (\xi + b_3 \xi^3 + b_5 \xi^5)$$

has been investigated. The parameter a_1 describes the "strength" of asymmetry and the polynomial in parenthesis its "character." Since $X_a(1) = 0$ we rewrite

$$X_a = a_1 \left[\xi + b_3 \xi^3 - (1 + b_3) \xi^5 \right]$$

By adding this expression to a symmetrical form we displace the maximum section from the origin (midship section) because of the linear term. This is advantageous when dealing with such high speed-types of ships as cross-channel steamers and destroyers.

For slower ships it is desirable to keep the maximum section at the origin. In such cases a polynomial

$$X_a = a_3 \left[\xi^3 + b_5 \xi^5 - (1+b_5) \xi^7 \right]$$

or of higher degree should be used.

ELEMENTARY SHIPS AND OTHER SIMPLIFIED SHIP FORMS

Let us start with a simplified hull form which is characterized by:

1. A rectangular centerplane contour,
2. The form of the equation

$$\eta(\xi, \zeta) = \eta = X(\xi)\, Z(\zeta) \qquad [40]$$

Here

$$\eta(\xi, 0) = X(\xi) \qquad X(0) = Z(0) = 1 \qquad [41]$$

$$\eta(0, \zeta) = Z(\zeta) \qquad X(\pm 1) = Z(1) = 0 \qquad [42]$$

are the equations of the load waterline and the midship section respectively. We call such bodies elementary ships.

Elementary ships have the following important properties:

a. all sections are affine to the midship section.

b. $$a^*(\xi) = \frac{1}{\beta} \int_0^1 \eta(\xi, \zeta)\, d\zeta = X(\xi) \qquad [43]$$

i.e., the dimensionless waterline and sectional area curve coincide.

c. hence $\phi = \alpha$; and $\delta = \alpha\,\beta$, or $\delta/\alpha\,\beta = 1$. [44]

The last coefficient $\delta/\alpha\,\beta$ was very popular with naval architects of the old school.

One great advantage of the elementary ship concept consists in the possibility of investigating waterlines and sections independently. The equation of the hull is immediately built up from ship lines by one multiplication.

The practical applicability of such elementary hulls is limited essentially by the occurence of high local curvatures in sections close to the bow and stern; these are unavoidable when the midship section is rather full.

For fine midship sections, however, very reasonable body plans may be obtained from Equation [40].

An example of a simple elementary ship is (see Figure 3)

$$\eta = (1-\xi^2)\,(1-\zeta^2) \qquad [45]$$

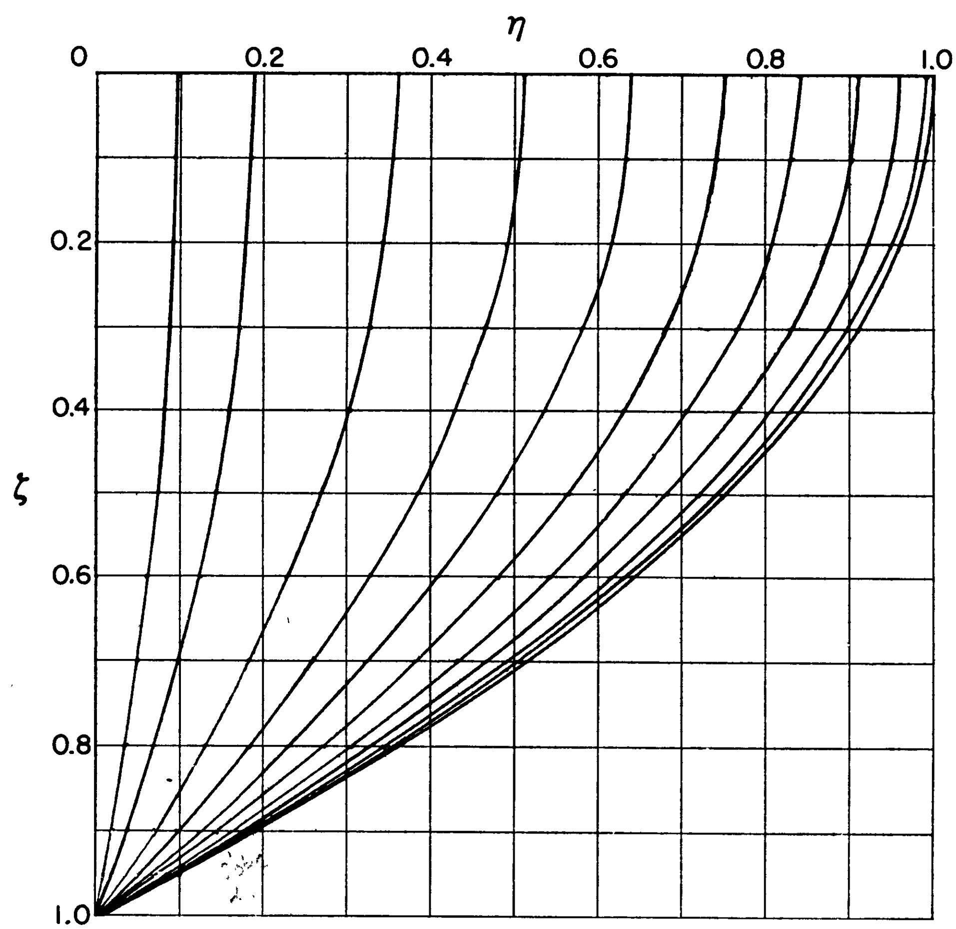

Figure 3 - Body Plans of an Elementary Ship where $\eta = (1-\xi^2)\,(1-\zeta^2)$ and $\delta = 4/9$

3. As the next somewhat more general equation we choose

$$\eta = [X(\xi) - v(\xi)\, v_1(\zeta)]\, Z(\zeta) \qquad [46]$$

where the "fining function" $v(\xi)$ complies with the condition

$$v(1) = v(-1) = v(0) = 0 \qquad [47]$$

and $v_1(\zeta)$ complies with the condition

$$v_1(0) = 0 \qquad [48]$$

Equation [46] can be interpreted as an elementary ship minus a layer $v(\xi)\, v_1(\zeta)\, Z(\zeta)$ which assumes zero values on the centerplane contour.

Examples of body plans are shown in Figures 4, 5, 6.

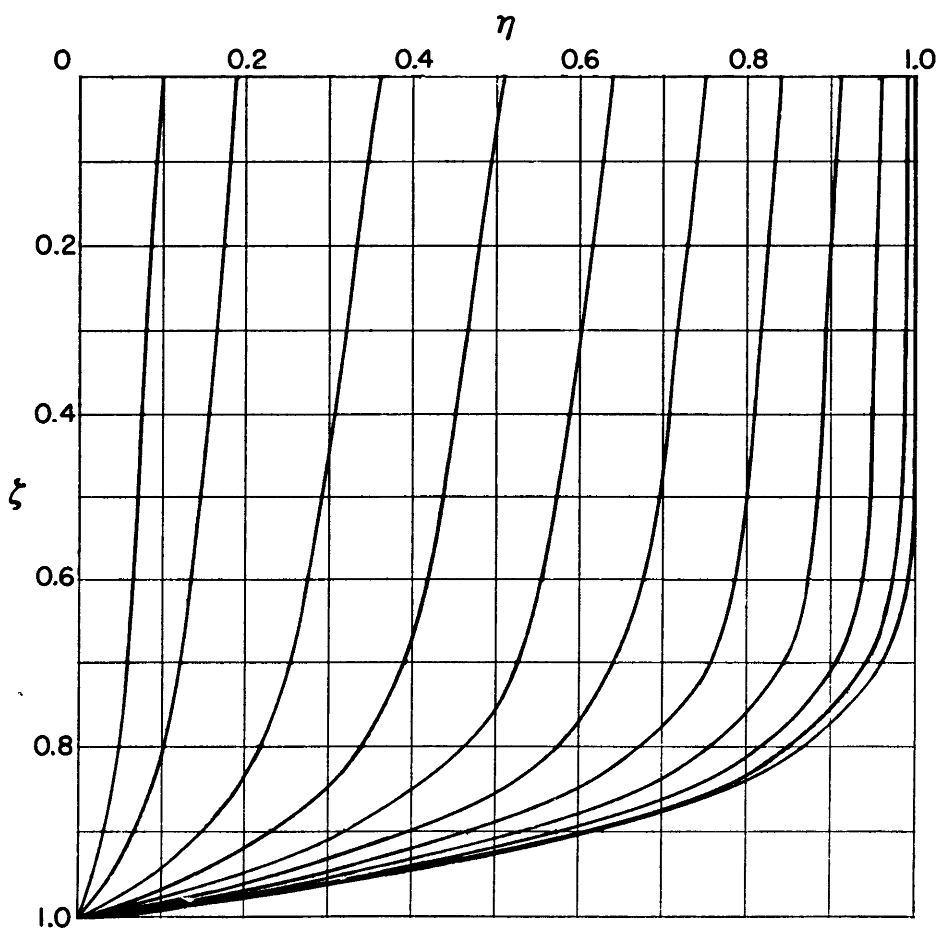

Figure 4 - Examples of Ship Lines
$\eta = [1 - \xi^2 - 0.5757\, (\xi^2 - \xi^4)\, \zeta]\, (1 - \zeta^9)$ and $\delta = 0.5689$

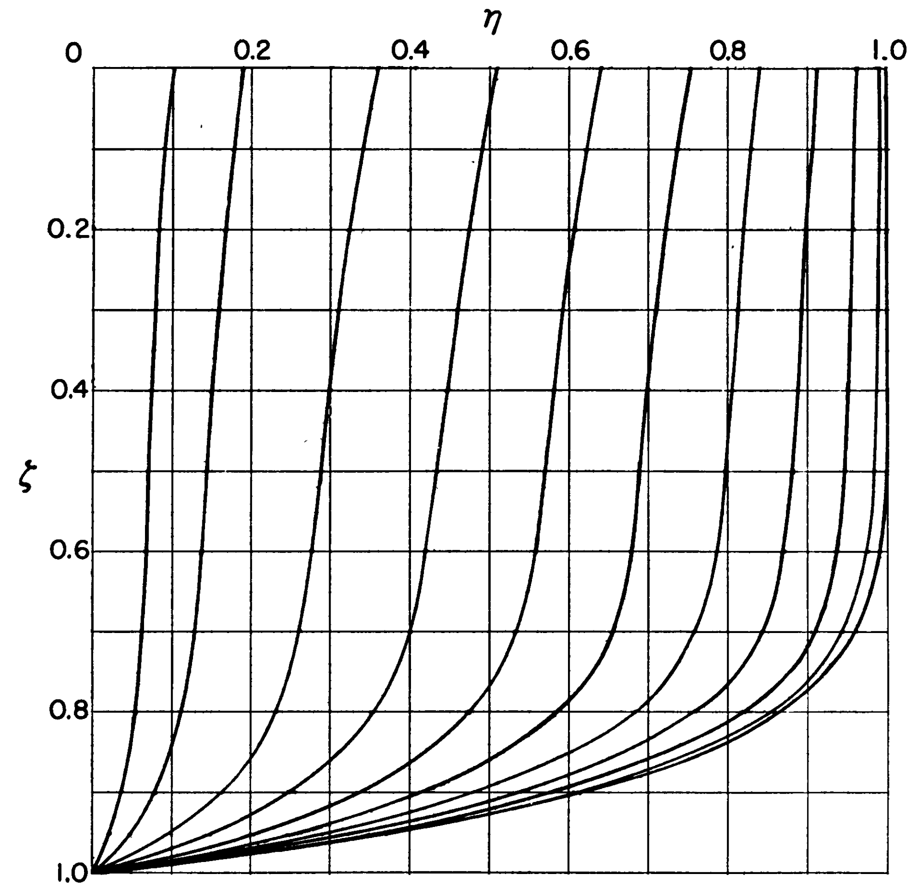

Figure 5 - Examples of Ship Lines
$\eta = [1 - \xi^2 - 0.4105\ (\xi^2 - \xi^4)\ (2\zeta - \zeta^2)]\ (1 - \zeta^9)$ and $\delta = 0.5689$

Assuming that the midship section is vertical at the load waterline $\partial Z(0)/\partial \zeta = 0$ $v_1(\zeta)$ can be used to obtain an inclination of the sections at $\zeta = 0$.

The sectional area curve becomes

$$a^*(\xi) = X(\xi) - \frac{\beta_1}{\beta}\ v(\xi) \qquad [49]$$

$$\phi = \alpha - \frac{\beta_1}{\beta}\ \alpha_1 \qquad [50]$$

with $\beta_1 = \int_0^1 Z(\zeta)\ v_1(\zeta)\ d\zeta \qquad \alpha_1 = \int_0^1 v(\xi)\ d\xi$

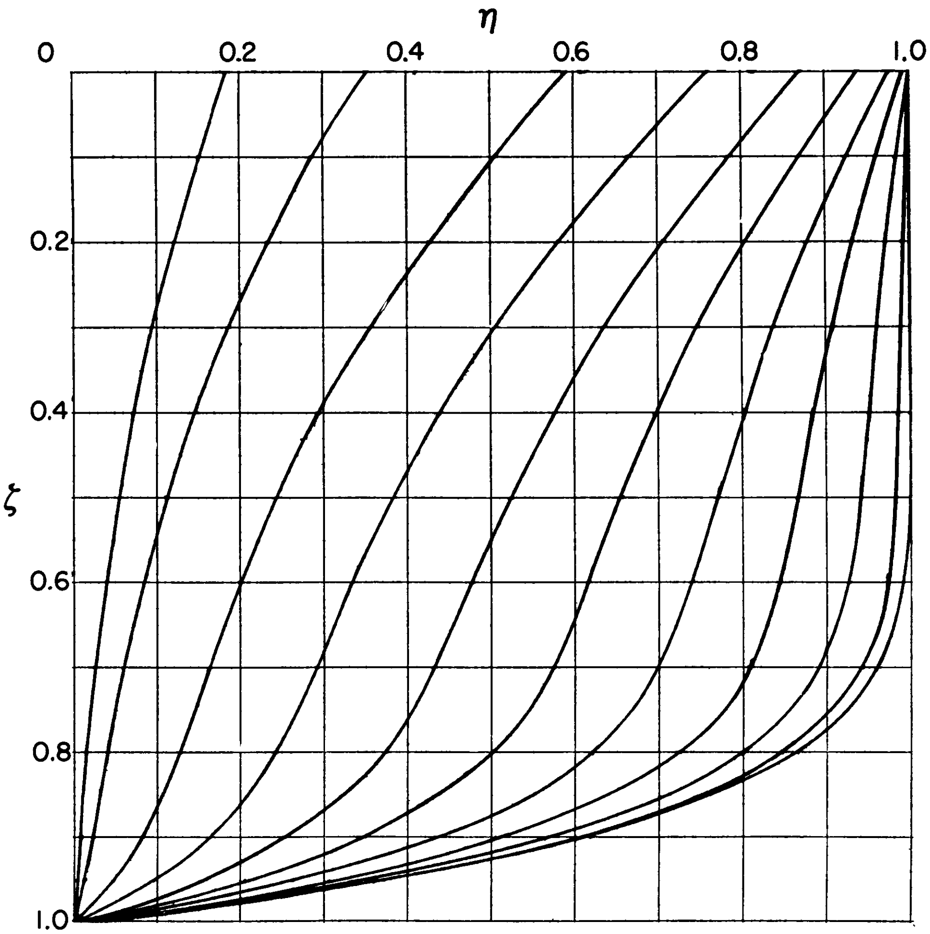

Figure 6 - Examples of Ship Lines

$\eta = [1 - \xi^4 - 2\,(\xi^2 - \xi^4)\,(2\zeta - \zeta^2)]\,(1 - \zeta^9)$ and $\delta = 0.5685$

The local sectional area coefficient $\beta(\xi)$ defined by $\beta \;\; a^*(\xi)/X(\xi)$ becomes

$$\beta(\xi) = \beta - \beta_1 \frac{v(\xi)}{X(\xi)} \qquad [51]$$

When $v(\xi) > 0$, $\beta_1 > 0$, as will be the usual case, the local coefficients

$$\beta(\xi) < \beta$$

Notwithstanding its simplicity, Equation [46] is general enough to yield definite conclusions as to the basic wave resistance properties of V-shaped versus U-shaped hulls.

As an illustrative example of the application of Equation [46] in the calculation of ship hulls, some cases are presented. The same procedure is applied in each case, the only variation occuring in the differences between the load waterline and sectional area curves.

For the first case let us assume the following data:

1. $a^*(\xi) = 1 - 1.5\,\xi^2 + 0.5\,\xi^4 \qquad \phi = 0.6$
2. $X(\xi) = 1 - \xi^2 \qquad \alpha = 2/3$
3. $Z(\zeta) = 1 - \zeta^9 \qquad \beta_1 = \int_0^1 (1 - \zeta^9)\,\zeta d\zeta = 9/22$ [52]
4. $v_1(\zeta) = \zeta \qquad \frac{\beta_1}{\beta} = 5/11$

The function $v(\xi)$ is immediately found from Equation [49].

$$1 - 1.5\,\xi^2 + 0.5\,\xi^4 = 1 - \xi^2 - \frac{\beta_1}{\beta}\,v(\xi) \qquad [53]$$

$$v(\xi) = 1.1\,(\xi^2 - \xi^4)$$

$$\eta = [1 - \xi^2 - 1.1\,(\xi^2 - \xi^4)\,\zeta]\,(1 - \zeta^9) \qquad [54]$$

The compatibility of the data can be checked using the condition $\eta(\xi, \zeta) > 0$ Letting $\zeta \to 1$, we obtain

$$\eta(\xi, \zeta) \doteq [1 - 2.1\,\xi^2 + 1.1\,\xi^4]\,(1 - \zeta^9)$$

from which it immediately follows that $\eta(\xi, \zeta)$ becomes < 0 close to the ends of the ship at the bottom (the easiest check is that the tangent value t becomes < 0 at the bottom).

Let us change the condition [52,4] into

$$v_1(\zeta) = \zeta - 0.5\,\zeta^3 \qquad [55]$$

leading to $\beta_1 = 0.3225$; $\beta_1/\beta = 0.359$ with $v(\xi) = 1.392\,(\xi^2 - \xi^4)$. Since the maximum of $v_1(\zeta)$, at $\zeta = \sqrt{2/3}$, amounts to $v_1(\sqrt{2/3}) = 0.545$ it is easily seen that the condition $\eta(\xi, \zeta) > 0$ is fulfilled. The same procedure is now applied when the difference between the load waterline and sectional area curve is larger.

Assume

$$v_1(\zeta) = 2\,\zeta - \zeta^2 \qquad [56]$$

with

$$\frac{\partial v_1(1)}{\partial \zeta} = 0 \qquad v_{1_{max}} = v_1(1) = 1 \qquad v_1(0) = 0$$

and

$$X(\xi) = 1 - \xi^4$$

while

$$Z(\zeta) = 1 - \zeta^9$$

and $a^*(\xi) = 1 - 1.5\,\xi^2 + 0.5\,\xi^4$ remain as before in Equation [52]
Further

$$\beta_1 = 0.568; \quad \beta_1/\beta = 0.632$$

Hence

$$v(\xi) = 2.37\,(\xi^2 - \xi^4) \qquad [57]$$

As

$$\zeta \to 1$$

$$\eta \to (1 - 2.37\,\xi^2 + 1.37\,\xi^4)\,(1 - \zeta^9)$$

$$t_{\zeta \to 1} < 0$$

The form Equation [56] is suitable only when the coefficient a in

$$v(\xi) = a(\xi^2 - \xi^4) \text{ is } \leq 2 \qquad [57a]$$

To $a = 2$ corresponds the equation

$$a^*(\xi) = 1 - \xi^4 - 2\,(\xi^2 - \xi^4)\,0.632 \qquad [58]$$

with

$$\phi = 0.632$$

When the difference between the load waterline and the sectional area curve is large, the form of $v_1(\zeta)$ must be such that it reaches its maximum at $\zeta = 1$.

Within the range of compatibility

$$\eta(\xi, \zeta) > 0 \quad |\xi| \leq 1 \quad \zeta \leq 1$$

the equations of the sectional area curve and of the waterline can be arbitrarily assumed.

4. Equation [46] can be generalized by introducing more terms of the type $v(\xi)$, $v_1(\zeta)$ complying with the same boundary conditions.

PART II

THE EVALUATION OF MICHELL'S INTEGRAL FOR SIMPLIFIED NORMAL SHIP FORMS

ELEMENTARY SHIPS

The basic importance of the sectional area curve in wave resistance research has been definitely established by numerous experimental and theoretical investigations. It is therefore advantageous to begin with systematic evaluations of the resistance integral for ship forms which are defined by their sectional area curve in the most straightforward way, i.e. for elementary ships following Equation [40].

Wave resistance values thus obtained are immediately applicable to U-shaped section forms, but may be used even in a more general way. In these cases the elementary ship concept leads, briefly speaking, to a substitution of the sectional area curve for the actual ship form.

Let the waterline equation be given as the sum of an even part $X_s(\xi)$ and odd part $X_a(\xi)$ with respect to ξ.

$$X(\xi) = X_s(\xi) + X_a(\xi) = 1 - \sum a_n \xi^n - \sum b_m \xi^m$$

Even exponents, $n = 2, 4, 6, 8, 10, 12$ will be used in our present evaluations; additionally the third power of the absolute value $|\xi|^3$ will be admitted as an even term. The resistance due to odd powers ξ^m will be investigated in a later report.

The equation of the midship section can be taken in a simple form. As such we choose

$$Z(\zeta) = 1 - e\zeta^4 \qquad [59]$$

where the parameter e can be varied between + 1 and any negative number $1 \geq e \geq -\infty$ (see Figure 7). In practice, clearly, negative values of e will be seldom used.

$e = 1$ corresponds to a $\beta = 0.8$

$e = 0.5$ corresponds to a $\beta = 0.9$

$e = 0$ corresponds to a $\beta = 1.0$

It is thought that values of β below 0.8 are of little interest. Besides in a publication by Wigley, Tr. I.N.A. 1942, the wave resistance has been calculated for a hull family

$$\eta(\xi, \zeta) = <2\ 4\ 6; \quad \alpha; \quad t > (1 - \zeta^2) \qquad [60]$$

i.e., this paper yields information on resistance properties of ships with very small values of β.

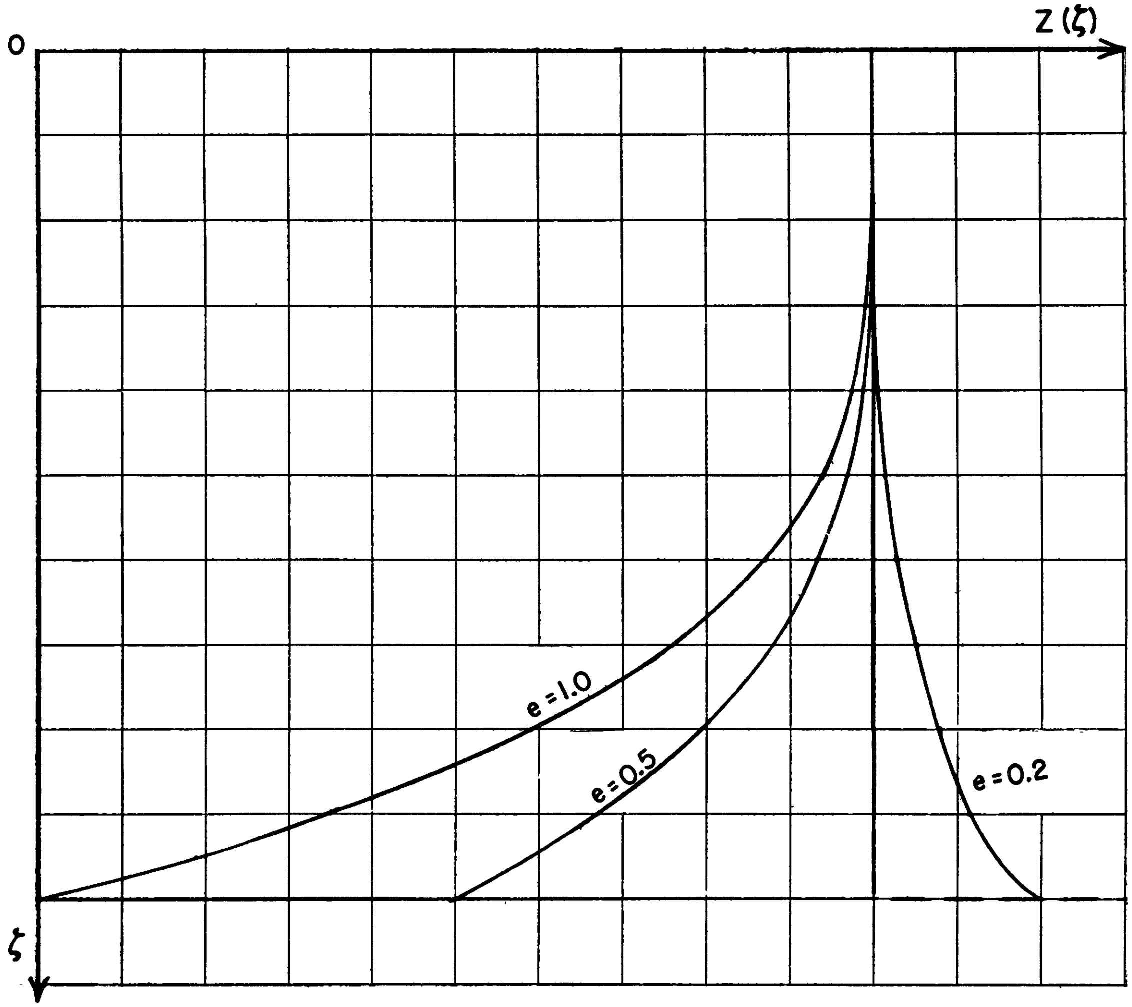

Figure 7 - Midship Equation $Z(\zeta) = 1 - e\,\zeta^4$

The evaluation of the resistance integral R follows closely the procedure given in TMB Report 758.[4]

One obtains for R, or a dimensionless value R^*

$$R^* = \frac{R}{\frac{8}{\pi}\rho g \frac{B^2 H^2}{L}}$$

a quadratic form in the parameters $(n_1 a_{n_1})\,(n_2 a_{n_2})\,\ldots$ with some auxiliary integrals as coefficients. These integrals were tabulated by the Bureau of Standards and are presented in Appendix II of this report.

We sketch briefly the derivation of an expression for R^* which leads to a simple symbolic connection with the ship form.

Differentiate the surface equation [40,10b] with respect to ξ

$$\frac{\partial X(\xi)}{\partial \xi} Z(\zeta) = \left(\frac{\partial X_s}{\partial \xi} + \frac{\partial X_a}{\partial \xi}\right) Z(\zeta) = -Z(\zeta)\left[\sum n a_n \xi^{n-1} + \sum m b_m \xi^{m-1}\right] \quad [61]$$

Insert now the expressions [59] and [61] in Michell's integral which is written in the form (Reference 2, Equation [16] of Appendix 2)

$$R^* = \int_{\gamma_0}^{\infty} f(\gamma)\left[J^{*2}(\gamma) + I^{*2}(\gamma)\right] d\gamma = \frac{R}{\frac{8}{\pi}\rho g \frac{B^2 H^2}{L}} \quad [62]$$

with

$$\gamma_0 = \frac{1}{2F^2}; \quad v = 2\frac{H}{L}\frac{\gamma^2}{\gamma_0} = K\frac{\gamma^2}{\gamma_0}; \quad f(\gamma) = \frac{\left(\frac{\gamma}{\gamma_0}\right)^2}{\sqrt{\left(\frac{\gamma}{\gamma_0}\right)^2 - 1}} \quad [62a]$$

$$J^*(\gamma) = \Phi(v)\, S_s(\gamma) = \int_0^1 e^{-v\zeta} Z(\zeta)\, d\zeta \int_0^1 \frac{\partial X_s}{\partial \xi} \sin \gamma\xi\, d\xi \quad [62b]$$

$$I^*(\gamma) = \Phi(v)\, S_a(\gamma) = \int_0^1 e^{-v\zeta} Z(\zeta)\, d\zeta \int_0^1 \frac{\partial X_a}{\partial \xi} \cos \gamma\xi\, d\xi \quad [62c]$$

In our case

$$\Phi(v) = \int_0^1 e^{-v\zeta}(1 - e\zeta^4)\, d\zeta = E_0(v) - e\,E_4(v);$$

$$E_0(v) = \int_0^1 e^{-v\zeta}\, d\zeta; \quad E_4(v) = \int_0^1 \zeta^4 e^{-v\zeta}\, d\zeta \quad [62d]$$

$E_0(v)$ corresponds to a rectangular midship section.

$$S_s(\gamma) = -\int_0^1 \sum n a_n \xi^{n-1} \sin \gamma\xi\, d\xi = -\sum n a_n M_{n-1}(\gamma) \quad [62e]$$

$$S_a(\gamma) = -\int_0^1 \sum m\, b_m\, \xi^{m-1} \cos \gamma \xi \, d\xi = -\sum m\, b_m\, M'_{m-1}(\gamma) \qquad [62f]$$

with

$$M_{n-1}(\gamma) = \int_0^1 \xi^{n-1} \sin \gamma \xi \, d\xi \qquad [62g]$$

$$M'_{m-1}(\gamma) = \int_0^1 \xi^{m-1} \cos \gamma \xi \, d\xi \qquad [62h]$$

Thus

$$R^* = R^*(\gamma_0) = \int_{\gamma_0}^{\infty} (E_0 - e\,E_4)^2 \left\{ \left[\sum n a_n M_{n-1}(\gamma) \right]^2 + \left[\sum m\, b_m\, M'_{m-1}(\gamma) \right]^2 \right\} f(\gamma)\, d\gamma \qquad [63]$$

Omitting the odd term $I^{*2}(\gamma)$ which will be treated in a following report and expanding the squared terms, we obtain

$$R^*(\gamma) = \int_{\gamma_0}^{\infty} \left[E_0^2 - 2\,e\,E_0 E_4 + e^2 E_4^2 \right] \left[4a_2^2\, M_1^2 + 9a_3^2\, M_2^2 + \ldots + 12 a_2 a_3\, M_1\, M_2 + \ldots \right] f(\gamma)\, d\gamma \qquad [64]$$

Thus the computation of R^* reduces to the summation of quadratures of the type

$$\int_{\gamma_0}^{\infty} E_0^2\, f(\gamma)\, M_{n_1-1}\, M_{n_2-1}\, d\gamma = \mathfrak{M}_{ij}\,[0\ 0;\ K;\ \gamma_0] \qquad [64a]$$

$$\int_{\gamma_0}^{\infty} E_0 E_4\, f(\gamma)\, M_{n_1-1}\, M_{n_2-1}\, d\gamma = \mathfrak{M}_{ij}\,[0\ 4;\ K;\ \gamma_0] \qquad [64b]$$

$$\int_{\gamma_0}^{\infty} E_4^2\, f(\gamma)\, M_{n_1-1}\, M_{n_2-1}\, d\gamma = \mathfrak{M}_{ij}\,[4\ 4;\ K;\ \gamma_0] \qquad [64c]$$

The whole procedure depends upon the availability of tables of $\mathfrak{M}$– functions.

For example, considering Equation [24], let

$$X(\xi) = 1 - a_2\,\xi^2 - a_4\,\xi^4 - a_6\,\xi^6$$

with

$$a_6 = 1 - a_2 - a_4$$

$$\frac{\partial X}{\partial \xi} = -2\,a_2\,\xi^1 - 4a_4\,\xi^3 - 6\,a_6\,\xi^5 \qquad [65]$$

and

$$Z(\zeta) = 1 = \zeta^0$$

Then

$$R^*(\gamma_0) = \int_{\gamma_0}^{\infty} E_0^2\, f(\gamma) \left[4\,a_2^2\, M_1^2 + 16\,a_4^2\, M_3^2 + \ldots + 16\,a_2\,a_4\,M_1\,M_3 + \ldots\right] d\gamma$$

$$= 4\,a_2^2\,\mathfrak{M}_{11} + 16\,a_4^2\,\mathfrak{M}_{33} + 36\,a_6^2\,\mathfrak{M}_{55} + 16\,a_2\,a_4\,\mathfrak{M}_{13} + 24\,a_2\,a_6\,\mathfrak{M}_{15}$$

$$+ 48\,a_4\,a_6\,\mathfrak{M}_{35} \qquad [66]$$

where, for example,

$$\mathfrak{M}_{13} = \mathfrak{M}_{13}\,[0\;0;\; K;\; \gamma_0]$$

The expression for R^* is immediately obtained when $(\partial X/\partial \xi)^2$ is written as

$$\left(\frac{\partial X}{\partial \xi}\right)^2 = 4\,a_2^2\,\xi^1\,\xi^1 + 16\,a_4^2\,\xi^3\,\xi^3 + 36\,a_6^2\,\xi^5\,\xi^5 + 16\,a_2 a_4\,\xi^1\,\xi^3 + \ldots . \qquad [67]$$

the exponents ij of $\xi^i\,\xi^j$ in Equation [67] become subscripts of the $\mathfrak{M}$– functions in the resistance formula [66] while the coefficients $4a_2^2 \ldots$ remain the same.

When $Z(\zeta) = 1 - e\,\zeta^4$, Equation [59], the amount of computations involved is slightly increased, compared with $Z(\zeta) = 1$ as follows from Equation [64]. Assume for the sectional area curve the commonon parabola $X(\xi) = 1 - \xi^2$; then the resistance is given by

$$R^*(\gamma) = 4\,\mathfrak{M}_{11}\;[0\;0;\;K;\;\gamma_0] - 8\,e\,\mathfrak{M}_{11}[0\;4;\;K;\;\gamma_0] + 4e^2\,\mathfrak{M}_{11}[4\;4;\;K;\;\gamma_0]$$

This procedure holds, clearly, for any polynomial.

Some general remarks on the functions $\mathfrak{M}$ are needed.

Assuming

$$Z(\zeta) = 1 - e\,\zeta^r \qquad [68]$$

$$\phi(v) = E_0 - e\,E_r \qquad [68a]$$

more general functions $\mathfrak{M}$ of the type $\mathfrak{M}_{ij}$ $[0\,r;\ K;\ \gamma_0]$ and $\mathfrak{M}_{ij}[r\,r;\ K;\ \gamma_0]$ can be obtained, but it seems that there is no need to go beyond Equation [59] for the purpose of our systematic investigation.

The shorthand notation $\mathfrak{M}_{ij}$ should be used with some care.

The full symbol, for example, $\mathfrak{M}_{ij}$ $[0\ 0;\ K;\ \gamma_0]$ indicates that

1. the value of our auxiliary integral depends upon the exponents of the product $\xi^i\,\xi^j$, for example of $\xi^1\,\xi^2$,

2. the index of the E-function is 0; to the square $E_0^{\,2} = E_0\,E_0$ corresponds in the bracket to the symbol 0 0;

3. that $\mathfrak{M}$ depends upon $K = 2\,H/L$ and

4. upon the Froude number, or $\gamma_0 = 1/2F^2$

From the form of the functions $\mathfrak{M}$ which depend on several parameters it is seen that a considerable amount of computations is necessary to cover the field. It appears therefore necessary to restrict the variations in the parameter without impairing too much the generality of the results.

We admit, as mentioned before, seven exponents $n = 2, 3, 4, 6, 8, 10, 12$ and consequently seven values of $i, j = 1, 2, 3, 5, 7, 9, 11$. From the $7 + \binom{7}{2} = 28$ possible products $M_i\,M_j$ we select 24 since such combinations as $M_1\,M_{11}$ are of minor interest.

We choose further as basic values of the parameter $K = 2\,H/L = 0.1, 0.2, 0.06$, and as equation of sections $Z(\zeta) = 1$, which corresponds to a rectangular distribution of singularities over the draft and simulates very full sections.

To obtain consistent plots of resistance curves, intervals of $\Delta\gamma_0 = 0.5$ are considered sufficient. We assume for normal displacement ships an upper speed limit of $F = 1$ or $\gamma_0 = 0.5$ and restrict the lower limit to $F = 1/\sqrt{30}$ or $\gamma_0 = 15$, since it is thought that below this Froude number the influence of viscosity on wave effects becomes excessively strong.

Thus about 30 speed values $\gamma_0 = 1/2\,F^2$ or Froude numbers F are needed within the range $0.5 \le \gamma_0 \le 15$. It is, however, permissible to choose as lower limit $\gamma = 5$ when dealing with high degree polynomials, say $n = 10$ and $n = 12$, since such forms are of no interest at high Froude numbers or low γ_0 values. Thus for the corresponding $\mathfrak{M}$- functions the range is reduced to $5 \le \gamma_0 \le 15$.

When using the generalized equation of sections $Z(\zeta) = 1 - e\,\zeta^4$ further reductions are made in the evaluation of the functions.

$\mathfrak{M}_{ij}$ [04; K; γ_0] and $\mathfrak{M}_{ij}$[44; K; γ_0] as follows:

1. as basic value of K we consider $K = 0.1$; only for $K = 0.1$ the interval $\Delta\gamma_0 = 0.5$ is retained while for $K = 0.2$ and $K = 0.06$ we admit $\Delta\gamma_0 = 1$

2. instead of 24 products $M_i M_j$ only 6 products are tentatively evaluated, i.e., we restrict ourselves to the family < 2 4 6; α; t > with the resulting functions $\mathfrak{M}_{11}$ $\mathfrak{M}_{33}$ $\mathfrak{M}_{55}$ $\mathfrak{M}_{13}$ $\mathfrak{M}_{15}$ $\mathfrak{M}_{35}$. It is thought that the dependence of the resistance upon the section form can be derived from a limited number of sectional area shapes.

So far 96 $\mathfrak{M}$ – functions have been computed. Using these results special investigations will be made to check if these values meet all needs envisaged by the present program.

Something may be said about an earlier approach of evaluating the wave resistance integral, which in general is superseded by the tabulation of the $\mathfrak{M}$ – functions, but, nevertheless, may be needed in special cases. This method relies on tables of the intermediate M-functions, Equation [62g], and of the functions E_0, E_2, etc. It involves one integration over γ. Such computations must be performed when the parameter K is abnormal, or peculiar features like the bulb are investigated. In addition, there may be exceptional cases when the accuracy of the tabulated $\mathfrak{M}$– functions is no more sufficient. This may arise when $X(\xi)$ consists of a considerable number of terms and the coefficients $n_1 a_{n_1}$, $n_2 a_{n_2}$ become very large.

Extended tables of M-functions, Equation [62g], are available at the TMB.

The wave resistance has been computed for seven simple ship forms with rectangular midship sections using the $\mathfrak{M}$– function tabulated in Appendix II of this report. The results are plotted in Figure 8.

SIMPLIFIED V-FORM HULLS

Former investigations have shown that the wave resistance is not sensitive to *small* changes in the form of the *sections*. It is therefore thought that basic information concerning the influence of the vertical displacement distribution on the wave resistance can be obtained from a surface equation of the type Equation [46]

$$\eta(\xi, \zeta) = [X(\xi) - \zeta\, v(\xi)]\, Z(\zeta) \qquad [69]$$

where $X(\xi)$ and $Z(\zeta)$ as before are the design waterline and the midship section.

The centerplane contour is again a rectangle. The term $\zeta\, v(\xi)$ "generates" V-shaped sections-

The "fining function" $v(\xi)$ is a polynomial complying with the conditions $v(0) = v(1) = 0$. For a given $X(\xi)$, $Z(\zeta)$ and sectional area curve $a^*(\xi)$

$$a^*(\xi) = \frac{1}{2\beta}\, a(\xi) = \frac{1}{\beta}\int_0^1 \eta(\xi, \zeta)\, d\zeta = X(\xi) - \frac{\beta_1}{\beta} v(\xi) \qquad [49]$$

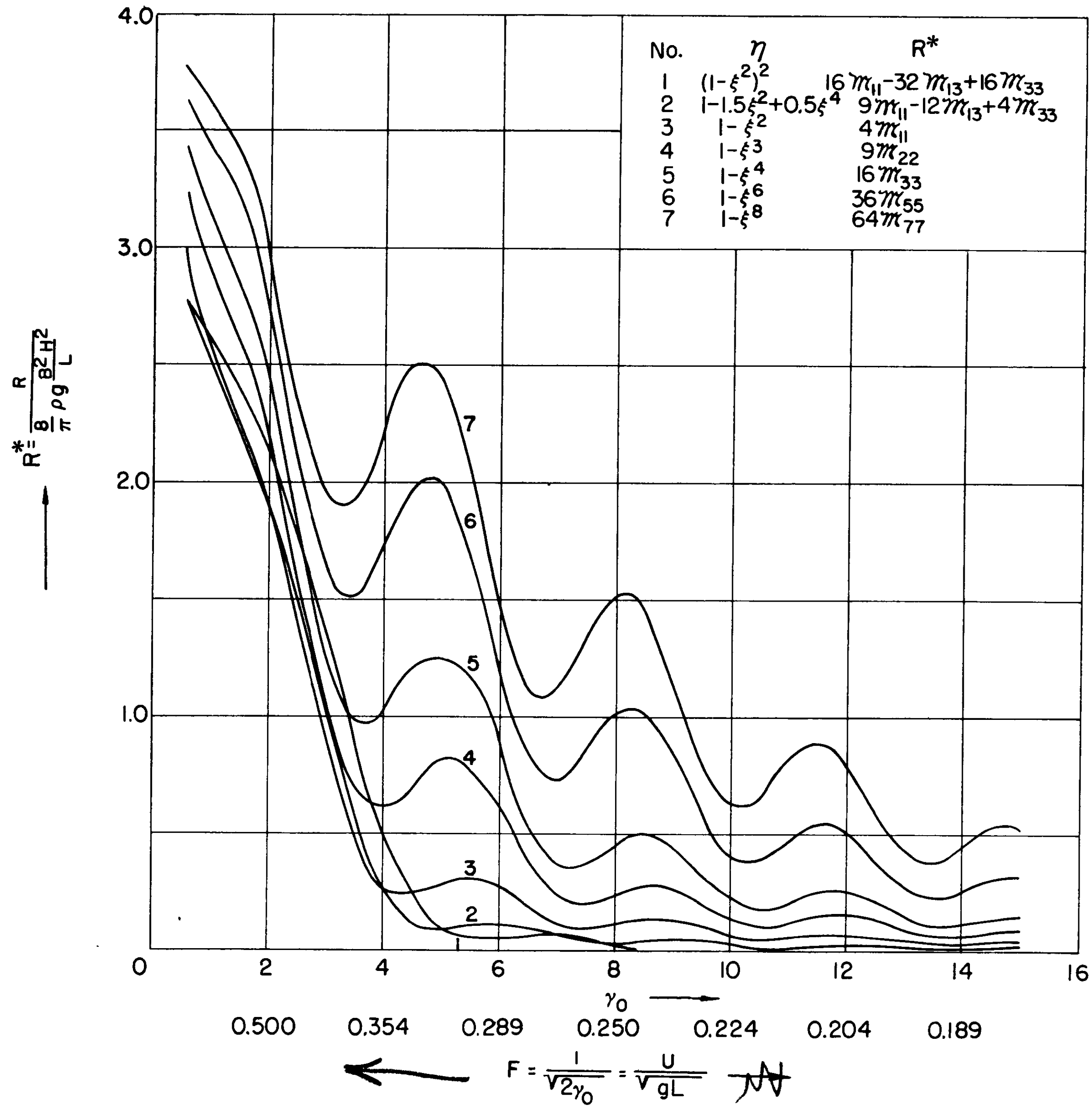

Figure 8 - Wave Resistance for Simple Ship Forms with Rectangular Midship Sections

where

$$\beta_1 = \int_0^1 \zeta Z(\zeta)\, d\zeta$$

$v(\xi)$ is completely determined.

$$v(\xi) = \frac{\beta}{\beta_1}[X(\xi) - a^*(\xi)] \qquad [49a]$$

Differentiating [69]

$$\frac{\partial \eta}{\partial \xi} = \frac{\partial X(\xi)}{\partial \xi} Z(\zeta) - \frac{\partial v}{\partial \xi} \zeta Z(\zeta) \qquad [70]$$

one obtains

$$J^*(\gamma) = \int_0^1 e^{-v\zeta} z(\zeta)\, d\zeta \int_0^1 \frac{\partial X(\xi)}{\partial \xi} \sin \gamma \xi \, d\xi - \int_0^1 e^{-v\zeta} \zeta Z(\zeta)\, d\zeta \int_0^1 \frac{\partial v(\xi)}{\partial \xi} \sin \gamma \xi \, d\xi$$

$$= \Phi(v)\, S(\gamma) - \Phi_1(v)\, S_1(\gamma) \qquad [71]$$

with $\Phi(v) = E_0 - e E_4$ as before $S(\gamma) = \int_0^1 \frac{\partial X(\xi)}{\partial \xi} \sin \gamma \xi \, d\xi$

$$\Phi_1(v) = E_1 - e E_5 \qquad S_1(\gamma) = \int_0^1 \frac{\partial v(\xi)}{\partial \xi} \sin \gamma \xi \, d\xi$$

The resistance integral can be written as

$$R^* = \int_{\gamma_0}^{\infty} \Phi^2 S^2 f(\gamma)\, d\gamma - 2 \int_{\gamma_0}^{\infty} \Phi \Phi_1 S S_1 f(\gamma)\, d\gamma + \int_{\gamma_0}^{\infty} \Phi_1^2 S_1^2 f(\gamma)\, d\gamma \qquad [72]$$

The first integral coincides with the even part of [63].
The other integrals contain the new functions

$$\Phi \Phi_1 = E_0 E_1 - e [E_1 E_4 + E_0 E_5] + e^2 E_4 E_5 \qquad [73]$$

$$\Phi_1^2 = E_1^2 - 2 e E_1 E_5 + e^2 E_5^2 \qquad [74]$$

which are readily computed.

The products S, S_1, and S_1^2 consist of terms $M_i M_j$ with i j integers for which tables are available.

As a first step we assume again

$$\Phi = E_0$$

$$\Phi_1 = E_1$$

in this case only two new functions, $E_0\,E_1$ and $E_1^{\,2}$, are involved therefore the evaluation of the corresponding M-functions does not present too much work.

The computation of functions $\mathfrak{M}[01;\, K;\, \gamma_0]$ and $\mathfrak{M}[11;\, K;\, \gamma_0]$ will present the next step in our systematic research. It is thought that by the tabulation of these functions and their application we may already exhaust to some degree the physical content of Michell's integral as far as the problem *U* versus *V* sections is concerned.

Reference is made, however, to a recent paper by Juin (Journal of Zôsen Kyôkai 1953) on exact hull forms which opens a promising outlook for further progress.

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APPENDIX I

(Prepared by Mr. Hirschberger of Bureau of Standards)

EVALUATION OF THE AUXILIARY INTEGRALS

The integral to be computed is given by:

$$I = \int_{\gamma_0}^{\infty} (E_h E_g) \frac{\left(\frac{\gamma}{\gamma_0}\right)^2}{\sqrt{\left(\frac{\gamma}{\gamma_0}\right)^2 - 1}} M_i(\gamma)\, M_j(\gamma)\, d\gamma$$

Under the transformation

$$\gamma = z^2 + \gamma_0$$

$$d\gamma = 2\, z\, d\, z$$

the integral becomes:

$$I = \frac{2}{\gamma_0} \int_0^{\infty} (E_h E_g) \frac{(z^2 + \gamma_0)^2}{\sqrt{z^2 + 2\,\gamma_0}} M_i(z^2 + \gamma_0)\, M_j(z^2 + \gamma_0)\, d\, z$$

In this form the integral behaves in such a manner that numerical integration is practical.

The numerical integration was performed and checked using Simpson's rule. The interval ΔZ was taken as 0.05 and the range extended from 0.00 to approximately 15.00.

The functions

$$M_n = \int_0^1 \xi^n \sin \gamma\, \xi\, d\, \xi$$

were computed by the form

$$M_n = P\left(\frac{1}{\gamma}\right) \cos(\gamma) + Q\left(\frac{1}{\gamma}\right) \sin(\gamma) + R\left(\frac{1}{\gamma}\right)$$

for $\gamma > 1$. For $\gamma < 1$, the M-functions were computed by the series

$$M_n = \sum_{i=0}^{\infty} (-1)^i \frac{(\gamma)^{2i+1}}{(2\,i+1)!\;(n+2+2\,i)}$$

The M-functions were checked by the recurrence relations

$$M_n = -\frac{\cos\gamma}{\gamma} + \frac{n}{\gamma} M'_{n-1}(\gamma)$$

$$M'_n = \frac{\sin\gamma}{\gamma} - \frac{n}{\gamma} M_{n-1}(\gamma)$$

$e^{-k\gamma/\gamma_0}$ was computed from the tables and the use of the approximation $e^{-x} = 1 - x + x^2/2$, $x < 0.01$.

The function

$$E_0 = \int_0^1 e^{-\frac{k\gamma}{\gamma_0}\zeta} \, d\zeta$$

and the algebraic functions in the integrand were computed straight-forwardly. They were checked by differencing.

The function

$$E_4 = \int_0^1 e^{-\frac{k\gamma}{\gamma_0}\zeta} \zeta^4 \, d\zeta$$

was computed by the form

$$E_4 = 24\left(\frac{\gamma_0}{k\gamma}\right)^4 E_0 - e^{-\frac{k\gamma}{\gamma_0}} \left[\left(\frac{\gamma_0}{k\gamma}\right) + 4\left(\frac{\gamma_0}{k\gamma}\right)^2 + 12\left(\frac{\gamma_0}{k\gamma}\right)^3 + 24\left(\frac{\gamma_0}{k\gamma}\right)^4\right]$$

for $k\gamma/\gamma_0 > 1$. For $k\gamma/\gamma_0 < 1$, the function was computed by the series

$$E_4 = \sum_{n=0}^{\infty} \frac{\left(\frac{k\gamma}{\gamma_0}\right)^n}{(n+5)\,(n!\,)}$$

All of this computation was done on the IBM electronic calculator (type 604), and the auxiliary IBM punch card equipment. All the IBM operations were checked.

APPENDIX II

(Computed by the Bureau of Standards)

Tables of Integrals $\mathfrak{M}_{ij}[0\ 0;\ K;\ \gamma_0]$, $\mathfrak{M}_{ij}[0\ 4;\ K;\ \gamma_0]$, $\mathfrak{M}_{ij}[4\ 4;\ K;\ \gamma_0]$

$\mathfrak{M}_{1j}[0\,0;\ 0.06;\ r_0]$

r_0	$\mathfrak{M}_{1,1}$	$\mathfrak{M}_{1,2}$	$\mathfrak{M}_{1,3}$	$\mathfrak{M}_{1,5}$	$\mathfrak{M}_{1,7}$	$\mathfrak{M}_{1,9}$
.5	1.00789	.69936	.52663	.34442	.25200	
1.0	.78640	.60509	.41755	.27344	.19940	
1.5	.66884	.47282	.35914	.23553	.17139	
2.0	.54919	.38711	.29284	.19018	.13695	
2.5	.40351	.28208	.21139	.13449	.094829	
3.0	.25594	.17988	.13454	.084282	.058036	
3.5	.14461	.11000	.086180	.056607	.039723	
4.0	.090826	.085389	.074778	.055622	.042116	
4.5	.085892	.094170	.087895	.069770	.054984	
5.0	.10039	.10982	.10221	.080839	.063613	.051123
5.5	.10561	.11006	.10008	.076968	.059403	.046996
6.0	.090596	.090039	.079977	.059636	.044898	.034713
6.5	.063950	.061389	.054137	.040220	.030105	.023044
7.0	.042112	.040290	.037107	.029975	.023864	.019128
7.5	.035039	.034729	.034697	.031706	.027347	.023239
8.0	.040011	.040326	.041374	.039061	.034311	.029534
8.5	.046340	.046101	.046505	.042827	.036989	.031437
9.0	.045181	.044022	.043285	.038401	.032297	.026863
9.5	.035945	.034442	.033129	.028539	.023524	.019227
10.0	.025032	.024042	.022980	.019989	.016794	.013974
10.5	.019313	.019241	.018798	.017536	.015808	.013963
11.0	.020463	.021033	.020972	.020522	.019247	.017524
11.5	.024491	.025086	.024926	.024207	.022542	.020397
12.0	.025941	.026117	.025596	.024176	.021988	.019513
12.5	.022578	.022331	.021587	.019827	.017610	.015317
13.0	.016675	.016316	.015704	.014278	.012616	.010950
13.5	.012392	.012218	.011982	.011266	.010343	.0093359
14.0	.012013	.012071	.012169	.011990	.011506	.010804
14.5	.014307	.014423	.014604	.014476	.013955	.013146
15.0	.016070	.016048	.016077	.015637	.014817	.013746

$\mathfrak{M}_{ij}$ [00; 0.06, r_0]

r_0	$\mathfrak{M}_{22}$	$\mathfrak{M}_{23}$	$\mathfrak{M}_{25}$	$\mathfrak{M}_{27}$	$\mathfrak{M}_{33}$	$\mathfrak{M}_{35}$
.5	.50285	.38783	.26128	.19451	.30400	.20900
1.0	.40889	.32060	.21912	.16375	.25725	.18089
1.5	.35617	.28172	.19395	.14513	.22879	.16272
2.0	.29431	.23371	.16102	.12009	.19155	.13728
2.5	.21861	.17510	.12110	.089995	.14631	.10666
3.0	.14855	.12290	.087488	.065612	.10775	.082175
3.5	.10633	.094721	.072409	.056215	.089641	.073237
4.0	.098592	.094060	.076601	.061467	.093144	.079087
4.5	.11187	.10772	.088511	.071431	.10568	.089023
5.0	.12306	.11583	.093039	.074128	.11039	.090473
5.5	.11576	.10596	.082452	.064320	.098236	.078238
6.0	.090022	.080584	.061064	.046679	.073492	.057703
6.5	.059416	.053164	.040707	.031327	.049155	.039950
7.0	.039139	.036971	.031238	.025772	.036551	.033098
7.5	.034991	.035703	.033670	.029693	.037507	.036733
8.0	.041026	.042491	.040729	.036201	.044498	.043293
8.5	.046124	.046747	.043468	.037882	.047604	.044618
9.0	.043123	.042568	.038153	.032437	.042165	.038081
9.5	.033262	.032177	.028169	.023635	.031272	.027705
10.0	.023409	.022603	.020201	.017452	.022003	.020062
10.5	.019484	.019264	.018462	.017041	.019226	.018792
11.0	.021821	.021909	.021735	.020615	.022124	.022188
11.5	.025800	.025721	.025143	.023545	.025731	.025307
12.0	.026355	.025890	.024566	.022443	.025508	.024334
12.5	.022137	.021459	.019819	.017708	.020884	.019431
13.0	.016023	.015495	.014221	.012695	.015084	.014020
13.5	.012115	.011965	.011400	.010604	.011925	.011547
14.0	.012190	.012356	.012292	.011900	.012603	.012670
14.5	.014581	.014806	.014753	.014290	.015078	.015101
15.0	.016057	.016115	.015731	.014959	.016199	.015866

$\mathfrak{M}_{ij}$ [00; 0.06; r_0]

r_0	$\mathfrak{M}_{3,7}$	$\mathfrak{M}_{3,9}$	$\mathfrak{M}_{5,5}$	$\mathfrak{M}_{5,7}$	$\mathfrak{M}_{5,9}$	$\mathfrak{M}_{5,11}$
.5	.15747		.14746	.11285		
1.0	.13746		.13183	.10238		
1.5	.12413		.12064	.094413		
2.0	.10485		.10347	.081556		
2.5	.081876		.083019	.066367		
3.0	.064306		.067874	.055733		
3.5	.059188		.064320	.054271		
4.0	.065140		.070556	.059970		
4.5	.073086		.077732	.065440		
5.0	.073176	.060151	.076716	.063654	.053356	.045413
5.5	.062162	.050393	.065019	.053396	.044417	.037557
6.0	.045401	.036455	.048357	.040014	.033424	.028305
6.5	.032212	.023518	.035856	.031020	.026673	.023038
7.0	.028604	.024457	.032978	.030231	.026948	.023861
7.5	.033148	.029151	.037776	.035142	.031596	.028145
8.0	.038873	.034066	.043094	.039365	.034979	.030900
8.5	.039149	.033747	.042503	.037858	.033069	.028837
9.0	.032628	.027621	.035060	.030645	.026423	.022802
9.5	.023546	.019822	.025335	.022252	.019304	.016738
10.0	.017682	.015369	.019200	.017702	.015980	.014300
10.5	.017642	.016112	.019127	.018554	.017381	.015988
11.0	.021235	.019657	.022707	.022088	.020716	.019067
11.5	.023833	.021784	.025172	.023945	.022084	.020064
12.0	.022353	.020032	.023446	.021753	.019685	.017615
12.5	.017500	.015442	.018333	.016757	.015009	.013325
13.0	.012684	.011279	.013342	.012367	.011260	.010155
13.5	.010908	.010112	.011500	.011151	.010578	.0098828
14.0	.012379	.011806	.012965	.012862	.012427	.011778
14.5	.014694	.013958	.015265	.014976	.014332	.013474
15.0	.015134	.014132	.015643	.015019	.014112	.013066

$\mathfrak{M}_{ij}[00;\ 0.06;\ r_0]$

r_0	$\mathfrak{M}_{7,7}$	$\mathfrak{M}_{7,9}$	$\mathfrak{M}_{7,11}$	$\mathfrak{M}_{9,9}$	$\mathfrak{M}_{9,11}$	$\mathfrak{M}_{11,11}$
.5	.087220					
1.0	.080606					
1.5	.075104					
2.0	.065585					
2.5	.054419					
3.0	.047100					
3.5	.046985					
4.0	.052019					
4.5	.056088					
5.0	.053847	.045809	.039446	.039419	.034251	.029976
5.5	.044981	.038159	.032772	.032866	.028565	.025062
6.0	.034366	.029524	.025561	.025896	.022783	.020291
6.5	.028097	.024959	.022100	.022675	.020416	.018612
7.0	.028691	.026193	.023614	.024305	.022182	.020432
7.5	.033343	.030420	.027414	.028059	.025506	.023347
8.0	.036461	.032771	.029227	.029733	.026727	.024184
8.5	.034208	.030260	.026676	.027064	.024085	.021607
9.0	.027336	.024002	.021044	.021415	.019035	.017117
9.5	.020188	.018010	.015995	.016446	.014890	.013690
10.0	.016966	.015790	.014480	.015039	.014041	.013288
10.5	.018458	.017622	.016453	.017062	.016106	.015335
11.0	.021767	.020630	.019155	.019721	.018443	.017352
11.5	.022984	.021372	.019560	.020022	.018448	.017099
12.0	.020384	.018627	.016820	.017181	.015651	.014372
12.5	.015553	.014144	.012739	.013053	.011916	.011013
13.0	.011741	.010932	.010061	.010387	.0097300	.0092508
13.5	.011063	.010703	.010165	.010523	.010129	.0098545
14.0	.012924	.012623	.012074	.012440	.011988	.011626
14.5	.014801	.014258	.013484	.013815	.013135	.012548
15.0	.014510	.013718	.012774	.013045	.012216	.011501

$\mathfrak{M}_{ij}[00;\ 0.1;\ r_0]$

r_0	$\mathfrak{M}_{11}$	$\mathfrak{M}_{12}$	$\mathfrak{M}_{13}$	$\mathfrak{M}_{15}$	$\mathfrak{M}_{17}$	$\mathfrak{M}_{19}$
.5	.72185	.50356	.38139	.25218	.18625	
1.0	.62857	.43752	.33010	.21634	.15844	
1.5	.55564	.38694	.29145	.18990	.13817	
2.0	.45972	.31742	.23695	.15172	.10867	
2.5	.33334	.22634	.16609	.10285	.071422	
3.0	.20404	.13723	.099160	.059000	.039153	
3.5	.10757	.077280	.057909	.035503	.023596	
4.0	.062129	.057205	.049131	.035411	.026191	
4.5	.058841	.065673	.061190	.048091	.037590	
5.0	.071696	.079532	.073851	.057841	.045153	.036096
5.5	.076521	.080269	.072580	.054997	.041925	.032864
6.0	.064732	.064412	.056622	.041199	.030343	.023031
6.5	.043757	.041831	.036258	.025896	.018673	.013813
7.0	.026809	.025454	.023059	.017980	.013861	.010797
7.5	.021464	.021271	.021302	.019396	.016605	.014006
8.0	.025317	.025596	.026446	.025043	.021939	.018818
8.5	.030111	.029976	.030346	.027915	.023984	.020272
9.0	.029319	.028507	.028037	.024710	.020573	.016937
9.5	.022678	.021615	.020727	.017605	.014247	.011425
10.0	.014924	.014225	.013513	.011526	.0094591	.0076854
10.5	.010917	.010862	.010584	.0098087	.0087681	.0076766
11.0	.011716	.012107	.012092	.011878	.011149	.010139
11.5	.014468	.014876	.014794	.014396	.013399	.012100
12.0	.015449	.015575	.015250	.014379	.013029	.011506
12.5	.013222	.013067	.012594	.011496	.010126	.0087217
13.0	.0093580	.0091305	.0087424	.0078624	.0068538	.0058591
13.5	.0065870	.0064783	.0063330	.0059104	.0053786	.0048085
14.0	.0063373	.0063768	.0064445	.0063640	.0061120	.0057362
14.5	.0077738	.0078498	.0079698	.0079219	.0076465	.0072035
15.0	.0088641	.0088545	.0088805	.0086391	.0081777	.0075720

$\mathcal{M}_{ij}[00;\ 0.1;\ r_0]$

r_0	$\mathcal{M}_{2,2}$	$\mathcal{M}_{2,3}$	$\mathcal{M}_{2,5}$	$\mathcal{M}_{2,7}$	$\mathcal{M}_{3,3}$	$\mathcal{M}_{3,5}$
.5	.35905	.27601	.18591	.13879	.21436	.14627
1.0	.31622	.24465	.16536	.12328	.19249	.13287
1.5	.28277	.21987	.14893	.11082	.17460	.12140
2.0	.23324	.18145	.12231	.090351	.14506	.10119
2.5	.16851	.13157	.088307	.064588	.10693	.075500
3.0	.10838	.087118	.059811	.043898	.074525	.055152
3.5	.073064	.064002	.047775	.036523	.060192	.048516
4.0	.067810	.064766	.052394	.041793	.064490	.054558
4.5	.080227	.077421	.063289	.050826	.076124	.063754
5.0	.090404	.084944	.067620	.053473	.080750	.065492
5.5	.084939	.077309	.059270	.045661	.071202	.055778
6.0	.064455	.057087	.042197	.031553	.051475	.039395
6.5	.040310	.035464	.026149	.019444	.032299	.025426
7.0	.024572	.022919	.018845	.015177	.022566	.020179
7.5	.021476	.022042	.020806	.018265	.023407	.023074
8.0	.026133	.027272	.026232	.023258	.028793	.028127
8.5	.030009	.030520	.028340	.024563	.031184	.029182
9.0	.027861	.027506	.024483	.020600	.027244	.024428
9.5	.020765	.020025	.017290	.014253	.019398	.016950
10.0	.013765	.013221	.011627	.0098574	.012810	.011518
10.5	.011017	.010884	.010412	.0095717	.010868	.010632
11.0	.012639	.012719	.012680	.012047	.012878	.012986
11.5	.015359	.015325	.015010	.014050	.015344	.015120
12.0	.015738	.015445	.014627	.013312	.015200	.014470
12.5	.012943	.012509	.011480	.010171	.012135	.011219
13.0	.0089415	.0086038	.0078142	.0068867	.0083379	.0076757
13.5	.0064118	.0063183	.0059853	.0055291	.0062915	.0060715
14.0	.0064517	.0065597	.0065464	.0063470	.0067158	.0067799
14.5	.0079497	.0080945	.0080884	.0078447	.0082667	.0083035
15.0	.0088620	.0089031	.0086917	.0082560	.0089591	.0087743

$\mathfrak{M}_{ij}[00;\ 0.1;\ r_0]$

r_0	$\mathfrak{M}_{3,7}$	$\mathfrak{M}_{3,9}$	$\mathfrak{M}_{5,5}$	$\mathfrak{M}_{5,7}$	$\mathfrak{M}_{5,9}$	$\mathfrak{M}_{5,11}$
.5	.11005		.10151	.077164		
1.0	.10029		.094199	.072261		
1.5	.091754		.087278	.067312		
2.0	.076317		.073740	.057123		
2.5	.056986		.056750	.044514		
3.0	.042421		.044429	.035936		
3.5	.038840		.042293	.035417		
4.0	.044729		.048485	.040966		
4.5	.052046		.055193	.046089		
5.0	.052513	.042909	.054771	.044927	.037350	.031610
5.5	.043714	.035076	.045441	.036718	.030175	.025288
6.0	.030326	.023927	.032161	.026051	.021406	.017900
6.5	.019951	.015911	.022358	.019012	.016127	.013783
7.0	.017218	.014562	.020218	.018489	.016418	.014486
7.5	.020795	.018237	.024006	.022350	.020062	.017837
8.0	.025201	.022013	.028114	.025616	.022679	.019967
8.5	.025460	.021816	.027735	.024543	.021294	.018460
9.0	.020707	.017344	.022311	.019280	.016439	.014046
9.5	.014158	.011716	.015301	.013228	.011302	.0096691
10.0	.0099891	.0085484	.010941	.0099946	.0089402	.0079339
10.5	.0099649	.0090712	.010895	.010596	.0099259	.0091193
11.0	.012454	.011525	.013377	.013046	.012236	.011250
11.5	.014230	.012979	.015064	.014317	.013172	.011932
12.0	.013239	.011802	.013908	.012846	.011561	.010285
12.5	.010019	.0087565	.010517	.0095321	.0084572	.007436
13.0	.0068643	.0060269	.0072470	.0066545	.0059983	.005356
13.5	.0057101	.0052654	.0060506	.0058620	.0055502	.005172
14.0	.0066395	.0063376	.0069770	.0069450	.0067212	.006373
14.5	.0080904	.0076860	.0084183	.0082700	.0079142	.007434
15.0	.0083599	.0077898	.0086490	.0082922	.0077736	.007176

$\mathcal{M}_{ij}\,[00;\ 0.1;\ r_0]$

r_0	$\mathcal{M}_{7,7}$	$\mathcal{M}_{7,9}$	$\mathcal{M}_{7,11}$	$\mathcal{M}_{9,9}$	$\mathcal{M}_{9,11}$	$\mathcal{M}_{11,11}$
.5	.059034					
1.0	.055993					
1.5	.052580					
2.0	.045006					
2.5	.035749					
3.0	.029919					
3.5	.030417					
4.0	.035260					
4.5	.039092					
5.0	.037482	.031564	.026983	.026843	.023124	.020042
5.5	.030373	.025417	.021609	.021568	.018539	.016072
6.0	.021917	.018536	.015858	.016015	.013928	.012266
6.5	.017029	.014986	.013170	.013518	.012099	.010973
7.0	.017576	.016018	.014409	.014849	.013526	.012434
7.5	.021228	.019331	.017382	.017788	.016124	.014709
8.0	.023648	.021159	.018793	.019095	.017080	.015368
8.5	.022010	.019321	.016917	.017132	.015131	.013462
9.0	.016991	.014746	.012797	.013000	.011434	.010173
9.5	.011836	.010423	.0091521	.0094140	.0084410	.0076972
10.0	.0095473	.0088460	.0080749	.0084139	.0078367	.0074098
10.5	.010598	.010133	.0094604	.0098348	.0092867	.0088455
11.0	.012892	.012218	.011331	.011676	.010903	.010240
11.5	.013725	.012726	.011609	.011883	.010907	.010067
12.0	.011979	.010881	.0097658	.0099726	.0090252	.0082309
12.5	.0087737	.0079053	.0070545	.0072307	.0065434	.0059974
13.0	.0062735	.0057975	.0052959	.0054799	.0051056	.0048369
13.5	.0058287	.0056417	.0053552	.0055599	.0053555	.0052190
14.0	.0070076	.0068585	.0065646	.0067745	.0065339	.0063417
14.5	.0081833	.0078818	.0074466	.0076347	.0072500	.0069161
15.0	.0079982	.0075418	.0070013	.0071515	.0066745	.0062611

$\mathfrak{M}_{ij}$ [00; 0.2; r_0]

r_0	$\mathfrak{M}_{1,1}$	$\mathfrak{M}_{1,2}$	$\mathfrak{M}_{1,3}$	$\mathfrak{M}_{1,5}$	$\mathfrak{M}_{1,7}$	$\mathfrak{M}_{1,9}$
.5	.39530	.28000	.21474	.14472	.10837	
1.0	.41383	.28958	.21988	.14596	.10811	
1.5	.39129	.27096	.20388	.13336	.097673	
2.0	.32701	.22215	.16437	.10458	.074964	
2.5	.23134	.15226	.10942	.066137	.045405	
3.0	.13355	.084666	.058441	.032493	.020516	
3.5	.062757	.040775	.028231	.015211	.0090156	
4.0	.030750	.026893	.022316	.015330	.010960	
4.5	.028792	.033066	.030894	.024209	.018885	
5.0	.037519	.042504	.039498	.030793	.023963	.019140
5.5	.040773	.043181	.038863	.029083	.021962	.017112
6.0	.033667	.033585	.029179	.020670	.014877	.011082
6.5	.021310	.020274	.017162	.011617	.0079562	.0056036
7.0	.011620	.010905	.0096028	.0070636	.0051695	.0038413
7.5	.0086347	.0085598	.0085972	.0078092	.0066478	.0055781
8.0	.010663	.010841	.011319	.010802	.0094741	.0081263
8.5	.013128	.013093	.013324	.012275	.010517	.0088635
9.0	.012718	.012341	.012151	.010650	.0087879	.0071710
9.5	.0094603	.0089603	.0085649	.0071626	.0056791	.0044599
10.0	.0057563	.0054286	.0051163	.0042518	.0033816	.0026607
10.5	.0038788	.0038486	.0037368	.0034326	.0030379	.0026340
11.0	.0042150	.0043857	.0043930	.0043427	.0040900	.0037244
11.5	.0054053	.0055841	.0055619	.0054322	.0050630	.0045718
12.0	.0058125	.0058707	.0057447	.0054129	.0048931	.0043063
12.5	.0048685	.0048092	.0046211	.0041943	.0036660	.0031294
13.0	.0032742	.0031847	.0030315	.0026938	.0023140	.0019765
13.5	.0021509	.0021089	.0020496	.0018988	.0017105	.0015220
14.0	.0020407	.0020570	.0020847	.0020654	.0019875	.0018670
14.5	.0025837	.0026141	.0026621	.0026557	.0025694	.0024236
15.0	.0029877	.0029859	.0029986	.0029194	.0027632	.0025564

$\mathfrak{M}_{ij}[00;\ 0.2;\ r_0]$

r_0	$\mathfrak{M}_{2,2}$	$\mathfrak{M}_{2,3}$	$\mathfrak{M}_{2,5}$	$\mathfrak{M}_{2,7}$	$\mathfrak{M}_{3,3}$	$\mathfrak{M}_{3,5}$
.5	.20058	.15500	.10545	.079391	.12042	.082474
1.0	.20663	.15899	.10728	.080226	.12345	.084280
1.5	.19288	.14788	.099028	.073532	.11485	.078185
2.0	.15708	.11950	.078798	.057724	.092671	.062636
2.5	.10725	.080922	.052279	.037520	.063136	.042620
3.0	.061880	.047423	.030767	.021860	.038817	.027375
3.5	.036408	.030914	.022266	.016680	.028784	.022880
4.0	.033264	.032006	.025937	.020700	.032365	.027558
4.5	.042264	.041122	.033707	.027097	.040746	.034145
5.0	.049338	.046400	.036770	.028977	.044080	.035479
5.5	.046098	.041746	.031581	.024079	.038181	.029409
6.0	.033670	.029452	.021166	.015450	.026172	.019408
6.5	.019429	.016688	.011672	.0082690	.014844	.011139
7.0	.010421	.0094994	.0074670	.0057932	.0092601	.0081117
7.5	.0086753	.0089798	.0085143	.0074630	.0096919	.0096791
8.0	.011135	.011749	.011391	.010108	.012548	.012360
8.5	.013125	.013416	.012468	.010768	.013774	.012893
9.0	.012033	.011888	.010515	.0087604	.011780	.010488
9.5	.0085510	.0082177	.0069831	.0056408	.0079311	.0068174
10.0	.0052034	.0049633	.0042692	.0035286	.0047783	.0042125
10.5	.0039076	.0038573	.0036821	.0033723	.0038544	.0037776
11.0	.0046150	.0046607	.0046822	.0044670	.0047372	.0048165
11.5	.0057920	.0057884	.0056894	.0053317	.0058036	.0057377
12.0	.0059419	.0058269	.0055130	.0050039	.0057293	.0054470
12.5	.0047603	.0045859	.0041837	.0036772	.0044344	.0040739
13.0	.0031090	.0029743	.0026699	.0023198	.0028665	.0026099
13.5	.0020822	.0020453	.0019240	.0017633	.0020333	.0019540
14.0	.0020859	.0021282	.0021328	.0020734	.0021880	.0022207
14.5	.0026526	.0027092	.0027170	.0026411	.0027754	.0027979
15.0	.0029896	.0030072	.0029377	.0027896	.0030296	.0029686

$\mathfrak{M}_{ij}\,[00;\ 0.2;\ \gamma_0]$

γ_0	$\mathfrak{M}_{3,7}$	$\mathfrak{M}_{3,9}$	$\mathfrak{M}_{5,5}$	$\mathfrak{M}_{5,7}$	$\mathfrak{M}_{5,9}$	$\mathfrak{M}_{5,11}$
.5	.062335		.056980	.043296		
1.0	.063454		.058407	.044379		
1.5	.058625		.054363	.041296		
2.0	.046583		.043714	.033162		
2.5	.031456		.030420	.023243		
3.0	.020530		.021223	.016831		
3.5	.018178		.019948	.016661		
4.0	.022639		.024629	.020803		
4.5	.027852		.029425	.024448		
5.0	.028282	.023032	.029270	.023758	.019614	.016527
5.5	.022749	.018088	.023401	.018570	.015066	.012511
6.0	.014557	.011253	.015277	.012030	.0096794	.0079663
6.5	.0084008	.0064863	.0094615	.0078420	.0065281	.0055006
7.0	.0068053	.0056800	.0082073	.0075098	.0066605	.0058700
7.5	.0087497	.0076790	.010271	.0096145	.0086457	.0076945
8.0	.011084	.0096774	.012449	.011341	.010025	.0088143
8.5	.011203	.0095598	.012239	.010770	.0092922	.0080168
9.0	.0087985	.0072948	.0094965	.0081069	.0068326	.0057792
9.5	.0055791	.0045252	.0060514	.0051287	.0043015	.0036197
10.0	.0035741	.0029965	.0039553	.0035677	.0031551	.0027725
10.5	.0035385	.0032153	.0039091	.0038223	.0035882	.0032995
11.0	.0046398	.0043022	.0050066	.0049062	.0046107	.0042427
11.5	.0054050	.0049269	.0057325	.0054501	.0050084	.0045294
12.0	.0049681	.0044112	.0052247	.0048077	.0043060	.0038124
12.5	.0036084	.0031242	.0037926	0034076	.0029944	.0026080
13.0	.0023037	.0019943	.0024406	.0022165	.0019750	.0017442
13.5	.0018285	.0016772	.0019482	.0018863	.0017835	.0016590
14.0	.0021825	.0020878	.0023007	.0023007	.0022330	.0021210
14.5	.0027322	.0025986	.0028466	.0028022	.0026842	.0025219
15.0	.0028272	.0026314	.0029267	.0028038	.0026245	.0024183

$\mathfrak{M}_{ij}[00;\ 0.2;\ \gamma_0]$

γ_0	$\mathfrak{M}_{7,7}$	$\mathfrak{M}_{7,9}$	$\mathfrak{M}_{7,11}$	$\mathfrak{M}_{9,9}$	$\mathfrak{M}_{9,11}$	$\mathfrak{M}_{11,11}$
.5	.032714					
1.0	.033915					
1.5	.031627					
2.0	.025472					
2.5	.018139					
3.0	.013765					
3.5	.014277					
4.0	.017852					
4.5	.020562					
5.0	.019541	.016295	.013837	.013692	.011696	.010037
5.5	.015031	.012384	.010410	.010324	.0087610	.0074895
6.0	.0098337	.0081444	.0068606	.0068930	.0059054	.0051251
6.5	.0069123	.0060114	.0052373	.0053817	.0047888	.0043252
7.0	.0071907	.0065673	.0059125	.0061084	.0055712	.0051277
7.5	.0091804	.0083700	.0075279	.0077046	.0069801	.0063586
8.0	.010454	.0093272	.0082626	.0083828	.0074706	.0066902
8.5	.0095892	.0083570	.0072729	.0073469	.0064416	.0056836
9.0	.0070466	.0060373	.0051815	.0052488	.0045625	.0040095
9.5	.0045055	.0039026	.0033786	.0034743	.0030779	.0027781
10.0	.0033934	.0031293	.0028440	.0029758	.0027686	.0026202
10.5	.0038564	.0037028	.0034641	.0036119	.0034192	.0032653
11.0	.0048710	.0046243	.0042905	.0044249	.0041320	.0038788
11.5	.0052234	.0048345	.0043999	.0045029	.0041213	.0037908
12.0	.0044631	.0040317	.0035985	.0036723	.0033031	.0029924
12.5	.0031083	.0027734	.0024515	.0025121	.0022522	.0020457
13.0	.0020717	.0018978	.0017194	.0017830	.0016516	.0015590
13.5	.0018812	.0018236	.0017319	.0018036	.0017409	.0017017
14.0	.0023336	.0022914	.0021974	.0022710	.0021945	.0021338
14.5	.0027779	.0026774	.0025294	.0025945	.0024629	.0023479
15.0	.0027014	.0025426	.0023551	.0024057	.0022395	.0020946

$\mathcal{M}_{ij}$ [04; 0.06; γ_0]

γ_0	$\mathcal{M}_{11}$	$\mathcal{M}_{13}$	$\mathcal{M}_{15}$	$\mathcal{M}_{33}$	$\mathcal{M}_{53}$	$\mathcal{M}_{55}$
1.0	.13508	.070155	.045679	.040574	.027853	.019615
2.0	.097650	.050268	.032062	.030990	.021584	.015696
3.0	.043835	.021461	.012750	.016319	.012056	.0096836
4.0	.013723	.010903	.0078219	.014234	.011987	.010611
5.0	.015835	.016271	.012700	.017763	.014375	.012004
6.0	.014328	.012521	.0090863	.011382	.0086973	.0070940
7.0	.0059645	.0051262	.0039825	.0050208	.0044822	.0044887
8.0	.0056670	.0059196	.0055957	.0064484	.0062922	.0062851
9.0	.0065923	.0062982	.0055380	.0061165	.0054733	.0049903
10.0	.0033253	.0030024	.0025475	.0028420	.0025464	.0024149
11.0	.0026059	.0026934	.0026449	.0028749	.0029006	.0029910
12.0	.0034811	.0034352	.0032342	.0034232	.0032545	.0031242
13.0	.0020741	.0019313	.0017279	.0018370	.0016839	.0015847
14.0	.0013776	.0014033	.0013862	.0014668	.0014829	.0015296
15.0	.0019750	.0019793	.0019243	.0019977	.0019554	.0019265

$\mathcal{M}_{ij}[04; 0.1; r_0]$

r_0	$\mathcal{M}_{1,1}$	$\mathcal{M}_{1,3}$	$\mathcal{M}_{1,5}$	$\mathcal{M}_{3,3}$	$\mathcal{M}_{3,5}$	$\mathcal{M}_{5,5}$
.5	.10400	.055848	.037440	.030634	.020776	.014179
1.0	.10270	.053597	.035228	.029674	.020111	.013843
1.5	.094441	.048630	.031551	.027446	.018650	.012967
2.0	.078698	.039390	.024925	.022546	.015281	.010714
2.5	.056321	.026770	.016150	.015907	.010816	.0077810
3.0	.033260	.014869	.0083293	.010300	.0073253	.0057008
3.5	.016253	.0076791	.0042433	.0079566	.0063033	.0054483
4.0	.0084259	.0062837	.0043256	.0088842	.0074956	.0066399
4.5	.0079820	.0084863	.0065985	.011025	.0091833	.0078774
5.0	.010261	.010727	.0083175	.011905	.0095497	.0078629
5.5	.011139	.010582	.0078897	.010378	.0079792	.0063473
6.0	.0092603	.0080145	.0056620	.0071911	.0053296	.0042002
6.5	.0059224	.0047701	.0032224	.0041357	.0031053	.0026429
7.0	.0032687	.0027050	.0019860	.0026170	.0022916	.0023182
7.5	.0024535	.0024443	.0022140	.0027589	.0027502	.0029149
8.0	.0030477	.0032347	.0030793	.0035870	.0035268	.0035477
8.5	.0037704	.0038240	.0035142	.0039520	.0036918	.0034985
9.0	.0036617	.0034933	.0030514	.0033829	.0030027	.0027108
9.5	.0027147	.0024497	.0020361	.0022622	.0019339	.0017084
10.0	.0016267	.0014372	.0011822	.0013373	.0011708	.0010957
10.5	.0010742	.0010331	.00094467	.0010690	.0010480	.0010886
11.0	.0011811	.0012358	.0012231	.0013397	.0013652	.0014232
11.5	.0015447	.0015929	.0015559	.0016658	.0016473	.0016463
12.0	.0016718	.0016519	.0015535	.0016470	.0015628	.0014960
12.5	.0013883	.0013138	.0011863	.0012569	.0011488	.0010640
13.0	.00090592	.00083276	.00073228	.00078251	.00070592	.00065515
13.5	.00056591	.00053681	.00049208	.00053080	.00050814	.00050711
14.0	.00053397	.00054805	.00054425	.00057948	.00059081	.00061589
14.5	.00070035	.00072490	.00072497	.00075938	.00076759	.00078310
15.0	.00082400	.00082801	.00080594	.00083752	.00082034	.00080834

$$\mathfrak{M}_{ij}[04;\ 0.2;\ \gamma_0]$$

γ_0	$\mathfrak{M}_{1,1}$	$\mathfrak{M}_{1,3}$	$\mathfrak{M}_{1,5}$
1.0	.060404	.032597	.021917
2.0	.050905	.025266	.016053
3.0	.019319	.0077132	.0040102
4.0	.0032112	.0020120	.0012652
5.0	.0041845	.0045317	.0035261
6.0	.0037137	.0031878	.0021907
7.0	.00097413	.00074529	.00048244
8.0	.00083627	.00091525	.00088705
9.0	.0010483	.0010009	.00087041
10.0	.00037541	.00031948	.00024881
11.0	.00022733	.00024346	.00024505
12.0	.00035083	.00034791	.00032755
13.0	.00016274	.00014672	.00012569
14.0	.000075133	.000078187	.000078493
15.0	.00012971	.00013098	.00012792

γ_0	$\mathfrak{M}_{3,3}$	$\mathfrak{M}_{3,5}$	$\mathfrak{M}_{5,5}$
1.0	.017928	.012178	.0083190
2.0	.013394	.0088272	.0059322
3.0	.0044218	.0029217	.0021420
4.0	.0034267	.0029577	.0026771
5.0	.0051854	.0041492	.0033786
6.0	.0028144	.0020078	.0015038
7.0	.00069528	.00058194	.00060139
8.0	.0010480	.0010483	.0010706
9.0	.00096727	.00085190	.00076035
10.0	.00028729	.00024044	.00021839
11.0	.00027149	.00028210	.00030011
12.0	.00034761	.00032968	.00031490
13.0	.00013509	.00011873	.00010746
14.0	.000084406	.000087453	.000092950
15.0	.00013301	.00013059	.00012887

$\mathfrak{M}_{ij}$ [44; 0.06; γ_0]

γ_0	$\mathfrak{M}_{1,1}$	$\mathfrak{M}_{1,3}$	$\mathfrak{M}_{1,5}$
1.0	.023917	.012386	.0080880
2.0	.017769	.0089811	.0056916
3.0	.0077495	.0036134	.0020773
4.0	.0021906	.0016891	.0011842
5.0	.0026048	.0026987	.0020951
6.0	.0023566	.0020471	.0014617
7.0	.00089957	.00075788	.00057126
8.0	.00084949	.00089491	.00084858
9.0	.0010075	.00096146	.00084159
10.0	.00047303	.00042176	.00035141
11.0	.00035616	.00037072	.00036557
12.0	.00049318	.00048695	.00045795
13.0	.00027749	.00025629	.00022678
14.0	.00017181	.00017584	.00017424
15.0	.00025851	.00025949	.00025240

γ_0	$\mathfrak{M}_{3,3}$	$\mathfrak{M}_{3,5}$	$\mathfrak{M}_{5,5}$
1.0	.0069417	.0047191	.0032723
2.0	.0053166	.0036453	.0025965
3.0	.0026286	.0019028	.0015017
4.0	.0022932	.0019285	.0017040
5.0	.0029706	.0023894	.0019779
6.0	.0018472	.0013868	.0011098
7.0	.00073805	.00065209	.00065642
8.0	.00098439	.00096405	.00096632
9.0	.00093199	.00082962	.00075181
10.0	.00039544	.00034965	.00032917
11.0	.00039931	.00040512	.00042035
12.0	.00048534	.00046072	.00044139
13.0	.00024197	.00021959	.00020490
14.0	.00018513	.00018813	.00019532
15.0	.00026224	.00025675	.00025294

$\mathfrak{M}_{ij}\,[44;\,0.1;\,\gamma_0]$

γ_0	$\mathfrak{M}_{1,1}$	$\mathfrak{M}_{1,3}$	$\mathfrak{M}_{1,5}$
.5	.015933	.0087277	.0059210
1.0	.017384	.0091606	.0060687
1.5	.016506	.0084922	.0055278
2.0	.013821	.0068476	.0043267
2.5	.0097856	.0045427	.0027134
3.0	.0056258	.0023868	.0012913
3.5	.0025958	.0011057	.00056105
4.0	.0012255	.00086566	.00057761
4.5	.0011538	.0012515	.00097297
5.0	.0015451	.0016364	.0012669
5.5	.0016947	.0016150	.0011967
6.0	.0013889	.0011959	.00083206
6.5	.00085276	.00067423	.00043908
7.0	.00043329	.00034734	.00024265
7.5	.00030571	.00030555	.00027651
8.0	.00039465	.00042438	.00040679
8.5	.00050137	.00051134	.00047077
9.0	.00048481	.00046233	.00040246
9.5	.00034839	.00031205	.00025618
10.0	.00019461	.00016883	.00013524
10.5	.00011748	.00011213	.00010150
11.0	.00013095	.00013856	.00013821
11.5	.00017830	.00018508	.00018155
12.0	.00019418	.00019215	.00018068
12.5	.00015761	.00014865	.00013350
13.0	.000096999	.000088194	.000076412
13.5	.000054949	.000051514	.000046544
14.0	.000050583	.000052316	.000052255
14.5	.000069587	.000072557	.000072969
15.0	.000083385	.000084010	.000081899

$M_{ij}\,[44;\,0.1;\,\gamma_0]$

γ_0	$M_{3,3}$	$M_{3,5}$	$M_{5,5}$
.5	.0048305	.0032961	.0022563
1.0	.0050056	.0033816	.0023080
1.5	.0046629	.0031405	.0021520
2.0	.0037685	.0025178	.0017300
2.5	.0025533	.0016978	.0011891
3.0	.0015431	.0010684	.00081348
3.5	.0011335	.00089178	.00077201
4.0	.0013055	.0011073	.00098512
4.5	.0016828	.0014035	.0012015
5.0	.0018371	.0014685	.0012004
5.5	.0015852	.0012078	.00094798
6.0	.0010643	.00077397	.00059583
6.5	.00057275	.00041550	.00034435
7.0	.00033193	.00028578	.00029177
7.5	.00035255	.00035546	.00038292
8.0	.00047722	.00047241	.00047809
8.5	.00053084	.00049624	.00047001
9.0	.00044700	.00039483	.00035416
9.5	.00028561	.00024083	.00020962
10.0	.00015467	.00013262	.00012252
10.5	.00011667	.00011480	.00012067
11.0	.00015230	.00015663	.00016488
11.5	.00019474	.00019328	.00019375
12.0	.00019173	.00018176	.00017370
12.5	.00014160	.00012861	.00011825
13.0	.000081962	.000072888	.000066760
13.5	.000050673	.000048184	.000048129
14.0	.000055951	.000057540	.000060615
14.5	.000076558	.000077797	.000079764
15.0	.000085153	.000083493	.000082313

$\mathcal{M}_{ij}[44;\ 0.2;\ \gamma_0]$

γ_0	$\mathcal{M}_{1,1}$	$\mathcal{M}_{1,3}$	$\mathcal{M}_{1,5}$
1.0	.0093205	.0051206	.0034797
2.0	.0081639	.0040733	.0026025
3.0	.0029271	.0011223	.00056729
4.0	.00037529	.00019893	.00011318
5.0	.00050078	.00055526	.00043500
6.0	.00043724	.00037473	.00025505
7.0	.000094031	.000067482	.000038716
8.0	.000073365	.000082292	.000080997
9.0	.000094212	.000090152	.000078379
10.0	.000028712	.000023724	.000017653
11.0	.000014318	.000015680	.000016048
12.0	.000023647	.000023541	.000022220
13.0	.0000096154	.0000085364	.0000071607
14.0	.0000033665	.0000035566	.0000036148
15.0	.0000064583	.0000065549	.0000064278

γ_0	$\mathcal{M}_{3,3}$	$\mathcal{M}_{3,5}$	$\mathcal{M}_{5,5}$
1.0	.0028392	.0019393	.0013283
2.0	.0021168	.0013847	.00091792
3.0	.00057804	.00036367	.00025516
4.0	.00039753	.00035056	.00032416
5.0	.00064787	.00051985	.00042259
6.0	.00032858	.00023045	.00016801
7.0	.000060561	.000048361	.000050855
8.0	.000096573	.000097960	.00010127
9.0	.000087126	.000076528	.000067934
10.0	.000020689	.000016603	.000014601
11.0	.000017951	.000018997	.000020586
12.0	.000023582	.000022391	.000021383
13.0	.0000077231	.0000066343	.0000058638
14.0	.0000039287	.0000041437	.0000044975
15.0	.0000066843	.0000065836	.0000065109

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