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A THEOREM ON BENDING STRESSES IN ROTATING SHAFTS

by

R.T. McGoldrick

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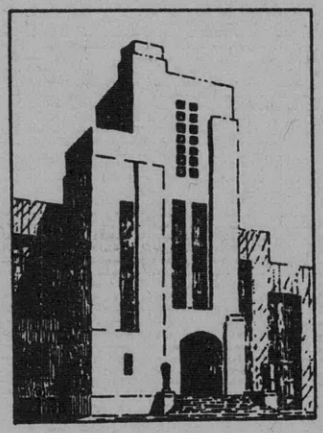

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A THEOREM ON BENDING STRESSES IN ROTATING SHAFTS

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ABSTRACT

It is shown that with the exception of the shaft fitted with a two-bladed propeller, the harmonic components of the varying bending stress set up in rotating propeller shafts due to periodic forces or moments acting on the shaft will have the same amplitudes regardless of the position of the strain gage on the circumference of the shaft.

INTRODUCTION

In connection with the investigation of failures of propeller tailshafts of single-screw merchant vessels, experimental determinations have been made of the bending stresses in the shafts under service conditions. In these investigations, it has been customary to install one set of metalectric strain gages in line with a propeller blade while other sets may be installed in planes making various angles with the plane of the first set. The question has arisen whether there is a preferred location for the gages or whether the desired information can be obtained for any gage location in the same fore-and-aft position. The analysis given in this report indicates that, in practice, with three or more blades, the same harmonic components of bending stress will be obtained regardless of the plane in which the gages are installed and, furthermore, that the peak bending stress due to superposition of the harmonic components of stress need not occur at a location in line with a propeller blade.

DERIVATION

The theory of shaft vibration is discussed rather thoroughly in Reference 1.* However, in demonstrating the theorem on bending stresses, it is not necessary to consider quantitatively the vibratory response of the propeller-shafting system to the moments applied at the propeller, a problem which involves uncertainties in the evaluation of some of the parameters. It is sufficient merely to make use of the fact that the bending stress due to vibration in any one plane is given by the simple beam formulas:

$$M = EI \frac{\partial \theta}{\partial x}$$

$$\sigma = \frac{M_c}{I}$$

where σ is the normal bending stress,

θ is the rotation of the plane about an axis normal to the longitudinal axis of the beam,

M is the bending moment,

EI is the bending rigidity,

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*References are listed on page 7.

E is Young's modulus,

I is the area moment of inertia of the section, and

c is the distance from the point of stress to the neutral axis.

It is to be noted that the strain gages give normal stress due to bending only and that normal stresses due to thrust variation are cancelled out.

Under operating conditions, the propeller shaft may execute complex vibratory or whirling motions, but whenever a steady state is reached, the orientation of the cross section at the location of the strain gages can be defined in terms of two angles, β and α. These angles are found by projecting the normal to the section on the vertical and horizontal planes passing through the rest position of the shaft axis (the fixed axis) as shown in Figure 1. The angle β is the angle which the projection of the normal makes with the fixed axis in the vertical (XY) plane, and the angle α is the angle which the projection makes with the fixed axis in the horizontal (XZ) plane. When the shaft is at rest and unstrained, both β and α are zero. The displacements of the center of the shaft section in the Y- and Z-directions are not involved in this derivation.

When the shaft is running, β, α, $\partial\beta/\partial x$, and $\partial\alpha/\partial x$ vary with time. A steady-state motion implies that all functions of time repeat whenever the shaft makes one complete revolution. Actually, if the propeller blades are perfectly uniform, it is clear that the excitation at the propeller has a fundamental period which is reduced to the time required for the propeller to rotate only through an angle equal to 2π/number of blades.

Let a fiduciary mark be made at point Q on the shaft surface not necessarily in line with a propeller blade, and let ω be the steady angular velocity of the shaft which may be assumed to be rotating clockwise as viewed in the positive direction of the X-axis as in Figure 2. Time t is taken zero when the fiduciary mark is at the top or vertical position. The angle measured clockwise from the fiduciary mark to an arbitrary strain gage location L on the shaft is designated as ϵ.

The instantaneous bending stress at the gage location L, defined by the angle ϵ, is then given by the equation

$$\sigma = ER\left[\frac{\partial\beta}{\partial x}\cos(\omega t + \epsilon) + \frac{\partial\alpha}{\partial x}\sin(\omega t + \epsilon)\right]$$

where R is the outer radius of the shaft. This equation is obtained by adding the stress components at the instantaneous position of the gage due to bending in the vertical and horizontal planes, and it is clear that β and α must be measured in such a way that when $\partial\beta/\partial x$ and $\partial\alpha/\partial x$ are both positive, they produce bending stresses of the same sign at points in the upper right-hand quadrant.

The quantities $\partial\beta/\partial x$ and $\partial\alpha/\partial x$ are then expressed in Fourier series

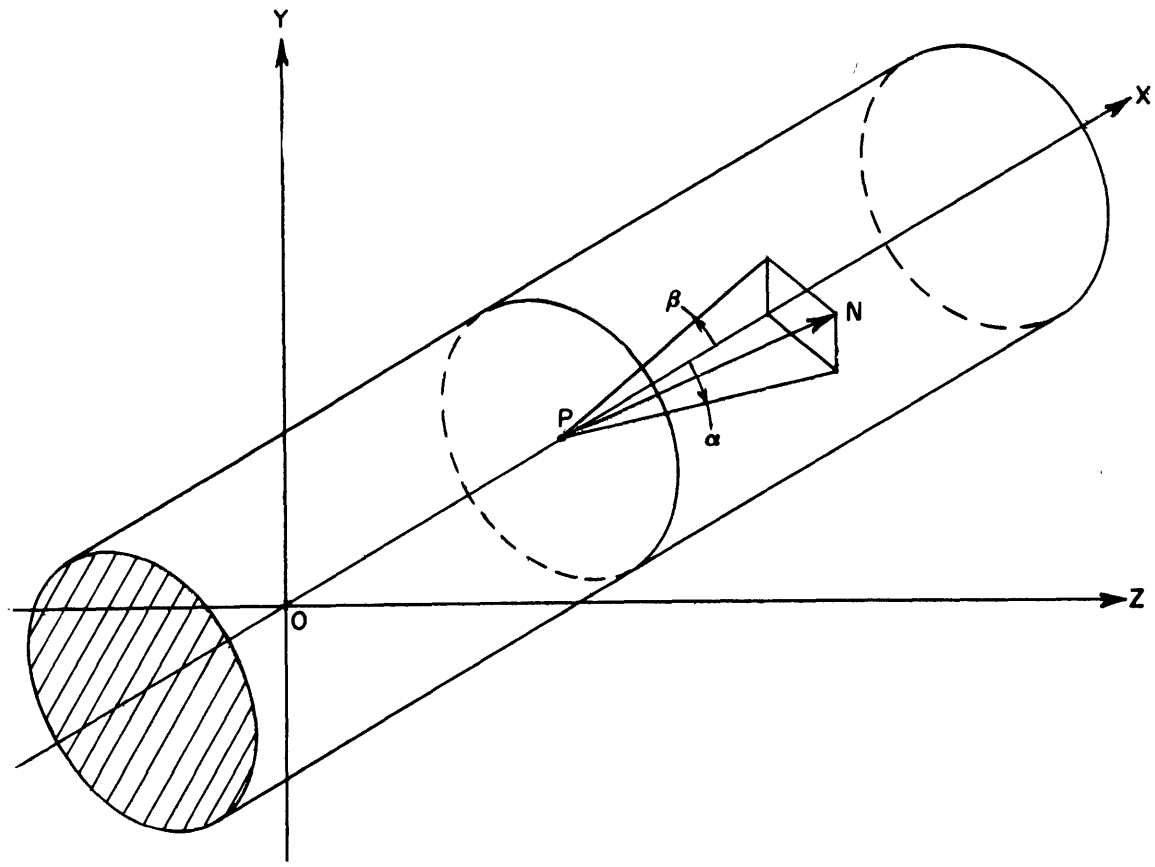

Figure 1 - Specification of Orientation of Section of Rotating Shaft at which Bending Stress is Measured

P is at the geometrical center of the shaft section, PN is normal to the section, OX is parallel to the longitudinal axis of the shaft when at rest, plane XY is vertical, plane XZ is horizontal, PN is projected on the XY- and XZ-planes, projection of PN on the XY-plane makes the angle α with OX, and the projection of PN on the XZ-plane makes the angle β with OX.

$$\frac{\partial \beta}{\partial x} = A_0 + A_1 \sin \omega t + A_2 \sin 2\omega t + \ldots + A_n \sin n\omega t + \ldots$$
$$+ B_1 \cos \omega t + B_2 \cos 2\omega t + \ldots + B_n \cos n\omega t + \ldots$$

$$\frac{\partial \alpha}{\partial x} = D_0 + D_1 \sin \omega t + D_2 \sin 2\omega t + \ldots + D_n \sin n\omega t + \ldots$$
$$+ E_1 \cos \omega t + E_2 \cos 2\omega t + \ldots + E_n \cos n\omega t + \ldots$$

Hence

$$\sigma = ER \left[\cos (\omega t + \epsilon) \left\{ A_0 + \sum_{n=1}^{n=\infty} A_n \sin n\omega t + \sum_{n=1}^{n=\infty} B_n \cos n\omega t \right\} \right.$$
$$\left. + \sin (\omega t + \epsilon) \left\{ D_0 + \sum_{n=1}^{n=\infty} D_n \sin n\omega t + \sum_{n=1}^{n=\infty} E_n \cos n\omega t \right\} \right]$$

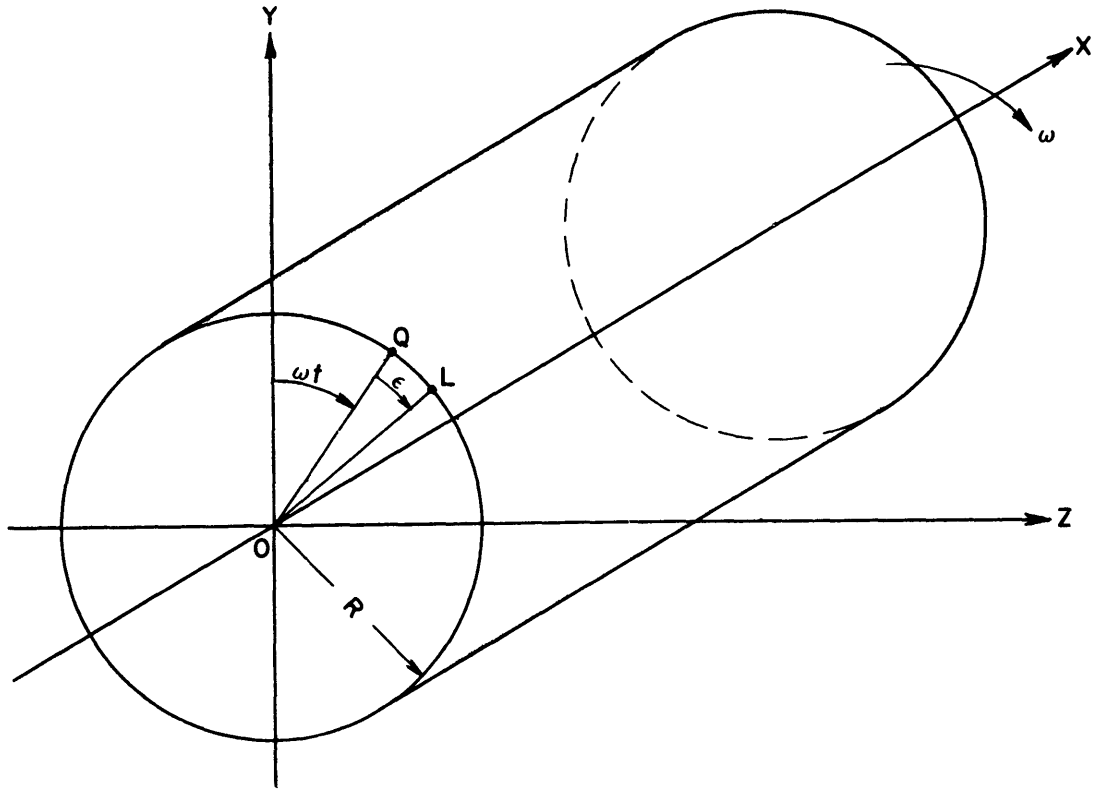

Figure 2 - Specification of Bending Stress at an Arbitrary Point on the Surface of the Shaft

The strain gage is at point L, the fiduciary mark is at point Q, OQ is vertical when $t = 0$, and the fixed angular distance between Q and L is ϵ.

whence

$$\sigma = ER\,[A_0 \cos(\omega t + \epsilon) + D_0 \sin(\omega t + \epsilon)$$
$$+ \Sigma A_n \sin n\omega t \cos(\omega t + \epsilon) + \Sigma B_n \cos n\omega t \cos(\omega t + \epsilon)$$
$$+ \Sigma D_n \sin n\omega t \sin(\omega t + \epsilon) + \Sigma E_n \cos n\omega t \sin(\omega t + \epsilon)]$$

Therefore

$$\sigma = ER\Big[A_0 \cos(\omega t + \epsilon) + D_0 \sin(\omega t + \epsilon)$$
$$+ \Sigma \frac{A_n}{2}\{\sin(n\omega t + \omega t + \epsilon) + \sin(n\omega t - \omega t - \epsilon)\}$$
$$+ \Sigma \frac{B_n}{2}\{\cos(n\omega t + \omega t + \epsilon) + \cos(n\omega t - \omega t - \epsilon)\}$$
$$- \Sigma \frac{D_n}{2}\{\cos(n\omega t + \omega t + \epsilon) - \cos(n\omega t - \omega t - \epsilon)\}$$
$$+ \Sigma \frac{E_n}{2}\{\sin(\omega t + \epsilon + n\omega t) + \sin(\omega t + \epsilon - n\omega t)\}\Big]$$

which reduces to

$$\sigma = ER \Big[A_0 \cos(\omega t + \epsilon) + D_0 \sin(\omega t + \epsilon)$$
$$+ \Sigma \frac{A_n + E_n}{2} \sin\{(n+1)\,\omega t + \epsilon\}$$
$$+ \Sigma \frac{B_n - D_n}{2} \cos\{(n+1)\,\omega t + \epsilon\}$$
$$+ \Sigma \frac{A_n - E_n}{2} \sin\{(n-1)\,\omega t - \epsilon\}$$
$$+ \Sigma \frac{B_n + D_n}{2} \cos\{(n-1)\,\omega t - \epsilon\} \Big]$$

DISCUSSION

The validity of the theorem that the amplitudes of the harmonic components of stress are independent of the position of the strain gage around the circumference of the shaft follows from the nature of the series in the final expression for σ in conjunction with certain considerations as to the nature of shaft excitation. The symbol n indicates the position of a general term in the Fourier series for $\partial\beta/\partial x$ and $\partial\alpha/\partial x$ (since it always accompanies a term having circular frequency $n\omega$) and it also indicates the "order" of the excitation which this term represents. The "order" as used in vibration terminology is the ratio of the circular frequency of a particular component to the angular velocity of the shaft in question. Thus an rth-order vibration has a frequency equal to r times the number of revolutions which the shaft makes in unit time, and an rth-order stress involves r stress cycles per revolution of the shaft.

In the case of a ship's propeller shaft, the orders of excitation present are usually the zero order due to gravity or to eccentricity of the center of thrust, the first order due to unbalance in the propeller or shafting or due to nonuniformity of the propeller blades (pitch unbalance), the blade frequency order, and integer multiples of the blade-frequency order. The last two are due to the moments acting on the blades as discussed in Reference 2.

The expression for σ shows that as long as identical orders of stress are not set up by different orders of vibration, the amplitudes of the stress components found by harmonic analysis of the strain oscillograms will be the same regardless of the position of the gage on the shaft. This is due to the fact that the shifting of the gage location changes the sine and cosine components of each harmonic by the same phase angle so that the resultant is always the square root of the sum of the squares of the sine and cosine terms.

Table 1 shows the orders of stress and vibration generally set up in a propeller shaft.

The case of a two-bladed propeller is an exception to the theorem. As may be seen from Table 1, if N equals 2, the blade-frequency vibration will set up first-order and third-order stress components. When this first-order component is combined with the first-order stress due to gravity or thrust eccentricity, the resultant will vary with ϵ, which depends on the gage location.

TABLE 1

Orders of Bending Stress and Vibration Set up in Propeller Shafts

(Here n is the general term in the Fourier series, N is the number of propeller blades per propeller, and q is a multiple of the blade frequency.)

Excitation	Order of Whirl or Vibration	Order of Bending Stress Component
Gravity or thrust eccentricity	$n = 0$	1
Mass unbalance or pitch unbalance	$n = +1$	0
Blade frequency moments	$n = N \begin{cases} \text{forward} \\ \text{counter} \end{cases}$	$N - 1$ $N + 1$
Moments which are multiples of the blade frequency by q	$n = qN \begin{cases} \text{forward} \\ \text{counter} \end{cases}$	$qN - 1$ $qN + 1$

A three-bladed propeller will give second- and fourth-order stress components due to shaft vibration of blade frequency. If either mass or pitch unbalance sets up a first-order counterwhirl, this would also give a second-order stress component. However it is not believed at the present time that first-order counterwhirls are set up. The theorem thus appears to be applicable to three-bladed propellers. It is to be noted, however, as indicated in Table 1, that higher order excitation due to pitch unbalance is considered negligible.

For propellers with four or more blades, the stress components due to blade frequency vibrations and to first-order excitation cannot coincide in frequency so that the theorem is applicable.

The most important case in practice is that in which there is a gravity or eccentric thrust effect and a blade-frequency vibration or whirl (the latter having either a circular or elliptical path). In this case there are three orders of stress variation, namely, 1, $N + 1$, and $N - 1$. The last two will beat twice per revolution. While the interval between beats will be the same for all gage locations, the peak stress evidently will depend upon whether the maximum beating stress amplitude coincides with the peak first-order stress. If the first-order stress component is absent (no eccentricity of the center of thrust and negligible gravity effect, so that $A_0 = D_0 = 0$), identical strain records will be found on all strain-gage channels although the beats will occur at different instants on different channels. Since the whirl of the shaft will shift phase relative to the exciting dynamic moments at the propeller as whirling resonance is approached, there is, in general, no guarantee that peak stresses will occur at points in line with a propeller blade.

If, however, as indicated in Reference 3, stress concentration due to keyways or other abrupt changes of shaft dimensions are present, recorded stresses will differ by amounts not accounted for by this analysis. It is to be observed that if there is a first-order forward whirl

($n = 1$) due to unbalance in the propeller or shafting in addition to higher order vibrations, the quantity $(n - 1)\ \omega t$ corresponding to $n = 1$ vanishes, and there is a constant stress component which does vary with the position of the strain gage. This will shift the mean position with respect to which variations appear on the strain oscillogram. In this case the value of the peak stress will depend on the location of the strain gage relative to the position of the unbalance in the propeller or shafting.

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