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ANALYSIS OF STRESS IN THE CONICAL ELEMENTS OF SHELL STRUCTURES

by

C.E. Taylor and E. Wenk, Jr.

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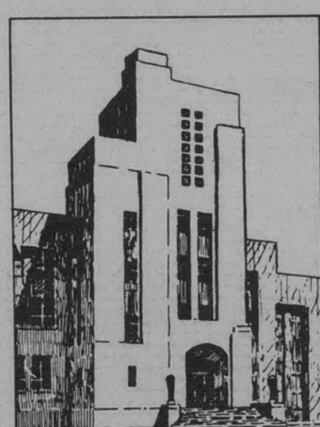

RESEARCH AND DEVELOPMENT REPORT

May 1956

Report 981

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Reprinted from the Proceedings
of the Second U.S. National Congress
of Applied Mechanics

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FOREWORD

Recent developments in submarine arrangements have included a trend toward large cylindrical pressure hull sections joined to smaller ones by a conical transition section. In order to design the intersection of these elements economically and safely, the naval architect must have some means of readily evaluating the stresses which exist in the shells at the juncture. Both theoretical and experimental investigations were undertaken by the Taylor Model Basin to study and improve the analysis of stresses at such intersections. This paper represents an extension and simplification of theoretical methods for calculating stresses.

The general procedure for determining stresses at any junction of axisymmetric shells is to consider first the membrane deformations which would occur in each member under pressure, assuming it to be completely separated from the others. These deformations may be computed by known methods of analysis. It will be found, in general, that the members do not deform the same amount, so that the edges at the intersection do not match. This discontinuity in displacements and rotations must then be theoretically eliminated by applying additional forces and moments at the edge of each member. In the course of such analysis, computations are facilitated if the distortions at the edge produced by unit values of force and moment are known. Such distortions, called influence coefficients, may be used by the principle of superposition, to determine the total displacements of the members.

The collective edge forces and moments must, of course, be in equilibrium. Furthermore, the condition that the ends of the members must meet at the joint must be satisfied. These requirements result in a set of equations involving the discontinuity forces, the solution of which yields the required unknown stress resultants, so that the total state of stress in the member can be found.

In TMB Report 826 influence coefficients were derived for the large end of unstiffened cones by approximate methods. A procedure for determining stresses everywhere in the cone in terms of the discontinuity forces and moments was also derived. Examples involving intersections stiffened by a ring were shown to be soluble by the use of these influence coefficients. Experimental verification of these results has been obtained and is presented in TMB Report 911.

In this paper, the authors have performed a more exact derivation of the coefficients, and the results are compared with the approximate ones of TMB Report 826. The errors in the approximation are given by special Ω functions and are found to be quite small for a considerable range of practical geometries. The derivation of these functions is an important step since it assures the designer of reasonable results without excessive labor. A second addition to the theory is the tabulation of influence coefficients for the small end of the unstiffened cone. This is of considerable interest for designs which involve a conical transition between two cylindrical shells.

Although the influence coefficients can be evaluated exactly, no short cuts have yet been found suitable for the computation of stresses from the known forces and moments. The formulas of TMB Report 826, however, are considered to be reasonably accurate and are reproduced here as Appendix B.

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ANALYSIS OF STRESSES IN THE CONICAL ELEMENTS OF SHELL STRUCTURES

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The Love-Meissner analysis for thin shells has previously been applied to cones of uniform wall thickness, and solutions for the stress resultants were given in terms of Kelvin's functions. Since tabulation of these functions for large arguments is not practical, considerable computation was still required. In the present paper, the authors define special functions which eliminate the necessity of evaluating Kelvin's functions and which may be used with simple algebraic and trigonometric functions to compute the boundary forces and displacements for cones for various loading conditions. These special functions also make clear the magnitude of errors which result from Geckeler's and other approximate solutions.

ıtroduction

The Love-Meissner [1,2]* equations for equilibrium of hin elastic shells were applied by Dubois [3] to the case f the cone and his solutions found by solving two second-rder differential equations were expressed in terms of pecial infinite series and asymptotic formulas. These quations were later solved by Flugge [4] directly in erms of Bessel functions of second order which he exressed as functions tabulated by Schleicher [5]. Watts nd Burrows [6] arrived at solutions identical to Flugge's, nd applied their results to the stress analysis of the ntersection of a cylinder with the large end of a truncated r complete cone.

To facilitate these otherwise intricate solutions, Watts nd Lang [7] derived coefficients for edge displacements nd rotations of the large opening in a truncated cone. ince these yet appeared too cumbersome, an extensive abulation of specific numerical solutions was also proided for practical design applications. Alternate numerial methods of the same problem have been presented by \u, Goodman and Newmark [8].

Another approach to simplifying the exact theory has een resort to approximate differential equations whose olutions were known and more readily susceptible to omputation. Moir [9], Wetterstrom [10], Rhys [11], and 3urrows [12] have indicated use of the Bauersfeld-Geckeler [13] type of approximation in which the cone element is replaced by an equivalent cylinder for which solutions are found in terms of trigonometric and exponential functions.

Hetenyi [14] obtained another set of approximate equations by considering the cone structure analogous to a set of tapered longitudinal beams on elastic foundations. In a further development, the present authors [15] have shown that Hetenyi's fourth-order equations may be obtained directly from Dubois' original expressions by first writing the two second-order equations as one of fourth order, and then omitting the one second-order term. It was thus possible to relate more closely the approximate and exact equations and to evaluate more definitely the error in the approximate analysis. In addition, there were extracted from these solutions a set of edge coefficients which appear in simple algebraic form readily amenable to general stress analysis of cone-shell intersections and in particular, that of reinforced intersections. In another paper, the present authors [16] extended the method to establish similar edge coefficients for the small as well as the large end of a truncated cone; similar equations have been employed by Linkous and Horvay [17].

In this paper, a new form of solution of the exact equation has been obtained which greatly facilitates numerical calculation. The Watts and Burrows form of solution in terms of ber_2, bei_2, ker_2 and kei_2 functions are given by a combination of simple algebraic expressions and special functions which are tabulated for wide ranges of geometry. These relationships are given explicitly for the coefficients of edge displacement and rotation for both ends of a truncated cone.

It is shown that when the special functions are assigned their asymptotic values of unity, the remaining terms, which may be regarded as an approximate solution to the exact equation, correspond to quasi-exact solutions of earlier published approximate equations of the Geckeler and Hetenyi form. Thus, by means of the special functions, the accuracy of approximate methods can be evaluated with ease. Of perhaps greater utility of this new form of solution is application of these special functions directly with the simple algebraic expressions for edge coefficients for an exact analysis of stresses at the reinforced intersection of conical and cylindrical shells subjected to normal pressure.

General Equations

The equilibrium conditions for axially symmetric shells have been expressed by two second order differential equations by E. Meissner[1]. Dubois [3] and others have

*Numbers in brackets refer to the bibliography at the end of he paper.

applied Meissner's general equations to the case of conical shells, but the form given by Watts and Burrows [6] was found most convenient for this analysis. These were expressed as:

$$L\,(V) + EhW \cot \alpha = 1.5\, px \tan \alpha$$

$$L\,(W) - \frac{V \cot \alpha}{D} = 0 \qquad (1)$$

where L is Meissner's operator, defined for a conical shell by

$$L\,(\quad) = x \frac{d^2}{dx^2}(\quad) + \frac{d}{dx}(\quad) - \frac{(\quad)}{x} \qquad (2)$$

and

p = external pressure

E = Young's modulus

ν = Poisson's ratio

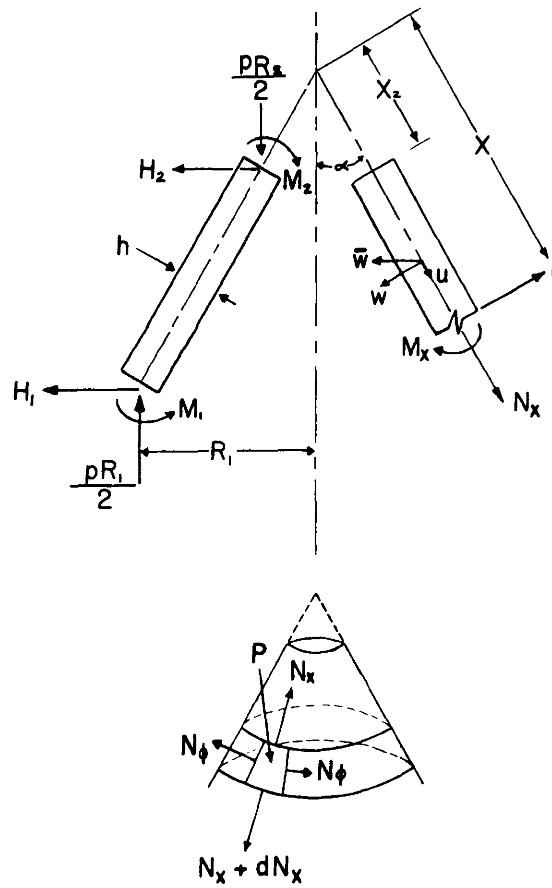


FIGURE 1. NOMENCLATURE AND SIGN CONVENTION USED FOR CONES

V, W, and D are defined as follows

$$V = Q_x x \tan \alpha$$

$$W = w_x \qquad (3)$$

$$D = \text{flexural rigidity} = \frac{Eh^3}{12\,(1-\nu^2)}$$

The subscript "x" on the symbol w denotes differentiation with respect to x, and the stress resultants and cone parameters are defined in Figure 1.

The two second-order differential equations (Eq. 1) may also be expressed as one fourth order differential equation in the variable W:

$$LL\,(W) + \lambda^4 W = \frac{1.5\, U^4 px}{E h^3} \qquad (4)$$

where

$$\lambda^4 = \frac{12\,(1-\nu^2)}{h^2 \tan^2 \alpha}$$

$$U^4 = 12\,(1-\nu^2) \qquad (5)$$

By integration, Eq. (4) reduces to the following equation in terms of the displacement w.

$$x^2 w_{xxxx} + 2\, x w_{xxx} - 2\, w_{xx} + \lambda^4 w = \frac{9\,(1-\nu^2)\, px^2}{Eh^3} + \text{const.} \qquad (6)$$

Since the constant in this equation represents translation parallel to the axis, it may be neglected without affecting the elastic analysis.

The form of Eq. (6) is used later when evaluating the Geckeler and other approximations, but solutions are more readily found by operations on the second order equations. Following known methods, the stress resultants, rotations, and radial displacements may be obtained directly. These were given by Watts and Burrows [6] as:

$$N_x = \frac{1}{x}\,(C_1\, ber_2\, \xi + C_2\, bei_2\, \xi + C_3\, ker_2\, \xi + C_4\, kei_2\, \xi) - \frac{px \tan \alpha}{2}$$

$$Q_x = \frac{\cot \alpha}{x}\,(C_1\, ber_2 \xi + C_2 bei_2 \xi + C_3 ker_2 \xi + C_4 kei_2 \xi)$$

$$M_x = \frac{h}{2\,U^2 x}\,[C_1(\xi\, bei_2'\, \xi + 2\nu\, bei_2\, \xi) - C_2\,(\xi\, ber_2'\, \xi + 2\nu ber_2\, \xi) + C_3\,(\xi\, kei_2'\, \xi + 2\nu\, kei_2\, \xi) -$$

$$C_4 (\lambda\, ker_2' \,\xi + 2\nu\, ker_2\, \xi)] + \frac{ph^2 \tan^2 \alpha}{8(1-\nu)}$$

$$\nu_x = \frac{U^2}{Eh^2} (C_1\, bei_2\, \xi - C_2\, ber_2\, \xi + C_3\, kei_2\, \xi - C_4\, ker_2\, \xi) + \frac{3\,px \tan \alpha}{2\,Eh}$$

$$\bar{\nu} = \frac{-\sin \alpha}{Eh} [C_1 (\frac{\xi}{2}\, ber_2' \,\xi - \nu\, ber_2\, \xi) + C_2 (\frac{\xi}{2}\, bei_2' \xi - \nu bei_2 \xi) + C_3 (\frac{\xi}{2}\, ker_2' \xi - \nu\, ker_2\, \xi) + C_4 (\frac{\xi}{2}\, kei_2' \,\xi - \nu\, kei_2\, \xi)] + \frac{px^2 \sin \alpha \tan \alpha}{Eh} (1 - \nu/2) \qquad (7)$$

C_1, C_2, C_3, and C_4 are constants of integration which correspond to any given boundary condition. The primes denote differentiation with respect to ξ, and

$$\xi = 2\lambda \sqrt{x} \qquad (8)$$

Edge Coefficients

For intersection analysis of shells, compatibility of contiguous edges requires development of relationships between edge loading (shear H_1 and moment M_1) and edge rotation θ and radial displacement $\bar{w}$ for all members of the intersection. These may be expressed in terms of edge coefficients as

$$\begin{aligned} \theta_1 &= a_1 M_1 + b_1 H_1 + c_1 p \\ \bar{w}_1 &= d_1 M_1 + e_1 H_1 + f_1 p \\ \theta_2 &= a_2 M_2 + b_2 H_2 + c_2 p \\ \bar{w}_2 &= d_2 M_2 + e_2 H_2 + f_2 p \end{aligned} \qquad (9)$$

Here the subscript "one" indicates the functions are evaluated at the large end of a truncated cone and "two" refers to the small end.
The quantities H_x and Q_x are related by:

$$Q_x = H_x \cos \alpha + \frac{pR}{2} \sin \alpha \qquad (10)$$

where the expression with the minus sign applies to the small-diameter end of a truncated cone and that with the plus sign to the large-diameter end.

Three appropriate loading conditions are now chosen so as to develop explicit expressions for edge coefficients for a cone, that of M_x at the edge, H_x at the edge, and p. These are represented by setting the quantities M_1, H_1, and p equal to M_1, 0, 0; 0, H_1, 0 and 0, 0, p respectively for which are found three corresponding sets of values of C_1 and C_2. By successive substitution into Eqs. (7), the following values are obtained for the edge coefficients:

$$a_1 = \frac{\theta_1^M}{M_1} = \frac{U^3}{E} \sqrt{\frac{2 R_1}{h^5 \cos \alpha}} \cdot \Omega_1$$

$$b_1 = \frac{\theta_1^H}{H} = -\frac{U^2}{E} \frac{R_1}{h^2} \cdot \Omega_2$$

$$c_1 = \frac{\theta_1^p}{p} = -\frac{U^2 R_1^2 \tan \alpha}{2\,Eh^2} \cdot \Omega_2 + \frac{3 R_1 \tan \alpha}{2\,Eh \cos \alpha} - \frac{3(1+\nu)\sqrt{R_1} \sin^2 \alpha}{\sqrt{2}\, UE \sqrt{h \cos^5 \alpha}} \cdot \Omega_1$$

$$d_1 = \frac{\bar{w}_1^M}{M_1} = \frac{U^2 R_1}{Eh^2} \cdot \Omega_2$$

$$e_1 = \frac{\bar{w}_1^H}{H_1} = -\frac{U}{E} \sqrt{\frac{2 R_1^3 \cos \alpha}{h^3}} \cdot \Omega_3$$

$$f_1 = \frac{\bar{w}_1^p}{p} = -\frac{U}{E} \sqrt{\frac{R_1^5 \sin^2 \alpha}{2 h^3 \cos \alpha}} \cdot \Omega_3 + \frac{(1-\nu/2) R_1^2}{Eh \cos \alpha} - \frac{U^2 R_1 \tan^2 \alpha}{8\,E\,(1-\nu)} \cdot \Omega_2$$

$$a_2 = \frac{\theta_2^M}{M_2} = -\frac{U^3}{E} \sqrt{\frac{2 R_2}{h^5 \cos \alpha}} \cdot \Omega_4$$

$$b_2 = \frac{\theta_2^H}{H_2} = \frac{U^2 R_2}{Eh^2} \cdot \Omega_5$$

$$c_2 = \frac{\theta_2^p}{p} = -\frac{U^2 R_2^2 \tan \alpha}{2\,Eh^2} \cdot \Omega_5 + \frac{3 R_2 \tan \alpha}{2\,Eh \cos \alpha} + \frac{3(1+\nu)\sqrt{R_2} \sin^2 \alpha}{UE \sqrt{2 h \cos^5 \alpha}} \cdot \Omega_4$$

$$d_2 = \frac{\bar{w}_2^M}{M_2} = \frac{U^2 R_2}{Eh^2} \cdot \Omega_5$$

$$e_2 = \frac{\bar{w}_2^H}{H_2} = -\frac{U}{E} \sqrt{\frac{2 R_2^3 \cos \alpha}{h^3}} \cdot \Omega_6$$

$$f_2 = \frac{\bar{w}_2^p}{p} = \frac{U}{E}\sqrt{\frac{R_2^5 \sin^2\alpha}{2h^3\cos\alpha}} \cdot \Omega_6$$

$$\frac{R_2^2 (1-\nu/2)}{Eh\cos\alpha} - \frac{U^2 R_2 \tan^2\alpha}{8E(1-\nu)} \cdot \Omega_5 \tag{11}$$

where

$$\Omega_1 = \frac{\xi_1 G}{\sqrt{2}(C + 2\nu G)}$$

$$\Omega_2 = \frac{-A}{C + 2\nu G}$$

$$\Omega_3 = \frac{\xi_1 B - \frac{4\nu^2 G}{\xi_1}}{\sqrt{2}(C + 2\nu G)}$$

$$\Omega_4 = \frac{-\xi_2 G_K}{\sqrt{2}(C_K + 2\nu G_K)} \tag{12}$$

$$\Omega_5 = \frac{-A_K}{C_K + 2\nu G_K}$$

$$\Omega_6 = -\frac{\xi_2 B_K - \frac{4\nu^2 G_K}{\xi_2}}{\sqrt{2}(C_K + 2\nu G_K)}$$

and

$$\begin{aligned}
A &= \xi_1 (ber_2' \xi_1\, bei_2 \xi_1 - bei_2' \xi_1\, ber_2 \xi_1) \\
B &= (ber_2' \xi_1)^2 + (bei_2' \xi_1)^2 \\
C &= \xi_1 (ber_2 \xi_1\, ber_2' \xi_1 + bei_2 \xi_1\, bei_2' \xi_1) \\
G &= (ber_2 \xi_1)^2 + (bei_2 \xi_1)^2 \\
A_K &= \xi_2 (ker_2' \xi_2\, kei_2 \xi_2 - kei_2' \xi_2\, ker_2 \xi_2) \\
B_K &= (ker_2' \xi_2)^2 + (kei_2' \xi_2)^2 \\
C_K &= \xi_2 (ker_2 \xi_2\, ker_2' \xi_2 + kei_2 \xi_2\, kei_2' \xi_2) \\
G_K &= (ker_2 \xi_2)^2 + (kei_2 \xi_2)^2
\end{aligned} \tag{13}$$

In this paper it is assumed that the cone is sufficiently long so that the boundary conditions at one edge do not appreciably affect the stresses near the opposite edge. This is discussed on page 477 of reference 18. Hence C_1 and C_2 are set equal to zero when evaluating the stresses near the small end of a truncated cone and C_3 and C_4 are set equal to zero when evaluating the stresses near the large end. If the differences in notation and sign convention are taken into account, the coefficients $a_1, b_1, \cdots f_1$ for the large end are found in agreement with those derived previously by Watts and Lang [7]. The authors are not aware of prior publication of the coefficients $a_2, b_2, \cdots f_2$ for the small end.

This particular form of the expression for the edge coefficients thus comprises only simple trigonometric and algebraic terms combined with the special functions. It was chosen when it was found that the ber_2 and bei_2 functions of ξ could be conveniently combined with powers of ξ to form functions which differ slightly from unity for small values of the argument, and approach unity asymptotically for large values. Because of their simplicity, the algebraic component of the coefficients may be regarded as approximate solutions to the exact equations whose accuracy is subject to the benefit of numerical evaluation by reference to the Ω terms. In other words, the approximation can be corrected to give results which are as exact as are desired. This will be discussed in more detail later.

Comparison with Approximate Analyses

As was noted in the introduction, earlier attempts were made to simplify the theory by reducing the exact differential equations to some approximate form in which it was recognized that the solution would be facilitated. For comparison of approximate and exact analyses, it is particularly convenient to use the fourth-order differential equation (Eq. 6). In that form, one approximation has involved neglecting the second-order term such that the remaining equation was:

$$x^2 w_{XXXX} + 2x w_{XXX} + \lambda^4 w = \frac{9(1-\nu^2)px^2}{Eh^3} \tag{14}$$

In earlier published work [15], the solution to this equation was given in terms of the ψ functions of Schleicher. However, these functions are directly related to ber_0, bei_0, ker_0, and kei_0 functions. Since it is desired to compare the solution of Eq. (14) with the solution of Eq. (6), and since the ber_2 and bei_2 solutions of Eq. (6) are related by recurrence formulas to the ber_0 and bei_0 functions, this latter form is employed here in preference to the ψ-functions of Schleicher. In the following the subscript "zero" will be omitted from Kelvin's functions and it will be understood that the functions will be of zero order unless otherwise indicated.

The solution for Eq. (14) is then:

$$w = \sqrt{x}(C_1\, ber\, \xi + C_2\, bei\, \xi + C_3\, ker\, \xi + C_4\, kei\, \xi) + \frac{3px^2\tan^2\alpha}{4Eh} \tag{15}$$

As was shown in Reference [15], for large values of the argument ξ, the shear and moments may be given by approximate expressions in lieu of Eq. (7). Thus, through the use of these, and Eq. (15) and its derivatives, a set f edge coefficients was derived which were analogous to hose in Eq. (11). They are presented as Eqs. (43) in Ref 15] for the large end and equations p. 6 in Reference [16] or both ends of a truncated cone.

By comparison of the edge coefficients resulting from oth the exact and approximate differential equations it an be noted that the short form of the coefficients of References [15] and [16] are almost identically equal to he results obtained by setting the Ω functions in Eq. (11) equal to unity. The exceptions are that the third terms in he exact equations for c_1, c_2, f_1 and f_2 are not present in the approximate coefficients. However, these omitted terms are generally negligible compared to the other erms if α is not nearly equal to $\pi/2$. In view of this correspondence between the approximate coefficients derived from the exact equation and the "exact" coefficients from solution of approximate equation, the error in use of he approximate equations which heretofore could only be estimated, can be evaluated explicitly; it appears simply in terms of the non-dimensional geometric parameter ξ as the special Ω functions given previously.

Although not discussed in Reference [15], the shorter forms of the coefficients also result from a Geckeler or equivalent cylinder* type of approximation. Here, the fourth-order differential equation is:

$$x_1^2\, w_{xxxx} + \lambda^4 w = \frac{9(1-\nu^2)\,px^2}{Eh^3} \qquad (16)$$

Inasmuch as practical solutions to the exact equations are developed in this present paper, evaluation of the various approximate solutions will not be pursued further. It should be noted, however, that their relationships can be extracted from the various forms of the differential equations of equilibrium given in expressions (6), (14) and (16).

Discussion of Ω Functions

The special Ω functions are defined by Eq.(12). All of these are expressed in terms of ber_2 and bei_2 or ker_2 and kei_2 functions and their derivatives. For numerical evaluation, the range of interest of the argument ξ was 10 to 170. Since tables for these functions with arguments of this size did not exist, resort has been made to the series expressions given by Dwight [19].

The resulting Ω functions are plotted in the accompanying graphs, Figures 2, 3, and 4. A sufficient number of terms were employed in the series expressions so as to develop six significant sigures in the results. For values of the argument greater than shown the asymptotic expressions given in Figures 2–4 may be applied with satisfactory accuracy.

From these plots it may be seen that these functions approach unity and that even for values of ξ as small as 10, the Ω-functions represent a maximum error in the approximate edge coefficients of only 10.5 per cent. Although these results may be applied to correct the ap-

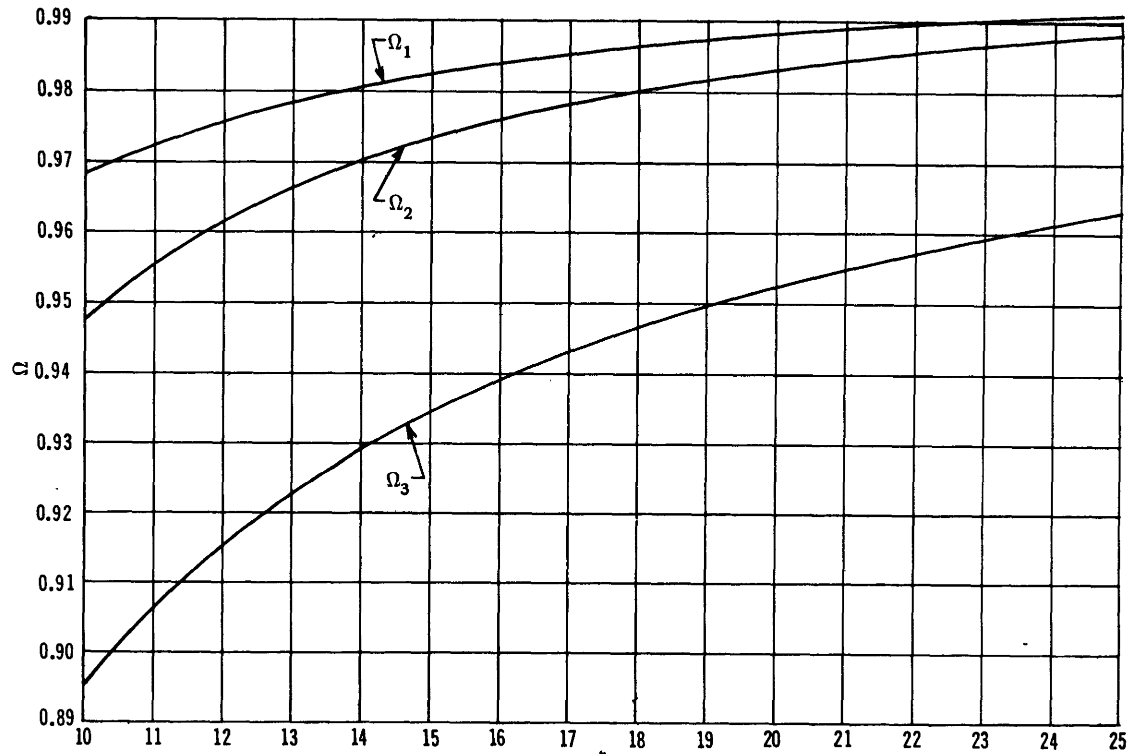


FIGURE 2. FUNCTIONS USED FOR COMPUTING EDGE ROTATION AND DISPLACEMENT COEFFICIENTS AT LARGE END OF A TRUNCATED CONE

*The exact equation governing the axisymmetric deformations for a thin cylindrical shell under uniform hydrostatic pressure p is $w_{xxxx} + \dfrac{\lambda^4 w}{R^2} = \dfrac{12(1-\nu^2)}{Eh^3}\,p$. In comparing this expression to Eq. (16) it appears that a discrepancy exists between the factors 12 nd 9 on the right-hand sides of their respective equations. It can be shown, however, that when $\alpha \to 0$ there is a discontinuous hange in the membrane term of Eq. (16).

proximate solution, their greatest value perhaps lies in their use to avoid intricate calculations with combinations of ber_2 and bei_2 functions when performing an exact analysis.

Stress Analysis of Cones

The meridianal and circumferential stresses in a conical shell are written in terms of stress resultants and moments as:

$$\sigma_x = \frac{N_x}{h} \pm \frac{6 M_x}{h^2}$$
$$\sigma_\varphi = \frac{N_\varphi}{h} \pm \frac{6 M_\varphi}{h^2} \tag{17}$$

Thus, as a first step in strength analysis, the stress resultants and moments given by Eq. (7), or in reference [6], must be determined from the constants of integration which reflect the physical boundary conditions. In this paper these are restricted to two cases of symmetrical edge loadings H_1, H_2, M_1, M_2 and to pressure p, but by suitable combination, virtually all important special boundary conditions can be accommodated. Thus, if M's, H's and p are known, stresses may be computed everywhere in the shell. If the cone is attached to any axisymmetric shell, the H's and M's represent discontinuity forces and moments at the intersection and may be computed in terms of the geometry and elastic constants of the two contiguous structures. This is accomplished by fulfilling requirements of compatibility of rotation and displacement at the joint of the two elements. For analysis, these pertinent quantities are $\bar{w}$ and θ which have already been related to the edge and pressure loadings as the previously derived edge coefficients.

Expressions for constants of integration for the exact theory are given in the appendix. No compact form has been discovered. However, interest in stress analysis is usually directed to maximum values and these maxima appear generally to occur at the edges of the cone. At the edges, computation of stress can be made without resort to the integration constants inasmuch as the M_x, Q_x, and N_x appearing in equations (7) can be determined directly from the values of edge loading H_1, H_2, M_1 and M_2.

This mode of analysis thus depends primarily on development of the quantities H and M from properties of the boundary conditions. Several examples of this development follow.

1. *Cone with Fixed Edges, Subject to Normal Pressure.* For a truncated cone with both edges completely fixed, the rotations and displacements at the large end are:

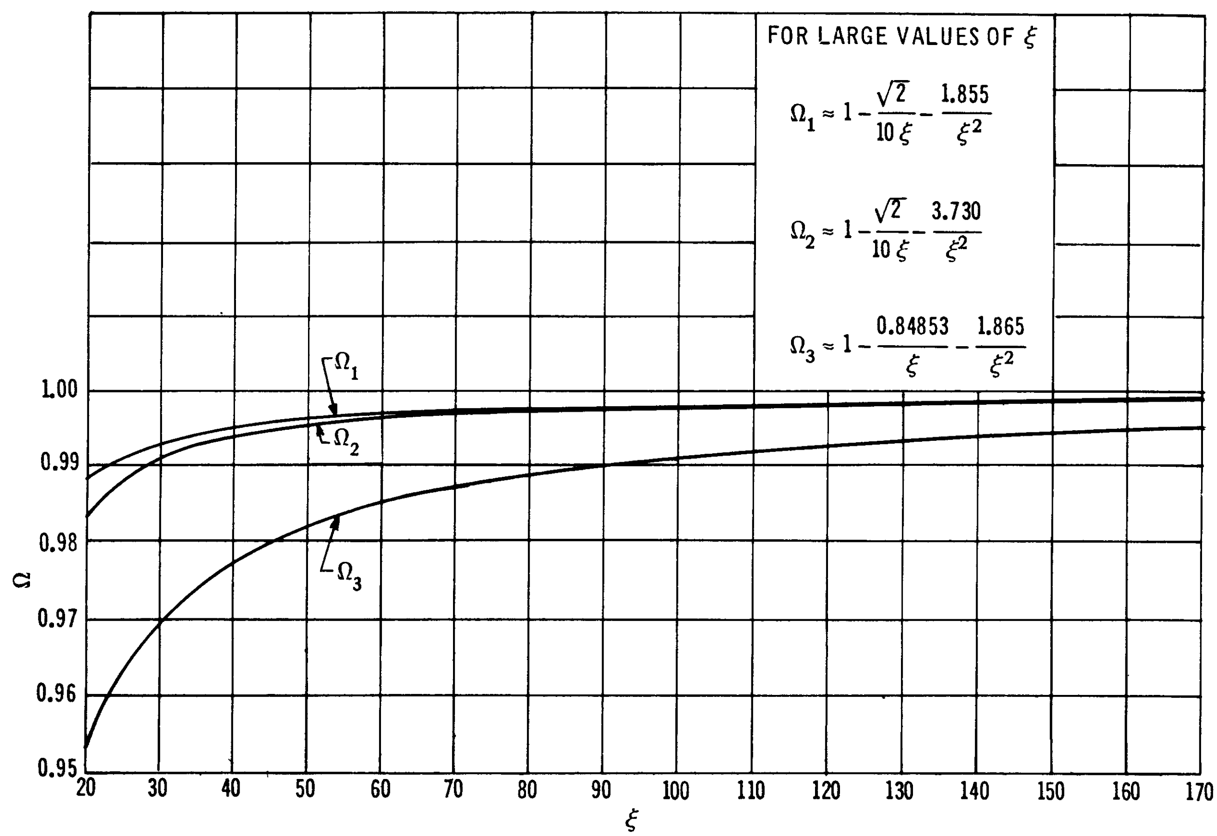


FIGURE 3. FUNCTIONS USED FOR COMPUTING EDGE ROTATION AND DISPLACEMENT COEFFICIENTS AT SMALL END OF A TRUNCATED CONE.

$$\theta_1 = 0$$
$$\bar{w}_1 = 0 \qquad (18)$$

m Eq. (9) and (18),

$$M_1 = \frac{b_1 f_1 - c_1 e_1}{a_1 e_1 - b_1 d_1}\, p$$
$$H_1 = \frac{c_1 d_1 - a_1 f_1}{a_1 e_1 - b_1 d_1}\, p \qquad (19)$$

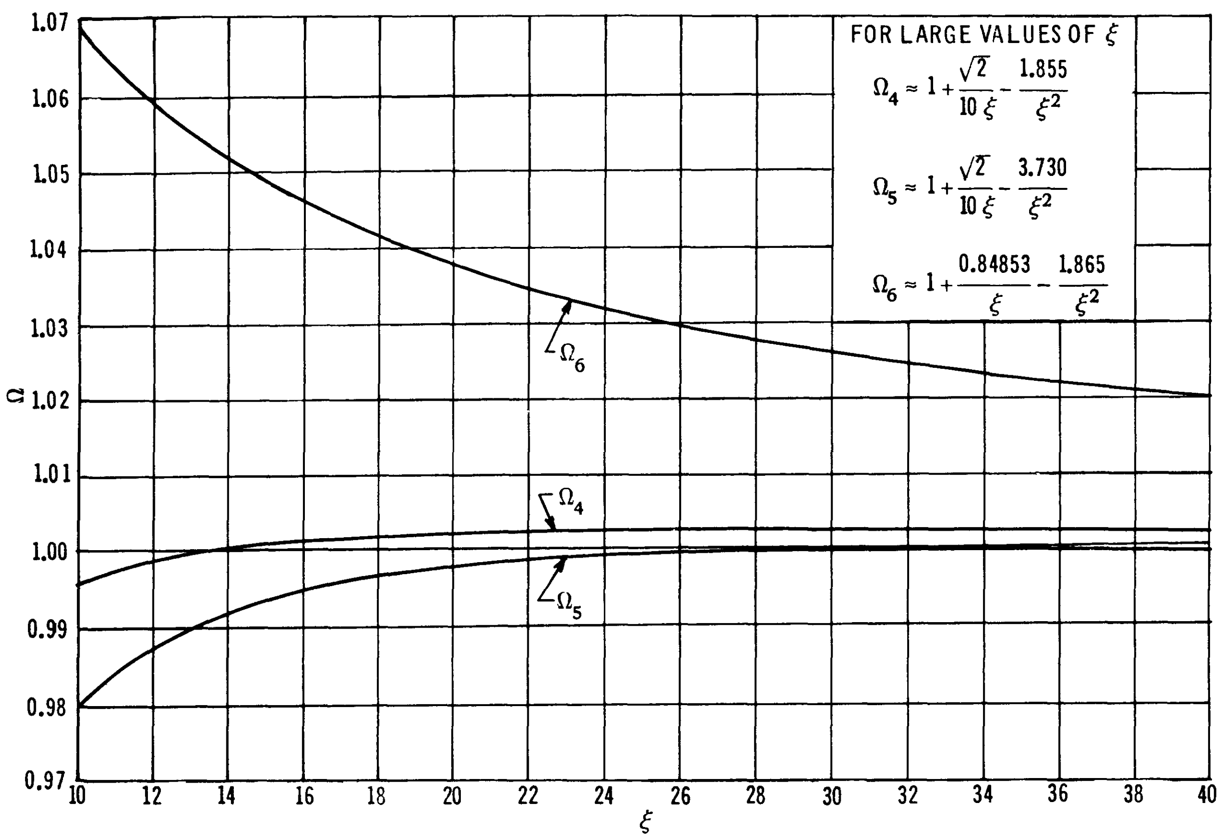
compute stresses, these may be substituted directly Equation (17). However, since errors can now be readi- determined for all edge coefficients, use of simple proximate forms for M_1 and H_1 as given in reference], are often justified. These are obtained by setting , Ω_2 and Ω_3 equal to unity and also neglecting the rd term in the expressions for c_1 and f_1 of Equation). These operations yield:

$$M_1 = \frac{(1-\nu/2)\,R_1 h}{\sqrt{12(1-\nu^2)}\cos\alpha}\, p - \left(\frac{3\sqrt{2}}{2\,[12(1-\nu^2)]^{3/4}}\sqrt{\frac{R_1 h^3 \tan^2\alpha}{\cos\alpha}}\right) p$$

$$H_1 = \frac{R_1 \tan\alpha}{2}\, p - \frac{3h\tan\alpha}{2\sqrt{12(1-\nu^2)}\cos\alpha}\, p + \left(\frac{1-\nu/2}{\sqrt[4]{12(1-\nu^2)}}\sqrt{\frac{2hR_1}{\cos^3\alpha}}\right) p \qquad (20)$$

By a similar procedure, approximate equations for M_2 and H_2 result:

$$M_2 = \frac{(1-\nu/2)\,R_2 h}{\sqrt{12(1-\nu^2)}\cos\alpha}\, p + \left(\frac{3\sqrt{2}}{2\,[12(1-\nu^2)]^{3/4}}\sqrt{\frac{R_2 h^3 \tan^2\alpha}{\cos\alpha}}\right) p$$



URE 4. FUNCTIONS USED FOR COMPUTING EDGE ROTATION AND DISPLACEMENT COEFFICIENTS AT LARGE END OF A TRUNCATED CONE

$$H_2 = \frac{R_2 \tan \alpha}{2} p + \frac{3h \tan \alpha}{2\sqrt{12(1-\nu^2)} \cos \alpha} p + \left(\frac{1-\nu/2}{\sqrt[4]{12(1-\nu^2)}} \sqrt{\frac{2hR_2}{\cos^3 \alpha}} \right) p \tag{21}$$

2. *Two Intersecting Cones, Subject to Normal Pressure.* The more general case of intersection of two cones subject to uniform external pressure is treated as follows: Two types of axially symmetric intersections are possible; one of these is shown in Figure 5. The requirements of compatibility and static equilibrium are

$$\begin{aligned} \overline{w}_{11} &= \overline{w}_{12} \\ \theta_{11} &= -\theta_{12} \\ H_{11} &= -H_{12} \\ M_{11} &= M_{12} \end{aligned} \tag{22}$$

In cases where two or more shells are involved, two subscripts are used. The first subscript refers to the appropriate end of the frustrum and the second subscript refers to the shells which have been designated 1 and 2 in Figure 5.

The first of Eq. (22) may be written:

$$(d_{11}-d_{12}) M_{11} + (e_{11}+e_{12}) H_{11} + (f_{11}-f_{12}) p = 0 \tag{23}$$

similarly, from the second of Eq. (22)

$$(a_{11}+a_{12}) M_{11} + (b_{11}-b_{12}) H_{11} + (c_{11}+c_{12}) p = 0 \tag{24}$$

For compactness define:

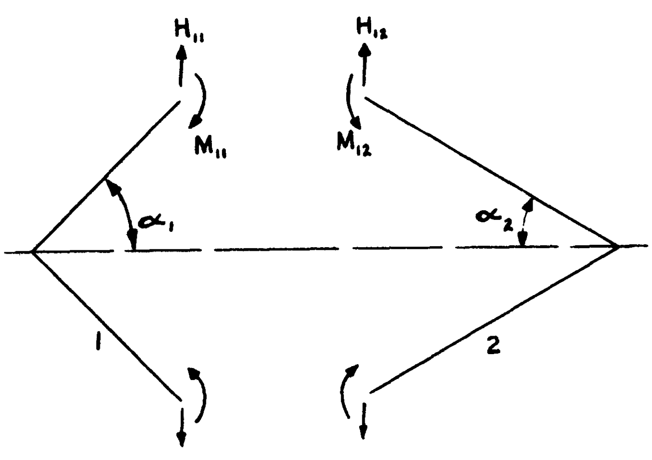


FIGURE 5. SCHEMATIC DIAGRAM SHOWING THE INTERSECTION OF TWO CONES

$$\begin{aligned} &\overline{a}_1 = a_{11} + a_{12} \quad , \quad \overline{b}_1 = b_{11} - b_{12} \\ &\overline{c}_1 = c_{11} + c_{12} \\ &\overline{d}_1 = d_{11} - d_{12} \quad , \quad \overline{f}_1 = f_{11} - f_{12} \quad , \\ &\overline{e}_1 = e_{11} + e_{12} \end{aligned} \tag{25}$$

Then the edge moment and radial force are:

$$\begin{aligned} M_{11} &= \frac{\overline{f}_1 \overline{b}_1 - \overline{c}_1 \overline{e}_1}{\overline{a}_1 \overline{e}_1 - \overline{b}_1 \overline{d}_1} p \\ H_{11} &= \frac{\overline{c}_1 \overline{d}_1 - \overline{a}_1 \overline{f}_1}{\overline{a}_1 \overline{e}_1 - \overline{b}_1 \overline{d}_1} p \end{aligned} \tag{26}$$

3. *Cylinder-Cone Intersection.* The case of a cylinder-conical intersection is developed by considering a cylinder as a limiting geometry of one of the cones, i.e., $\alpha_2 = 0^\circ$. Again, determination of discontinuity shears and moments can be facilitated by use of the approximate forms, wherever the Ω-functions indicate acceptably small error in results. Here, if the Ω terms are again set equal to unity and the approximate form of c_1 and f_1 are employed and, furthermore, if the thickness of the shells is the same, $b_1 = b_2$; then $\overline{b}_1 = \overline{d}_1 = 0$.

$$M_{11} = \frac{-c_{11} p}{a_{11}+a_{12}} = \frac{\sqrt{R_1 h} \tan \alpha_1 p}{\sqrt{8} \sqrt[4]{12(1-\nu^2)} \left[1+\sqrt{\cos \alpha_1}\right]} \times \left[R_1 \sqrt{\cos \alpha_1} - \frac{3h}{\sqrt{12(1-\nu^2) \cos \alpha_1}} \right]$$

$$H_{11} = \frac{(-f_{11}+f_{12})}{e_{11}+e_{12}} p = - \frac{R_1 \sin \alpha_1}{2(\cos \alpha_1 + \sqrt{\cos \alpha_1})} p + \frac{(1-\nu/2)\sqrt{R_1 h}}{\sqrt{2} \sqrt[4]{12(1-\nu^2)}} \left(\frac{1-\sqrt{\cos \alpha_1}}{\cos \alpha_1} \right) p \tag{27}$$

Following the same set of assumptions, solutions for a reinforced cone-cylinder intersection as given in reference [15] may be similarly employed.

Discussion

From the development in this paper, exact expressions have been developed for edge coefficients of a truncated or complete conical shell. These appear in the form of simple algebraic expressions combined with special Ω functions which are given for all values of the geometric parameter ξ greater than 10. Because of their simplicity, these edge coefficients are preferred to results of approximate solution for computation of discontinuity shears and moments at shell intersections. These special functions can then be employed to determine for which geometries the approximate expressions for stress given in earlier papers may then be satisfactorily employed. These functions appear to confirm estimates of error given previously [15].

From Figure 2-4 these are

ξ	Maximum Error of Ω Function at Small End, %	Maximum Error of Ω Function at Large End, %
10	6.9	10.5
20	3.8	4.8
30	2.6	3.1
40	2.0	2.3
60	1.4	1.5
100	.8	.9
170	.2	.5

From this tabulation, it would appear that the approximate solutions are sufficiently accurate for all but very thick or very flat cones.

ACKNOWLEDGMENTS

Assistance in numerical computation was ably provided by M. R. Overby and in checking equations, by M. F. Borg.

The authors also appreciate the helpful comments of W. R. Burrows of Standard Oil Company of Indiana which stimulated development of this analysis.

The results presented in this paper were obtained in the course of research conducted at the David Taylor Model Basin to study the stresses near intersections of conical and cylindrical shells.

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APPENDIX A

The constants of integration which appear in Eq. (7) may be expressed in terms of the boundary shears and moments as follows:

$$C_1 = [Q_1 x_1 b \tan\alpha\,(\xi_1\, ber_2'\,\xi_1 + 2\,\nu\, ber_2\,\xi_1) + 2\,M_1\,U^2\,x_1\, bei_2\,\xi_1]\,[b\,(C + 2\,\nu\,G)]^{-1}$$

$$C_2 = [Q_1 x_1 b \tan\alpha\,(\xi_1\, bei_2'\,\xi_1 + 2\,\nu\, bei_2\,\xi_1) - 2\,M_1\,U^2\,x_1\, ber_2\,\xi_1]\,[b\,(C + 2\,\nu\,G)]^{-1}$$

$$C_3 = [Q_2 x_2 b \tan\alpha\,(\xi_2\, ker_2'\,\xi_2 + 2\,\nu\, ker_2\,\xi_2) + 2\,M_2\,U^2\,x_2\, kei_2\,\xi_2]\,[b\,(C_K + 2\,\nu\,G_K)]^{-1}$$

$$C_4 = [Q_2 x_2 b \tan\alpha\,(\xi_2\, kei_2'\,\xi_2 + 2\,\nu\, kei_2\,\xi_2) - 2\,M_2\,U^2\,x_2\, ker_2\,\xi_2]\,[b\,(C_K + 2\,\nu\,G_K)]^{-1}$$

APPENDIX B

The stress equations derived in Reference [15] are as follows:*

$$\sigma_x = \frac{Q_x \tan\alpha}{h} - \frac{px}{2h}\tan\alpha \pm \frac{6M_x}{h^2}$$

$$\sigma_\phi = \frac{-E\,\overline{w}_a}{R} + \nu\sigma_x$$

The following values are derived in Reference [15]:

$$M_x = \frac{\eta}{\eta_0 \xi}\left\{\left[2\sqrt{2}\,x_0 H\cos\alpha + \sqrt{2}\,x_0^2\,p\sin^2\alpha - \sqrt{x_0}\,\lambda\left(2 - \frac{4\sqrt{2}}{\xi_0}\right)M\right]\sin\frac{(\xi_0-\xi)}{\sqrt{2}}\right.$$

$$\left. - 2\sqrt{x_0}\,\lambda M\cos\left(\frac{\xi_0-\xi}{\sqrt{2}}\right)\right\}$$

$$Q_x = \frac{\eta}{\eta_0\xi^2}\left\{\left[2\lambda^2\left(x_0^2\,p\sin^2\alpha + 2x_0 H\cos\alpha\right)\right]\left[\cos\left(\frac{\xi_0-\xi}{\sqrt{2}}\right) + \left(\frac{2\sqrt{2}}{\xi} - 1\right)\sin\left(\frac{\xi_0-\xi}{\sqrt{2}}\right)\right]\right.$$

$$\left. + 4\sqrt{2}\,\sqrt{x_0}\,\lambda^3 M\left[\left(1 - \frac{\sqrt{2}}{\xi_0} - \frac{\sqrt{2}}{\xi} + \frac{4}{\xi_0\xi}\right)\sin\left(\frac{\xi_0-\xi}{\sqrt{2}}\right) + \left(\frac{\sqrt{2}}{\xi_0} - \frac{\sqrt{2}}{\xi}\right)\cos\left(\frac{\xi_0-\xi}{\sqrt{2}}\right)\right]\right\}$$

$$\overline{w}_a = \frac{\left(1-\frac{\nu}{2}\right)R^2 p}{Eh\cos\alpha} - \frac{\sqrt{2x}}{D\lambda^3}\left(\frac{\eta}{\eta_0}\cos\alpha\right)\left\{\left[x_0 H\cos\alpha + \frac{x_0^2\sin^2\alpha}{2}p\right.\right.$$

$$\left.\left. + \sqrt{x_0}\,\lambda\left(\frac{2}{\xi_0} - \frac{\sqrt{2}}{2}\right)M\right]\cos\left(\frac{\xi_0-\xi}{\sqrt{2}}\right) + \frac{\sqrt{2}}{2}\sqrt{x_0}\,\lambda M\sin\left(\frac{\xi_0-\xi}{\sqrt{2}}\right)\right\}$$

$$\eta = \frac{e^{\xi/\sqrt{2}}}{\sqrt{2\pi\xi}}$$

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*Compare Equation (17) of this paper.

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