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A THEORETICAL ANALYSIS OF THE DYNAMICAL STABILITY OF TOWED MODELS

by

W.H. Roach

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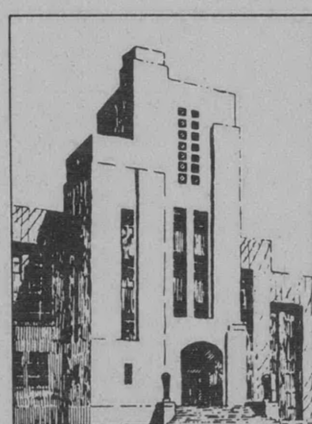

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NOTATION

a	Linear acceleration
a_1, a_2	Distance from center of gravity of model to struts
F, F_1, F_2	Strut force on model
h	Distance from top of strut to waterline
I	Mass moment of inertia of model
K	Lateral spring constant of strut
K_T	Torsional spring constant of strut
l	Length of strut
M	Hydrodynamic moment on model
M_0	Strut restoring moment (total)
M_s	Strut moment at any point
m	Mass of model
R	Reaction in y' direction at top of strut
S	Shear force at any point on strut
s	Distance between struts in cases of two struts or length of model in cases of one strut
T	Strut twisting moment on body
u	Velocity in x' direction
$\dot{u}$	Acceleration in x' direction
u_0	Velocity in x_0 direction
$\dot{u}_0$	Acceleration in x_0 direction
v	Velocity in y' direction
$\dot{v}$	Acceleration in y' direction
v_0	Velocity in y_0 direction
$\dot{v}_0$	Acceleration in y_0 direction
V	Absolute velocity vector
x', y', z'	Axes in moving coordinate system (body axis)
x_0, y_0, z_0	Axes in fixed (basin) coordinate system

Y	Hydrodynamic force on model
Y_S	Hydrodynamic force on strut per unit length
y	Displacement in fixed system of center of gravity of moving system
$\dot{y}$	v_0
$\ddot{y}$	$\dot{v}_0$
y_1, y_2	Displacement in fixed system of ends of struts
$\dot{y}_1, \dot{y}_2$	Velocity in fixed system of ends of struts
$\ddot{y}_1, \ddot{y}_2$	Acceleration in fixed system of ends of struts
α	Angular acceleration
θ	Angle between fixed coordinate system and moving coordinate system
$\dot{\theta}$	Angular velocity of moving coordinate system
$\ddot{\theta}$	Angular acceleration of moving coordinate system
ρ	Mass of water
ρ_S	Strut mass per unit length (variable)
ρ_S'	Virtual mass of strut/unit length (variable)

A THEORETICAL ANALYSIS OF THE DYNAMICAL STABILITY OF TOWED MODELS

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ABSTRACT

The equations of motion of an underwater model with several different strut combinations are developed. All forces including strut and hydrodynamic forces are enumerated, and the criteria for determining dynamical stability are indicated. The methods for obtaining all the constants necessary for a stability determination are outlined. Recommendations are made for carrying out a test program to check the validity of the theory involved.

INTRODUCTION

The purpose of this report is to present a theoretical treatment of the dynamic stability of towed underwater models. It has often been assumed that the necessary condition for stability is met by designing the towing struts strong enough to achieve static stability. However, recent opinion is that static stability does not assure dynamic stability, particularly when the model is towed at high speeds. This study includes all of the terms which are pertinent to the motion, including strut effects. Several model and strut combinations are analyzed to determine their criteria for stability. In each case the model is assumed to be neutrally buoyant.

Case I is a most general case involving a model supported by two struts with the center of gravity of the model arbitrarily located between the two struts.

Case II involves a model supported by two struts with the center of gravity of the model at the forward strut.

Case III involves a model supported by one strut fastened at the center of gravity of the model.

The method of attack is to consider the model as a free body under the influence of external hydrodynamic and strut forces. The dynamical stability of the model and strut combination is determined by assuming that its steady-state motion is subjected to a disturbance of sufficiently small amplitude to allow the use of linear theory. This method of disturbance has the advantage of leading to linear differential equations which can be solved easily and from which the stability of the model and strut combination can be determined.

CASE I

STABILITY DETERMINATION

The most general case involves a model supported by two struts with the center of gravity of the model located any place between the two struts (see Figure 1).

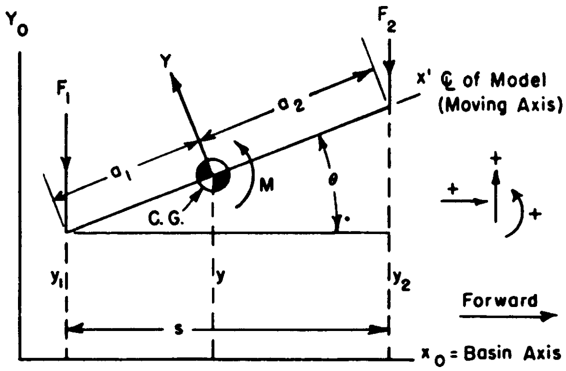

Figure 1 - Model Suspended on Two Struts with Center of Gravity of the Model any Place between the Two Struts

Forces on Body	Y	= Hydrodynamic Force on Model
	F_1, F_2	= Strut Forces on Model
Moment on Body	M	= Hydrodynamic Moment on Model

The assumption is made that the torsion of struts is negligible so that only forces F_1 and F_2 are applied to model.

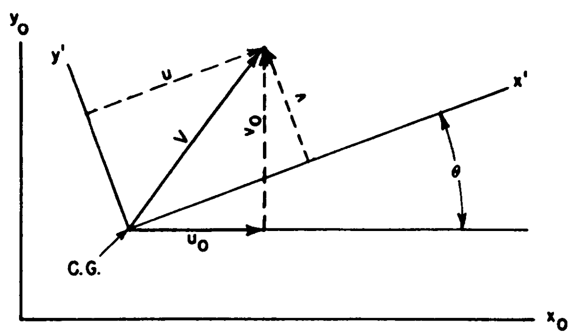

Figure 2 - Velocity Diagram Showing Relation between Fixed Coordinate System (Basin Axes) x_0, y_0 and Moving Coordinate System (Model Axes) x', y'

From Figure 2 we obtain

$$v = v_0 \cos\theta - u_0 \sin\theta$$

$$\dot{v} = \dot{v}_0 \cos\theta - v_0\dot{\theta}\sin\theta - \dot{u}_0 \sin\theta - u_0\dot{\theta}\cos\theta$$

$$\theta = \frac{y_2 - y_1}{s} \qquad y = \frac{a_1 y_2 + a_2 y_1}{s}$$

$$\dot{\theta} = \frac{\dot{y}_2 - \dot{y}_1}{s} \qquad \dot{y} = \frac{a_1 \dot{y}_2 + a_2 \dot{y}_1}{s} = v_0$$

$$\ddot{\theta} = \frac{\ddot{y}_2 - \ddot{y}_1}{s} \qquad \ddot{y} = \frac{a_1 \ddot{y}_2 + a_2 \ddot{y}_1}{s} = \dot{v}_0$$

Since θ = small angles, we say $a_1 + a_2 = s$

Applying $\Sigma \dot{F} = ma$ and $\Sigma T = I\ddot{\theta}$

$$Y \cos\theta - F_1 - F_2 = m \frac{a_1 \ddot{y}_2 + a_2 \ddot{y}_1}{s} \qquad [1]$$

$$M - (F_2 a_2 - F_1 a_1) \cos\theta = I\ddot{\theta} \qquad [2]$$

Evaluating Strut Effect

The presentation which follows contains several assumptions:

1. The deflection curve of the strut is a straight line.
2. $M_0 = K y_l l$ where K = spring constant of the strut.
3. The hydrodynamic twisting moment applied to the strut produces a neglibible change in the angle of attack.
4. The upper end of the strut is 100 percent fixed.
5. The angle of twist of the strut is a linear function of z.
6. There is no flow interference between struts.

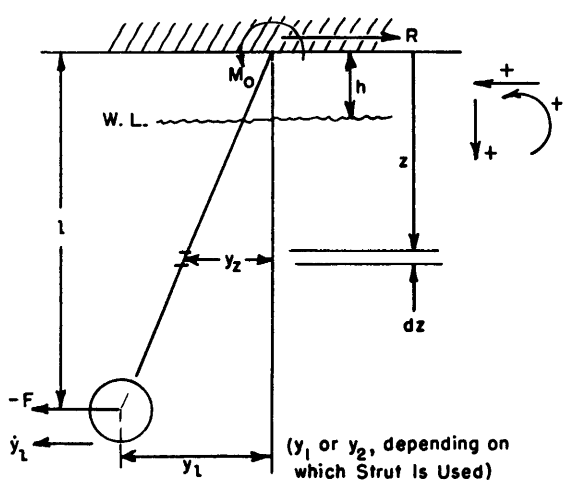

Figure 3 - Evaluating Strut Effect

Forces shown are those applied to the strut.

Consider the element of strut of length dz (see Figure 4).

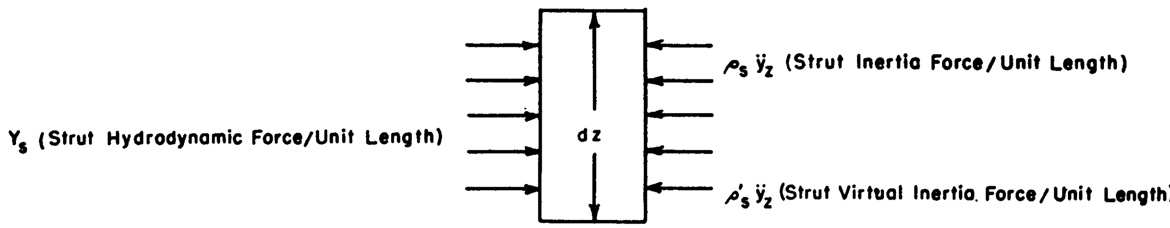

Figure 4 - Element of Strut of Length dz

Applying $\Sigma H = 0$ to the strut

$$- F - R - \int_h^l Y_s \, dz + \int_0^l \rho_s \ddot{y}_z dz + \int_h^l \rho_s' \ddot{y}_z dz = 0$$

Applying $\Sigma M = 0$ about the bottom of strut

$$M_0 - Rl - \int_h^l Y_s \, dz(l - z) + \int_0^l \rho_s \ddot{y}_z dz(l - z) + \int_h^l \rho_s' \ddot{y}_z dz(l - z) = 0$$

$$R = \frac{M_0}{l} - \frac{1}{l} \int_h^l Y_s \, dz(l - z) + \frac{1}{l} \int_0^l \rho_s \ddot{y}_z dz(l - z) + \frac{1}{l} \int_h^l \rho_s' \ddot{y}_z dz(l - z)$$

$$\begin{aligned} F = -\frac{1}{l} \Big[& K y_l \, l - \int_h^l Y_s dz(l - z) + \int_0^l \rho_s \ddot{y}_z dz(l - z) + \int_h^l \rho_s' \ddot{y}_z dz(l - z) \\ & + l \int_h^l Y_s dz - l \int_0^l \rho_s \ddot{y}_z dz - l \int_h^l \rho_s' \ddot{y}_z dz \Big] \end{aligned} \qquad [3]$$

<u>Evaluating Y_s</u>

From Figures 5 and 6 it is evident that

$$Y_s = C_L \, \frac{1}{2} \rho u_0^2 \, f(z)$$

Therefore

$$\begin{aligned} - F = {} & K y_l - \frac{1}{l} \int_h^l C_L \, \frac{1}{2} \rho u_0^2 \, f(z) \, dz(l - z) \\ & + \frac{1}{l} \int_0^l \rho_s \ddot{y}_z dz(l - z) + \frac{1}{l} \int_h^l \rho_s' \ddot{y}_z dz(l - z) \\ & + \int_h^l C_L \, \frac{1}{2} \rho u_0^2 \, f(z) \, dz \\ & - \int_0^l \rho_s \ddot{y}_z dz - \int_h^l \rho_s' \ddot{y}_z dz \end{aligned}$$

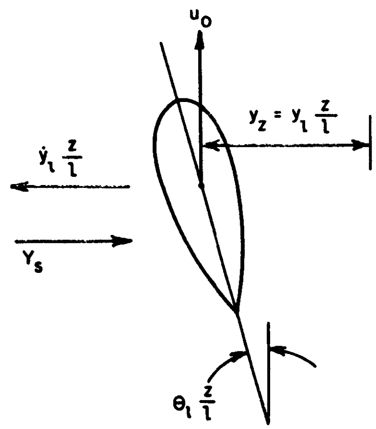

Figure 5 - Section Through the Strut

Let width = f(z)
Area of cross section = $\phi(z)$

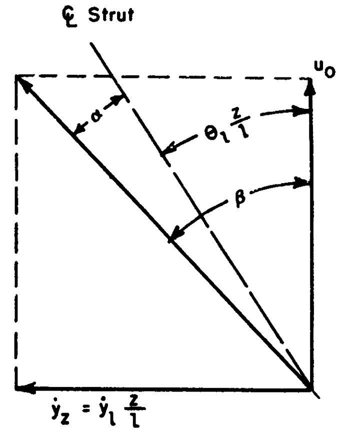

Figure 6 - Velocity Diagram Showing Angle of Attack

$$\alpha = \beta - \theta_l \times \frac{z}{l} = \frac{\dot{y}_l \times z}{u_0 l} - \theta_l \times \frac{z}{l}$$

θ = linear function of y_1 and y_2

Lift coefficient $C_L = \psi(\alpha) = \psi(\dot{y}_l, y_1, y_2)$ where function is linear in $\dot{y}_l, y_1$, and y_2.

Now

$$-\frac{1}{l}\int_h^l C_L \; \frac{1}{2}\rho u_0^2 \; f(z)\; dz(l - z) + \int_h^l C_L \; \frac{1}{2}\rho u_0^2 \; f(z)\; dz$$

$$= \text{a linear function of } \dot{y}_l,\; y_1 \text{ and } y_2$$

$$= A\dot{y}_l + Hy_1 + Jy_2$$

Also

$$\frac{1}{l}\int_0^l \rho_s \ddot{y}_z dz(l - z) + \frac{1}{l}\int_h^l \rho_s' \ddot{y}_z dz(l - z) - \int_0^l \rho_s \ddot{y}_z dz - \int_h^l \rho_s' \ddot{y}_z dz$$

$$= \text{a linear function of } \ddot{y}_l$$

$$= B\ddot{y}_l$$

Moreover, Ky_l = a linear function of y_l. Therefore,

$$F = B\ddot{y}_l + A\dot{y}_l - Ky_l + Hy_1 + Jy_2$$

$$F_1 = B_1\ddot{y}_1 + A_1\dot{y}_1 + (H_1 - K_1)y_1 + J_1 y_2 \qquad [4]$$

$$F_2 = B_2\ddot{y}_2 + A_2\dot{y}_2 + (J_2 - K_2)y_2 + H_2 y_1$$

<u>Evaluating the Hydrodynamic Force on Model</u>

$$Y = Y(u, v, \dot{u}, \dot{v}, \dot{\theta}, \ddot{\theta})$$

By a Taylor expansion and linearization and by expressing $\frac{\partial Y}{\partial u}$ as Y_u, etc., we get

$$Y = Y_u u + Y_v v + Y_{\dot{u}}\dot{u} + Y_{\dot{v}}\dot{v} + Y_{\dot{\theta}}\dot{\theta} + Y_{\ddot{\theta}}\ddot{\theta}$$

In evaluating these terms note that by symmetry

$Y_u u = 0$, $Y_{\dot{u}}\dot{u} = 0$, and $Y_{\ddot{\theta}}\ddot{\theta} = 0$ or is inconsequentially small.

$$Y_v v = Y_v \left[\frac{a_1\dot{y}_2 + a_2\dot{y}_1}{s}\cos\theta - u_0 \sin\theta\right]$$

$$Y_{\dot{\theta}}\dot{\theta} = Y_{\dot{\theta}}\frac{\dot{y}_2 - \dot{y}_1}{s}$$

Considering $Y_{\dot{v}}$ intrinsically positive, the virtual mass term equals

$$-Y_{\dot{v}}\dot{v} = -Y_{\dot{v}}\left[\frac{a_1\ddot{y}_2 + a_2\ddot{y}_1}{s}\cos\theta - \left(\frac{a_1\dot{y}_2 + a_2\dot{y}_1}{s}\right)\left(\frac{\dot{y}_2 - \dot{y}_1}{s}\right)\sin\theta - \dot{u}_0 \sin\theta - u_0\left(\frac{\dot{y}_2 - \dot{y}_1}{s}\right)\cos\theta\right]$$

Since θ = small angles, let $\cos\theta = 1$, and $\sin\theta = \theta = \frac{y_2 - y_1}{s}$.
By substitution in Equation [1]

$$Y_v\left[\frac{a_1\dot{y}_2 + a_2\dot{y}_1}{s} - u_0\left(\frac{y_2 - y_1}{s}\right)\right] + Y_{\dot{\theta}}\left(\frac{\dot{y}_2 - \dot{y}_1}{s}\right) - Y_{\dot{v}}\left[\frac{a_1\ddot{y}_2 + a_2\ddot{y}_1}{s} - \left(\frac{a_1\dot{y}_2 + a_2\dot{y}_1}{s}\right)\left(\frac{\dot{y}_2 - \dot{y}_1}{s}\right)\left(\frac{y_2 - y_1}{s}\right) - \dot{u}_0\left(\frac{y_2 - y_1}{s}\right) - u_0\left(\frac{\dot{y}_2 - \dot{y}_1}{s}\right)\right]$$

$$- (H_1 - K_1)y_1 - A_1\dot{y}_1 - B_1\ddot{y}_1 - J_1 y_2 - (J_2 - K_2)y_2 - A_2\dot{y}_2 - B_2\ddot{y}_2 - H_2 y_1$$
$$= m\left(\frac{a_1\ddot{y}_2 + a_2\ddot{y}_1}{s}\right)$$

Several operations are performed to simplify the equation. The equation is rendered nondimensional by means of the relations shown in Appendix 1. Although the coefficients appear unchanged in the equations which follow, it is to be understood that they have been rendered nondimensional. The nonlinear term is dropped because it is a higher order term and small in comparison to the other terms. Rewriting, we have

$$Y_{\dot{v}}\ [\dot{v}_0 - v_0\dot{\theta}\theta - \dot{u}_0\theta - u_0\dot{\theta}]$$

We can form the ratio

$$\frac{v_0\dot{\theta}\theta}{u_0\dot{\theta}} = \frac{v_0\theta}{u_0}$$

Now the limit $\frac{v_0\theta}{u_0} = 0$ as $v_0 \rightarrow 0$, therefore, the numerator is of higher order than the denominator. Since the formation of the ratio $\frac{\dot{v}_0}{u_0\theta}$ does not justify drawing any limit conclusions, neither term can be dropped. The term containing $\dot{u}_0$ is dropped, however, since the condition being investigated concerns towing the model forward at constant velocity, i.e., $\dot{u}_0 = 0$. After these operations are performed, terms are collected and the following equation is obtained:

$$\left[\frac{-Y_{\dot{v}}a_2}{s} - B_1 - \frac{ma_2}{s}\right]\ddot{y}_1 + \left[\frac{-Y_{\dot{v}}a_1}{s} - B_2 - \frac{ma_1}{s}\right]\ddot{y}_2 + \left[\frac{Y_v a_2}{s} - \frac{Y_{\dot{\theta}}}{s} - \frac{Y_{\dot{v}}}{s} - A_1\right]\dot{y}_1$$

$$+\left[\frac{Y_v a_1}{s} + \frac{Y_{\dot{\theta}}}{s} + \frac{Y_{\dot{v}}}{s} - A_2\right]\dot{y}_2 + \left[\frac{Y_v}{s} - (H_1 - K_1) - H_2\right]y_1$$

$$+\left[\frac{-\dot{Y}_v}{s} - (J_2 - K_2) - J_1\right]y_2 = 0$$

This may be written

$$C_1\ddot{y}_1 + C_2\ddot{y}_2 + C_3\dot{y}_1 + C_4\dot{y}_2 + C_5y_1 + C_6y_2 = 0 \qquad [5]$$

In evaluating the hydrodynamic moment on the model

$$M = M(u, v, \dot{u}, \dot{v}, \dot{\theta}, \ddot{\theta})$$

By Taylor expansion

$$M = M_u u + M_v v + M_{\dot{u}}\dot{u} + M_{\dot{v}}\dot{v} + M_{\dot{\theta}}\dot{\theta} + M_{\ddot{\theta}}\ddot{\theta}$$

By symmetry

$M_u u = 0$, $M_{\dot{u}}\dot{u} = 0$, and $M_{\dot{v}}\dot{v} = 0$, or is inconsequentially small.

$$M_V v = M_V\left[\frac{a_1\dot{y}_2 + a_2\dot{y}_1}{s}\cos\theta - u_0 \sin\theta\right]$$

$$M_{\dot{\theta}}\dot{\theta} = M_{\dot{\theta}}\frac{\dot{y}_2 - \dot{y}_1}{s}$$

Considering $M_{\ddot{\theta}}$ intrinsically positive, the virtual moment of inertia term equals

$$-M_{\ddot{\theta}}\ddot{\theta} = -M_{\ddot{\theta}}\frac{\ddot{y}_2 - \ddot{y}_1}{s}$$

Let $\cos\theta = 1$, and $\sin\theta = \theta = \frac{y_2 - y_1}{s}$. By substitution in Equation [2]

$$M_V\left[\frac{a_1\dot{y}_2 + a_2\dot{y}_1}{s} - u_0\left(\frac{y_2 - y_1}{s}\right)\right] + M_{\dot{\theta}}\frac{\dot{y}_2 - \dot{y}_1}{s} - M_{\ddot{\theta}}\frac{\ddot{y}_2 - \ddot{y}_1}{s}$$

$$- H_2 y_1 a_2 - (J_2 - K_2)a_2 y_2 - A_2 a_2 \dot{y}_2 - B_2 a_2 \ddot{y}_2 + (H_1 - K_1)a_1 y_1 + A_1 a_1 \dot{y}_1$$

$$+ B_1 a_1 \ddot{y}_1 + J_1 y_2 a_1 = I\,\frac{\ddot{y}_2 - \ddot{y}_1}{s}$$

Putting into dimensionless form by relations of Appendix 1 and collecting terms gives

$$\left[\frac{M_{\ddot{\theta}}}{s} + B_1 a_1 + \frac{I}{s}\right]\ddot{y}_1 + \left[\frac{-M_{\ddot{\theta}}}{s} - B_2 a_2 - \frac{I}{s}\right]\ddot{y}_2 + \left[\frac{M_V a_2}{s} - \frac{M_{\dot{\theta}}}{s} + A_1 a_1\right]\dot{y}_1$$

$$+\left[\frac{M_V a_1}{s} + \frac{M_{\dot{\theta}}}{s} - A_2 a_2\right]\dot{y}_2 + \left[\frac{M_V}{s} + (H_1 - K_1)a_1 - H_2 a_2\right]y_1$$

$$+\left[\frac{-M_V}{s} - (J_2 - K_2)a_2 + J_1 a_1\right]y_2 = 0$$

This may be written

$$D_1\ddot{y}_1 + D_2\ddot{y}_2 + D_3\dot{y}_1 + D_4\dot{y}_2 + D_5 y_1 + D_6 y_2 = 0 \qquad [6]$$

A solution to the differential equations [5] and [6] is

$y_1 = Ee^{\sigma t}$	$y_2 = Fe^{\sigma t}$
$\dot{y}_1 = E\sigma e^{\sigma t}$	$\dot{y}_2 = F\sigma e^{\sigma t}$
$\ddot{y}_1 = E\sigma^2 e^{\sigma t}$	$\ddot{y}_2 = F\sigma^2 e^{\sigma t}$

Substituting the assumed solutions into [5] and [6], we get

$$C_1E_n\sigma_n^2e^{\sigma t} + C_2F_n\sigma_n^2e^{\sigma} + C_3E_n\sigma_ne^{\sigma t} + C_4F_n\sigma_ne^{\sigma t}$$

[5a]

$$+ C_5E_ne^{\sigma t} + C_6F_ne^{\sigma t} = 0$$

$$D_1E_n\sigma_n^2e^{\sigma t} + D_2F_n\sigma_n^2e^{\sigma t} + D_3E_n\sigma_ne^{\sigma t} + D_4F_n\sigma_ne^{\sigma t}$$

[6a]

$$+ D_5E_ne^{\sigma t} + D_6F_ne^{\sigma t} = 0$$

where n = 1 to 4, i.e., 1,2,3, or 4.

$$E_n(C_1\sigma_n^2 + C_3\sigma_n + C_5) + F_n(C_2\sigma_n^2 + C_4\sigma_n + C_6) = 0 \qquad [5b]$$

$$E_n(D_1\sigma_n^2 + D_3\sigma_n + D_5) + F_n(D_2\sigma_n^2 + D_4\sigma_n + D_6) = 0 \qquad [6b]$$

These equations are compatible if

$$(C_1\sigma_n^2+C_3\sigma_n+C_5)(D_2\sigma_n^2+D_4\sigma_n+D_6) - (D_1\sigma_n^2+D_3\sigma_n+D_5)(C_2\sigma_n^2+C_4\sigma_n+C_6) = 0 \qquad [7]$$

This fourth-degree equation yields the four roots $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. If complex, the roots of the characteristic equation will have the form

$$\sigma_1 = \alpha_1 + j\beta_1$$

$$\sigma_2 = \alpha_1 - j\beta_1$$

$$\sigma_3 = \alpha_2 + \beta_2$$

$$\sigma_4 = \alpha_2 - \beta_2$$

or

$$y_1 = E_1e^{\sigma_1 t} + E_2e^{\sigma_2 t} + E_3e^{\sigma_3 t} + E_4e^{\sigma_4 t}$$

$$y_2 = F_1e^{\sigma_1 t} + F_2e^{\sigma_2 t} + F_3e^{\sigma_3 t} + F_4e^{\sigma_4 t}$$

From the above relationships it would appear that there are eight arbitrary coefficients necessary to define the motion in a given case. Physically, however, only four boundary conditions exist, namely, the initial values of y_1 and $\dot{y}_1$ and the initial values of y_2 and $\dot{y}_2$. Therefore, a relationship must exist between the coefficients such that only four are actually arbitrary. Since Equations [5b] and [6b] must be valid for all four values of σ_n, a relation may, in fact, be obtained between E_n and F_n, thus reducing the number of arbitrary constants from eight to four. That is,

$$\frac{E_n}{F_n} = \frac{-(C_2\sigma_n^2 + C_4\sigma_n + C_6)}{C_1\sigma_n^2 + C_3\sigma_n + C_5} = \frac{-(D_2\sigma_n^2 + D_4\sigma_n + D_6)}{D_1\sigma_n^2 + D_3\sigma_n + D_5} = R_n$$

Our equations of motion may be written in the form

$$y_1 = R_1F_1e^{\sigma_1 t} + R_2F_2e^{\sigma_2 t} + R_3F_3e^{\sigma_3 t} + R_4F_4e^{\sigma_4 t}$$

$$y_2 = F_1e^{\sigma_1 t} + F_2e^{\sigma_2 t} + F_3e^{\sigma_3 t} + F_4e^{\sigma_4 t}$$

The arbitrary constants are now only F_1, F_2, F_3, and F_4.

Expanding and collecting terms in Equation [7], we get

$$\sigma^4 + \left[\frac{C_3D_2 + C_1D_4 - D_3C_2 - D_1C_4}{C_1D_2 - D_1C_2}\right]\sigma^3$$

$$+\left[\frac{C_5D_2 + C_3D_4 + C_1D_6 - D_5C_2 - D_3C_4 - D_1C_6}{C_1D_2 - D_1C_2}\right]\sigma^2$$

$$+\left[\frac{C_5D_4 + C_3D_6 - D_5C_4 - D_3C_6}{C_1D_2 - D_1C_2}\right]\sigma + \left[\frac{C_5D_6 - D_5C_6}{C_1D_2 - D_1C_2}\right] = 0$$

This is called the stability equation and may be written in the form

$$\sigma^4 + P\sigma^3 + Q\sigma^2 + R\sigma + S = 0$$

For stability $P, Q, R, S > 0$ and $P^2S - PQR + R^2 < 0$

($P^2S < PQR - R^2$ known as Routh's discriminant)

CASE II

This case concerns a model which is supported by two struts with the center of gravity at the forward strut and is identical with Case I with $a_2 = 0$ and $a_1 = s$. Similarly, if the center of gravity is located at aft strut, then $a_1 = 0$ and $a_2 = s$.

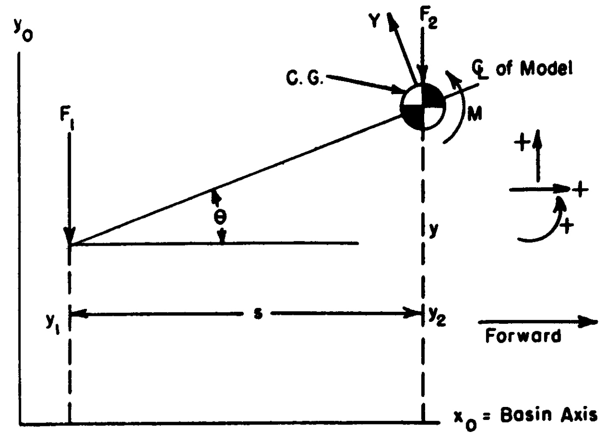

Figure 7 - Model Suspended on Two Struts with Center of Gravity of the Model at the Forward Strut

CASE III

A model supported by one strut fastened at the center of gravity constitutes the third case, see Figure 8.

It will be recalled from Figure 2 that the relation between a fixed and a moving coordinate system is

$$v = v_0 \cos\theta - u_0 \sin\theta$$

$$\dot{v} = \dot{v}_0 \cos\theta - v_0\dot{\theta}\sin\theta - \dot{u}_0 \sin\theta - u_0\dot{\theta}\cos\theta$$

Applying $\Sigma F = Ma$ and $\Sigma T = I\alpha$

$$Y \cos\theta - F = m\ddot{y} \qquad [8]$$

$$M - T = I\ddot{\theta} \qquad [9]$$

From strut analysis (Figure 6), we see that

$$\alpha = \beta - \theta_l \times \frac{z}{l} = \frac{y_l \times z}{u_0 \times l} - \theta_l \times \frac{z}{l}$$

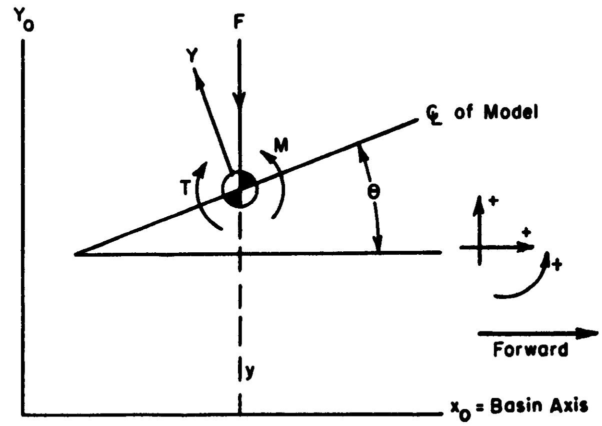

Figure 8 - Model Suspended on One Strut Fastened at the Center of Gravity of the Model

Forces on Body	Y = Hydrodynamic Force on Model
	F = Strut Force on Model
Moment on Body	M = Hydrodynamic Moment on Body
	T = Strut Moment on Body

Therefore, the lift coefficient at any station z equals

$$C_L = \psi(\alpha) = \psi(\dot{y}_l, \theta)$$

where the function is linear in $\dot{y}_l$ and θ.

By a similar process to that shown following Figure 6, lift integrals are found to be linear in $\dot{y}$ and θ and may be written $A\dot{y} + N\theta$. Inertia integrals are linear in $\ddot{y}$ and may be written $B\ddot{y}$. Spring force = Ky. Therefore

$$F = B\ddot{y} + A\dot{y} - Ky + N\theta \qquad [10]$$

From hydrodynamic considerations

$$Y = Y_v v + Y_{\dot{v}}\dot{v} + Y_{\dot{\theta}}\dot{\theta}$$

$$Y_v v = Y_v(\dot{y}\cos\theta - u_0 \sin\theta)$$

$$Y_{\dot{\theta}}\dot{\theta} = Y_{\dot{\theta}}\dot{\theta}$$

Considering $Y_{\dot{v}}$ intrinsically positive, the virtual mass term equals

$$-Y_{\dot{v}}\dot{v} = -Y_{\dot{v}}[\ddot{y}\cos\theta - \dot{y}\dot{\theta}\sin\theta - \dot{u}_0 \sin\theta - u_0\dot{\theta}\cos\theta]$$

Since θ = small angles, let $\cos\theta = 1$, $\sin\theta = \theta$. Substituting in Equation [8]

$$Y_v(\dot{y} - u_0\theta) + Y_{\dot{\theta}}\dot{\theta} - Y_{\dot{v}}(\ddot{y} - \dot{y}\dot{\theta}\theta - \dot{u}_0\theta - u_0\dot{\theta})$$

$$+ Ky - A\dot{y} - B\ddot{y} - N\theta = m\ddot{y}$$

Performing the same simplifying operations as were applied on the equation from which Equation [5] was derived, we get

$$(-Y_{\dot{v}} - B - m)\ddot{y} + (Y_v - A)\dot{y} + [Y_{\dot{\theta}} + Y_v]\dot{\theta} + Ky - Y_v\theta - N\theta = 0$$

This may be written in the form

$$C_1\ddot{y} + C_2\ddot{\theta} + C_3\dot{y} + C_4\dot{\theta} + C_5 y + C_6\theta = 0 \qquad [11]$$

The hydrodynamic moment on the model is

$$M = M_v v + M_{\dot{\theta}}\dot{\theta} + M_{\ddot{\theta}}\ddot{\theta}$$

$$M_v v = M_v(\dot{y}\cos\theta - u_0 \sin\theta)$$

$$M_{\dot{\theta}}\dot{\theta} = M_{\dot{\theta}}\dot{\theta}$$

Considering $M_{\ddot{\theta}}$ intrinsically positive, the virtual moment of inertia term equals

$$-M_{\ddot{\theta}}\ddot{\theta} = -M_{\ddot{\theta}}\ddot{\theta}$$

Let $\cos\theta = 1$, $\sin\theta = 0$, torque $T = K_T\theta$, where K_T equals torsional spring constant of strut. Substituting in Equation [9], we get

$$M_v(\dot{y} - u_0\theta) + M_{\dot{\theta}}\dot{\theta} - M_{\ddot{\theta}}\ddot{\theta} - K_T\theta = I\ddot{\theta}$$

Putting into dimensionless form by relations of Appendix 1 and collecting terms, we get

$$[-M_{\ddot{\theta}} - I]\ddot{\theta} + M_v\dot{y} + M_{\dot{\theta}}\dot{\theta} + [-M_v - K_T]\theta = 0$$

This may be written in the form

$$D_1\ddot{y} + D_2\ddot{\theta} + D_3\dot{y} + D_4\dot{\theta} + D_5 y + D_6\theta = 0 \qquad [12]$$

Roots of Equations [11] and [12] are of the form

$$y = Ee^{\sigma t}$$

$$\theta = Fe^{\sigma t}$$

Equations [11] and [12] are the same form as Equations [5] and [6] and the same solution is applicable, viz.,

$$\sigma^4 + \left[\frac{C_3D_2 + C_1D_4 - D_3C_2 - D_1C_4}{C_1D_2 - D_1C_2}\right]\sigma^3$$

$$+ \left[\frac{C_5D_2 + C_3D_4 + C_1D_6 - D_5C_2 - D_3C_4 - D_1C_6}{C_1D_2 - D_1C_2}\right]\sigma^2$$

$$+ \left[\frac{C_5D_4 + C_3D_6 - D_5C_4 - D_3C_6}{C_1D_2 - D_1C_2}\right]\sigma + \left[\frac{C_5D_6 - D_5C_6}{C_1D_2 - D_1C_2}\right] = 0$$

This is the stability equation and may be written in the form

$$\sigma^4 + P\sigma^3 + Q\sigma^2 + R\sigma + S = 0$$

For stability, P, Q, R, S > 0 and $P^2S - PQR + R^2 < 0$.

RESULTS

It is apparent that the stability equations take the same general form regardless of the strut and model configuration. By definition the motion is stable if neither θ nor y grows indefinitely large or undergoes a steady oscillation.

The form of the stability equation is

$$\sigma^4 + P\sigma^3 + Q\sigma^2 + R\sigma + S = 0$$

and

$$y_1 = Ee^{\sigma t} \qquad y_2 = Fe^{\sigma t}$$

$$\theta = \frac{y_1 - y_2}{s}$$

For stability, the values of σ, if real, must be negative or zero, and, if complex, must have their real parts negative. This may be determined by Routh's method, without actually solving for the roots.

Routh's criterion for stability states that the following relations must hold: $P>0$, $Q>0$, $R>0$, $S>0$, and $P^2S - PQR + R^2 < 0$.

Whether the motion is stable or unstable, the real part of the root is a measure of the damping, and the imaginary part of the root divided by 2π gives the absolute value of the period of oscillation.

CONCLUSIONS

In theory the calculations indicated by this report should be sufficient to determine whether dynamic stability is present in a particular case. The same methods could be extended to obtain the coefficients for the stability equation regardless of the strut and model configuration. It must be remembered that all terms are in the dimensionless form. The transformations necessary to arrive at the dimensionless forms are given in Appendix 1.

Methods for determining stability derivatives are given in TMB Report 553, "The Dynamical Stability of Torpedoes." As the nomenclature is different from that used in this report, Appendix 2 gives the relationship between the two nomenclatures.

If it can be proven by experiment that the stability equations derived in this report are reasonably valid, then this analysis should prove a valuable tool in checking the dynamic stability of a proposed towing arrangement. The cost and delay in testing which result from instability failures would thus be avoided and the necessary criteria for rational strut design would be provided.

It should be pointed out that the application of this analysis to a particular towing problem is likely to involve considerable time and effort, particularly if it should be necessary to determine experimentally the stability derivatives required to evaluate hydrodynamic loadings before proceeding with the analysis. Under some such circumstances it may well be that the work involved in such an analysis may not be justified and the risk of strut failure should be taken deliberately. However, where large, expensive and hard to replace struts are involved and where damage may be very extensive in case of failure, the problem of stability should be carefully considered.

It is recommended that the following test program be carried out in order to check the validity of the stability equations:

1. A model should be tested on the 3-component dynamometer and the oscillator to determine its stability derivatives.

2. The stability equations for Case II should be solved for several velocities to determine the calculated damping and frequency of oscillation.

3. The model should be towed on the oscillator at the velocities at which the calculations have been made. The model should be given a small disturbance and its response recorded to compare the observed damping and frequencies of oscillation with the calculated ones.

APPENDIX 1

RELATIONS FOR RENDERING EQUATIONS NONDIMENSIONAL

The final equations are written in the dimensionless form. The following relations must be used to transform the quantities involved before they may be substituted in the equations. All factors in the final equations are understood to be in prime (') form even though the prime signs are omitted.

ρ = Mass density of water

L = Length of model

u_0 = Forward velocity of model (x_0-direction)

$$I' = \frac{I}{\frac{1}{2}\rho L^5} \qquad M'_{\ddot{\theta}} = \frac{M_{\ddot{\theta}}}{\frac{1}{2}\rho L^5}$$

$$m' = \frac{m}{\frac{1}{2}\rho L^3} \qquad H', J', K' = \frac{H, J, K}{\frac{1}{2}\rho u_0^2 L}$$

$$M'_{\dot{\theta}} = \frac{M_{\dot{\theta}}}{\frac{1}{2}\rho u_0 L^4} \qquad s' = \frac{s}{L}$$

$$Y'_{\dot{\theta}} = \frac{Y_{\dot{\theta}}}{\frac{1}{2}\rho u_0 L^3} \qquad a' = \frac{a}{L}$$

$$M'_v = \frac{M_v}{\frac{1}{2}\rho u_0 L^3} \qquad A' = \frac{A}{\frac{1}{2}\rho u_0 L^2}$$

$$Y'_v = \frac{Y_v}{\frac{1}{2}\rho u_0 L^2} \qquad B' = \frac{B}{\frac{1}{2}\rho L^3}$$

$$Y'_{\dot{v}} = \frac{Y_{\dot{v}}}{\frac{1}{2}\rho L^3} \qquad u'_0 = \frac{u_0}{u_0} = 1$$

APPENDIX 2

NOMENCLATURE RELATIONSHIP BETWEEN THIS REPORT AND TMB REPORT 553,* "THE DYNAMICAL STABILITY OF TORPEDOES"

$$Y_\theta = \frac{\partial z}{\partial \theta}$$

$$M_\theta = \frac{\partial m}{\partial \theta}$$

$$Y_{\dot\theta} = \frac{\partial z}{\partial q}$$

$$M_{\dot\theta} = \frac{\partial m}{\partial q}$$

$$M_{\ddot\theta} = \frac{\partial m}{\partial \dot q}$$

$$Y_{\dot v} = \frac{\partial z}{\partial \dot w}$$

$$M_{\dot v} = \frac{\partial m}{\partial \dot w}$$

$$M_v = \frac{\partial m}{\partial \dot w} = \frac{1}{u} M_\theta$$

$$Y_v = \frac{\partial z}{\partial w} = \frac{1}{u} \quad (Y_\theta + X)$$

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*Garstens, M.A., Sc.D., Revised by Landweber, L. and Volta, A.J., "The Dynamical Stability of Torpedoes," TMB RESTRICTED Report 553, February 1949.

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