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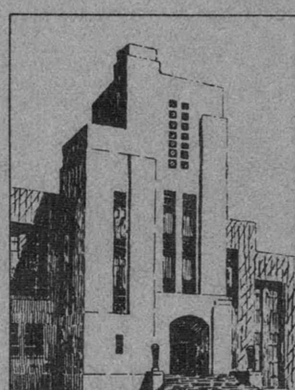
UNITED STATES NAVY

STRESSES AND DEFORMATIONS IN KNEES OF FRAMES AND IN SHARPLY CURVED BEAMS WITH VARIABLE CROSS SECTION

BY COMDR. H.M. WESTERGAARD, CEC, USNR



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JANUARY 1945

Report of a Study Made at the
BUREAU OF YARDS AND DOCKS
in Collaboration with the
DAVID TAYLOR MODEL BASIN

REPORT 529

NAVY DEPARTMENT
DAVID TAYLOR MODEL BASIN
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TABLE OF CONTENTS

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SYNOPSIS

A procedure is presented by which one may compute definite values of stresses and deformations in knees of frames or in sharply curved beams with variable cross section, with the influence of stiffeners taken into account. The stresses include fiber stresses, shearing stresses, and radial normal stresses. The values computed are approximate, and the analysis is classified not as mathematical theory of elasticity but under "strength of materials." The supporting evidence is that the procedure agrees satisfactorily with the principle of minimum of energy; agrees with and covers as a special case the standard analysis of sharply curved circular beams with constant cross section; agrees with known facts about wedge-shaped beams; and agrees with general information about the nature of stress concentration and the relief of stress concentration by well-placed stiffeners. The stresses found are consistent with the load and mutually consistent.

The procedure is stated first, and some applications are shown. The evidence is presented afterward.

The basic device of the analysis is the establishing of curved cross sections along lines of flow in an imagined temperature potential which is created by maintaining one constant temperature at the inner flange and another constant temperature at the outer flange. The corresponding isotherms will define curved fibers in which the stresses are particularly important; and the temperature gradients will serve to define rates of participation in the structural action. Plane cross sections are drawn by connecting the ends of the curved sections, and properties of the plane cross sections, such as the effective area, effective center, and effective moment of inertia, are determined. Thereafter the stresses and deformations may be computed by fairly simple formulas.

The procedure will lend itself to the study of types of design and to the evaluation of results of tests. The procedure is definite to such a degree that if two investigators apply it independently to the same case, they will be expected to obtain the same results. The simpler formulas may be found useful in the design of individual structures. Knowledge of the procedure should be found helpful in forming qualitative judgments of features of design.

A. NOTATION

1. - The letter symbols used are defined in the course of the presentation. Those recurring regularly and the numbers of the equations defining them are summarized as follows:

(a) Coordinates and Related Quantities:

x, y are rectangular coordinates; unless a specific temporary exception is made, y is measured along a plane cross section, as in Figure 2;

x, w are rectangular coordinates, as in Figure 2, unless an exception is stated;

r, θ are polar coordinates;

u, v are curvilinear coordinates; a constant value of u defines a curved cross section; a constant value of v defines a curved fiber;

$g = \partial v/\partial y$ is the gradient of v along a plane cross section;

α is the angle between plane and curved cross sections, Figure 2;

ρ is the radius of a curvature of a fiber, see Section 8.

(b) Properties of a Plane Cross Section, see Sections 8 to 10:

a, b are distances defining the moment center for shear, point i in Figure 2, by Equations [9] and [10];

A is the area of cross section, Equation [2];

B is the effective area of cross section, Equation [4];

B_1, B_2 are portions of B for the flanges only, Equations [20];

B_y is a part of the effective area, Equation [21];

B_s is the effective area for shear deformation, Equation [22];

B_v is the effective area for radial strains, Equation [24];

F_1, F_2 are flange areas, see Equations [20];

j is the distance from the center of gravity to the effective center, Equation [7];

J is the effective moment of inertia, Equation [8];

k is a ratio defined by Equation [15];

K is the effective moment of inertia for shear, Equation [12];

K_w is the value of K for the web only, Equation [19];

Q is the first moment of a part of A, Equation [11];

Q_w is the value of Q for the web only, see Equation [18];

R is the effective radius, Equation [17];

S is the first moment of a part of B, Equation [13];

t is thickness or width.

(c) Loads on Plane Cross Section:

M, N, V are bending moment, normal force, and total shear respectively, and are positive as in Figure 4;

M_i is the moment about the moment center for shear, point i, Equation [52].

(d) Stresses, see Figure 5:

σ_x is normal stress on a plane cross section;

σ is fiber stress;

σ_v is radial normal stress across the fibers;

τ is shearing stress.

(e) Elasticity:

E, G are moduli of elasticity in tension and shear respectively.

(f) Motions of a Cross Section Relative to an Adjacent Cross Section:

$\delta\xi$, $\delta\xi_c$, $\delta\eta$ are translations defined in Section 19;

$\delta\omega$, $\delta\omega_c$, $\delta\omega_i$ are rotations defined in Section 19.

(g) General Rules:

Integrals in which limits are not stated are extended over the whole area or over the whole depth of the cross section.

Logarithms used are natural logarithms.

The symbol $\approx$ is used occasionally to represent "approximately equal to."

B. OBJECTIVE

2. Scope - This report deals with stresses and deformations in knees of frames, in sharply curved beams, and in wedge-shaped parts of the variety of forms suggested in Figure 1. To avoid unnecessary complication of the discussion, each structural part will be assumed to have a plane of symmetry, even though most of the results will be applicable to other cases as well;

and the plane of symmetry will be assumed to be vertical. The loads will be assumed to be contained in the plane of symmetry, so as to produce bending but no twisting. The stresses will be assumed to be within the range of Hooke's law. Buckling of portions of the web or of flanges in compression will be assumed not to have begun, and is considered only indirectly by noting that the stresses without buckling must be known before the tendencies to buckling can be evaluated. As to buckling of the web or flanges, reference is made to a paper, which includes an extensive bibliography, by Leon S. Moisseiff and Frederick Lienhard,* 1941, and a publication by F. Bleich,** 1943. Similarly, incompleteness of the participation in the resistance by broad flanges in curved portions will not be considered directly; this effect may be assumed to have been taken into account in advance by stating properly reduced values of the cross-sectional areas of such flanges; concerning that effect, reference may be made to the publication by F. Bleich just mentioned.

A standard method, adequate for its purpose, is available for computing the fiber stresses in sharply curved circular beams with constant cross section.† The present objective is to extend that method and to establish a procedure by which stresses and deformations in knees, sharply curved beams, and wedge-shaped parts of such variety as is suggested by Figure 1 may be computed at many points and in detail if desired; and by which the motions of one cross section relative to another may be determined if desired. The computations will not be required to be exact as in solutions in the mathematical theory of elasticity, but are permitted to be approximate as characteristic of the subject of strength of materials, and as illustrated by the theory of curved beams with constant cross section. The procedure will be required to be definite so that independent investigators will obtain the same results.

The procedure presented differs from that proposed by William R. Osgood,†† 1939, and from that proposed by F. Bleich in the publication just mentioned.

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* "Theory of Elastic Stability Applied to Structural Design," by Leon S. Moisseiff and Frederick Lienhard, Transactions, American Society of Civil Engineers, Vol. 106, 1941, pp. 1052-1091.

** "Design of Rigid Frame Knees," by Friedrich Bleich, Committee on Steel Structures Research, American Institute of Steel Construction, July 1943, 30 pp.

† See, for example, Marks' Mechanical Engineers' Handbook, 4th Edition, McGraw-Hill Book Company, Inc., New York, N.Y., 1941, pp 499-500.

†† "A Theory of Flexure for Beams with Nonparallel Extreme Fibers," by William R. Osgood, Journal of Applied Mechanics, Transactions, American Society of Mechanical Engineers, September 1939, pp. A-122 to A-126.

3. Purposes - Availability of the procedure proposed here will serve the following purposes:

(a) Quick estimates of the structural behavior, including features of stress concentration and stress dispersion, by contemplating the general features of the procedure;

(b) Computation of stresses at a small number of important places during the design of a structure, by use of the least complex of the formulas;

(c) Detailed analyses, with stresses computed at many points, for the purpose of studying a type of design or of interpreting the results of tests;

(d) Computation of stresses preliminary to the analysis of buckling, it being noted that the analysis of buckling requires knowledge of the stresses that may cause it to impend.

C. NETWORK OF FIBERS AND CROSS SECTIONS

4. Required Properties of Network - If it is desired to obtain either a quick qualitative estimate or a detailed analysis of the structural action in bending of a knee, a sharply curved beam, a wedge-shaped piece, or a cracked beam, as sketched in Figure 1, it will be found expedient to draw a curved

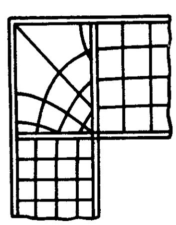

Figure 1a - Knee

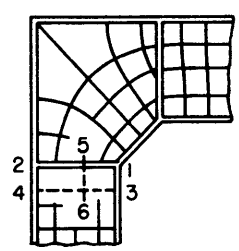

Figure 1b - Knee

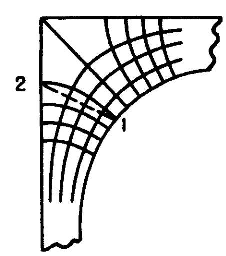

Figure 1c - Knee

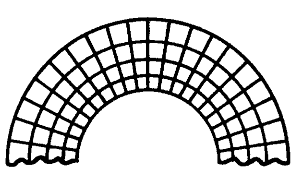

Figure 1d - Sharply Curved Beam

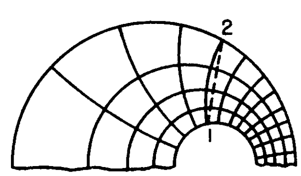

Figure 1e - Sharply Curved Beam

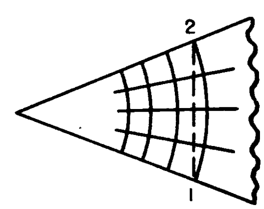

Figure 1f - Wedge-Shaped Beam

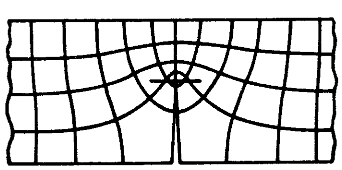

Figure 1g - Cracked Beam

Figure 1 - Networks of Fibers and Cross Sections in Knees and Beams

network of fibers and cross sections as indicated in Figure 1. It can be assumed that the normal stresses σ along such fibers will be of importance similar to that of the fiber stresses in an ordinary straight beam; and that the shearing stress τ along and across the fibers in Figure 1 and the normal stresses σ_v across the fibers will also be worthy of consideration.

If the network of fibers and cross sections is drawn correctly the closeness of the mesh at each place will serve to define a rate of participation by the material at the place in the structural action. For example, the open mesh at the outer corners of the knees in Figures 1a, 1b, and 1c indicates rightly a low rate of effectiveness of the material near these corners in resisting bending of the knee as a whole. On the other hand, the close mesh at the inside of the curved beams in Figures 1d and 1e is suggestive of the tendency to stress concentration known to exist at that place.

5. **Network of Fibers and Cross Sections in Region without Stiffeners** - Assume first that the knee or beam has no transverse stiffener serving the purpose of reducing stress concentration at a re-entrant corner or of increasing the participation in the structural action by the material near a convex corner. Then the following method of drawing the curved network of fibers and cross sections will be adopted:

A metal plate is considered which has the same side view as the knee or beam, has a constant thickness, and has no ribs. It is imagined that one constant temperature is maintained at the inner edge of this plate and that another temperature, also constant but lower, is maintained at the outer edge. It is imagined that no heat is transferred through the sides of the plate. When the resulting flow of heat from the inner to the outer edge has become steady a definite network of isotherms and of lines of flow intersecting the isotherms at right angles will be established. The fibers in the knee or beam will be drawn along the isotherms and the cross sections along the lines of flow.

The temperatures v will constitute a temperature potential, which may be interpreted instead, if one wishes, as an electric potential. The network may be drawn, therefore, in accordance with the theory of potentials* or in accordance with the theory of functions of a complex variable.** The

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* See for example, "Foundations of Potential Theory," by Oliver Dimon Kellogg, Julius Springer, Berlin, 1929, Chapter XII.

** See for example, "Functions of a Complex Variable," by E.J. Townsend, Henry Holt and Co., New York, N.Y., 1915, especially Chapter IV.

temperature v will be the imaginary part of an analytic function of a complex variable $z = x + iy$, in which x and y are rectangular coordinates in the plane of the plate; and each line of flow will be the locus of a constant value of the real part u of the same analytic function. If the network is drawn for equal small increments of u and v, the mesh will consist of small squares. Except for an unimportant uncertainty about the precise values of v at the ends of the area considered, the network will be defined completely by these rules.

As examples, the network consists of hyperbolas in Figure 1c, circles and straight lines in Figures 1d and 1f, and circles in Figure 1e.

6. Network of Fibers and Cross Sections with Influence of Stiffeners Taken into Account - Assume next that the knee or beam has one or more transverse stiffeners serving the purpose of reducing stress concentration at a re-entrant corner or of increasing the participation in the structural action by the material at a convex corner. In this case the same conductor plate will be used as already described, except that the plate will have a rib, made of the same metal, along the line of each of these stiffeners and extending all the way across the plate. Each of these ribs will participate in the conduction of heat, will tend toward making the temperature gradient along it more nearly uniform, and will make it possible for the isotherms to change direction as they cross the rib, as indicated in Figures 1a and 1b. While the cross-sectional area of such a rib should not necessarily be constant, it is proposed for the present, in the interest of definiteness, to make this area constant. Furthermore, it is proposed that the volume of the rib be made equal to the volume of the stiffener times the ratio of the thickness of the conductor plate to the average thickness of the web of the knee or beam in the region of the stiffener. These rules for the ribs are somewhat arbitrary and should be considered open to amendment when more quantitative information becomes available about the mechanical action of the stiffeners; the rules as proposed will produce results of the right trend.

An approximate determination of the influence that a rib such as 1-2 in Figure 1b has on the network may be obtained as follows: The rectangle 1-2-3-4 is drawn so that the volume of the web within that rectangle is equal to the volume of the stiffener; temperatures are determined with the influence of the rib left out of account, especially the average temperatures on lines such as 5-6 drawn across the rectangle; the true temperature at point 5 with the rib included will be approximately equal to the average temperature over the distance 5-6 when the rib is omitted. When the temperatures along the rib have been determined in this way, the final network can be drawn. It will be noted that if the rib is fairly substantial, a minor change of its size will have only a very small influence on the final network.

D. PROPERTIES OF PLANE CROSS SECTIONS

7. **Definition of Plane Cross Section** - The curved cross sections will serve to define significant plane cross sections at which the stresses along and across the fibers may be determined. Each of the plane cross sections is obtained by connecting the ends of one of the curved sections by a straight line. The dotted lines 1-2 in Figures 1c, 1e, and 1f are examples of such plane cross sections.

8. **Properties of Plane Cross Section Used in Formulas for Stresses in First Approximation** - Coordinates and dimensions are introduced as shown in Figure 2. These include the coordinates w and y measured along the cross section, and the angle α, positive as shown, at which a curved cross section, u = constant, intersects the plane cross section at each point. The following quantities are also used:

g is the gradient in the temperature potential in the direction of y (increase of temperature $\partial v/\partial y$ per unit of length in the direction of y). Since the temperatures are imagined quantities, the gradient may always be stated as a pure number; this will be assumed to be done unless a specific exception is stated;

ρ is the radius of curvature of any fiber at the point of intersection with the cross section; positive when the center of curvature lies on the side of the greater values of y, otherwise negative; and

is the thickness or width at any point of the cross section.

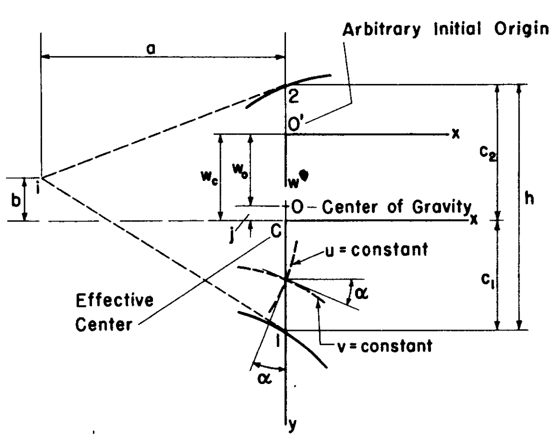

Figure 2 - Coordinates and Dimensions Related to a Plane Cross Section 1-2

The remaining quantities needed for the first approximation of the stresses are defined by the formulas that follow; these formulas are stated in a sequence in which they may be used; all integrals are extended over the whole area of the cross section, or over the whole depth, unless otherwise stated:

$$dA = t\,dw = t\,dy \qquad [1]$$

is an element of area of the cross section.

$$A = \int dA = \int t\,dw = \int_{-c_2}^{c_1} t\,dy \qquad [2]$$

is the total area of the cross section.

$$dB = g \cos^2 \alpha\ dA = g \cos^2\alpha\ t dw = g \cos^2\alpha\ t dy \qquad [3]$$

is an element of the effective area of the cross section.

$$B = \int dB = \int g \cos^2 \alpha\, dA \qquad [4]$$

is the effective area of the cross section.

$$w_o = \frac{1}{A} \int w\,dA \qquad [5]$$

is the distance from the initial arbitrarily chosen origin O' of the coordinates x, w to the center of gravity O of the area A.

$$w_c = \frac{1}{B} \int w\,dB = \frac{1}{B} \int w g \cos^2 \alpha\, dA \qquad [6]$$

is the distance from the initial origin O' to the effective center C, which is the center of gravity of the effective area B, this center of gravity being the crigin of the coordinates x, y.

$$j = w_c - w_o \qquad [7]$$

is the distance from the center of gravity O to the effective center C.

$$J = \int y^2 dB = \int y^2 g \cos^2 \alpha\, dA \qquad [8]$$

is the effective moment of inertia, which is the moment of inertia of the effective area about an axis through the effective center.

$-a$, $-b$ are x, y-coordinates of the "moment center for shear," point i in Figure 2; for an approximate determination, which will usually be sufficiently accurate, this point may be taken as the point of intersection of the tangents to the extreme fibers at points 1 and 2; the precise formulas defining point i, which need be used only in unusual cases, are

$$a \int y g \sin 2\alpha\ dA = 2J \qquad [9]$$

$$2bB = a \int g \sin 2\alpha\, dA \qquad [10]$$

Minor errors in the values of a and b will affect the computed stresses only slightly.

$$Q = \int_{y}^{c_1} (y+j)\, t\, dy \qquad [11]$$

is the first moment of the part of the cross section lying below the depth defined by y, with respect to an axis through the center of gravity of the true area.

$$K = \int_{-c_2}^{c_1} g^2 Q\, dy \qquad [12]$$

is the effective moment of inertia for shear.

$$S = \int_{y}^{c_1} y g \cos^2\alpha\, t\, dy \qquad [13]$$

is the first moment of the effective area of the part of the cross section lying below the depth defined by y, with respect to an axis through the effective center.

It will be observed that

$$J = \int_{-c_2}^{c_1} S\, dy \qquad [14]$$

9. **Additional Properties of Plane Cross Section Used in Formulas of Improved Accuracy for Stresses** - Two additional quantities identified with the cross section appear in the formulas of improved accuracy for the fiber stresses:

$$k = \frac{\int y^4 dB}{3Ja^2} \qquad [15]$$

a ratio, which need not be determined with great relative accuracy. Unless the circumstances are unusual, the approximate substitute formula

$$k = \frac{\int (y+j)^4 dA}{3a^2 \int (y+j)^2 dA} \qquad [16]$$

will be adequate. Equation [16] gives $k = h^2/20a^2$ for a rectangular section; $k = h^2/12a^2$ for an I-section with thin web and heavy flanges; and intermediate values for ordinary I-sections.

The other quantity is

$$R = \frac{J}{Bj} \qquad [17]$$

which will be called the effective radius. (It will be shown in Section 11 that when the beam is circular and the cross section constant, R is the radius of the effective centerline containing the effective centers).

The following further quantities are used in the formulas of improved accuracy for the shearing stresses along and across the fibers and the radial normal stresses acrc3s the fibers:

Q_w is the first moment of the part of the web lying below the depth defined by y, with respect to an axis through the center of gravity of the web; if the thickness t of the web is constant, Q_w may be computed as

$$Q_w = \frac{1}{2} t (c_1 - y)(y + c_2) \qquad [18]$$

in that special case.

$$K_w = \int g^2 Q_w \, dy \qquad [19]$$

is the effective moment of inertia for shear for the web only.

F_1, F_2 are the cross-sectional areas of the flanges at points 1 and 2 in Figure 2, measured in the direction perpendicular to the fibers.

$$B_1 = F_1 g_1 \cos \alpha_1, \; B_2 = F_2 g_2 \cos \alpha_2 \qquad [20]$$

are the effective areas of the flanges; g_1 and g_2 being the values of g, and α_1 and α_2 the values of α, at points 1 and 2, respectively, in Figure 2.

$$B_y = \int_y^{c_1} g \cos^2 \alpha \; t \, dy \qquad [21]$$

is that part of the effective area, including B_1, which lies below the depth defined by y.

10. Further Properties of Plane Cross Section Used in Formulas for Deformations - If it is desired to determine the motions of cross sections relative to one another, the following further properties of the cross sections will be needed:

$$B_s = \frac{K^2}{\int_{-c_2}^{c_1} g^3 \cos^2\alpha\, Q^2 t^{-1} dy} \qquad [22]$$

is the effective area for shear deformation. If the thickness of the web does not vary excessively, and if there are no other unusual circumstances, the following approximate formula will give results fairly close to those obtained by [22],* and may therefore ordinarily be used instead:

$$B_s = \frac{B_w}{1+\left(\frac{K_w}{K}\right)^2 \left[\frac{1}{5}+\frac{3(c_1-c_2)^2}{h^2}\right]} \qquad [23]$$

in which B_w is the portion of B in [4] accounting for the web only. Equation [23] gives $B_s = \frac{5}{6} B_w$ for a rectangular cross section; and very nearly $B_s = B_w$ if the beam has substantial flanges.

Finally,

$$B_v = \frac{J^2}{\int \frac{R^2 S^2 dy}{\rho^2 t g}} \qquad [24]$$

is the effective area for radial strains. Unless the circumstances are unusual, B_v will not differ greatly from B_s, and the value

$$B_v = B_s \qquad [25]$$

computed by [22] or [23] may be used without serious loss of accuracy.

11. **Derivation of the Properties of the Cross Section in the Special Case of a Circular Beam with Constant Cross Section** - Figure 3 shows the polar coordinates r, θ, and other dimensions that are naturally used in this case. R_o is the radius measured to the true center of gravity of the cross section. The radius R is introduced in this case not as a quantity defined by [17] but as the radius measured to the effective center.

The consideration of flow of heat gives radial lines of flow and circular isotherms, that is, radial cross sections, u = constant, and circular fibers, v = constant. This conclusion is also reached by use of a suitable function of the complex variable $re^{i\theta}$, in which $i = \sqrt{-1}$; the function is in this case

$$u + iv = -iR \log\left(\frac{r}{R} e^{i\theta}\right) = R\theta + iR \log \frac{R}{r} \qquad [26]$$

---

* Numbers in brackets indicate equations and formulas.

In [26] R might very well be replaced by any other constant distance. It is convenient, however, to use R for this purpose, and this will be done here. Equation [26] serves to define the gradient g, which becomes

$$g = \frac{dv}{dy} = -\frac{dv}{dr} = \frac{R}{r} \qquad [27]$$

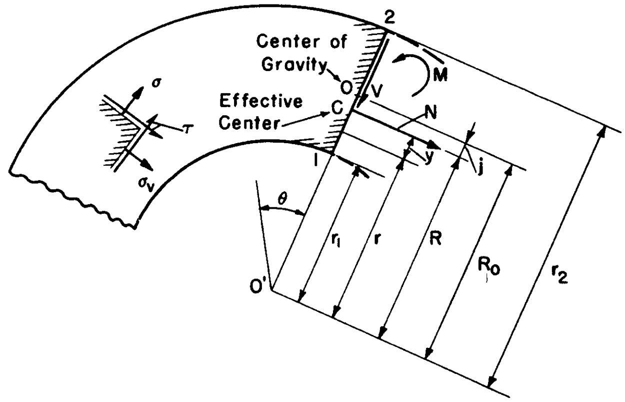

Figure 3 - Circular Beam with Constant Cross Section

Next, by expressing the moment of the effective area about the center O' of the circles, which is equivalent to using [6] though for a purpose other than its regular purpose, one finds

$$B = A \qquad [28]$$

By expressing B in accordance with [4], one finds

$$\frac{A}{R} = \int \frac{dA}{r} \qquad [29]$$

Equation [29] defines R and thereby

$$j = R_o - R \qquad [30]$$

thus locating the effective center C.

According to [8] the effective moment of inertia becomes

$$J = \int \frac{R(R-r)^2 dA}{r} = R^3 \int \frac{dA}{r} - 2AR^2 + ARR_o \qquad [31]$$

which gives, by use of [29],

$$J = AR(R_o - R) = ARj \qquad [32]$$

Equation [32] shows that the radius R defined by [17] is, in fact, the same as the radius R measured to the effective center.

According to [11],

$$Q = \int_{r_1}^{r} (R_o - r)\, t\, dr \qquad [33]$$

Then by [12],

$$K = \int_{r_1}^{r_2} \frac{R^2 Q\,dr}{r^2} = \int \frac{R^2 dQ}{r} = \int_{r_1}^{r_2} \frac{R^2 (R_o - r)\,t\,dr}{r} \qquad [34]$$

which gives, by use of [29] and [32],

$$K = J \qquad [35]$$

Finally, by [13],

$$S = \int_{r_1}^{r} \frac{R(R-r)\,t\,dr}{r} \qquad [36]$$

The derivations that follow correspond closely to those usually given in presenting the standard theory of sharply curved beams with constant cross section.*

Since the appearance of r in the denominator in [29] is likely to make the integration inconvenient by direct use of the formula, an approximate computation of j and J may be found advantageous. Let w be measured from the center of gravity, so that

$$w = R_o - r \qquad [37]$$

Let the following further notation be used:

$$I = \int w^2 dA \qquad [38]$$

is the ordinary moment of inertia of the cross section;

$$I_n = \int w^n dA \qquad \text{where} \qquad n = 3,\ 4,\ 5, \ldots\ldots \qquad [39]$$

and

$$q = \frac{I}{AR_o^2} + \frac{I_3}{AR_o^3} + \frac{I_4}{AR_o^4} + \ldots\ldots \qquad [40]$$

Then by substituting the series

$$\frac{1}{r} = \frac{1}{R_o}\left[1 + \frac{w}{R_o} + \left(\frac{w}{R_o}\right)^2 + \left(\frac{w}{R_o}\right)^3 + \left(\frac{w}{R_o}\right)^4 + \ldots\ldots\right] \qquad [41]$$

in [29] one finds

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* See for example, Marks' Mechanical Engineers' Handbook, 4th Edition, McGraw-Hill Book Company, Inc., New York, N.Y., 1941, pp. 499-500.

$$\frac{1}{R} = \frac{1+q}{R_o} \qquad [42]$$

and thereafter by [30] and [32]

$$j = \frac{R_o q}{1+q} = R_o(q - q^2 + q^3 - q^4 + \ldots.) \qquad [43]$$

and

$$J = \frac{A R_o^2 q}{(1+q)^2} = A R_o^2 (q - 2q^2 + 3q^3 - 4q^4 + \ldots.) \qquad [44]$$

The approximate formulas for j and J are obtained by retaining in [43] and [44] only the most important terms that may be substituted from [40]. Thus one finds the following approximate formulas which, in fact, will be found to have a high degree of accuracy in most practical cases:

$$j = \frac{I}{A R_o} - \frac{I^2}{A^2 R_o^3} + \frac{I_3}{A R_o^2} + \frac{I_4}{A R_o^3} \qquad [45]$$

and

$$J = I - \frac{2I^2}{A R_o^2} + \frac{I_3}{R_o} + \frac{I_4}{R_o^2} \qquad [46]$$

For S in [36] a good approximation can be shown to be

$$S = \frac{1}{r}\left[Q(r-j) - A'Rj + I'\right] \qquad [47]$$

in which

$$A' = \int_{r_1}^{r} t\,dr \qquad \text{and} \qquad I' = \int_{r_1}^{r} w^2 t\,dr \qquad [48]$$

The adequacy of the approximation by [47] is verified by noting first that the formula gives satisfactory values next to the flanges, and second that the value of the derivative $d(Sr)/dr$ obtained from [47] is $R(R - r)t + Q$ which differs only moderately from the true value.

In the example of a rectangular cross section of depth h Equations [45] to [47] give

$$j = \frac{I}{A R_o}\left(1 + \frac{h^2}{15 R_o^2}\right), \qquad J = I\left(1 - \frac{h^2}{60 R_o^2}\right) \qquad [49]$$

and

$$S = \frac{Q}{3r}(r + 2R_o - 3j) \qquad [50]$$

Equations [49] supply evidence that ordinarily the sum of the last three terms in [45] and in [46] will be sufficiently insignificant to permit the use of the still simpler approximate formulas

$$j = \frac{I}{AR_o} \qquad \text{and} \qquad J = K \approx I \qquad [51]$$

On the other hand, Equation [50] shows that [47] should not be replaced by such simpler forms as $S = Q$ or $S = gQ$ without careful consideration of the intended use.

E. STATEMENT OF FORMULAS FOR STRESSES

12. **Load on Plane Cross Section** - Figure 4 supplements Figure 2 by showing the load on the plane cross section. This load has three components:

- M, the bending moment, positive as shown;
- N, the normal axial force, applied at the effective center, positive as tension;
- V, the total transverse shear acting along the section, positive as shown.

In addition it is desirable to use the moment

- M_i which is the moment of the load on the cross section about point i, the moment center for shear defined approximately as the point of intersection of the tangents to the extreme fibers, and defined precisely by [9] and [10]. The value is

$$M_i = M + Nb - Va \qquad [52]$$

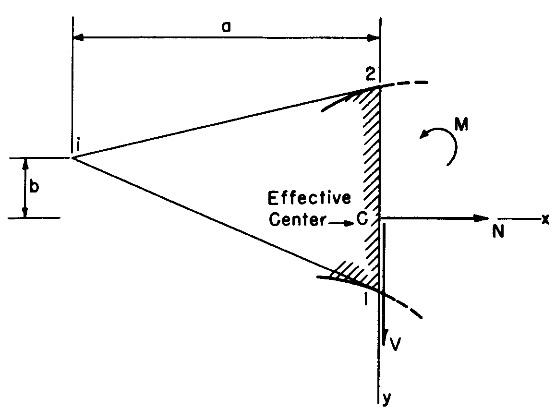

Figure 4 - Load on Plane Cross Section

13. **Reservation** - The formulas for stresses about to be stated, in Sections 14 and 15, are not generally applicable within a V-shaped region between two stiffeners that extend from a re-entrant corner and function to reduce the stress concentration at such a corner (a method of dealing with such a region is described in Section 18). In the immediate vicinity of a concentrated external load the formulas should be applied only with caution and with realization that terms may have to be added similar in character to the corresponding corrections for a straight beam.

14. **First Approximation for Stresses** - Figure 5 shows the stresses that are considered at any one cross section. In terms of the loads shown in Figure 4 and explained in Section 12, and in terms of the properties of the cross section defined in Section 8, a first approximation for these stresses may now be stated:

The normal stress on the cross section, positive in tension, is computed as

$$\sigma_x = \frac{Myg\cos^2\alpha}{J} + \frac{Ng\cos^2\alpha}{B} \qquad [53]$$

The fiber stress, that is, the normal stress in the direction of the fiber, positive in tension, is computed as

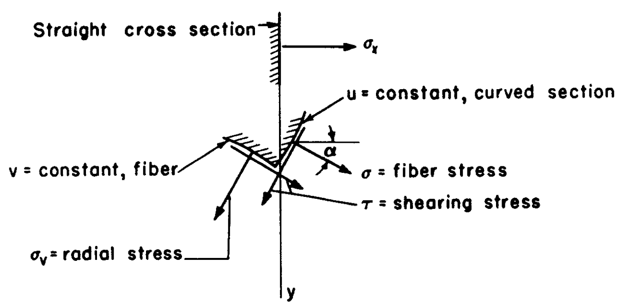

Figure 5 - Stresses at Plane Cross Section

$$\sigma = \frac{Myg}{J} + \frac{Ng}{B} \qquad [54]$$

The shearing stress along and across the fibers, positive as shown in Figure 5, is computed as

$$\tau = -\frac{M_i Q g^2}{aKt} \qquad [55]$$

in the general case of variable cross section. In the special case of a constant cross section or of a slowly varying cross section the distance a becomes infinite or very great. Then $-M_i/a$ is replaced by V. In that special case [55] becomes

$$\tau = \frac{VQg^2}{Kt} \quad \text{(special case)} \qquad [56]$$

The radial stress, that is, the normal stress across the fibers, positive in tension, is computed as

$$\sigma_v = \frac{MS}{Jt\rho\cos\alpha} \tag{57}$$

15. **Formulas of Improved Accuracy for Stresses** - While the values of stresses defined by [53], [54], [55], and [57] will be adequately accurate for most purposes, formulas of improved accuracy will be needed occasionally for special purposes. The improved approximation requires the use of the additional properties of the cross section defined in Section 9. The formulas follow:

The fiber stress replacing the value in [54] is

$$\sigma = \frac{Myg}{J} + \frac{Ng}{B} + \frac{M_i yg}{J}(k - \sin^2\alpha) - \frac{Mkg}{RB} \tag{58}$$

The formula of improved accuracy for the shearing stress in the web, replacing [55], is given under the assumption that in passing from the particular cross section to neighboring cross sections the cross-sectional areas F_1 and F_2 of the flanges remain constant, while the cross-sectional area of the web changes in proportion to the total depth h.

The value is

$$\tau = \tau' + \tau'' + \tau''' \tag{59}$$

in which τ' is the same as τ in [55] so that

$$\tau' = -\frac{M_i Q g^2}{aKt} \tag{60}$$

$$\tau'' = \frac{Mg^2}{aK^2 t}(KQ_w - K_w Q) \tag{61}$$

and

$$\tau''' = \frac{N}{aB^2 t}\left[B_y(B_1 + B_2) - BB_1\right] \tag{62}$$

The formula of improved accuracy replacing [57] is

$$\sigma_v = \frac{1}{Jt\rho\cos\alpha}\left[MS + NR(Qg\cos^2\alpha - S)\right] \tag{63}$$

The values of τ'', τ''', and σ_v at the junctions of the flanges with the web are of special interest. With subscripts 1 and 2 identifying the values at flanges 1 and 2 respectively, Equation [61] gives (by use of [11] and the definitions of F_1 and F_2 immediately before [20])

$$\tau_1'' = -\frac{MK_w F_1 (c_1 + j) g_1^2}{aK^2 t \cos \alpha_1} \qquad [64]$$

at flange 1, and

$$\tau_2'' = -\frac{MK_w F_2 (c_2 - j) g_2^2}{aK^2 t \cos \alpha_2} \qquad [65]$$

at flange 2. By observing that B_y is equal to B_1 next to flange 1 and equal to $B - B_2$ next to flange 2, and with B_w denoting the part of B accounting for the web only (so that $B_w = B - B_1 - B_2$), one finds by [62]

$$\tau_1''' = -\frac{NB_1 B_w}{aB^2 t} \quad \text{and} \quad \tau_2''' = \frac{NB_2 B_w}{aB^2 t} \qquad [66]$$

at the two flanges. Finally, from [63], by referring to [11], [13], and [17], one obtains the radial stress

$$\sigma_{v,1} = \frac{F_1 g_1}{\rho_1 t}\left(\frac{Mc_1}{J} + \frac{N}{B}\right) \qquad [67]$$

next to flange 1, and

$$\sigma_{v,2} = \frac{F_2 g_2}{\rho_2 t}\left(\frac{Mc_2}{J} - \frac{N}{B}\right) \qquad [68]$$

next to flange 2.

F. APPLICATIONS OF FORMULAS FOR STRESSES

16. **Stresses in Circular Beam with Constant Cross Section** - The standard theory of curved beams deals with this case and supplies values of the fiber stresses. These values will be found to agree with the results of the procedures presented here.

Figure 3 shows the case. The properties of the cross section were derived in Section 11. It will be remembered that the gradient was stated, in [27], in the particular form

$$g = \frac{R}{r} \qquad [69]$$

(in preference to the general form $g = l/r$ in which l is any distance). With g as in [69], the advantage is obtained that, as stated in [28] and [35],

$$B = A \qquad \text{and} \qquad J = K \qquad [70]$$

Then [54] gives the fiber stress

$$\sigma = \frac{R}{r}\left[\frac{M(R-r)}{J} + \frac{N}{A}\right] \qquad [71]$$

in which R is defined by [29]. R may be computed with very good approximation as $R_o - j$ through [45], and with ordinarily adequate approximation through [51] as

$$R = R_o - \frac{I}{AR_o} \qquad [72]$$

Furthermore, in [71], J is defined by [32] and may be computed with good approximation by [46] and with ordinarily adequate approximation through [51] as

$$J \approx I = \int (R_o - r)^2 \, dA \qquad [73]$$

Use of the formula of improved accuracy for the fiber stress, Equation [58], adds nothing in this case because the last two quantities in [58] vanish when the cross section is constant; Equation [58] reproduces [71].

If one prefers, Equation [71] may be restated by use of [32] in the form

$$\sigma = \frac{M(R-r)}{Ajr} + \frac{NR}{Ar} \qquad [74]$$

It is customary in presentations of the standard theory of circular beams to introduce the bending moment M_o about the center of gravity of the cross section instead of the bending moment M about the effective center. The two have the relation

$$M = M_o - Nj \qquad [75]$$

Substitution from [75] in [74] gives the equivalent formula

$$\sigma = \frac{M_o(R_o - r)}{Ajr} - \frac{M_o}{Ar} + \frac{N}{A} \qquad [76]$$

in which j may be taken from [45], or, with less but usually sufficient accuracy, according to [51], as

$$j \approx \frac{I}{A R_o} \qquad [77]$$

Equation [76] will be recognized as the formula usually given in presentation of the standard theory of curved beams with constant cross section.

Equations [56] and [70] give the shearing stress

$$\tau = \frac{V Q R^2}{J t r^2} \qquad [78]$$

Again, because the cross section is constant, the formulas of improved accuracy, Equations [59] to [62], give the same result. It is noted that the concentration factor by comparison with a straight beam is $(R/r)^2$ for the shearing stresses.

The radial stress becomes by the first approximation, Equation [57],

$$\sigma_v = \frac{M S}{J t r} \qquad [79]$$

but by the formula of improved accuracy, Equation [63],

$$\sigma_v = \frac{1}{J t r}\left[M S + N R\left(\frac{Q R}{r} - S\right)\right] \qquad [80]$$

By substituting S from the approximate formula [47] in [80] one obtains the formula

$$\sigma_v = \frac{M}{t r^2}\left[\frac{Q(r-j)}{J} - \frac{A'}{A} + \frac{I'}{J}\right] + \frac{N R}{t r^2}\left[\frac{Q(R_o - r)}{J} + \frac{A'}{A} - \frac{I'}{J}\right] \qquad [81]$$

which gives a good approximation and may be preferred to [80] because of the inconvenience of the direct computation of S. It will be observed that [81] gives values next to the flanges that agree satisfactorily with [67] and [68].

17. Examples of Stresses in Knee - In each of the examples shown in Figure 6 the network of curved cross sections, u = constant, and fibers, v = constant, will be defined by a function

$$Z = Z(z) = u + i v \qquad [82]$$

of the complex variable

$$z = x + iw \qquad [83]$$

with x and w as shown in Figure 6.

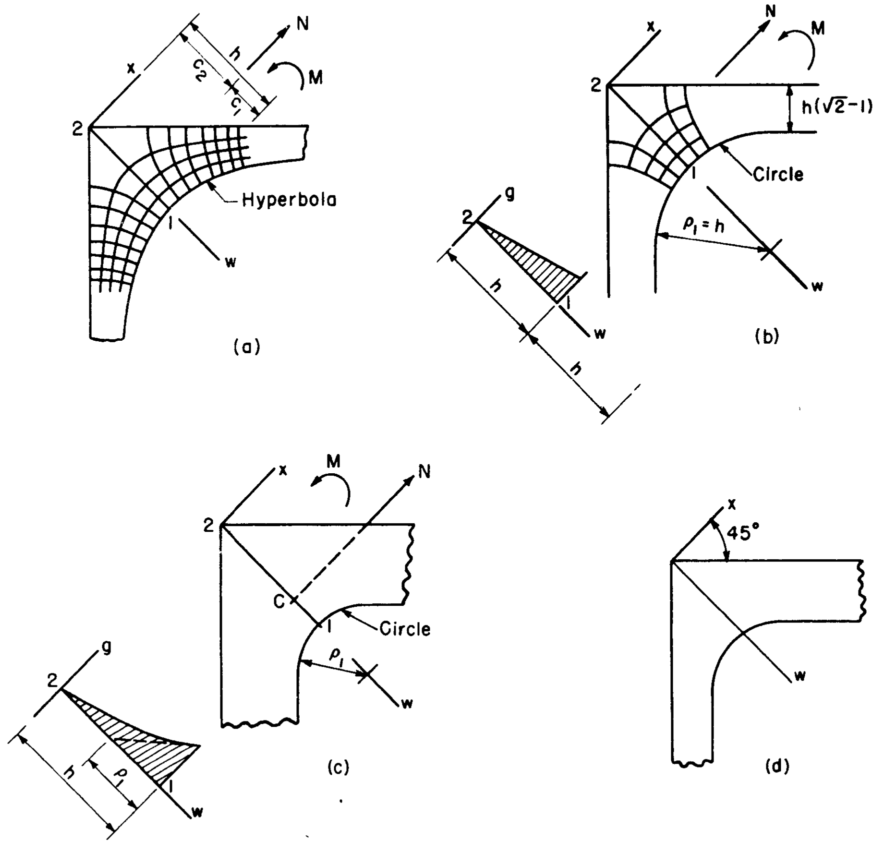

Figure 6 - Examples of Knees

In Figure 6a the function Z is

$$Z = -\frac{iz^2}{2h} = \frac{xw}{h} + \frac{i(w^2 - x^2)}{2h} \qquad [84]$$

The network defined by this function consists of two sets of hyperbolas, as shown. At section 1-2

$$u = 0, \; v = \frac{w^2}{2h}, \qquad \text{and} \qquad g = \frac{\partial v}{\partial w} = \frac{w}{h} \qquad [85]$$

The radius of curvature of the fibers at section 1-2 becomes

$$\rho = \frac{g}{\frac{\partial g}{\partial w}} = w, \ \rho = \rho_1 = h \text{ at point } 1 \qquad [86]$$

Since the radius of curvature at point 1 in Figure 6b is the same, it stands to reason that the network in the immediate vicinity of section 1-2 in Figure 6b will be so nearly the same as in Figure 6a that the stresses at that section will be practically the same in the two cases under the same load. The following computations therefore apply equally to Figure 6a and Figure 6b.

To make the example as simple as possible the cross section is assumed to be rectangular, without flanges. Then one finds, for section 1-2 in Figure 6a and 6b:

$$B = \frac{1}{2} A, \ c_1 = \frac{1}{3} h, \ c_2 = \frac{2}{3} h \qquad [87]$$

and the effective moment of inertia about point 2

$$J_2 = \frac{1}{4} A h^2 \qquad [88]$$

which gives

$$J = J_2 - B c_2^2 = \frac{1}{36} A h^2 \qquad [89]$$

By use of these properties of the cross section one finds the fiber stress at section 1-2 by [54]

$$\sigma = \frac{12 M (3w - 2h) w}{A h^3} + \frac{2 N w}{A h} \qquad [90]$$

At point 1 this stress is

$$\sigma_1 = \frac{12 M}{A h} + \frac{2 N}{A} \qquad [91]$$

which is twice the value that would be found if the rectangular section were counted fully effective, with g constant and N at the true center of gravity. The formula of improved accuracy, Equation [58], gives the same results because [9] and [10] give $a = \infty$ and $b/a = 0$ as they must at a section of symmetry.

At point 2, Equation [90] gives $\sigma = 0$, which is correct. Actually, in the knee, $\partial\sigma/\partial w$ must also vanish at point 2. While the latter feature is not brought out by [90], the discrepancy is minor; the important fact is that the stresses are zero at point 2 and very small in the immediate vicinity.

To obtain the shearing stress on section 1-2 in Figures 6a and 6b, Q and K are determined according to [11] and [12], which give

$$Q = \frac{Aw(h-w)}{2h} \qquad \text{and} \qquad K = \frac{Ah^2}{40} \tag{92}$$

Then by [56] the shearing stress becomes

$$\tau = \frac{20V}{A}\left[\left(\frac{w}{h}\right)^3 - \left(\frac{w}{h}\right)^4\right] \tag{93}$$

With S in [13] computed from the formula

$$\frac{dS}{dw} = t(c_2 - w)g \tag{94}$$

as

$$S = \frac{A}{3h^2}(hw^2 - w^3) \tag{95}$$

and with R determined from [17] as $R = h/3$, the radial stress in [63] becomes, by use of [86],

$$\sigma_v = \frac{1}{Ah^3}(12M + 2Nh)(hw - w^2) \tag{96}$$

The approximate formula [57] gives the same result except that the term containing N is omitted. The criticism stated for [90] applies also to [96]: $\partial\sigma_v/\partial w$ should, in fact, be zero at point 2. Again the discrepancy is minor.

For the knee in Figure 6c the network of fibers and sections in the immediate vicinity of section 1-2 will be defined adequately by the function

$$Z = -\frac{iz^2}{2h} - \frac{i(h-\rho_1)z^6}{6(5\rho_1 - h)h^5} = u + iv \tag{97}$$

in which again $z = x + iw$. Equation [97] leads to the proper radius of curvature ρ_1 at point 1 by a computation as in [86]. Assume as an example

$$\rho_1 = \frac{1}{2} h \qquad [98]$$

Then at section 1-2

$$g = \frac{w}{h} + \frac{w^5}{3h^5} \qquad [99]$$

As in the previous example, assume again that the cross section is rectangular, without flanges. Then one finds by computations as in the previous example

$$B = \frac{5}{9} A, \ c_2 = \frac{24}{35} h, \ c_1 = \frac{11}{35} h, \ J = \frac{179}{5880} A h^2 \qquad [100]$$

and the fiber stress at point 1

$$\sigma_1 = \frac{2464 M}{179 A h} + \frac{12 N}{5A} = \frac{13.8 M}{A h} + \frac{2.4 N}{A} \qquad [101]$$

Both terms on the right side in [101] are greater than the corresponding terms in [91], as should be expected because of the smaller radius of curvature.

For the knee in Figure 6d a corresponding solution may be obtained by defining the network of fibers and sections by a function of the form

$$Z = \sum_{1,2,3..}^{n} i^n C_n z^{2n} \qquad [102]$$

with a limited number of constant coefficients C_n. Equations [84] and [97] are special cases of [102].

18. Stresses in Rectangular Corner Panel in Knee with Stiffeners - Figure 7a shows a possible network of fibers and sections in a knee of this type. This network may be applied to determine the stresses at a vertical cross section immediately to the right of the stiffener 1-3 and at a horizontal cross section immediately below the stiffener 1-4. The network, however, does not lend itself directly to the determination of the stresses within the corner panel itself; in accordance with the reservation made in Section 13 the formulas that have been stated for the stresses do not apply generally within the V-shaped area between the two stiffeners.

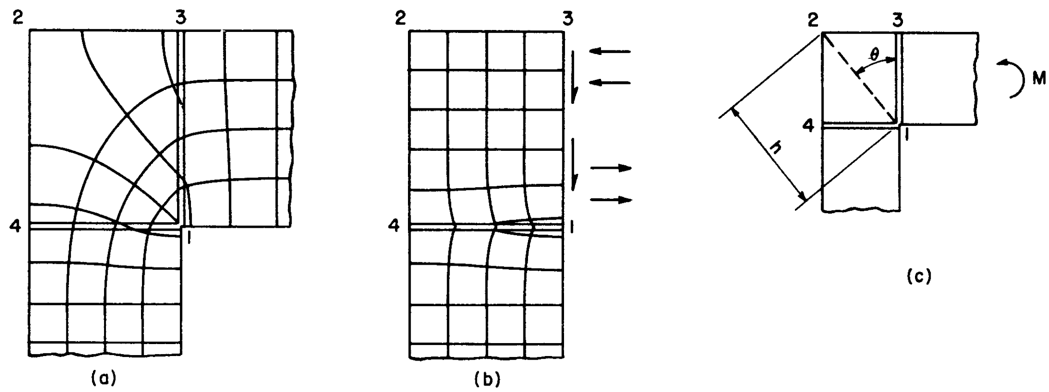

Figure 7 - Rectangular Corner Panel in Knee, with Stiffeners 1-3 and 1-4

A solution is suggested by Figure 7b, which shows the corner panel as a part of a vertical beam with a network below the stiffener 1-4 as in Figure 7a and with the network above drawn so that the fibers above the stiffener join those below. Stresses next to the vertical stiffener in Figure 7a become known external loads in Figure 7b. The normal stresses on horizontal sections may be computed by [53]; and there will be no particular difficulty about computing plausible values of the horizontal and vertical shearing stresses, with the shear loads at 1-3 taken into account. In the same way the corner panel may be considered as a part of a horizontal beam ending at 2-4, and thereby the horizontal normal stresses may be computed, again by [53]. Two papers by William R. Osgood* on stresses in such knees will also be found useful.

While the formulas for stresses, in Sections 14 and 15, are not generally directly applicable within the corner panel, one of the formulas, Equation [53], may be used nevertheless with fair results to determine the normal stresses on the diagonal section 1-2. This will be illustrated by an example which is made as simple as possible by considering an extreme case in which the stiffeners are fully effective, so that the network has no distortion below 1-4 and on the right of 1-3 but consists of straight lines as in a long straight beam. Then the network will consist of hyperbolas as in Figure 6a. The fibers intersect the diagonal section at a constant angle, and $g \cos^2\alpha$ at that section will be proportional to the distance from point 2. Assume further, for the sake of simplicity of the example, that there are no flanges and that the thickness is constant within the panel; and that

* "Stresses in a Rectangular Knee of a Rigid Frame," by William R. Osgood, Journal of Research of the National Bureau of Standards, Vol. 27, November 1941, pp. 443-448.

"Rectangular Plate Loaded along Two Adjacent Edges by Couples in Its Own Plane," by William R. Osgood, ibid., Vol. 28, February 1942, pp. 159-163.

the load consists of the bending moment M only, as indicated in Figure 7c. Then the stress on section 1-2 at point 1, defined by [53], will be the same as the stress in [91] with $N = 0$; that is,

$$\sigma_1 = \frac{12M}{h^2 t} \qquad [103]$$

With θ as in Figure 7c, the horizontal and vertical normal stresses at point 1, computed by considering sections 1-3 and 1-4 respectively, become

$$\sigma_x = \frac{6M}{(h\cos\theta)^2 t} \quad \text{and} \quad \sigma_y = \frac{6M}{(h\sin\theta)^2 t} \qquad [104]$$

Since the three stresses in [103] and [104] occur at the same point, and since the horizontal and vertical shearing stress at that point must be assumed to be zero, the stresses should obey the relation

$$\sigma_1 = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta \qquad [105]$$

They do, and the three normal stresses are consistent; Equation [53] gives a satisfactory result in this application.

G. STATEMENT OF FORMULAS FOR DEFORMATIONS

19. Rotation and Translation of a Cross Section Relative to an Adjacent Cross Section - The two adjacent plane cross sections 1-2 and 3-4 in Figure 8a are identified by values of the coordinate u: u at points 1 and 2; and $u + \delta u$ at points 3 and 4. The motion of section 3-4 relative to section 1-2 under the influence of the loads on the cross section shown in Figure 8b can be stated most

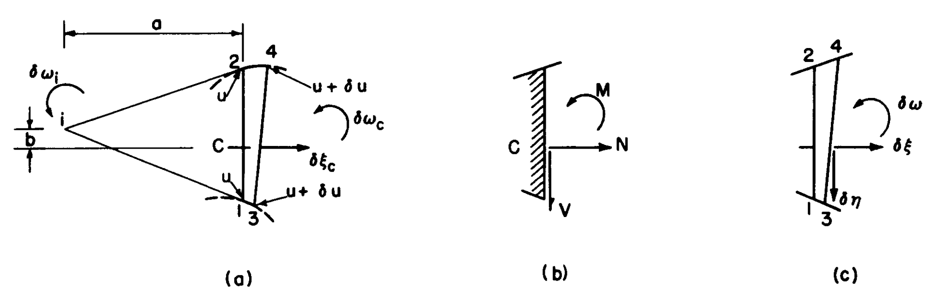

Figure 8 - Motion of Cross Section 3-4 Relative to an Adjacent Cross Section 1-2

conveniently in terms of three motions which take place at the same time and are combined by superposition. These motions are:

$\delta\omega_c$, a rotation about the effective center C, positive counterclockwise;

$\delta\xi_c$, a translation in the direction normal to the cross section, positive toward the right; and

$\delta\omega_i$, a rotation, positive counterclockwise, about point i, the moment center for shear, the coordinates of which are defined by [9] and [10].

The amounts of these motions can be stated as

$$\delta\omega_c = \frac{d\omega_c}{du}\,\delta u,\quad \delta\xi_c = \frac{d\xi_c}{du}\,\delta u,\quad \text{and} \quad \delta\omega_i = \frac{d\omega_i}{du}\,\delta u \qquad [106]$$

Then the problem is to state values of the three derivatives in [106]. Adequate approximation is obtained by the following three simple formulas, the derivation of which from laws of energy will be shown in Section 29:

$$\frac{d\omega_c}{du} = \frac{M}{E}\left(\frac{1}{J} + \frac{1}{B_v R^2}\right) \qquad [107]$$

$$\frac{d\xi_c}{du} = \frac{N}{EB} \qquad [108]$$

and

$$\frac{d\omega_i}{du} = \frac{M_i}{G B_s a^2} \qquad [109]$$

in which $M_i = M + Nb - Va$ (Equation [52]) is the moment about point i, with a and b defined by [9] and [10];

E is the modulus of elasticity; and

G is the modulus of elasticity in shear.

The quantities representing properties of the cross section are defined by previous equations: J by [8]; B by [4]; B_s by [22] and in some important cases by [23]; B_v by [24] or approximately by [25]; and R by [17].

If it is preferred, the motion of section 3-4 in Figure 8a relative to section 1-2 may be stated as a combination of another set of three motions; namely, those indicated in Figure 8c:

$\delta\omega$, the total rotation about C, positive counterclockwise;

$\delta\xi$, the translation in the direction of the normal, positive toward the right; and

$\delta\eta$, the translation in the direction of y, positive downward.

Simple geometrical considerations show that these motions have the following relations to those introduced first:

$$\delta\omega = \delta\omega_c + \delta\omega_i, \quad \delta\xi = \delta\xi_c + b\,\delta\omega_i, \quad \text{and} \quad \delta\eta = -a\,\delta\omega_i \qquad [110]$$

If one writes, in analogy with [106],

$$\delta\omega = \frac{d\omega}{du}\,\delta u, \quad \delta\xi = \frac{d\xi}{du}\,\delta u, \quad \text{and} \quad \delta\eta = \frac{d\eta}{du}\,\delta u \qquad [111]$$

Equations [106] and [111] give

$$\frac{d\omega}{du} = \frac{d\omega_c}{du} + \frac{d\omega_i}{du}, \quad \frac{d\xi}{du} = \frac{d\xi_c}{du} + b\,\frac{d\omega_i}{du}, \quad \text{and} \quad \frac{d\eta}{du} = -a\,\frac{d\omega_i}{du} \qquad [112]$$

Substitution from [107], [108], and [109] gives the following three formulas which are to be used in conjunction with [111]; these formulas are equivalent to [107], [108], and [109], and while they are in fact less convenient for most purposes, they may be preferred in some circumstances.

$$\frac{d\omega}{du} = M\left(\frac{1}{EJ} + \frac{1}{EB_vR^2} + \frac{1}{GB_sa^2}\right) + \frac{Nb}{GB_sa^2} - \frac{V}{GB_sa} \qquad [113]$$

$$\frac{d\xi}{du} = \frac{Mb}{GB_sa^2} + N\left(\frac{1}{EB} + \frac{b^2}{GB_sa^2}\right) - \frac{Vb}{GB_sa} \qquad [114]$$

$$\frac{d\eta}{du} = -\frac{M}{GB_sa} - \frac{Nb}{GB_sa} + \frac{V}{GB_s} \qquad [115]$$

Equations [113], [114], and [115] will be observed to possess symmetry of coefficients with respect to the diagonal downward to the right; this symmetry is in agreement with Maxwell's law of reciprocal displacements.

When the cross section is constant, $a = \infty$, $b/a = 0$, M_i/a is replaced by $-V$, and $\delta\omega_i/a$ is replaced by $-\delta\eta$. Then either set of formulas, Equations [107] to [109] or [113] to [115], assumes the form

$$\frac{d\omega}{du} = M\left(\frac{1}{EJ} + \frac{1}{EB_vR^2}\right), \quad \frac{d\xi}{du} = \frac{N}{EB}, \quad \text{and} \quad \frac{d\eta}{du} = \frac{V}{GB_s} \qquad [116]$$

In numerical computations it may be desirable to conceive of δu in [106] or [111] as a finite increment rather than an infinitesimal increment of u. In such computations the degree of accuracy of the formulas obviously will be improved by reinterpreting the values representing loads and properties of the cross section as values for the cross section midway between the two adjacent sections, instead of interpreting them as applying to the cross section at the left. Accordingly one may identify the two adjacent cross sections by the values $u - \frac{1}{2}\delta u$ and $u + \frac{1}{2}\delta u$, and introduce in the equations the loads and properties for the cross section identified by the value u.

20. Elastic Weights and Couples - An adaption of the method of elastic weights lends itself to the determination of the total motion of any cross section.

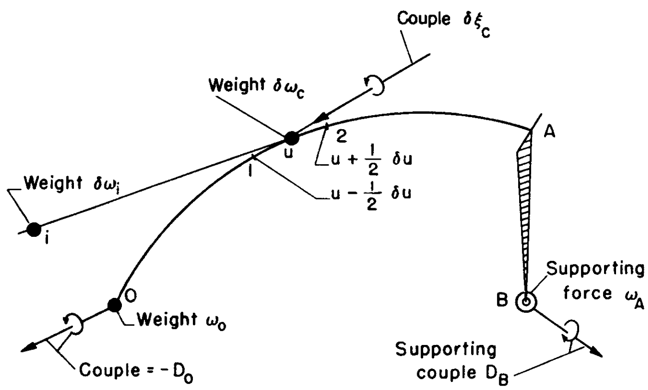

Figure 9 - Elastic Weights and Couples

In Figure 9 the curve OA represents a part of the effective centerline (the locus of the effective centers of the cross sections). Assume that the displacement D_o of the effective center of the cross section at O is already known in magnitude and direction, as a vector; and that the rotation ω_o of the cross section at O, positive in counterclockwise direction, is already known. Let the curve OA be divided into elements such as 1-2. The plane cross sections intersecting the centerline at points 1 and 2 will be identified by the values $u - \frac{1}{2}\delta u$ and $u + \frac{1}{2}\delta u$ of the curvilinear coordinate u; and the particular value u defines a point u and a cross section u that may be assumed to be midway between points 1 and 2. Point i is the moment center for shear for section u. For the element 1-2 the loads on the cross section and the properties of the cross section will be taken as those applying to cross section u. When these loads and properties are substituted in [107], [108], and [109], these equations and [106] will define the motion of section 2 relative to section 1. When the relative motions have been determined for all the elements into which the arc OA has been divided, the basis is ready for determining the total motion of cross section A.

The motion of cross section A may be described as consisting of two components; a translation with displacement D_B which is a vector having magnitude and direction; and a rotation ω_A, positive counterclockwise, about an arbitrarily chosen point B. If point B is attached to cross section A by an imagined rigid arm, as indicated in Figure 9, point B will move with the cross section; D_B will be the displacement of point B; and ω_A will be the rotation of the cross section and of the arm AB. The problem, then, is to determine the displacement D_B of point B, and the rotation ω_A of AB.

Before the elastic weights and couples are defined, it will be convenient to conceive of the picture in Figure 9 as tilted from the original vertical plane into a horizontal plane. Then all the elastic weights will be vertical forces positive downward, and all the elastic couples will be couples that act in vertical planes about horizontal axes. The couples may be represented by horizontal vectors contained in the plane of the drawing. The arc OA, all lines such as u-i, and the arm AB are imagined as solid and rigid and as joined together rigidly. They constitute a "conjugate structure," a purely imagined structure which imitates the original structure only by the shape of the centerline and in no other respect. The conjugate structure is loaded by the elastic weights and couples; it is supported at point B only.

Figure 9 shows the elastic weights and couples. They are: (a) At point O, a weight ω_0; the statement that this weight is positive downward like the other elastic weights means that the force exerted on the conjugate structure at O is actually downward if the rotation ω_o is positive, that is, actually counterclockwise. (b) At point O, a couple $-D_o$; the line vector representing this couple is laid off in the direction opposite to that of the displacement D_o; the round arrow is drawn according to the right-hand rule (fingers in direction of round arrow, thumb in the direction of the vector), and it shows the direction in which the couple turns. (c) For the element 1-2 a weight $\delta\omega_c$, defined by [107], at u, a weight $\delta\omega_i$, defined by [109], at point i, and a couple of magnitude $\delta\xi_c$, defined by [108], turning in the plane of the original cross section, and represented by a line vector normal to the cross section and pointing backward. (d) Similar weights and couples for all other elements like 1-2.

The reaction created at B by the loads that have been described will consist of a vertical force through B and a couple. By simple considerations of geometry applied to the original beam and of statics applied to the conjugate structure it can be seen that the rotation ω_A, positive counterclockwise, is equal to the supporting force at B, positive upward; and that the displacement D_B is equal in magnitude and direction to the line vector representing the supporting couple. Thus the total motion of the original cross section A and the attached point B is determined as reactions supporting the conjugate structure.

If one prefers, the rigid piece *OA* may be considered a part of a curved conjugate beam extending beyond *O* and *A* along the effective centerline of the original beam. Then the forces and couples at *O* and *B* in Figure 9 are reinterpreted as internal forces and couples acting on the cross sections *O* and *A* in the conjugate beam; the rotation of a cross section in the original beam becomes minus the vertical shear on the corresponding section in the conjugate beam; and the displacements in the original beam become bending and twisting moments in the conjugate beam.

H. EVIDENCE AND DERIVATIONS

21. Summary of Evidence - The following is a summary of the evidence that will be presented concerning the adequacy of the basic formulas that were stated for stresses in Sections 14 and 15 and for deformations in Section 19:

(a) A redistribution of the stresses would increase the most important item in the energy of the fiber stresses; a relative minimum of that item is attained; see Section 22.

(b) The formulas for fiber stresses, when applied to a circular beam with constant cross section, agree with those given in the standard theory of curved beams with constant cross section. The shearing stresses and radial stresses obtained for circular beams with constant cross section are derived from the fiber stresses; see Section 23.

(c) The formulas for stresses agree with facts that are known or can be derived for wedge-shaped beams; see Section 24.

(d) General features of the state of stress in the immediate vicinity of the end of a crack, as derived by use of the network of fibers and cross sections, agree with facts known about stresses at cracks; see Section 25.

(e) In the general case under consideration the shearing stresses can be derived from the fiber stresses with acceptable approximation; see Section 26.

(f) In the general case under consideration the radial stresses, i.e., the normal stresses across the fibers, as defined by the formulas, meet the following requirements: near the extreme fibers or next to the flanges the radial stresses are consistent with the fiber stresses; at intermediate points the variation is smooth, takes the change of radius of curvature into account, and agrees with the results derived for the circular beam with constant cross section. In addition, the formula for the radial stress can be derived by an approximate process; see Section 27.

(g) The normal stresses on the plane cross section derived from the formulas of improved accuracy for the fiber stresses, with shearing stresses and radial stresses taken into account, are consistent with the load on the cross section; see Section 28.

(h) The formulas for the motion of a cross section relative to an adjacent cross section are derived by use of Castigliano's law of derivatives of energy; see Section 29.

22. **Minimum of Energy of the Fiber Stresses** - Let the fiber stress at the cross section 1-2 in Figure 4 be stated in the form

$$\sigma_u = \sigma + \phi \qquad [117]$$

in which σ is the value of the fiber stress given in [54],

$$\sigma = \frac{Myg}{J} + \frac{Ng}{B} \qquad [118]$$

while ϕ represents whatever correction may be needed.

The normal stress on the cross section may be stated in terms of the stresses along and across the fibers; see Figure 5. The value is

$$\sigma_x = \sigma_u \cos^2\alpha + \sigma_v \sin^2\alpha - \tau \sin 2\,\alpha \qquad [119]$$

Since σ_v and τ are generally less important than the fiber stress σ_u and are multiplied by the small factors $\sin^2\alpha$ and $\sin 2\alpha$, the last two terms in [119] may be ignored for the present purpose. Then [117], [118], and [119] give

$$\sigma_x = \frac{Myg\cos^2\alpha}{J} + \frac{Ng\cos^2\alpha}{B} + \phi\cos^2\alpha \qquad [120]$$

By referring to the definitions of J and B in [8] and [4] it will be seen that the first two terms on the right side in [120] account fully for M and N. It follows that the supplementary stresses $\phi\cos^2\alpha$ are self-balancing. Then

$$\int \phi\cos^2\alpha\,dA = \int \phi y \cos^2\alpha\,dA = 0 \qquad [121]$$

and consequently, by use of [118],

$$\int \sigma\phi g^{-1}\cos^2\alpha\,dA = 0 \qquad [122]$$

It is concluded, that so far as requirements of statics are concerned, the correction ϕ in [117] may be introduced as any function of y satisfying [121]; ϕ will then also satisfy [122].

The curvilinear coordinates u and v define curved cross sections by u = constant, and fibers by v = constant. In terms of these coordinates the equation of the plane cross section 1-2 in Figure 4 may be stated in the form

$$u = u' - u_o \qquad [123]$$

in which u' is the value of u at the extreme points 1 and 2, and u_o is a function of v. A sequence of "usually slightly curved cross sections" is obtained by assigning a sequence of values to u' in [123]. It will be observed that in the two important cases of a circular beam with constant cross section and of a wedge-shaped beam these "usually slightly curved cross sections" are, in fact, plane. The sequence of the "usually slightly curved cross sections" will divide the beam into a series of usually slightly curved and usually wedge-shaped slices, each bounded by two successive cross sections in the sequence. It is important that the combined space occupied by these slices is substantially the whole volume of the beam, and that, consequently, the sum of energy stored in these slices will be substantially the total energy stored in the beam. The slight curvature of the slices will not interfere materially with the applicability of [117] as a formula for the fiber stresses in any one of the slices.

A cross section, usually slightly curved, adjacent to the plane cross section 1-2 in Figure 4 is obtained by replacing u' in [123] by $u' + \delta u$. Since the gradient $\partial v/\partial y$ along the plane cross section is g, the gradient along the fiber will be $g \sec \alpha$, and the distance measured along a fiber from the plane cross section 1-2 to the adjacent section will be $g^{-1} \cos \alpha \, \delta u$. It follows that the distance measured normal to the plane cross section, the thickness of the slice at the particular place, will be $g^{-1} \cos^2 \alpha \, du$.

The energy of the fiber stresses within the slice just defined may now be stated as

$$\delta U = \frac{\delta u}{2E} \int g^{-1} \cos^2 \alpha \, \sigma_u^2 \, dA \qquad [124]$$

in which $\sigma_u = \sigma + \phi$ according to [117]. By referring to [122] it will be seen that [124] may be restated in the form

$$\delta U = \frac{\delta u}{2E} \int g^{-1} \cos^2 \alpha \, (\sigma^2 + \phi^2) \, dA \qquad [125]$$

Clearly, this δU attains a minimum when

$$\phi = 0 \tag{126}$$

that is, when σ in [118] (or in [54]) is the whole fiber stress.

Assume that it can be declared that when the fibers are drawn as specified, the energy of the fiber stresses will be of dominating importance compared with the energy contributed by the shearing stresses along and across the fibers and by the radial normal stresses across the fibers. Then $\phi = 0$ and $\sigma_u = \sigma$ will make practically the whole stress energy a minimum. This result can be considered as evidence of the rationality of [118], or [54], as a formula for the fiber stress.

The argument that has been presented does not preclude improvements either by small supplementary terms in the formula for the fiber stress as in [58], or by changing the specification for drawing the network of fibers and curved sections. The supplementary terms in [58] will be discussed in Sections 24 and 28. If the network were drawn by a different specification, but still with intersections of fibers and curved cross sections at right angles, the argument that was presented remains valid if g is redefined so that $g \sec \alpha$ retains its meaning as the gradient of u along the fibers. Reasons can be given, however, for retaining the specification of the network: (a) the general plausibility of the pattern; (b) the definiteness of the specification; and (c) the fact that right results are obtained, as shown in Sections 23 to 25, in important cases about which information is available.

23. **Circular Beam with Constant Cross Section** - It was shown in Section 16 that the general formulas for fiber stresses, Equations [54] and [58], give the same results in this case as the well-established standard theory of such beams. It remains therefore to show the validity of the formulas for shearing stresses and radial stresses.

With coordinates and positive directions as in Figure 3 the stresses acting on an element bounded by circles with radii r and $r + dr$ and by radial lines θ and $\theta + d\theta$ must obey the following two equations of equilibrium, the first of which expresses the balance of moments about O', and the second the balance of radial forces.

$$\frac{\partial(t\tau r^2)}{\partial r} = \frac{\partial(t\sigma r)}{\partial\theta} \tag{127}$$

and

$$\frac{\partial(t\sigma_v r)}{\partial r} = \frac{\partial(t\tau)}{\partial\theta} + t\sigma \tag{128}$$

in which t is the thickness or width. Furthermore, with directions as in Figure 3, and with no external forces in the immediate vicinity, equilibrium requires that

$$\frac{dM}{d\theta} = RV, \quad \frac{dN}{d\theta} = V, \quad \text{and} \quad \frac{dV}{d\theta} = -N \qquad [129]$$

With the fiber stress σ defined by [71], and with [129] taken into account, it is found that the shearing stress τ in [78] satisfies [127]; it is found furthermore that the radial stress σ_v in [80] and τ in [78] satisfy [128]; and that this solution for τ and σ_v is the only one that also satisfies the conditions at or next to the extreme fibers. This proves the correctness of [78], which is consistent with the general formula [55] for τ, and of [80], which is consistent with the general formula [63] for σ_v.

24. **Wedge-Shaped Beam** - In the wedge-shaped beam in Figure 10a the fibers are straight lines, θ = constant, radiating from the vertex i; the curved sections are circles, r = constant; also, $\alpha = \theta$; and the gradient in

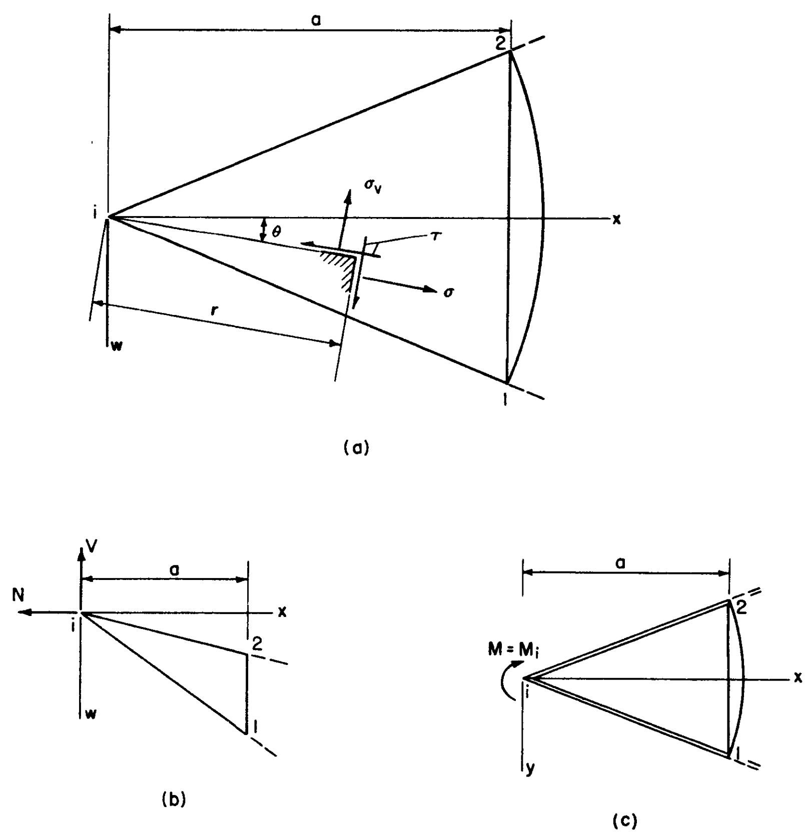

Figure 10 - Wedge-Shaped Beams

the direction of the fiber, as may be seen by comparison with a circular beam, can be stated as a/r. Then the gradient along a straight cross section defined by x = constant becomes

$$g = \frac{a \cos\theta}{r} = \frac{a \cos^2\theta}{x} \qquad [130]$$

At the particular section 1-2

$$g = \cos^2\theta \qquad [131]$$

A familiar type of solution in the theory of elasticity is the following, which covers as an example Boussinesq's problem in two dimensions and applies generally to a plate the thickness of which is a function of θ only, under a proper load contained in the plane of symmetry:

$$\sigma = \frac{C \sin\theta}{r} + \frac{D \cos\theta}{r} \;, \quad \sigma_v = \tau = 0 \qquad [132]$$

in which C and D are constants and the stresses are those shown in Figure 10a. If the wedge-shaped beam in Figure 10b has a thickness or width that is a function of θ only, for example, a web of constant thickness, and flanges with cross-sectional area proportional to r, the stresses will be as in [132] provided that the resultant load is a force through the vertex as indicated in Figure 10b Then, using [131] and [132], one may state the fiber stress and normal stress at section 1-2 as follows:

$$\sigma = \frac{Cwg}{a^2} + \frac{Dg}{a} \qquad [133]$$

and, since $\theta = \alpha$,

$$\sigma_x = \frac{Cwg \cos^2\alpha}{a^2} + \frac{Dg \cos^2\alpha}{a} \qquad [134]$$

When values are assigned to C and D so as to account for the loads N and V, Equations [133] and [134] assume precisely the form of [53] and [54], even though the cross section be not drawn according to the specified rules but in an arbitrary direction; and they assume that form no matter how great the angle is at the vertex. Also, the result $\tau = \sigma_v = 0$ agrees with [55] and [63].

If the wedge-shaped beam in Figure 10c has two equal flanges with cross-sectional areas proportional to r, and has a web of constant thickness, and if the load is a couple $M = M_i$ applied at the vertex, the following solution, known from the theory of elasticity, will apply in the web:

$$\sigma = \frac{2C \sin 2\theta}{r^2}, \quad \tau = \frac{1}{r^2}(-C\cos 2\theta + D), \ \sigma_v = 0 \qquad [135]$$

C and D being appropriate constants. The solution in [135] may be obtained by means of an Airy function f which defines the stresses by the general formulas

$$\sigma = \sigma_r = \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}, \quad \sigma_v = \sigma_\theta = \frac{\partial^2 f}{\partial r^2} \qquad [136]$$

and

$$\tau = \tau_{r\theta} = -\frac{\partial^2}{\partial r \partial \theta}\left(\frac{f}{r}\right) \qquad [137]$$

In this case the Airy function is

$$f = -\frac{1}{2} C \sin 2\theta + D\theta \qquad [138]$$

At section 1-2 [131] and [135] give

$$\sigma = \frac{4Cyg}{a^3} - \frac{4Cyg \sin^2\alpha}{a^3} \qquad [139]$$

and

$$\tau = \left[(C+D)y^2 - (C-D)a^2\right]\frac{g^2}{a^4} \qquad [140]$$

Equation [139] would be of the form of [54] if the second term on the right side were omitted. That second term is ordinarily small; it represents a correction which is taken into account in the formula of improved accuracy, Equation [58]. In the present case, Equation [58] becomes

$$\sigma = \frac{M(1+k)yg}{J} - \frac{Myg\sin^2\alpha}{J} \qquad [141]$$

Equation [141] would be exactly of the form of [139] if the last term were multiplied by $1 + k$, but since k and $\sin^2\alpha$ are both small this correction is unnecessary. Thus [139] serves toward confirming [58].

Equation [140] will be recognized as being of the form of [55] and serves toward confirming that equation. Finally, $\sigma_v = 0$, Equation [135], agrees with [57] because in this case $\rho = \infty$.

25. Cracked Beam - Let l denote the length of the crack in Figure 1g, and h the depth of the beam. With the origin of coordinates x, y chosen at the upper end of the crack, and the axis of x in this particular case pointing upward, the network of fibers and cross sections is defined by the function

$$Z = -i\,\frac{2h}{\pi}\cos^{-1}\frac{\cos\dfrac{\pi(z+l)}{2h}}{\cos\dfrac{\pi l}{2h}} = u + iv \qquad [142]$$

The corresponding gradients on the cross section directly above the crack may be expressed as follows for small values of the distance r from the end of the crack:

$$g = C_1 r^{-0.5} + C_2 r^{0.5} + C_3 r^{1.5} + \cdots \qquad [143]$$

in which the first term dominates. If the fiber stress on that section obeys [54], its value may be written as

$$\sigma = (D_0 + Dr)\,g \qquad [144]$$

By this formula the fiber stress at $r = 0$ is infinite, which means that the stress is very great, unless $D_0 = 0$, in which case the dominating term in the fiber stress, for small values of r, becomes

$$\sigma = C_1 D\, r^{0.5} \qquad [145]$$

It is entirely possible that D_0 may be zero, because the length of the crack will adjust itself to the load on the cross section owing to the very fact that the stress at $r = 0$ must be either very great or zero.

These general conclusions, derived in accordance with the formulas, agree with facts that are known about cracks.*

26. Shearing Stresses in a General Case - The following derivation of the shearing stresses in a general case is approximate only. The aim is to account approximately for the total amount of shear and to establish approximately the right distribution.

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* See "Bearing Pressures and Cracks," by H.M. Westergaard, Journal of Applied Mechanics, Transactions, American Society of Mechanical Engineers, June 1939, pp. A-49 to A-53.

In terms of the stresses along and across the fibers the shearing stress on the plane cross section 1-2 in Figure 4 is

$$\tau_{xy} = \frac{1}{2}(\sigma - \sigma_v)\sin 2\alpha + \tau\cos 2\alpha \qquad [146]$$

For the present purpose it will be permissible to simplify [146] by omitting the least important features and write.

$$\tau_{xy} = \frac{1}{2}\sigma\sin 2\alpha + \tau = \frac{1}{2}\left(\frac{Myg}{J} + \frac{Ng}{B}\right)\sin\ 2\alpha + \tau \qquad [147]$$

By [147] τ is interpreted as a part of the shearing stress on the plane cross section. By referring to [9] and [10] it will be seen that the sum of the moments of these stresses τ about the moment center for shear, point i, will be

$$-a\int\tau\, dA = M + Nb - Va = M_i \qquad [148]$$

Equation [148] expresses the requirement that the total amount of shear be right. By referring to the definitions of K and K_w in [12] and [19], it will be seen immediately that τ in [55] and τ' in [60] account exactly and fully for M_i; and that τ'' in [61] adds nothing to M_i. On the other hand τ''' in [62] may contribute a small amount to M_i; but τ''' is the least important part of τ in [59], and the possible discrepancy is very unimportant. The formulas account satisfactorily for the total amount of shear.

Figure 11 shows a small element bounded by curved cross sections u and $u + du$ and by fibers v and $v + dv$, and shows the forces acting on the element. The quantity g_o denotes the gradient in the direction of the fibers or in the direction of the curved cross sections, so that

$$g_o = g\ \sec\alpha \qquad [149]$$

Inspection of Figure 11 shows that balance of forces in the direction of the fibers requires that

$$\frac{\partial(t\sigma g_o^{-1})}{\partial u} + \frac{\partial(t\tau g_o^{-1})}{\partial v} + t\tau\ \frac{\partial g_o^{-1}}{\partial v} = t\sigma_v\frac{\partial g_o^{-1}}{\partial u} \qquad [150]$$

In [150] the second and third terms may be combined. Furthermore, because of the slow variation of $\cos\alpha$, it will be permissible in [150] to replace g_o by g. Moreover, the term on the right side may be ignored on the

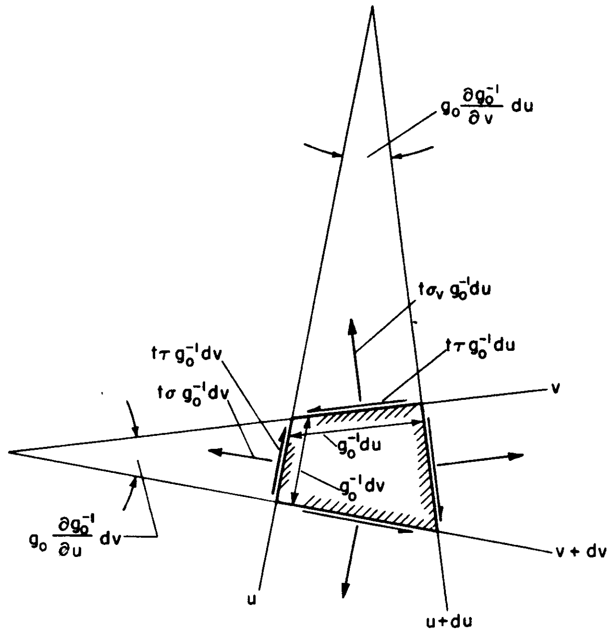

Figure 11 - Stresses on Small Element

assumption that either σ_v or the derivative of $g_o{}^{-1}$ will be sufficiently small. Thus [150] is replaced by

$$-g\,\frac{\partial(t\tau g^{-2})}{\partial v} = \frac{\partial(t\sigma g^{-1})}{\partial u} \qquad [151]$$

Let s denote distance measured along the effective centerline, and let this s be used as a coordinate identifying each of the plane or slightly curved cross sections discussed in Section 22 after [123]. Then, in view of the moderate departure of the curved from the plane sections, it will be permissible, as an approximation, to replace ∂u in [151] by $B/A\ \partial s$, and ∂v by gdy. Thereby, and with σ substituted from [54], Equation [151] is changed to

$$\begin{aligned} -\frac{\partial(t\tau g^{-2})}{\partial y} &= \frac{A}{B}\,\frac{d}{ds}\left(\frac{Myt}{J} + \frac{Nt}{B}\right) \\ &= \frac{A}{B}\left[\frac{yt}{J}\,\frac{dM}{ds} + \frac{t}{B}\,\frac{dN}{ds} + M\,\frac{d}{ds}\left(\frac{yt}{J}\right) + N\,\frac{d}{ds}\left(\frac{t}{B}\right)\right] \qquad [152] \end{aligned}$$

In [152] one may subsitute (see [129] and [17])

$$\frac{dM}{ds} = V - \frac{Nb}{a} \qquad \text{and} \qquad \frac{dN}{ds} = \frac{V}{R} = \frac{BVj}{J} \qquad [153]$$

In the further derivations the subscripts 1, 2, and w serve to designate values contributed by flange 1, flange 2, and the web, respectively. (If there are no flanges, $B_w = B$, $K_w = K$, etc., and the formulas may still be used). As one passes from cross section 1-2 in Figure 4 to an adjacent section by increasing s by ds, it can be assumed that the flange areas A_1 and A_2 remain constant, and that A_w increases by the ratio $\frac{a + ds}{a}$; g will generally decrease with the result that B_w remains nearly constant and $B_1 + B_2$ decreases. Accordingly one may assume that

$$\frac{d}{ds}\left(\frac{B}{t}\right) = -\frac{B_1 + B_2}{at} \qquad \text{and} \qquad \frac{d}{ds}\left(\frac{t}{B}\right) = \frac{(B_1 + B_2)\,t}{aB^2} \tag{154}$$

In the same way, and with consideration of the increase of y by moving along a fiber to the adjacent section, one arrives at the formula

$$\frac{d}{ds}\left(\frac{yt}{J}\right) = -\frac{J_w yt}{aJ^2} \approx -\frac{K_w yt}{aJK} \tag{155}$$

One more approximate substitution will be made; namely,

$$\frac{BJ}{A} = K \tag{156}$$

By writing

$$\tau = \tau' + \tau'' + \tau''' + \tau'''' \tag{157}$$

and dividing [152] accordingly into four equations, and by substituting from [153] to [156], one obtains the following equations which, although they are only approximate, will nevertheless define the variation of the shearing stress in the web reasonably well.

$$-\frac{\partial(t\tau' g^{-2})}{\partial y} = \frac{1}{aK}(Va - Nb - M)(y + j)\,t = -\frac{M_i(y + j)\,t}{aK} \tag{158}$$

$$-\frac{\partial(t\tau'' g^{-2})}{\partial y} = \frac{M(y + j)\,t}{aK} - \frac{MK_w(y + j)\,t}{aK^2} \tag{159}$$

$$-\frac{\partial(t\tau''' g^{-2})}{\partial y} = \frac{NA\,(B_1 + B_2)\,t}{aB^3} \tag{160}$$

and

$$-\frac{\partial(t\tau'''' g^{-2})}{\partial y} = \frac{MK_w jt}{aK^2} + \frac{Nbjt}{aK} \tag{161}$$

By noting that according to [11]

$$\frac{dQ}{dy} = -(y + j)t \qquad [162]$$

it will be seen that τ' and τ'' in [60] and [61] satisfy [158] and [159] exactly. Furthermore, it will be seen that the relatively unimportant stress τ''' in [62] satisfies [161] nearly. Finally, the right side in [161] is so small that it will be justified to ignore τ''''.

When the evidence thus obtained by approximate derivations is added to the results of precise derivations in Section 23 and 24 for the special cases of circular beams and wedge-shaped beams, the conclusion is justified that the shearing stresses defined in [59] to [62] are approximately right in amount and distribution. Since the shearing stresses are relatively less important than the fiber stresses, the degree of approximation that has been attained will be ample for the present purposes.

27. **Radial Stresses in a General Case** - If the fiber stress in flange 1 is σ_1, and if flange 1 has the cross-sectional area F_1 normal to the fibers and the radius of curvature ρ_1, equilibrium requires that the radial stress directly inside the flange be

$$\sigma_{v,1} = \frac{F_1 \sigma_1}{\rho_1 t} \qquad [163]$$

With σ_1 defined by [54], Equation [163] leads directly to [67]. Equation [68] is verified in the same way.

Equations [67] and [68] for the radial stress next to the flanges were obtained in Section 15 as special cases of [63] for the radial stress at any place. Equations [67] and [68] therefore serve as evidence in support of [63]. Further evidence is that [63] agrees perfectly with [80] for the radial stress in a circular beam, which was derived in Section 23 by means of [128]. Moreover it is observed that [63] defines a smooth variation which takes into account reasonably the variation of the radius of curvature of the fibers. Since the radial stresses are relatively less important than the fiber stresses, these observations may be considered sufficient evidence.

It is possible, however, to add the following which will serve to confirm the variation of the radial stress between the flanges as defined by [63]. First, an equation is written which is a companion of [150], expresses the balance of forces in the direction of the curved cross sections, and is of the same form as [150] except that the two middle terms have been combined into one. The equation is

$$\frac{\partial(t\sigma_v g_o^{-1})}{\partial v} + g_o \frac{\partial(t\tau g_o^{-2})}{\partial u} = t\sigma \frac{\partial g_o^{-1}}{\partial v} \qquad [164]$$

in which, as before, $g_o = g \sec \alpha$; see Equation [149].

Because of the relatively small departure of the curved from the plane cross section it will be permissible here, as in obtaining [151], to convert [164] into an equation in terms y and s by replacing ∂v by $g\partial y$ and ∂u by $\frac{B}{A}\partial s$, s being measured along the effective centerline and identifying the usually slightly curved sections introduced in Section 22. Inspection of Figure 11 shows that the radius of curvature ρ of the fibers is defined by

$$\frac{1}{\rho} = -g_o^2 \frac{\partial(g_o^{-1})}{\partial v} = \frac{\partial g_o}{\partial v} \qquad [165]$$

in which the minus sign means that the value is positive when the center of curvature lies on the side of the greater values of v. Accordingly Equation [164] will be replaced by the nearly equivalent equation

$$\frac{\partial(t\sigma_v g^{-1}\cos\alpha)}{\partial y} = -\frac{t\sigma\cos^2\alpha}{\rho g} - \frac{Ag^2}{B\cos\alpha} \frac{\partial(t\tau g^{-2}\cos^2\alpha)}{\partial s} \qquad [166]$$

The last term in [166] will be examined first. Let τ be taken from [55]. Then

$$t\tau g^{-2} = -\frac{M_i Q}{aK} = -\frac{(M + Nb - Va)Q}{aK} \qquad [167]$$

In passing from one plane cross section to an adjacent section it can be assumed that Q varies approximately in proportion to a^2 and K approximately in proportion to a so that Q/aK remains nearly constant. Since the angle between the two sections is approximately ds/R, point i for the adjacent section may be assumed to lie at a distance ads/R in the direction of $-y$ above point i for the original section. M_i for the adjacent section may be computed as the moment of the load on the original section about the new point i. Accordingly one may write

$$\frac{dM_i}{ds} = \frac{Na}{R} \qquad [168]$$

With the derivative of $\cos^2\alpha$ considered unimportant and with the permissible replacement $AK = BJ$ the result is that the last term in [166] will be replaced by the last term in the next equation, Equation [169].

Then, with σ as in [54], Equation [166] becomes

$$\frac{\partial(t\sigma_v g^{-1}\cos\alpha)}{\partial y} = -\frac{t\cos^2\alpha}{\rho}\left(\frac{My}{J} + \frac{N}{B}\right) + \frac{A^2NQg^2\cos\alpha}{B^2JR} \qquad [169]$$

Equation [63] for σ_v will be tested by [169]. For this purpose [63] is rewritten, with use of [17], in the form

$$t\sigma_v g^{-1}\cos\alpha = \frac{1}{\rho g}\left[\frac{MS}{J} + \frac{N}{Bj}(Qg\cos^2\alpha - S)\right] \qquad [170]$$

By referring to [11] and [13] it is seen that

$$\frac{\partial}{\partial y}(Qg\cos^2\alpha - S) = -tjg\cos^2\alpha + Q\,\frac{\partial(g\cos^2\alpha)}{\partial y} \qquad [171]$$

in which according to [165] the last term may be replaced approximately by $Qg\cos^3\alpha/\rho$. The product ρg is constant in a circular beam; let it be assumed that ρg varies sufficiently slowly to permit ignoring the derivative $\frac{\partial}{\partial y}\left(\frac{1}{\rho g}\right)$. Then it will be found that substitution from [170] in [169] will make all terms cancel except a residue which, when gathered on the right side, becomes

$$\text{Residue} = \frac{NQ\cos\alpha}{JR}\left(\frac{A^2g^2}{B^2} - \frac{R^2\cos^2\alpha}{\rho^2}\right) \qquad [172]$$

The residue in [172] vanishes for a circular beam. In other cases it will invariably be very small compared with the right side in [169]. The derivation confirms [63] as an approximate formula for the radial stress.

28. **Formula of Improved Accuracy for the Fiber Stresses** - In the formula referred to, Equation [58], the first two terms alone represent the first approximation in [54]; the remaining terms represent corrections, which will generally be small.

The term in [58] containing $\sin^2\alpha$ was justified in Section 24 through [139] and [141] as applicable to a wedge-shaped beam; general applicability of such a term can be assumed. It will be shown now that if the remaining terms are to be a linear function of y, they must have values that are very close to those given in [58] if the resultant of the normal stresses on the plane cross section is to be accounted for accurately. In the computation of the supplementary terms a moderate degree of relative accuracy is sufficient because the corrective terms are themselves small.

The normal stress on cross section 1-2 in Figure 4 is

$$\sigma_x = \sigma \cos^2\alpha + \sigma_v \sin^2\alpha - \tau \sin 2\alpha \qquad [173]$$

With σ as in [58], σ_x may be stated in the form

$$\sigma_x = \sigma_x' + \sigma_x'' + \sigma_x''' \qquad [174]$$

in which

$$\sigma_x' = \frac{Myg\cos^2\alpha}{J} + \frac{Ng\cos^2\alpha}{B} \qquad [175]$$

(the same as σ_x in the first approximation, Equation [53]),

$$\sigma_x'' = \frac{M_i y g\cos^2\alpha}{J}(k - \sin^2\alpha) - \tau\sin 2\alpha \qquad [176]$$

and

$$\sigma_x''' = -\frac{Mkg\cos^2\alpha}{RB} + \sigma_v\sin^2\alpha \qquad [177]$$

The stress σ_x' in [175] accounts fully for the bending moment M and the normal force N. It can be seen that the contributions of σ_x'' to N and of σ_x''' to M will be negligible. It remains to show, then, that σ_x'' adds nothing to M, and that σ_x''' adds nothing to N.

With k as in [15], it is noted that

$$\int y^2 S\,dy = \frac{1}{3}\int y^4 dB = a^2 J k \qquad [178]$$

Furthermore, the following approximate statements will be admissible in the present application in integrations:

$$\sin^2\alpha \approx \frac{y^2}{a^2}, \quad \sin 2\alpha \approx \frac{2y}{a} \qquad [179]$$

$$\tau \approx -\frac{M_i S}{aJt}, \qquad \text{and} \qquad \sigma_v \approx \frac{MS}{JRt} \qquad [180]$$

(Compare with [55] and [57]). Then, by substituting from [179] and [180] in [176] and [177], and by use of [178], one finds

$$\int \sigma_x'' y\,dA = 0 \qquad \text{and} \qquad \int \sigma_x''' dA = 0 \qquad [181]$$

This completes the verification of [58] for the fiber stress.

29. **Deformations** - The shearing stresses generally keep the originally plane cross section from remaining perfectly plane under load. Nevertheless one may speak rationally of rotations and translations of cross sections. These motions are interpreted as average rotations and average translations. The averages can be determined by use of Castigliano's law of derivatives of the stress energy.

For the purpose of the present derivation the cross section 3-4 in Figure 8 will be defined as in Section 22; it is a "usually slightly curved section" with each point at the distance $g^{-1} \cos^2 \alpha \, \delta u$ from section 1-2. The stress energy in the slice between the two sections may be expressed as

$$\delta U = \delta U_E + \delta U_G \qquad [182]$$

in which, with μ denoting Poisson's ratio,

$$\delta U_E = \frac{\delta u}{2E} \int g^{-1} \cos^2 \alpha \, (\sigma^2 - 2\mu \, \sigma \, \sigma_v + \sigma_v^2) \, dA \qquad [183]$$

and

$$\delta U_G = \frac{\delta u}{2G} \int g^{-1} \cos^2 \alpha \; \tau^2 \, dA \qquad [184]$$

In applications of [183] the term containing μ, because of the distribution of positive and negative values of the product $\sigma\sigma_v$, will contribute only slightly to the several integrals, and may therefore be ignored.

Then the three motions defined at the beginning of Section 19 can be derived as follows: Imagine first, and temporarily only, that G is infinite for the directions along and across the fibers, but that E remains finite. Under these circumstances δU_E will be the whole stress energy in the slice, and the first two of the motions will be, according to Castigliano's law,

$$\delta \omega_c = \frac{\partial \delta U_E}{\partial M} = \frac{\delta u}{E} \int g^{-1} \cos^2 \alpha \left(\sigma \frac{\partial \sigma}{\partial M} + \sigma_v \frac{\partial \sigma_v}{\partial M} \right) dA \qquad [185]$$

and

$$\delta \xi_c = \frac{\partial \delta U_E}{\partial N} = \frac{\delta u}{E} \int g^{-1} \cos^2 \alpha \left(\sigma \frac{\partial \sigma}{\partial N} + \sigma_v \frac{\partial \sigma_v}{\partial N} \right) dA \qquad [186]$$

and these two motions will constitute the complete motion. Imagine next, and temporarily only, that E is infinite for the directions along and across the fibers, but that G is finite. Under these circumstances δU_G will be the whole stress energy in the slice, and the last of the three motions will be

$$\delta \omega_i = \frac{\partial \delta U_G}{\partial M_i} = \frac{\delta u}{G} \int g^{-1} \cos^2 \alpha \; \tau \frac{\partial \tau}{\partial M_i} \; dA \qquad [187]$$

and this will be the complete motion.

Accordingly, Equations [185] and [186] can be said to represent the motions caused by the elasticity measured by $1/E$, and [187] can be said to represent the motion caused by the elasticity measured by $1/G$. When both E and G have their proper finite values, at the same time, the three motions defined by [185] to [187] will occur at the same time, and will superimpose on each other.

With σ, τ, and σ_v as in [54], [55], and [57], and with B, J, a, R, B_s, and B_v as in [4], [8], [9], [17], [22], and [24] respectively, Equations [185], [186], and [187] lead directly to [107], [108], and [109]. Equations [113], [114], and [115] were derived from [107], [108], and [109].

This completes the verification of the formulas for the motions of a cross section relative to an adjacent cross section.

I. CONCLUSIONS

30. - It has been shown that stresses in knees and sharply curved beams under loads producing bending without twisting may be computed at any point and in any direction; and that rotations and translations of a cross section relative to any other cross section in such a knee or beam may be determined. Networks of fibers and cross sections serve these purposes. In its entirety the problem is fairly complex, yet the formulas representing the first approximation are quite simple, and the formulas of improved accuracy are not excessively complex.

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