THE DAVID W. TAYLOR MODEL BASIN

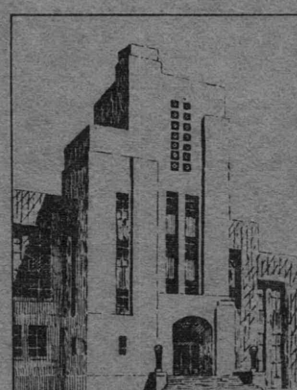
UNITED STATES NAVY

INTRODUCTION TO NON-LINEAR MECHANICS

PART II

ANALYTICAL METHODS OF NON-LINEAR MECHANICS

BY N. MINORSKY, Ph.D.



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BY N. MINORSKY, Ph.D.

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DAVID TAYLOR MODEL BASIN

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FOREWORD

The report on the introduction to non-linear mechanics as a whole falls into four major divisions.

Part I, published as David Taylor Model Basin Report 534 under date of December 1944, is concerned with the topological methods; its presentation substantially follows the "Theory of Oscillations" by Andronow and Chaikin. The material is slightly rearranged, the text is condensed, and a number of figures in this report were taken from the book. Chapter V, concerning Liénard's analysis, was added since it constitutes an important generalization and establishes a connection between the topological and the analytical methods, which otherwise might appear as somewhat unrelated.

Part II, published here, gives an outline of the three principal analytical methods, those of Poincaré, Van der Pol, and Kryloff-Bogoliuboff.

Part III, to be published soon, deals with the complicated phenomena of non-linear resonance with its numerous ramifications such as internal and external sub-harmonic resonance, entrainment of frequency, parametric excitation, and the like. This subject is still in a state of development, and the classification of the numerous experimental phenomena is far from being definitely established. Much credit for the experimental discoveries and theoretical studies of these phenomena is due to Mandelstam and Papalexi, and to the school of physicists under their leadership. The first four chapters of Part III will represent the application of the quasi-linear theory of Kryloff and Bogoliuboff to these problems and the last three will concern the developments of Mandelstam, Papalexi, Andronow, Witt, and others, following the classical theory of Poincaré.

Finally, Part IV will review the interesting developments of Mandelstam, Chaikin, and Lochakow in the theory of relaxation oscillations for large values of the parameter μ. This theory is based on the existence of quasi-discontinuous solutions of differential equations at the point of their "degeneration," that is, when one of the coefficients approaches zero so that the differential equation "degenerates" into one of lower order. A considerable number of experimental facts will be explained on the basis of this theoretical idealization.

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INTRODUCTION TO NON-LINEAR MECHANICS

PART II

ANALYTICAL METHODS OF NON-LINEAR MECHANICS*

44. INTRODUCTORY REMARKS

It is apparent that for practical applications the general qualitative methods reviewed in Part I are not sufficient, and that quantitative methods capable of yielding numerical solutions of differential equations are necessary. Thus, for example, a physicist or an engineer may wish to determine the amplitude and phase of an oscillatory process with a certain prescribed accuracy once the general qualitative aspects of the phenomenon have been ascertained.

In general, there exist no methods capable of yielding exact solutions of non-linear differential equations, and the only methods available are those of approximations. A typical and very general class of non-linear differential equations encountered in applications is represented by the equation

$$\ddot{x} + \omega^2 x = \mu f(x, \dot{x}, t) \tag{44.1}$$

The solutions of this equation are periodic with frequency ω if $\mu = 0$. Poincaré (1)** has shown that very near some of these solutions, when $\mu = 0$, periodic solutions of Equation [44.1] may exist for very small values of the parameter μ. The search for these solutions is the object of *the method of small parameters* of Poincaré. This method, with its various ramifications, constitutes the principal subject of Part II.

The scope of the quantitative methods available at present is rather limited. It is restricted, in fact, to the class of non-linear differential equations of the type [44.1], and for this class it is further restricted by the condition that the parameter μ should be very small, that is, $\mu \ll 1$. In spite of these limitations, the usefulness of the method is very great and its applications in various branches of applied science are extensive. Only in special cases when the parameter μ is large does the theory

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* The text of Part II follows the presentation contained in the two treatises on Non-Linear Mechanics: "Theory of Oscillations," by A. Andronow and S. Chaikin, Moscow, (Russian), 1937, Chapters VII and VIII. "Introduction to Non-Linear Mechanics," by N. Kryloff and N. Bogoliuboff, Kieff, (Russian), 1937, Chapters X, XI, and XIII.

** Numbers in parentheses indicate references on page 112 of this report.

of Poincaré cease to be applicable, and for such cases analytical methods are practically unexplored.*

We shall frequently refer to Equation [44.1], with $\mu \ll 1$, as *quasi-linear*, which means that the solutions of this equation do not differ appreciably from solutions of the corresponding linear equation when $\mu = 0$. The important point to be noted is that, although the solutions of [44.1] do not differ much from the solutions of the corresponding linear equation if μ is very small, these periodic solutions do not exist in the neighborhood of *any* periodic solution of the corresponding linear equation but only in the neighborhood of *certain special* solutions of that equation. The establishment of conditions for the existence of periodic solutions of [44.1] when $\mu \ll 1$ thus constitutes the crux of the theory of Poincaré.

Another important point is the effect of the so-called *secular terms* in the approximate solutions obtained by these quantitative methods. Poincaré's method consists in substituting certain power series in Equation [44.1] and in determining the coefficients of these series by a recurrence procedure. As a result of this, there may appear terms such as $t^n \sin \omega t$ and $t^n \cos \omega t$, the *secular terms,* in which the time t appears *explicitly* in the expansions. It is apparent that the existence of these terms, which grow beyond any bound as $t \to \infty$, destroys the periodicity one is seeking. This can be illustrated by the following argument.

Let $\omega = \alpha + \beta$. Then

$$\sin \omega t = \sin \alpha t \cos \beta t + \cos \alpha t \sin \beta t$$

$$= \left[\alpha t - \frac{(\alpha t)^3}{3!} + \cdots\right] \cos \beta t + \left[1 - \frac{(\alpha t)^2}{2!} + \cdots\right] \sin \beta t \qquad [44.2]$$

The terms $\alpha t \cos \beta t$, $\frac{(\alpha t)^3}{3!} \cos \beta t$, and so on, in this expression are secular terms. It is obvious that if the series expansion of $\sin \alpha t$ and $\cos \alpha t$ is limited to a finite number of terms one cannot speak of the "periodicity" of the expression on the right side of Equation [44.2] since the polynominals in the parentheses of this expression will increase indefinitely as $t \to \infty$. If, however, $n \to \infty$, the expressions in parentheses approach $\sin \alpha t$ and $\cos \alpha t$, and the whole expression [44.2] then becomes a periodic function of time with period $T = 2\pi/\omega$.

In practice it is necessary to stop the series expansions at a certain finite number of terms; hence one is generally confronted with secular terms. In the original work of Poincaré this difficulty was obviated by limiting the expansions to a certain finite time interval sufficient for

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* See, for example, the paper by J.A. Shohat (2).

astronomical purposes. Gylden and Lindstedt (3) have avoided this difficulty by eliminating the secular terms in each step of the recurrence procedure by which the coefficients of the expansion are determined. These methods will be mentioned briefly in Chapter XI.

For this reason the transfer of the methods of celestial mechanics to the problems of non-linear oscillations presented certain difficulties. In fact, if one attempts to apply Poincaré's method, for example, to a thermionic generator oscillating with a frequency of several megacycles per second, it is apparent that in a few seconds such a generator will pass through stages corresponding to those through which an astronomical system passes in many millions of years. The effect of secular terms in such a case should be felt within a few seconds. Nothing of the kind, however, is observed.

In view of this it becomes necessary, in adapting these methods to the theory of non-linear oscillations, to follow the method of Lindstedt, which eliminates the secular terms in each step of the recurrence procedure. This was accomplished by Kryloff and Bogoliuboff. We shall return to this question in Section 58.

In spite of the difficulty of using the original theory of Poincaré for studies of non-linear oscillation, this theory is still capable of yielding a considerable amount of information. Chapter VIII outlines the salient points of Poincaré's method as outlined in "Theory of Oscillations," by A. Andronow and S. Chaikin, (4).

Chapter IX is devoted to the Van der Pol method; its presentation follows closely the text of Andronow and Chaikin. Chapter X concerns the theory of the first approximation of Kryloff and Bogoliuboff (5).

These two methods of Van der Pol and of Kryloff and Bogoliuboff are analogous in some respects and follow a method similar to the *method of variation of constants* of Lagrange.

Chapter XI deals with Lindstedt's method, as applied by Kryloff and Bogoliuboff to approximations of orders higher than the first.

Chapter XII is devoted to the *method of equivalent linearization* of Kryloff and Bogoliuboff, which is an attempt to simplify the problem by reducing the given non-linear differential equation to an equivalent linear one.

CHAPTER VIII

METHOD OF POINCARÉ

45. CONDITION OF PERIODICITY

Consider a system of differential equations

$$\dot{x} = ax + by + \mu f_1(x,y); \quad \dot{y} = cx + dy + \mu f_2(x,y) \qquad [45.1]$$

where f_1 and f_2 are the non-linear elements of the system and μ is a parameter. It will be assumed that f_1 and f_2 are analytic functions of their variables in certain intervals under consideration.

For $\mu = 0$ the system becomes linear. In general, we shall be interested in periodic solutions of the non-linear system. Let us consider, first, the periodic solutions of the system [45.1] when $\mu = 0$. Forming the characteristic equation, we obtain

$$S^2 - (a + d)S + (ad - bc) = 0 \qquad [45.2]$$

The periodic solutions of [45.1] for $\mu = 0$ correspond clearly to purely imaginary roots of Equation [45.2]. We thus obtain the following conditions of periodicity

$$a + d = 0; \quad ad - bc > 0 \qquad [45.3]$$

Under these conditions the linear system admits an infinity of periodic solutions of the form

$$x = K\cos(\omega t + \phi); \quad y = gK\sin(\omega t + \phi + X) \qquad [45.4]$$

where $\omega = \sqrt{ad - bc}$ and g is a determined constant. Obviously, the phase angle ϕ is arbitrary and can be made equal to zero by a suitable choice of the origin of time. K and the relative phase angle X appear as the integration constants determined by the initial conditions.

The general form of periodic solutions of [45.1] for $\mu = 0$ is then

$$x = x_0(t,K) = K\cos\omega t; \quad y = y_0(t,K) = gK\sin(\omega t + X) \qquad [45.5]$$

so that $x_0(t,K)$ and $y_0(t,K)$ are periodic functions with period $2\pi/\omega$. Let us assume now that periodic solutions exist for small values of $\mu \neq 0$ and let $x = x(t,\mu,K)$ and $y = y(t,\mu,K)$ be these solutions. For $t = 0$ the solutions are $x(0,\mu,K)$ and $y(0,\mu,K)$ and we can write

$$x(0,\mu,K) = x_0(0,K) + \beta_1 \quad \text{and} \quad y(0,\mu,K) = y_0(0,K) + \beta_2 \qquad [45.6]$$

which defines the functions $\beta_1(\mu)$ and $\beta_2(\mu)$. It is obvious that $\beta_1(0) = \beta_2(0) = 0$.

The method of Poincaré consists in developing the solutions $x(t,\mu,K)$ and $y(t,\mu,K)$ as power series in μ, β_1, and β_2. Poincaré shows (6) that the expansions converge if the values of $|\mu|$, $|\beta_1|$, and $|\beta_2|$ are sufficiently small. Moreover, this convergence is uniform for any finite time interval $0 < t < t_1$. The coefficients of the expansions so obtained are functions of time. By substituting these expansions in the differential equations [45.1] it is possible to determine these coefficients by equating like powers of μ, β_1, and β_2. One obtains in this manner a system of differential equations subject to certain initial conditions.

Let us write the solutions of [45.1] in the form

$$x = x(t,\mu,\beta_1,\beta_2,K); \quad y = y(t,\mu,\beta_1,\beta_2,K)$$

Since we are looking for periodic solutions of [45.1] in the neighborhood of a known periodic solution with period T, when $\mu = 0$, it is logical to assume that in this neighborhood the period of the solution [45.1] will be $T + \tau$, where $\tau(\mu)$ is a small correction which approaches zero as μ approaches zero. Our chief objective is to show that under certain conditions periodic solutions *may* exist provided μ is small. The condition for the periodicity of [45.1] is clearly

$$\begin{aligned} x(T + \tau,\mu,\beta_1,\beta_2,K) - x(0,\mu,\beta_1,\beta_2,K) &= 0 \\ y(T + \tau,\mu,\beta_1,\beta_2,K) - y(0,\mu,\beta_1,\beta_2,K) &= 0 \end{aligned} \qquad [45.7]$$

For given values of μ and K we must select functions $\beta_1(\mu)$, $\beta_2(\mu)$, and $\tau(\mu)$ so as to satisfy Equations [45.7]. Furthermore, the non-linear equation becomes linear when $\mu = 0$ so that

$$\tau(0) = \beta_1(0) = \beta_2(0) = 0 \qquad [45.8]$$

The phase is arbitrary, however, so that it is possible to assume that one of the β's, say β_1, equals zero. Putting $\beta_2 = \beta$, the conditions [45.7] can be written as

$$\begin{aligned} x(T + \tau,\mu,0,\beta,K) - x(0,\mu,0,\beta,K) &= \phi(\tau,\mu,\beta,K) \\ y(T + \tau,\mu,0,\beta,K) - y(0,\mu,0,\beta,K) &= \psi(\tau,\mu,\beta,K) \end{aligned} \qquad [45.9]$$

It is apparent that, when $\mu = 0$ and hence $\tau(0) = \beta(0) = 0$, the system [45.1], with the conditions expressed by [45.3], has an infinity of periodic solutions corresponding to the arbitrary values of the integration constants K and X in Equations [45.4]. In such a case Equations [45.9] become identically satisfied for any value of K. One can express this by writing

$$\begin{aligned} \phi(\tau,\mu,\beta,K) &= \mu\phi_1(\tau,\mu,\beta,K) = 0 \\ \psi(\tau,\mu,\beta,K) &= \mu\psi_1(\tau,\mu,\beta,K) = 0 \end{aligned} \qquad [45.10]$$

The right-hand side of these equations represents a straight line $\mu = 0$ in the (μ,K)-plane and a point represented by the intersection of the curves $\bar{\phi}_1(\mu,K) = 0$ and $\bar{\psi}_1(\mu,K) = 0$, where $\bar{\phi}_1$ and $\bar{\psi}_1$ are the functions ϕ_1 and ψ_1 in which $\tau(\mu)$ and $\beta(\mu)$ have been expressed in terms of μ. We can, for instance, represent $\tau(\mu)$ and $\beta(\mu)$ by power series

$$\tau(\mu) = d\mu + e\mu^2 + \cdots; \quad \beta(\mu) = d_1\mu + e_1\mu^2 + \cdots \qquad [45.11]$$

Expanding the functions $\phi_1(\tau,\mu,\beta,K)$ and $\psi_1(\tau,\mu,\beta,K)$ we get

$$\begin{aligned} \phi_1 &= \phi_{01} + a\mu + b\tau + c\beta + \cdots = 0 \\ \psi_1 &= \psi_{01} + a_1\mu + b_1\tau + c_1\beta + \cdots = 0 \end{aligned} \qquad [45.12]$$

Substituting in these expressions the values of $\tau(\mu)$ and $\beta(\mu)$ from [45.11] and considering μ as a small quantity of the first order, one obtains

$$\begin{aligned} \phi_1 &= \phi_{01} + \mu(a + bd + cd_1) = 0 \\ \psi_1 &= \psi_{01} + \mu(a_1 + b_1 d + c_1 d_1) = 0 \end{aligned} \qquad [45.13]$$

These equations hold only when μ is very small so that the terms containing powers of μ greater than the first are negligible.

Since these two equations must be satisfied for a sufficiently small μ, two conditions must be fulfilled:

$$\phi_{01} = \phi_{01}(K) = 0; \quad \psi_{01} = \psi_{01}(K) = 0 \qquad [45.14]$$

$$a + bd + cd_1 = 0; \quad a_1 + b_1 d + c_1 d_1 = 0 \qquad [45.15]$$

The condition [45.14] states that the terms independent of μ must be equal to zero and [45.15] that the system of the two equations must yield the values of d and d_1 which determine the quantities $\tau(\mu)$ and $\beta(\mu)$ to the first order. It is apparent that this is possible whenever the determinant $\begin{vmatrix} b & c \\ b_1 & c_1 \end{vmatrix} \neq 0$. With Expressions [45.12] taken into account, this is equivalent to the condition

$$J = \begin{vmatrix} \dfrac{\partial\phi_1}{\partial\tau} & \dfrac{\partial\phi_1}{\partial\beta} \\ \dfrac{\partial\psi_1}{\partial\tau} & \dfrac{\partial\psi_1}{\partial\beta} \end{vmatrix} = \frac{\partial(\phi_1,\psi_1)}{\partial(\tau,\beta)} \neq 0 \qquad [45.16]$$

Hence, whenever the Jacobian [45.16] is different from zero, periodic solutions of the non-linear problem exist since it is possible then to determine the functions $\tau(\mu)$ and $\beta(\mu)$ provided μ is sufficiently small and provided the conditions [45.14] are fulfilled. If Equations [45.14] can be solved, one obtains one or several values of K; hence the problem is solved.

We can recapitulate the problem somewhat differently using the terminology of the phase plane. For $\mu = 0$ there is a continuum of closed trajectories corresponding to different values of the integration constants, as was shown in Section 1. For $\mu \neq 0$ but very small, closed trajectories may exist in certain restricted regions of the phase plane in which the condition [45.16] is fulfilled; the value of the integration constant K_1, the amplitude, is determined by solving Equations [45.14]. The solution of [45.1], when $\mu = 0$ and $K = K_1$ as just explained, is called the *generating solution.* We shall see numerous examples of this procedure in what follows.

46. EXPANSIONS OF POINCARÉ; GENERATING SOLUTIONS; SECULAR TERMS

Instead of Equations [45.1], we shall consider now a non-linear differential equation of the form

$$\ddot{x} + x = \mu f(x, \dot{x}) \qquad [46.1]$$

Let $x = x(t, \mu, \beta_1, \beta_2, K)$ be its periodic solution in the neighborhood of $\mu = 0$. Expanding this solution into a power series of μ, β_1, and β_2, we know by the theorem of Poincaré that this expansion converges in any arbitrary but finite time interval provided these quantities are sufficiently small in absolute value. We obtain

$$x = \phi_0(t) + A\beta_1 + B\beta_2 + C\mu + D\beta_1\mu + E\beta_2\mu + F\mu^2 + \cdots \qquad [46.2]$$

where ϕ_0, A, B, $\cdots$ are functions of t. Our purpose will be to identify the expansion [46.2] with a periodic solution of [46.1] provided $|\mu|$, $|\beta_1(\mu)|$, and $|\beta_2(\mu)|$ are small. Differentiating [46.2] with respect to t, we obtain the following equations

$$\dot{x} = \dot{\phi}_0(t) + \dot{A}\beta_1 + \dot{B}\beta_2 + \cdots; \quad \ddot{x} = \ddot{\phi}_0(t) + \ddot{A}\beta_1 + \ddot{B}\beta_2 + \cdots \qquad [46.3]$$

Expanding $f(x, \dot{x})$ in a Taylor series around the values x_0, $\dot{x}_0$ we get

$$f(x,\dot{x}) = f(x_0,\dot{x}_0) + (x - x_0)\left(\frac{\partial f}{\partial x}\right)_0 + (\dot{x} - \dot{x}_0)\left(\frac{\partial f}{\partial \dot{x}}\right)_0 + \frac{1}{2}(x - x_0)^2\left(\frac{\partial^2 f}{\partial x^2}\right)_0 +$$

$$+ \frac{1}{2}(\dot{x} - \dot{x}_0)^2\left(\frac{\partial^2 f}{\partial \dot{x}^2}\right)_0 + (x - x_0)(\dot{x} - \dot{x}_0)\left(\frac{\partial^2 f}{\partial x\,\partial \dot{x}}\right)_0 + \cdots \qquad [46.4]$$

Substituting $x - x_0 = x - \phi_0(t)$; $\dot{x} - \dot{x}_0 = \dot{x} - \dot{\phi}_0(t)$ as given by [46.2] and [46.3] into [46.4] and replacing x, $\ddot{x}$, and $f(x,\dot{x})$ by their values [46.2], [46.3], and [46.4] in the differential equation [46.1], one obtains a series arranged in terms of μ, β_1, β_2, $\beta_1\mu$, $\beta_2\mu$, μ^2, $\cdots$ which by the theorem of Poincaré (6) converges. By equating the coefficients of μ, β_1, $\cdots$, one obtains a set of differential equations. If the expansion is limited to the second order, one obtains nine differential equations, of which three are identically satisfied and the remaining six are as follows:

$$\ddot{A} + A = 0; \quad \ddot{B} + B = 0 \qquad [46.5]$$

$$\ddot{C} + C = f(x_0, \dot{x}_0); \quad \ddot{D} + D = \left(\frac{\partial f}{\partial x}\right)_0 A + \left(\frac{\partial f}{\partial \dot{x}}\right)_0 \dot{A}$$

$$\ddot{E} + E = \left(\frac{\partial f}{\partial x}\right)_0 B + \left(\frac{\partial f}{\partial \dot{x}}\right)_0 \dot{B}; \quad \ddot{F} + F = \left(\frac{\partial f}{\partial x}\right)_0 C + \left(\frac{\partial f}{\partial \dot{x}}\right)_0 \dot{C} \qquad [46.5]$$

Here the symbols $\left(\frac{\partial f}{\partial x}\right)_0$ and $\left(\frac{\partial f}{\partial \dot{x}}\right)_0$ designate the partial derivatives of f with respect to the variables x and $\dot{x}$ in which the generating solutions $x_0 = \phi_0(t)$ and $\dot{x}_0 = \dot{\phi}_0(t)$ have been substituted after differentiation.

Writing [46.2] and its derived equation for $t = 0$ in the form

$$x - x_0 = \beta_1 = A\beta_1 + B\beta_2 + C\mu + D\beta_1\mu + \cdots$$

$$\dot{x} - \dot{x}_0 = \beta_2 = \dot{A}\beta_1 + \dot{B}\beta_2 + \dot{C}\mu + \dot{D}\beta_1\mu + \cdots \qquad [46.6]$$

one obtains the following initial conditions

$$A(0) = 1; \quad \dot{B}(0) = 1$$

$$B(0) = C(0) = D(0) = E(0) = F(0) = \dot{A}(0) = \dot{C}(0) = \dot{D}(0) = \dot{E}(0) = \dot{F}(0) = 0 \qquad [46.7]$$

With these initial conditions the first two equations [46.5] have the solutions

$$A = \cos t; \quad B = \sin t$$

with period 2π.

The remaining four equations [46.5] are of the form $\ddot{v} + v = V(t)$, having the initial conditions $v(0) = \dot{v}(0) = 0$. The solution of this equation is

$$v = \int_0^t V(u) \sin(t - u)\, du \qquad [46.8]$$

Replacing $V(u)$ in Equation [46.8] by the right-hand terms of the last four equations [46.5], one obtains the following expressions:

$$A = \cos t; \quad \dot{A} = -\sin t$$

$$B = \sin t; \quad \dot{B} = \cos t$$

$$C = \int_0^t f\left[\phi_0(u), \dot{\phi}_0(u)\right] \sin(t - u)\, du; \quad \dot{C} = \int_0^t f\left[\phi_0(u), \dot{\phi}_0(u)\right] \cos(t - u)\, du \qquad [46.9]$$

$$D = \int_0^t \left[\frac{\partial f}{\partial x_0} \cos u - \frac{\partial f}{\partial \dot{x}_0} \sin u\right] \sin(t - u)\, du; \quad \dot{D} = \int_0^t \left[\frac{\partial f}{\partial x_0} \cos u - \frac{\partial f}{\partial \dot{x}_0} \sin u\right] \cos(t - u)\, du$$

$$E = \int_0^t \left[\frac{\partial f}{\partial x_0} \sin u + \frac{\partial f}{\partial \dot{x}_0} \cos u\right] \sin(t - u)\, du; \quad \dot{E} = \int_0^t \left[\frac{\partial f}{\partial x_0} \sin u + \frac{\partial f}{\partial \dot{x}_0} \cos u\right] \cos(t - u)\, du$$

$$F = \int_0^t \left[\frac{\partial f}{\partial x_0} C + \frac{\partial f}{\partial \dot{x}_0} \dot{C}\right] \sin(t - u)\, du; \quad \dot{F} = \int_0^t \left[\frac{\partial f}{\partial x_0} C + \frac{\partial f}{\partial \dot{x}_0} \dot{C}\right] \cos(t - u)\, du$$

where $x_0 = K \cos t$ and $\dot{x}_0 = -K \sin t$ are the generating solutions in which the phase is taken equal to zero.

Inasmuch as only periodic solutions are of interest here, it is important to know the values of A, B, $\cdots F$ after one period. Replacing t by 2π in Expressions [46.9], one obtains

$$A(2\pi) = 1; \quad \dot{A}(2\pi) = 0$$

$$B(2\pi) = 0; \quad \dot{B}(2\pi) = 1$$

$$C(2\pi) = -\int_0^{2\pi} f(x_0, \dot{x}_0) \sin u \, du; \quad \dot{C}(2\pi) = \int_0^{2\pi} f(x_0, \dot{x}_0) \cos u \, du$$

$$D(2\pi) = \int_0^{2\pi} \left[-\frac{1}{2} \frac{\partial f}{\partial x_0} \sin 2u + \frac{\partial f}{\partial \dot{x}_0} \sin^2 u \right] du; \quad \dot{D}(2\pi) = \int_0^{2\pi} \left[\frac{\partial f}{\partial x_0} \cos^2 u - \frac{1}{2} \frac{\partial f}{\partial \dot{x}_0} \sin 2u \right] du$$

[46.10]

$$E(2\pi) = \int_0^{2\pi} \left[-\frac{\partial f}{\partial x_0} \sin^2 u - \frac{1}{2} \frac{\partial f}{\partial \dot{x}_0} \sin 2u \right] du; \quad \dot{E}(2\pi) = \int_0^{2\pi} \left[\frac{1}{2} \frac{\partial f}{\partial x_0} \sin 2u + \frac{\partial f}{\partial \dot{x}_0} \cos^2 u \right] du$$

$$F(2\pi) = \int_0^{2\pi} \left[-\frac{\partial f}{\partial x_0} C(u) \sin u - \frac{\partial f}{\partial \dot{x}_0} \dot{C}(u) \sin u \right] du$$

$$\dot{F}(2\pi) = \int_0^{2\pi} \left[\frac{\partial f}{\partial x_0} C(u) \cos u + \frac{\partial f}{\partial \dot{x}_0} \dot{C}(u) \cos u \right] du$$

The expressions for D and E can be further simplified by expressing the values of $\frac{1}{K} \frac{d}{du}(f \cos u)$ and $\frac{1}{K} \frac{d}{du}(f \sin u)$ differently.

We have

$$\frac{d}{du}(f \cos u) = \frac{df}{du} \cos u - f \sin u; \quad \frac{df}{du} = \frac{\partial f}{\partial x_0} \frac{\partial x_0}{\partial u} + \frac{\partial f}{\partial \dot{x}_0} \frac{\partial \dot{x}_0}{\partial u}$$

Since the generating solutions x_0 and $\dot{x}_0$ are

$$x_0 = K \cos u; \quad \dot{x}_0 = -K \sin u$$

$$\frac{\partial x_0}{\partial u} = -K \sin u; \quad \frac{\partial \dot{x}_0}{\partial u} = -K \cos u$$

hence

$$\frac{df}{du} = -K \frac{\partial f}{\partial x_0} \sin u - K \frac{\partial f}{\partial \dot{x}_0} \cos u$$

and thus

$$\frac{1}{K} \frac{d}{du}(f \cos u) = -\frac{\partial f}{\partial x_0} \sin u \cos u - \frac{\partial f}{\partial \dot{x}_0} \cos^2 u - \frac{f}{K} \sin u$$

or

$$\frac{1}{K} \frac{d}{du}(f \cos u) = -\frac{1}{2} \frac{\partial f}{\partial x_0} \sin 2u - \frac{\partial f}{\partial \dot{x}_0} \cos^2 u - \frac{f}{K} \sin u \qquad [46.11]$$

and similarly

$$\frac{1}{K}\frac{d}{du}(f\sin u) = -\frac{\partial f}{\partial x_0}\sin^2 u - \frac{1}{2}\frac{\partial f}{\partial \dot{x}_0}\sin 2u + \frac{f}{K}\cos u \qquad [46.12]$$

Expressions [46.10] for D, $\dot{D}$, E, and $\dot{E}$, taking into account the expression for C, are simplified by means of Equations [46.11] and [46.12] and assume the following symmetrical form:

$$D(2\pi) = \int_0^{2\pi}\frac{\partial f}{\partial \dot{x}_0}du - \frac{C(2\pi)}{K}; \quad \dot{D}(2\pi) = \int_0^{2\pi}\frac{\partial f}{\partial x_0}du - \frac{\dot{C}(2\pi)}{K}$$

$$E(2\pi) = -\frac{1}{K}\dot{C}(2\pi); \quad \dot{E}(2\pi) = \frac{1}{K}C(2\pi) \qquad [46.13]$$

If $C(2\pi) = 0$ and $\dot{C}(2\pi) \neq 0$, Equations [46.13] become

$$D(2\pi) = \int_0^{2\pi}\frac{\partial f}{\partial \dot{x}_0}du; \quad \dot{D}(2\pi) = \int_0^{2\pi}\frac{\partial f}{\partial x_0}du - \frac{\dot{C}(2\pi)}{K}$$

$$E(2\pi) = -\frac{1}{K}\dot{C}(2\pi); \quad \dot{E}(2\pi) = 0 \qquad [46.14]$$

If $C(2\pi) = \dot{C}(2\pi) = 0$, one has

$$D(2\pi) = \int_0^{2\pi}\frac{\partial f}{\partial \dot{x}_0}du; \quad \dot{D}(2\pi) = \int_0^{2\pi}\frac{\partial f}{\partial x_0}du$$

$$E(2\pi) = 0; \quad \dot{E}(2\pi) = 0 \qquad [46.15]$$

From the latter form of the expressions for $D(2\pi)$ and $\dot{D}(2\pi)$, it is apparent that they represent the constant terms in the Fourier expansion of $\frac{\partial f}{\partial \dot{x}_0}$ and $\frac{\partial f}{\partial x_0}$, multiplied by 2π. On the other hand, $-C(2\pi)$ and $\dot{C}(2\pi)$, as given by Equation [46.10], are the coefficients of $\sin t$, $\cos t$ in the expansion of $f(x_0, \dot{x}_0)$, multiplied by π. Hence, if $f(x_0, \dot{x}_0)$ is given, these coefficients can be calculated directly from Equations [46.10].

We are now in a position to write Equations [45.10] in a new form, expressing the existence of periodic solutions. It is apparent that by the choice of generating solutions in the form $x_0(t) = K\cos t$ and $\dot{x}_0(t) = -K\sin t$, the amplitude K is already contained in the expressions for β_1 and β_2 so that Equations [45.10] can be written as

$$\phi(\tau, \mu, \beta_1, \beta_2) = 0 \quad \text{and} \quad \psi(\tau, \mu, \beta_1, \beta_2) = 0 \qquad [46.16]$$

These equations express sufficient conditions for the existence of periodic solutions. There are thus two equations with three unknowns, τ, β_1, and β_2. One of the β's, however, is arbitrary and can be taken equal to zero, as previously mentioned. If, therefore, Equations [46.16] can be solved giving τ and β_1 as functions of μ in such a manner that for $\mu \to 0$, $\tau(\mu) \to 0$, and $\beta_1(\mu) \to 0$, the problem is solved. If this is impossible, there is still another alternative. We may put $\beta_1 = 0$ and try to solve for τ and β_2 as unknown functions of μ.

The left-hand terms of Equations [46.16] represent the differences $x(2\pi + \tau) - x(0)$ and $\dot{x}(2\pi + \tau) - \dot{x}(0)$. Expanding $x(2\pi + \tau)$ and $\dot{x}(2\pi + \tau)$ in a Taylor series in which τ is considered small, we have

$$\begin{aligned} x(2\pi + \tau) &= x(2\pi) + \tau\dot{x}(2\pi) + \cdots \\ \dot{x}(2\pi + \tau) &= \dot{x}(2\pi) + \tau\ddot{x}(2\pi) + \cdots \end{aligned} \qquad [46.17]$$

Here we substitute the series expansions [46.2] and [46.3]. The coefficients $A(2\pi)$, $B(2\pi)$, $\cdots$ have already been calculated in Equations [46.10]. Considering τ and μ as small quantities of the first order and carrying out the expansions to the second order, one has

$$\begin{aligned} x(2\pi + \tau) = {} & x_0(2\pi) + A(2\pi)\beta_1 + B(2\pi)\beta_2 + C(2\pi)\mu + D(2\pi)\beta_1\mu + E(2\pi)\beta_2\mu + \\ & + F(2\pi)\mu^2 + \tau\dot{x}_0(2\pi) + \tau\dot{A}(2\pi)\beta_1 + \tau\dot{B}(2\pi)\beta_2 + \tau\dot{C}(2\pi)\mu + \frac{\tau^2}{2}\ddot{x}_0(2\pi) \end{aligned} \qquad [46.18]$$

$$\begin{aligned} \dot{x}(2\pi + \tau) = {} & \dot{x}_0(2\pi) + \dot{A}(2\pi)\beta_1 + \dot{B}(2\pi)\beta_2 + \dot{C}(2\pi)\mu + \dot{D}(2\pi)\beta_1\mu + \dot{E}(2\pi)\beta_2\mu + \\ & + \dot{F}(2\pi)\mu^2 + \tau\ddot{x}_0(2\pi) + \tau\ddot{A}(2\pi)\beta_1 + \tau\ddot{B}(2\pi)\beta_2 + \tau\ddot{C}(2\pi)\mu + \frac{\tau^2}{2}\dddot{x}_0(2\pi) \end{aligned} \qquad [46.19]$$

But $x_0(2\pi) = x_0(0)$ and $\dot{x}_0(2\pi) = \dot{x}_0(0)$. Furthermore, $A(2\pi) = 1$, $\dot{A}(2\pi) = 0$, $B(2\pi) = 0$, and $\dot{B}(2\pi) = 1$. With these values of the coefficients, Equations [46.18] and [46.19] become

$$\begin{aligned} x(2\pi + \tau) - x(0) = {} & -K\frac{\tau^2}{2} + \tau\beta_2 + C(2\pi)\mu + \dot{C}(2\pi)\tau\mu + \\ & + D(2\pi)\beta_1\mu + E(2\pi)\beta_2\mu + F(2\pi)\mu^2 = 0 \end{aligned} \qquad [46.20]$$

and

$$\begin{aligned} \dot{x}(2\pi + \tau) - \dot{x}(0) = {} & -K\tau - \tau\beta_1 + \dot{C}(2\pi)\mu + \ddot{C}(2\pi)\tau\mu + \\ & + \dot{D}(2\pi)\beta_1\mu + \dot{E}(2\pi)\beta_2\mu + \dot{F}(2\pi)\mu^2 = 0 \end{aligned} \qquad [46.21]$$

One of the parameters β can be fixed as we please. Thus for a given value of one β these equations determine the other β and the correction τ for the period. Since μ, β_1, β_2, and τ are small quantities of the first order, one can obtain different conditions according to the order of the approximation. The simplest case is that in which one considers the first-order solution, dropping terms of the second order. The only term of the first order in [46.20] is $C(2\pi)\mu$, and in [46.21] there are two terms of the first order, $-K\tau$ and $\dot{C}(2\pi)\mu$. By equating these terms to zero, we obtain the following two equations

$$C(2\pi) = -\int_0^{2\pi} f(K\cos u, -K\sin u)\sin u\,du \equiv \phi(K) = 0 \qquad [46.22]$$

$$\tau = \frac{\dot{C}(2\pi)\mu}{K} = \frac{\mu}{K}\int_0^{2\pi} f(K\cos u, -K\sin u)\cos u\,du \equiv \mu\psi(K) \qquad [46.23]$$

Equation [46.22] determines the amplitude K of the generating solution in the neighborhood of which exist periodic solutions of [46.1], and Equation [46.23] gives the correction τ for the period, provided $C(2\pi) \neq 0$. If $\dot{C}(2\pi) = 0$, from [46.15], $E(2\pi) = \dot{E}(2\pi) = 0$. Hence, Equation [46.20] reduces to

$$D(2\pi)\beta_1 + F(2\pi)\mu = 0 \qquad [46.24]$$

Since τ in this case is zero to the first order, we may proceed to the second order and put $\tau = \sigma\mu^2$. From [46.21], in which we can put $\beta_2 = 0$, and where $\dot{C}(2\pi) = 0$, $\ddot{C}(2\pi) = 0$, and $\dot{E}(2\pi) = 0$, we obtain

$$-K\sigma\mu^2 + \dot{D}(2\pi)\beta_1\mu + \dot{F}(2\pi)\mu^2 = 0$$

Dividing by μ and substituting the value of β_1 derived from [46.24] into this expression, we obtain

$$\sigma = \frac{\dot{F}(2\pi)D(2\pi) - F(2\pi)\dot{D}(2\pi)}{K \cdot D(2\pi)} \qquad [46.25]$$

The correction $\tau = \sigma\mu^2$ must be introduced each time the motion is isochronous to the first order. Thus, if $K \neq 0$ and $D(2\pi) \neq 0$, Equations [46.24] and [46.25] determine β_1 and $\tau = \sigma\mu^2$, the amplitude K having been determined from Equation [46.22]. Substituting the values of A, B, C, and β_1 in Equation [46.2], we obtain

$$x = K\cos t + \mu\left[\int_0^t f(K\cos u, -K\sin u)\sin(t-u)\,du - \frac{F(2\pi)}{D(2\pi)}\cos t\right] \qquad [46.26]$$

where K has been calculated from Equation [46.22].*

It is to be noted that the term $\frac{2}{T+\tau}\int_0^{T+\tau} f(x_0, \dot{x}_0)\sin(t-u)\,du$ represents the first term of the Fourier expansion of the function appearing on the right side of Equation [46.1]. Moreover, the period has been changed because of the presence of the term τ, the correction for the period. The function $x(t)$ given by Equation [46.26] remains periodic.

It should be mentioned here again that the presence of secular terms does not destroy the periodicity but merely accounts for a modification of the period, as was explained in connection with Equation [44.2]. The effect of the appearance of secular terms can also be ascertained from the following example. Assume that we have a periodic function

$$x(t) = \sum_{K=0}^{\infty}\left[a_K(\mu)\cos K\omega(\mu)t + b_K(\mu)\sin K\omega(\mu)t\right] \qquad [46.27]$$

in which both the amplitudes and frequencies are functions of a parameter μ. Expansion of this function in a power series of μ gives

$$x(t) = \sum_{K=0}^{\infty}\left[a_K(0)\cos K\omega(0)t + b_K(0)\sin K\omega(0)t\right] +$$

$$+ \mu\sum_{K=0}^{\infty}\left[a_K'(0)\cos K\omega(0)t + b_K'(0)\sin K\omega(0)t -\right.$$

$$\left. - a_K(0)\omega'(0)Kt\sin K\omega(0)t + b_K(0)\omega'(0)Kt\cos K\omega(0)t\right] + \mu^2\sum_{K=0}^{\infty}\cdots \qquad [46.28]$$

where a_K', b_K', and ω' designate the derivatives of the functions $a_K(\mu)$, $b_K(\mu)$, and $\omega(\mu)$ with respect to μ in which the value $\mu = 0$ has been substituted after differentiation. It is observed that, since the function $x(t)$ is periodic, the appearance of secular terms does not destroy the periodicity in view of the summation of these terms from 0 to ∞. This might not be the case if only a few secular terms were considered in the expansion.

With reference to [46.26], it is to be noted that the secular terms do not appear in the expansion for $x(t)$ if the correction τ can be calculated first, which requires that $D(2\pi) \neq 0$. It is sufficient then to use as the period over which the functions A, B, $\cdots$ of Poincaré are determined, the corrected period $T + \tau$, which amounts to the choice of generating solutions x_0 and $\dot{x}_0$ in the form $K\cos\left[1 - \frac{\sigma\mu^2}{2\pi}\right]t$ and $-K\sin\left[1 - \frac{\sigma\mu^2}{2\pi}\right]t$, instead of $K\cos t$ and $-K\sin t$. The question of secular terms will be discussed in more detail in Section 58.

* In order to re-establish the arbitrariness of the phase, t should be replaced by $t + \delta$, where δ is an arbitrary phase.

47. SYSTEMS WITH TWO DEGREES OF FREEDOM

In the preceding sections we have been concerned with the establishment of conditions for periodicity of solutions of a single non-linear differential equation [46.1] of the second order, which generally represents in applications a dynamical system with one degree of freedom. The only stationary solutions in this case are periodic ones, and the topological representation of such motions in the phase plane does not present any particular difficulty, as was shown in Part I.

Although theoretically the extension of Poincaré's method to systems with several degrees of freedom follows the same argument, the practical difficulties rapidly increase and the benefit derived from topological considerations disappears.* Moreover, the stationary motions in systems of more than one degree of freedom are not necessarily periodic. It thus becomes necessary to restrict the analysis somewhat by endeavoring to formulate only the conditions of stability, which in applications is equivalent to the physical possibility of a particular motion. Systems with two degrees of freedom play an important role in applications, and for that reason it may be of interest to give a brief outline of the method of Poincaré in connection with such systems, omitting the details which have already been explained. The calculations of Andronow and Witt (7) (8) are given in this and the following sections.

Consider a quasi-linear system of two differential equations of the second order

$$\begin{aligned} \ddot{x} + \omega_1^2 x &= \mu f(x, \dot{x}, y, \dot{y}; \mu) \\ \ddot{y} + \omega_2^2 y &= \mu g(x, \dot{x}, y, \dot{y}; \mu) \end{aligned} \qquad [47.1]$$

where μ is a small positive number and the functions f and g are analytic functions of the indicated variables. Since there are two degrees of freedom, one has a greater variety of limit conditions for $\mu = 0$ than in the previously discussed case of a single equation [46.1]. Thus we can write the the limit conditions either as

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* In fact, a dynamical system with two degrees of freedom is generally reducible to a system of four differential equations of the first order, and its representation in a phase plane becomes generally impossible. Only in very special cases of the so-called "degeneration" defined in Part IV is a planar representation possible, but such "degenerate" systems possess entirely new features which are not investigated here.

$$x = R\cos\omega_1 t = \phi_0(t); \quad y = 0 \quad \text{with period} \quad T = \frac{2\pi}{\omega_1}$$

or [47.2]

$$x = 0; \quad y = R\cos\omega_2 t = \psi_0(t) \quad \text{with period} \quad T = \frac{2\pi}{\omega_2}$$

The question as to which of these two generating solutions the dynamical system will "select" will form an important object of a later study.

The procedure initially follows the pattern outlined in connection with a single equation [46.1]. Let us assume that we select the first alternative of [47.2] and apply the perturbation method by putting

$$x = \phi_0(t) + \xi; \quad y = 0 + \eta \qquad [47.3]$$

In terms of the perturbations ξ and η, Equations [47.1] become

$$\ddot{\xi} + \omega_1^2\xi = \mu\left[f(\phi_0,\dot{\phi}_0,0,0;0) + f_x\xi + f_{\dot{x}}\dot{\xi} + f_y\eta + f_{\dot{y}}\dot{\eta} + f_\mu\mu + O_2(\xi,\dot{\xi},\eta,\dot{\eta},\mu)\right]$$

[47.4]

$$\ddot{\eta} + \omega_2^2\eta = \mu\left[g(\phi_0,\dot{\phi}_0,0,0;0) + g_x\xi + g_{\dot{x}}\dot{\xi} + g_y\eta + g_{\dot{y}}\dot{\eta} + g_\mu\mu + O_2(\xi,\dot{\xi},\eta,\dot{\eta},\mu)\right]$$

where the quantity O_2 contains terms of a degree higher than the first in $\xi, \cdots \mu$. According to Poincaré, the solutions of these equations can be taken as power series

$$\xi = \beta_1 A + \beta_2 B + \beta_3 C + \beta_4 D + \mu\big[E + \beta_1 F + \beta_2 G + \beta_3 H + \beta_4 K + \mu L + O_2(\beta_1,\beta_2,\beta_3,\beta_4,\mu)\big] + \overline{O}_2(\beta_1,\beta_2,\beta_3,\beta_4)$$

[47.5]

$$\eta = \beta_1\overline{A} + \beta_2\overline{B} + \beta_3\overline{C} + \beta_4\overline{D} + \mu\big[\overline{E} + \beta_1\overline{F} + \beta_2\overline{G} + \beta_3\overline{H} + \beta_4\overline{K} + \mu\overline{L} + O_2(\beta_1,\beta_2,\beta_3,\beta_4,\mu)\big] + \overline{O}_2(\beta_1,\beta_2,\beta_3,\beta_4)$$

where $A, \cdots L$ and $\overline{A}, \cdots \overline{L}$ are functions of time and

$$\beta_1 = \xi(0); \quad \beta_2 = \dot{\xi}(0); \quad \beta_3 = \eta(0); \quad \beta_4 = \dot{\eta}(0) \qquad [47.6]$$

One of the β's, as will be seen, can be assumed to be equal to zero; for example, $\beta_2 = 0$. If one substitutes the expressions [47.5] into Equations [47.4] and equates the coefficients of like powers of $\beta_1, \cdots \mu$, a system of differential equations results from which the functions $A, \cdots L$ and $\overline{A}, \cdots \overline{L}$ can be determined. One obtains the following expressions:

$$A = \cos\omega_1 t; \quad C = 0; \quad D = 0; \quad O_2(\beta_1, \beta_3, \beta_4) = 0$$

$$E = \frac{1}{\omega_1}\int_0^t f(\phi_0, \dot{\phi}_0, 0, 0; 0) \sin\omega_1(t - u)\, du$$

$$F = \frac{1}{\omega_1}\int_0^t (\cos\omega_1 u \cdot f_x - \omega_1 \sin\omega_1 u \cdot f_{\dot{x}}) \sin\omega_1(t - u)\, du$$

$$H = \frac{1}{\omega_1}\int_0^t (\cos\omega_2 u \cdot f_y - \omega_2 \sin\omega_2 u \cdot f_{\dot{y}}) \sin\omega_1(t - u)\, du$$

$$K = \frac{1}{\omega_1}\int_0^t \left(\frac{1}{\omega_2} \sin\omega_2 u \cdot f_y + \cos\omega_2 u \cdot f_{\dot{y}}\right) \sin\omega_1(t - u)\, du$$

$$L = \frac{1}{\omega_1}\int_0^t \left[E f_x + \dot{E} f_{\dot{x}} + \overline{E} f_y + \dot{\overline{E}} f_{\dot{y}} + f_\mu\right] \sin\omega_1(t - u)\, du$$

[47.7]

$$\overline{A} = 0; \quad \overline{C} = \cos\omega_2 t; \quad \overline{D} = \frac{1}{\omega_2}\sin\omega_2 t; \quad \overline{O}_2(\beta_1, \beta_3, \beta_4) = 0$$

$$\overline{E} = \frac{1}{\omega_2}\int_0^t g(\phi_0, \dot{\phi}_0, 0, 0; 0) \sin\omega_2(t - u)\, du$$

$$\overline{F} = \frac{1}{\omega_2}\int_0^t (\cos\omega_1 u \cdot g_x - \omega_1 \sin\omega_1 u \cdot g_{\dot{x}}) \sin\omega_2(t - u)\, du$$

$$\overline{H} = \frac{1}{\omega_2}\int_0^t (\cos\omega_2 u \cdot g_y - \omega_2 \sin\omega_2 u \cdot g_{\dot{y}}) \sin\omega_2(t - u)\, du$$

$$\overline{K} = \frac{1}{\omega_2}\int_0^t \left(\frac{1}{\omega_2} \sin\omega_2 u \cdot g_y + \cos\omega_2 u \cdot g_{\dot{y}}\right) \sin\omega_2(t - u)\, du$$

$$\overline{L} = \frac{1}{\omega_2}\int_0^t \left[E g_x + \dot{E} g_{\dot{x}} + \overline{E} g_y + \dot{\overline{E}} g_{\dot{y}} + g_\mu\right] \sin\omega_2(t - u)\, du$$

The conditions of periodicity for the solutions [47.3] are

$$[x] = x\left(\frac{2\pi}{\omega_1} + \tau\right) - x(0) = 0; \quad [\dot{x}] = \dot{x}\left(\frac{2\pi}{\omega_1} + \tau\right) - \dot{x}(0) = 0$$

[47.8]

$$[y] = y\left(\frac{2\pi}{\omega_1} + \tau\right) - y(0) = 0; \quad [\dot{y}] = \dot{y}\left(\frac{2\pi}{\omega_1} + \tau\right) - \dot{y}(0) = 0$$

These four conditions, with the use of the symbols [ ] defined by [47.8], become

$$R[\cos\omega_1 t] + \beta_1[A] + \mu\Big([E] + \beta_1[F] + \beta_3[H] + \beta_4[K] + \mu[L] + [O_2(\beta_1,\beta_3,\beta_4,\mu)]\Big) = 0$$

$$R[-\omega_1\sin\omega_1 t] + \beta_1[\dot{A}] + \mu\Big([\dot{E}] + \beta_1[\dot{F}] + \beta_3[\dot{H}] + \beta_4[\dot{K}] + \mu[\dot{L}] + [O_2(\beta_1,\beta_3,\beta_4,\mu)]\Big) = 0$$

[47.9]

$$\beta_3[\bar{C}] + \beta_4[\bar{D}] + \mu\Big([\bar{E}] + \beta_1[\bar{F}] + \beta_3[\bar{H}] + \beta_4[\bar{K}] + \mu[\bar{L}] + [O_2(\beta_1,\beta_3,\beta_4,\mu)]\Big) = 0$$

$$\beta_3[\dot{\bar{C}}] + \beta_4[\dot{\bar{D}}] + \mu\Big([\dot{\bar{E}}] + \beta_1[\dot{\bar{F}}] + \beta_3[\dot{\bar{H}}] + \beta_4[\dot{\bar{K}}] + \mu[\dot{\bar{L}}] + [O_2(\beta_1,\beta_3,\beta_4,\mu)]\Big) = 0$$

Developing the expressions indicated by the symbols [ ] in power series of τ, the correction for the period, one obtains

$$[A] = A\Big(\frac{2\pi}{\omega_1} + \tau\Big) - A(0) = a_0 + a_1\tau + a_2\tau^2 + \cdots$$

[47.10]

$$[\dot{A}] = \dot{A}\Big(\frac{2\pi}{\omega_1} + \tau\Big) - \dot{A}(0) = \dot{a}_0 + \dot{a}_1\tau + \dot{a}_2\tau^2 + \cdots$$

. . . .

where $a_0, \cdots$ and $\dot{a}_0, \cdots$ can be calculated by Equations [47.7]. One obtains from [47.10] the following equations

$$-\frac{R\omega_1^2\tau}{2} + \beta_1(a_0 + a_1\tau) + \mu(e_0 + e_1\tau + \beta_1 f_0 + \beta_3 h_0 + \beta_4 k_0 + \mu l_0) + O_3(\beta_1,\beta_3,\beta_4,\mu,\tau) = 0$$

$$-R\omega_1^2\tau + \beta_1(\dot{C}_0 + a_1\tau) + \mu(\dot{e}_0 + \dot{e}_1\tau + \beta_1\dot{f}_0 + \beta_3\dot{h}_0 + \beta_4\dot{k}_0 + \mu\dot{l}_0) + O_3(\beta_1,\beta_3,\beta_4,\mu,\tau) = 0$$

[47.11]

$$\beta_3(\bar{C}_0 + \bar{C}_1\tau) + \beta_4(\bar{d}_0 + \bar{d}_1\tau) + \mu(\bar{e}_0 + \bar{e}_1\tau + \beta_1\bar{f}_0 + \beta_3\bar{h}_0 + \beta_4\bar{k}_0 + \mu\bar{l}_0) + O_3(\beta_1,\beta_3,\beta_4,\mu,\tau) = 0$$

$$\beta_3(\dot{\bar{C}}_0 + \dot{\bar{C}}_1\tau) + \beta_4(\dot{\bar{d}}_0 + \dot{\bar{d}}_1\tau) + \mu(\dot{\bar{e}}_0 + \dot{\bar{e}}_1\tau + \beta_1\dot{\bar{f}}_0 + \beta_3\dot{\bar{h}}_0 + \beta_4\dot{\bar{k}}_0 + \mu\dot{\bar{l}}_0) + O_3(\beta_1,\beta_3,\beta_4,\mu,\tau) = 0$$

In these equations certain coefficients do not depend on the choice of the functions f and g. Thus, one always has

$$a_0 = 0;\quad a_1 = 0;\quad \dot{a}_0 = 0;\quad \dot{a}_1 = -\omega_1^2$$

[47.12]

$$\bar{C}_0 = \cos\gamma - 1;\quad \bar{d}_0 = \frac{1}{\omega_2}\sin\gamma;\quad \dot{\bar{C}}_0 = -\omega_2\sin\gamma;\quad \dot{\bar{d}}_0 = \cos\gamma - 1$$

where $\gamma = 2\pi\frac{\omega_2}{\omega_1}$.

The second equation [47.11], taking into consideration [47.12] and $O_3(\beta_1,\beta_3,\beta_4,\mu)$, gives τ, viz.,

$$\tau = \mu\big[\alpha_0 + \beta_1\alpha_1 + \beta_3\alpha_3 + \beta_4\alpha_4 + \mu\alpha_5 + O_2(\beta_1,\beta_3,\beta_4,\mu)\big] \qquad [47.13]$$

where

$$\alpha_0 = \frac{\dot{e}_0}{R\omega_1^2}; \quad \alpha_1 = \frac{R\dot{f}_0 - \dot{e}_0}{R^2\omega_1^2}; \quad \alpha_3 = \frac{\dot{h}_0}{R\omega_1^2}; \quad \alpha_4 = \frac{\dot{k}_0}{R\omega_1^2}; \quad \alpha_5 = \frac{R\omega_1^2\dot{l}_0 + \dot{e}_0\dot{e}_1}{R^2\omega_1^4}$$

Introducing the value of τ given in Equation [47.13] into the remaining equations [47.11], one obtains the following three equations:

$$\begin{aligned} &e_0 + \beta_1 f_0 + \beta_3 h_0 + \beta_4 k_0 + \mu\left(l_0 + a_0 e_1 - \frac{R}{2}\omega_1^2 a_0^2\right) + O_2(\beta_1,\beta_3,\beta_4,\mu) = 0 \\ &\beta_3(\cos\gamma - 1) + \beta_4\left(\frac{1}{\omega_2}\sin\gamma\right) + \mu\bar{e}_0 + O_2(\beta_1,\beta_3,\beta_4,\mu) = 0 \\ &\beta_3(-\omega_2\sin\gamma) + \beta_4(\cos\gamma - 1) + \mu\dot{\bar{e}}_0 + O_2(\beta_1,\beta_3,\beta_4,\mu) = 0 \end{aligned} \qquad [47.14]$$

Since we are looking for a periodic solution of the system [47.1], which reduces for $\mu = 0$ to the first generating solution [47.2], it is necessary that the functions $\beta_1(\mu)$, $\beta_3(\mu)$, and $\beta_4(\mu)$ approach zero as μ approaches 0. Hence, in view of [47.12], the necessary condition for the existence of a periodic solution is

$$e_0 = \int_0^{\frac{2\pi}{\omega_1}} f(R\cos\omega_1 u, -\omega_1 R\sin\omega_1 u, 0, 0; 0)\sin\omega_1 u\, du = 0 \qquad [47.15]$$

From this equation one obtains the amplitude of the periodic solution in the neighborhood of $\mu = 0$. The sufficient condition for the existence of a periodic solution is

$$\begin{vmatrix} f_0 & h_0 & k_0 \\ 0 & \cos\gamma - 1 & \frac{1}{\omega_2}\sin\gamma \\ 0 & -\omega_2\sin\gamma & \cos\gamma - 1 \end{vmatrix} = 2f_0(1 - \cos\gamma) \neq 0 \qquad [47.16]$$

In this condition the value of R, and hence of f_0, is the one which satisfies the amplitude equation [47.15].

If $\omega_2 \neq n\omega_1$, that is, if $\gamma \neq 2\pi n$, where n is an integer, the condition for the existence of a periodic solution can be written as

$$f_0 = \int_0^{\frac{2\pi}{\omega_1}} (\cos\omega_1 u \cdot f_x - \omega_1\sin\omega_1 u \cdot f_{\dot{x}})\sin\omega_1 u\, du \neq 0 \qquad [47.17]$$

If the determinant [47.16] is zero, it is sufficient for the existence of a periodic solution that one of the determinants, obtained by equating to zero a β other than β_2 as assumed here, be different from zero. If all determinants are zero, a special study is required.

The periodic solution thus obtained is of the form

$$x = \phi(t) = R\cos\omega_1 t + \beta_1 A + \mu\left[E + \beta_1 F + \beta_3 H + \beta_4 K + \mu L + O_2(\beta_1, \beta_3, \beta_4, \mu)\right]$$
$$y = \overline{\phi}(t) = \beta_3 \overline{C} + \beta_4 \overline{D} + \mu\left[\overline{E} + \beta_1 \overline{F} + \beta_3 \overline{H} + \beta_4 \overline{K} + \mu \overline{L} + O_2(\beta_1, \beta_3, \beta_4, \mu)\right] \quad [47.18]$$

where R is determined by [47.15]; A, E, F, $\cdots$ by [47.7]; and β_1, β_3, and β_4 by [47.14]. Replacing β_1, β_3, and β_4 by their values in [47.18] and arranging the terms of the series according to the powers of μ, one obtains

$$x = \phi(t) = \phi_0 + \mu\phi_1 + \mu^2\phi_2 + \mu^3\phi_3 + \cdots$$
$$y = \overline{\phi}(t) = \mu\overline{\phi}_1 + \mu^2\overline{\phi}_2 + \mu^3\overline{\phi}_3 + \cdots \quad [47.19]$$

It is apparent that similar results can be obtained if one starts with the second generating solution [47.2].

48. STABILITY OF A PERIODIC SOLUTION

The stability of the periodic solutions [47.18] can be investigated by the perturbation method. Consider the perturbed solution

$$x = \phi(t) + u; \quad y = \overline{\phi}(t) + v \quad [48.1]$$

The variational equations obtained from [47.1] are

$$\ddot{u} + \omega_1^2 u = \mu(f_x u + f_{\dot{x}}\dot{u} + f_y v + f_{\dot{y}}\dot{v})$$
$$\ddot{v} + \omega_2^2 u = \mu(g_x u + g_{\dot{x}}\dot{u} + g_y v + g_{\dot{y}}\dot{v}) \quad [48.2]$$

In these equations the non-linear terms in u and v are left out and the functions f_x, $\cdots$ $g_{\dot{y}}$ are the derivatives of f and g with respect to the indicated variables in which x, $\dot{x}$, y, and $\dot{y}$ are replaced by $\phi(t)$, $\dot{\phi}(t)$, $\overline{\phi}(t)$ and $\dot{\overline{\phi}}(t)$ respectively. These equations have periodic coefficients, and we can expect solutions of the form

$$u = A\overline{\beta}_1 + B\overline{\beta}_2 + C\overline{\beta}_3 + D\overline{\beta}_4$$
$$v = \overline{A}\overline{\beta}_1 + \overline{B}\overline{\beta}_2 + \overline{C}\overline{\beta}_3 + \overline{D}\overline{\beta}_4 \quad [48.3]$$

where A, $\cdots$ $\overline{D}$ are unknown functions of time and the $\overline{\beta}$'s are initial values

$$u(0) = \overline{\beta}_1; \quad \dot{u}(0) = \overline{\beta}_2; \quad v(0) = \overline{\beta}_3; \quad \dot{v}(0) = \overline{\beta}_4 \quad [48.4]$$

Since the functions f_x, $\cdots$ $g_{\dot{y}}$ appearing in [48.2] can be developed in power series in terms of μ, the functions A, $\cdots$ $\bar{D}$ can be assumed to be also of the form

$$A = A_0 + \mu A_1 + \mu^2 A_2 + \cdots$$
$$\bar{A} = \bar{A}_0 + \mu \bar{A}_1 + \mu^2 \bar{A}_2 + \cdots \qquad [48.5]$$

Substituting the expressions [48.3] into [48.2], taking into account their form [48.5], and comparing the terms with the same powers of μ, one obtains a number of differential equations from which the coefficients A_0, A_1, $\cdots$ $\bar{A}_0$, $\bar{A}_1$ can be determined. One obtains in this manner the following expressions:

$$A_0 = \cos\omega_1 t; \quad A_1 = F; \text{ etc.}$$
$$B_0 = \frac{1}{\omega_1}\sin\omega_1 t; \quad B_1 = \frac{1}{\omega_1}\int_0^t \left(\frac{1}{\omega_1}\sin\omega_1 u \cdot f_x + \cos\omega_1 u \cdot f_{\dot{x}}\right)\sin\omega_1(t-u)\,du; \text{ etc.}$$
$$C_0 = 0; \quad C_1 = H; \text{ etc.} \qquad [48.6]$$
$$D_0 = 0; \quad D_1 = K; \text{ etc.}$$
$$\bar{A}_0 = 0; \quad \bar{A}_1 = \bar{F}; \text{ etc.}$$
$$\cdots\cdots\cdots\cdots\cdots$$

Introducing the notations of Poincaré, viz.,

$$u(T) - u(0) = [u] = \psi_1; \quad \dot{u}(T) - \dot{u}(0) = [\dot{u}] = \psi_2$$
$$v(T) - v(0) = [v] = \psi_3; \quad \dot{v}(T) - \dot{v}(0) = [\dot{v}] = \psi_4 \qquad [48.7]$$

where $T = \frac{2\pi}{\omega_1} + \tau$, τ being the correction for the period given by [47.13], one obtains the equation for the determination of the characteristic exponents in the form

$$\Delta = \begin{vmatrix} \frac{\partial\psi_1}{\partial\beta_1} + 1 - e^{\alpha T} & \frac{\partial\psi_1}{\partial\beta_2} & \frac{\partial\psi_1}{\partial\beta_3} & \frac{\partial\psi_1}{\partial\beta_4} \\ \frac{\partial\psi_2}{\partial\beta_1} & \frac{\partial\psi_2}{\partial\beta_2} + 1 - e^{\alpha T} & \frac{\partial\psi_2}{\partial\beta_3} & \frac{\partial\psi_2}{\partial\beta_4} \\ \frac{\partial\psi_3}{\partial\beta_1} & \frac{\partial\psi_3}{\partial\beta_2} & \frac{\partial\psi_3}{\partial\beta_3} + 1 - e^{\alpha T} & \frac{\partial\psi_3}{\partial\beta_4} \\ \frac{\partial\psi_4}{\partial\beta_1} & \frac{\partial\psi_4}{\partial\beta_2} & \frac{\partial\psi_4}{\partial\beta_3} & \frac{\partial\psi_4}{\partial\beta_4} + 1 - e^{\alpha T} \end{vmatrix} = 0 \qquad [48.8]$$

This reduces to the form

$$\Delta = \begin{vmatrix} [A] + 1 - e^{\alpha T} & [B] & [C] & [D] \\ [\dot{A}] & [\dot{B}] + 1 - e^{\alpha T} & [\dot{C}] & [\dot{D}] \\ [\bar{A}] & [\bar{B}] & [\bar{C}] + 1 - e^{\alpha T} & [\bar{D}] \\ [\dot{\bar{A}}] & [\dot{\bar{B}}] & [\dot{\bar{C}}] & [\dot{\bar{D}}] + 1 - e^{\alpha T} \end{vmatrix} = 0 \qquad [48.9]$$

with $[A] = A\left(\frac{2\pi}{\omega_1} + \tau\right) - A(0)$, etc. Putting $1 - e^{\alpha T} = \rho$, this equation can be written in the form

$$\Delta(\rho) = \begin{vmatrix} a_{11} + \rho & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} + \rho & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} + \rho & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} + \rho \end{vmatrix} = 0 \qquad [48.10]$$

which reduces to the quartic equation

$$\rho^4 + a\rho^3 + b\rho^2 + c\rho + d = 0 \qquad [48.11]$$

with

$$a = a_{11} + a_{22} + a_{33} + a_{44}; \quad b = A^{11}_{22} + A^{11}_{33} + A^{11}_{44} + A^{22}_{33} + A^{22}_{44} + A^{33}_{44}$$
$$c = A_{11} + A_{22} + A_{33} + A_{44}; \quad d = \Delta(0) \qquad [48.12]$$

where $A_{11}, \cdots A_{44}$ and $A^{11}_{22}, \cdots A^{33}_{44}$ are the diagonal minors of Δ.

Since one of the characteristic exponents is always zero (9) because the equations are autonomous, $d = 0$, and the quartic equation thus reduces to a cubic one

$$\rho^3 + a\rho^2 + b\rho + c = 0 \qquad [48.13]$$

If the motion is stable in the sense of Liapounoff, the remaining three characteristic exponents must have negative real parts, which means that the moduli of the quantities $e^{\alpha T}$ must be less than one. This means that the complex number $\rho = 1 - e^{\alpha T}$ must be represented in the complex plane ρ by points situated inside a circle of radius 1 whose center is on the real axis at a unit distance from the origin, see Figure 48.1.

By means of the function $\rho = \frac{2}{1 - z}$, the interior of the circle in the (ρ_1, ρ_2)-plane is mapped into a half plane (z_1, z_2) so that the circles, see broken line in Figure 48.1, transform into straight lines, see broken line in Figure 48.2, parallel to the z_2-axis on the axis of the negative z_1.

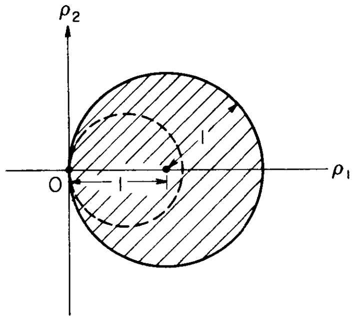

Figure 48.1

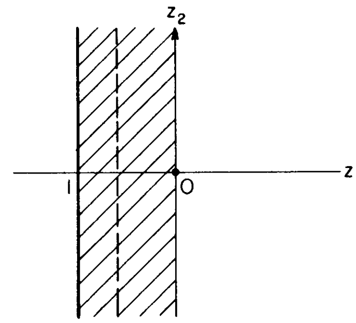

Figure 48.2

By this transformation the problem of finding the roots of [48.13] with moduli situated inside the circle of the (ρ_1, ρ_2)-plane is reduced to that of determining the roots of the transformed equation having negative real parts. This brings the problem within the scope of the Routh-Hurwitz criteria. If this transformation is carried out, Equation [48.13] becomes

$$z^3(-c) + z^2(2b + 3c) + z(-4a - 4b - 3c) + (8 + 4a + 2b + c) = 0 \qquad [48.14]$$

The Routh-Hurwitz criteria of stability (10) (11) are

$$\frac{2b + 3c}{-c} > 0; \quad b^2 + c^2 + 2bc + ab + ac - c < 0; \quad \frac{8 + 4a + 2b + c}{-c} > 0 \; [48.15]$$

These are the necessary and sufficient conditions for the roots of Equation [48.14] to have negative real parts, or, which is the same, for the roots of [48.13] to have moduli less than two, which assures the stability of the periodic motion. The conditions [48.15] can be written also in the form

$$2b + 3c > 0; \quad c < 0; \quad (b + c)^2 + a(b + c) - c < 0; \quad 8 + 4a + 2b + c > 0 \; [48.16]$$

In order to apply these conditions of stability, it is necessary to calculate the determinant [48.9]. The quantities $[A]$, $[B]$, $\cdots$ can be developed in power series in terms of μ, for example

$$a_{11} = [A] = [A_0] + \mu[A_1] + \mu^2[A_2] + \cdots = b_{11}\mu + O_2(\mu)$$

$$a_{12} = [B] = [B_0] + \mu[B_1] + \mu^2[B_2] + \cdots = b_{12}\mu + O_2(\mu)$$

. . . .

The value of the determinant $\Delta(0)$, in which are written only terms containing linear terms in μ and terms independent of μ, is

$$\Delta(0) = \begin{vmatrix} b_{11}\mu + \cdots & b_{12}\mu + \cdots & b_{13}\mu + \cdots & b_{14}\mu + \cdots \\ b_{21}\mu + \cdots & b_{22}\mu + \cdots & b_{23}\mu + \cdots & b_{24}\mu + \cdots \\ b_{31}\mu + \cdots & b_{32}\mu + \cdots & (\cos\gamma - 1) + b_{33}\mu + \cdots & \frac{\sin\gamma}{\omega_2} + b_{34}\mu + \cdots \\ b_{41}\mu + \cdots & b_{42}\mu + \cdots & -\omega_2\sin\gamma + b_{43}\mu + \cdots & (\cos\gamma - 1) + b_{44}\mu + \cdots \end{vmatrix} \quad [48.17]$$

The values of b_{11}, b_{12}, $\cdots$ are determined from Equations [48.6]. Taking into account [48.12], one obtains the following relations

$$c = \mu 2(1 - \cos\gamma)(b_{11} + b_{22}) + O_2(\mu)$$

$$(b + c)^2 + a(b + c) - c = \mu 2(1 - \cos\gamma)\left[\cos\gamma(b_{33} + b_{44}) + \sin\gamma\left(\omega_2 b_{34} - \frac{b_{43}}{\omega_2}\right)\right] + O_2(\mu)$$

$$8 + 4a + 2b + c = 4(1 + \cos\gamma) + O_1(\mu) \quad [48.18]$$

$$2b + 3c = 4(1 - \cos\gamma) + O_1(\mu)$$

If we consider $\mu \ll 1$, $\gamma \neq 2\pi n$, and $\gamma \neq 2(n + 1)\pi$, n being an integer, the stability conditions reduce to the following

$$b_{11} + b_{22} < 0$$

$$\cos\gamma(b_{33} + b_{44}) + \sin\gamma\left(\omega_2 b_{34} - \frac{b_{43}}{\omega_2}\right) < 0 \quad [48.19]$$

The values of b_{11}, b_{22}, $\cdots$ are given by the expressions

$$b_{11} = \frac{1}{\omega_1}\int_0^{\frac{2\pi}{\omega_1}}(\cos\omega_1 u \cdot f_x - \omega_1 \sin\omega_1 u \cdot f_{\dot{x}}) \sin\omega_1\left(\frac{2\pi}{\omega_1} - u\right) du$$

$$b_{22} = \int_0^{\frac{2\pi}{\omega_1}}\left(\frac{1}{\omega_1}\sin\omega_1 u \cdot f_x + \cos\omega_1 u \cdot f_{\dot{x}}\right)\cos\omega_1\left(\frac{2\pi}{\omega_1} - u\right) du$$

$$b_{33} = \int_0^{\frac{2\pi}{\omega_1}}\left(\cos\omega_2 u \cdot g_y - \omega_2 \sin\omega_2 u \cdot g_{\dot{y}}\right)\sin\omega_2\left(\frac{2\pi}{\omega_1} - u\right) du - a_0\omega_2\sin\gamma$$

$$b_{44} = \int_0^{\frac{2\pi}{\omega_1}}\left(\frac{1}{\omega_2}\sin\omega_2 u \cdot g_y + \cos\omega_2 u \cdot g_{\dot{y}}\right)\cos\left(\frac{2\pi}{\omega_1} - u\right) du - a_0\omega_2\sin\gamma \quad [48.20]$$

$$b_{34} = \frac{1}{\omega_2}\int_0^{\frac{2\pi}{\omega_1}}\left(\frac{1}{\omega_2}\sin\omega_2 u \cdot g_y + \cos\omega_2 u \cdot g_{\dot{y}}\right)\sin\omega_2\left(\frac{2\pi}{\omega_1} - u\right) du + a_0\cos\gamma$$

$$b_{43} = \int_0^{\frac{2\pi}{\omega_1}}\left(\cos\omega_2 u \cdot g_y - \omega_2\sin\omega_2 u \cdot g_{\dot{y}}\right)\cos\omega_2\left(\frac{2\pi}{\omega_1} - u\right) du - a_0\omega_2^2\cos\gamma$$

Introducing these values into the inequalities [48.19], one obtains the following conditions of stability

$$\int_0^{\frac{2\pi}{\omega_1}} f_x \, du < 0; \quad \int_0^{\frac{2\pi}{\omega_1}} g_{\dot{y}} \, du < 0 \qquad [48.21]$$

In applying these criteria one has to take $x = R \cos \omega_1 u$; $\dot{x} = - R\omega_1 \sin \omega_1 u$; $y = 0$; $\dot{y} = 0$; and $\mu = 0$. R is determined by [47.15].

We shall return to this matter in Part III in connection with the question of stability of coupled electronic oscillators. This problem is expressible in terms of two non-linear differential equations of the second order.

49. LIMIT CYCLE AND FREQUENCY OF A THERMIONIC GENERATOR

We propose to apply the preceding theory to Equation [30.9] of a thermionic generator. The quadratic term $\gamma_1 v^2$ in Expression [30.5] of the characteristic will be dropped, inasmuch as this term accounts for only a slight assymmetry of the characteristic and has no effect on the calculation of the stationary motion.*

The simplified equation [30.9] can be written as

$$\ddot{v} + v = \mu(\beta - 3\delta v^2)\dot{v} \qquad [49.1]$$

This equation is dimensionless; $\beta > 0$ and $\delta > 0$. The small parameter μ is introduced so that we may consider the oscillation in the quasi-linear range and therefore be able to apply the preceding theory.

For $\mu = 0$ the generating solutions are of the form

$$v_0 = \phi_0(t) = K \cos t; \quad \dot{v}_0 = \dot{\phi}_0(t) = - K \sin t \qquad [49.2]$$

In this case $f(v,\dot{v}) = \beta\dot{v} - 3\delta v^2 \dot{v}$ and hence

$$f(v_0, \dot{v}_0) = - \beta K \sin t + 3\delta K^3 \cos^2 t \sin t \qquad [49.3]$$

Making use of the condition of Poincaré, Equation [46.22], upon integrating Equation [49.3] we obtain $\beta K - \frac{3}{4}\delta K^3 = 0$ and hence

$$K^2 = \frac{4\beta}{3\delta} \qquad [49.4]$$

Thus the amplitude of the generating solution to the first approximation depends on the ratio $\sqrt{\frac{\beta}{\delta}}$. In other words, the amplitude of oscillation reached

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* The reader will note that by retaining the quadratic term $\gamma_1 v^2$ in Equation [30.5] one would have Equation [30.9] instead of Equation [49.1]. The integral from 0 to 2π of the term $2\gamma v \dot{v}$ in Equation [46.22] vanishes, which proves that the term $\gamma_1 v^2$ has no effect on the calculation of the stationary motion. This remark applies also to all *even* terms in the polynomial [30.4]. See also Section 54.

by the self-excited process will be greater as the value of δ is smaller. This is physically obvious, for as $\delta \to 0$ the self-excitation would build up indefinitely since the factor that eventually limits it is precisely the non-linearity of the characteristic expressed by the term $-\delta_1 v^3$ in Equation [30.5]. Returning to the non-linear term $f(v,\dot{v})$, we find that $\frac{\partial f}{\partial v} = -6\delta v\dot{v}$ and $\frac{\partial f}{\partial \dot{v}} = \beta - 3\delta v^2$, so that

$$\frac{\partial f}{\partial v_0} = 6\delta K^2 \cos t \sin t = 3\delta K^2 \sin 2t$$

and

$$\frac{\partial f}{\partial \dot{v}_0} = \beta - 3\delta K^2 \cos^2 t$$

From [46.14] we have, upon taking account of [49.4],

$$D(2\pi) = \int_0^{2\pi} (\beta - 3\delta K^2 \cos^2 t)\, dt = 2\pi\left(\beta - \frac{3\delta K^2}{2}\right) = -2\pi\beta \qquad [49.5]$$

One finds also that $\dot{D}(2\pi) = 0$. From Equation [46.26] the correction for the period $\tau = \sigma\mu^2$ is

$$\tau = \frac{\dot{F}(2\pi)}{K}\mu^2 \qquad [49.6]$$

Calculating $\dot{F}(2\pi)$ from the last equation [46.10], one obtains

$$\tau = \pi\mu^2\beta^2 \qquad [49.7]$$

The coefficient $C(t)$ given by Equation [46.9] after a calculation is

$$C(t) = -\frac{3\delta K^3}{32}\sin 3t + \frac{15\delta K^3}{32}\sin t = -\frac{\beta}{4}\sqrt{\frac{\beta}{3\delta}}\sin 3t + \frac{5\beta}{4}\sqrt{\frac{\beta}{3\delta}}\sin t$$

so that the periodic solution without secular terms is then

$$v = 2\sqrt{\frac{\beta}{3\delta}}\cos\left[\left(1 - \frac{\mu^2\beta^2}{2}\right)t + \psi\right] + \mu\left(-\frac{\beta}{4}\right)\sqrt{\frac{\beta}{3\delta}}\sin 3\left[\left(1 - \frac{\mu^2\beta^2}{2}\right)t + \psi\right] +$$

$$+ \mu\frac{5\beta}{4}\sqrt{\frac{\beta}{3\delta}}\sin\left[\left(1 - \frac{\mu^2\beta^2}{2}\right)t + \psi\right] + \mu^2\left[\qquad\right] + \cdots \qquad [49.8]$$

where ψ is an arbitrary phase angle. It is clear that the periodic solution occurs in the neighborhood of the amplitude $2\sqrt{\frac{\beta}{3\delta}}$. The correction for the period is of the second order and hence can be neglected for small values of μ. The secular terms do not appear here in view of the fact that the correction τ for the period has been calculated first.

50. BIFURCATION THEORY FOR QUASI-LINEAR SYSTEMS

The results obtained in Section 29 can be somewhat extended by means of the quantitative method of Poincaré, see Section 46. In order to obtain this extension, we shall consider instead of Equation [46.1] the equation

$$\ddot{x} + x = \mu f(x, \dot{x}, \lambda) \qquad [50.1]$$

in which there appears an additional parameter λ which we have encountered in the bifurcation theory. The procedure remains substantially the same as before; that is, there appear certain generating solutions in the neighborhood of which periodic solutions exist when μ is small.

We propose to investigate now what happens to these generating solutions when the parameter λ determining the state of the system varies and reaches a critical value.

It was shown in Section 46 that the amplitude K of the generating solutions is given by Equation [46.22], which can be written here in the form

$$\frac{C(2\pi)}{2\pi} = -\frac{1}{2\pi}\int_0^{2\pi} f(K\cos u, -K\sin u, \lambda)\sin u\, du = 0 \qquad [50.2]$$

Putting $K^2 = \rho$ and multiplying Equation [50.2] by $2\sqrt{\rho}$, we have

$$\frac{C(2\pi)}{\pi}\sqrt{\rho} \equiv \phi(\rho,\lambda) = -\frac{1}{\pi}\int_0^{2\pi} f(\sqrt{\rho}\cos u, -\sqrt{\rho}\sin u, \lambda)\sqrt{\rho}\sin u\, du \qquad [50.3]$$

Differentiating this equation with respect to ρ, we have

$$\phi_\rho(\rho,\lambda) = -\frac{1}{2\pi\rho}\int_0^{2\pi}\left[f_x\sqrt{\rho}\sin u\cos u + f_{\dot{x}}\sqrt{\rho}\cos^2 u\right]du +$$

$$+\frac{1}{2\pi}\int_0^{2\pi} f_{\dot{x}}\, du - \frac{1}{2\pi\sqrt{\rho}}\int_0^{2\pi} f\sin u\, du \qquad [50.4]$$

The first term on the right side of Equation [50.4] is equal to

$$\frac{1}{2\pi\sqrt{\rho}}\Big[f\cos u\Big]_0^{2\pi} + \frac{1}{2\pi\sqrt{\rho}}\int_0^{2\pi} f\sin u\, du$$

as is easily verified by integrating the term $\frac{1}{2\pi\sqrt{\rho}}\int_0^{2\pi} f\sin u\, du$ by parts. Equation [50.4] reduces then to a simple form

$$\phi_\rho(\rho,\lambda) = \frac{1}{2\pi}\int_0^{2\pi} f_{\dot{x}}\, du \qquad [50.5]$$

In Section 13 it was shown that, in a conservative system containing a parameter λ, the conditions for stable equilibrium are

$$f(x,\lambda) = 0; \quad f_x(x,\lambda) < 0 \qquad [50.6]$$

Suppose we now impose the condition that $\int_0^{2\pi} f_{\dot{x}}\, du < 0$ so that $\phi_\rho(\rho,\lambda) < 0$. Associating the function $\phi(\rho,\lambda)$ with $f(x,\lambda)$, we may infer that the limit cycles are stable if

$$\phi(\rho,\lambda) = 0; \quad \phi_\rho(\rho,\lambda) < 0 \qquad [50.7]$$

This is by no means a proof, but merely a plausible deduction. A proof for the conditions under which stationary motion is stable may be found in Chapter III of Liapounoff's treatise (12).

The remainder of the bifurcation theory applies directly to limit cycles, the coordinate x of equilibrium being replaced by $\rho = K^2$, the square of the amplitude of the limit cycle.

In a number of problems of non-linear mechanics, the function $f(x,\dot{x},\lambda)$ is of the form

$$f(x,\dot{x},\lambda) = f_1(x,\lambda)\dot{x} \qquad [50.8]$$

where $f_1(x,\lambda)$ is a polynomial of the form

$$f_1(x,\lambda) = a_0(\lambda) + a_1(\lambda)x + a_2(\lambda)x^2 + \cdots \qquad [50.9]$$

To this class belong equations of the generalized Van der Pol type. Substituting $f = f_1\dot{x}$ in Equation [50.3], with the generating solutions $x = \sqrt{\rho}\cos u$ and $\dot{x} = -\sqrt{\rho}\sin u$, and carrying out the integrations, one has

$$\phi(\rho,\lambda) = \frac{1}{2}\left[a_0\rho + \frac{a_2\rho^2}{4} + \frac{a_4\rho^3}{8} + \cdots\right]$$

$$\phi_\rho(\rho,\lambda) = \frac{1}{2}\left[a_0 + \frac{a_2\rho}{2} + \frac{3a_4\rho^2}{8} + \cdots\right] \qquad [50.10]$$

In these equations the coefficients a_0, a_2, a_4, $\cdots$ are functions of λ.

51. "SOFT" AND "HARD" SELF-EXCITATION OF THERMIONIC GENERATORS; OSCILLATION HYSTERESIS

It is well known that there are two kinds of self-excitation of thermionic circuits designated as "soft" and "hard." It is observed that by increasing the coefficient of mutual induction λ between the anode and the grid circuits, self-excitation starts smoothly as soon as a critical value $\lambda = \lambda_0$ of this parameter is reached; for $\lambda > \lambda_0$ the amplitude of oscillations steadily increases with increasing λ, as shown in Figure 51.1 representing this "soft" case of excitation; upon decreasing λ the phenomenon takes place in the opposite direction, as shown by the arrows. The theory of soft self-excitation has been studied in Section 29.

In some cases, however, a different type of self-excitation occurs, as shown in Figure 51.2. With increasing λ it is observed that the self-excitation starts *abruptly* with a finite amplitude for $\lambda = \lambda_1$ and increases

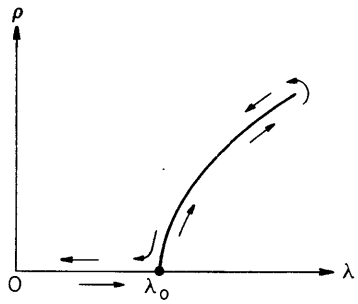

Figure 51.1

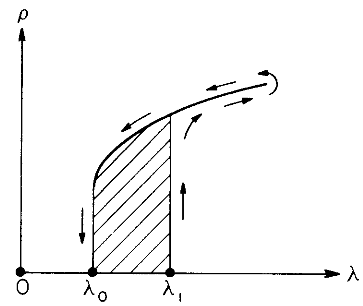

Figure 51.2

smoothly for $\lambda \geqq \lambda_1$. For decreasing λ it is observed, however, that the phenomenon is different; namely, for $\lambda = \lambda_1$ the self-excitation does not disappear; it disappears at $\lambda = \lambda_0 < \lambda_1$. There exists a kind of "hysteresis cycle" shown by the shading in Figure 51.2. This type of self-excitation is called "hard." These phenomena are due to the non-linearity of the system, and the hysteresis cycle referred to above is sometimes called "oscillation hysteresis" (13). We have already analyzed this situation qualitatively in Section 24. In this section we propose to investigate this effect utilizing the theory of Poincaré.

Consider the circuit shown in Figure 51.3 with positive directions indicated. The differential equation of the oscillating circuit is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t (i - I_a)\,dt = 0 \qquad [51.1]$$

where $I_a = f(V)$ is the non-linear function expressing the anode current I_a as a function of grid voltage V. Let us approximate this function by a power series in V limited to terms through V^5 for reasons which will appear later. We have

$$I_a = f(V) = \alpha_0 V + \beta_0 V^2 + \gamma_0 V^3 + \delta_0 V^4 + \epsilon_0 V^5 \qquad [51.2]$$

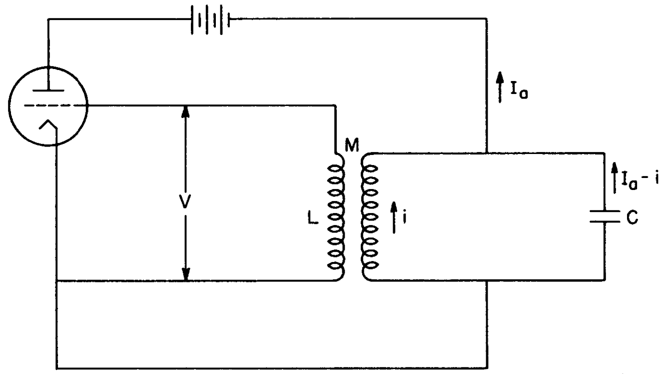

Figure 51.3

It is convenient to introduce a dimensionless variable $x = V/V_s$ where V_s is the "saturation voltage," that is, a sufficiently high grid voltage beyond which the current I_a does not change appreciably. Since $V = M\frac{di}{dt}$ we can write

$$x = \frac{M}{V_s}\frac{di}{dt} \qquad [51.3]$$

From this equation we obtain

$$\frac{di}{dt} = \frac{V_s}{M}x; \quad \frac{d^2i}{dt^2} = \frac{V_s}{M}\dot{x}; \quad i = \frac{V_s}{M}\int x\,dt$$

Substituting these values in [51.1] and differentiating we get, after a few simplifications,

$$LC\ddot{x} + RC\dot{x} + x = M\left[\alpha_0 + 2\beta_0V_sx + 3\gamma_0V_s^2x^2 + 4\delta_0V_s^3x^3 + 5\epsilon_0V_s^4x^4\right]\frac{dx}{dt} \qquad [51.4]$$

Introducing the new independent variable $\tau = \omega_0 t$, where $\omega_0 = 1/\sqrt{LC}$, the preceding equation becomes

$$\frac{d^2x}{d\tau^2} + x = \omega_0M\left[\left(\alpha_0 - \frac{RC}{M}\right) + 2\beta_0V_sx + 3\gamma_0V_s^2x^2 + 4\delta_0V_s^3x^3 + 5\epsilon_0V_s^4x^4\right]\frac{dx}{d\tau}$$

The condition of quasi-linearity is fulfilled if we assume that the coefficients $\alpha_0, \cdots \epsilon_0$ of [51.2] are small. One can take one of these coefficients, for example β_0, as a factor and write

$$\frac{d^2x}{d\tau^2} + x = \beta_0V_s\omega_0M\left[\frac{\alpha_0M - RC}{M\beta_0V_s} + 2x + \frac{3\gamma_0V_s}{\beta_0}x^2 + \frac{4\delta_0V_s^2}{\beta_0}x^3 + \frac{5\epsilon_0V_s^3}{\beta_0}x^4\right]\frac{dx}{d\tau} \qquad [51.5]$$

By introducing the notations

$$\beta_0 V_s \omega_0 = \mu; \quad \frac{\alpha_0 M - RC}{\beta_0 V_s} = a(M); \quad 2M = b(M);$$

$$\frac{3\gamma_0 V_s M}{\beta_0} = c(M); \quad \frac{4\delta_0 V_s^2 M}{\beta_0} = d(M); \quad \frac{5\epsilon_0 V_s^3 M}{\beta_0} = e(M)$$

Equation [51.5] can be written as

$$\frac{d^2x}{d\tau^2} + x = \mu\left[a(M) + b(M)x + c(M)x^2 + d(M)x^3 + e(M)x^4\right]\frac{dx}{d\tau} = \mu f_1(x;M)\frac{dx}{d\tau} \quad [51.6]$$

that is, the function $f(x,\dot{x},M) = f_1(x;M)\frac{dx}{d\tau}$ has the form [50.8].

The function $\phi(\rho,M)$ given by [50.10] can be written, after certain transformations, as

$$\phi(\rho,M) = a_1\rho + a_2'\gamma_0 M\rho^2 + a_3\epsilon_0 M\rho^3 \quad [51.7]$$

where

$$a_1 = \frac{\alpha_0 M - RC}{\beta_0 V_s}; \quad a_2' = \frac{3V_s}{4\beta_0}; \quad a_3 = \frac{5V_s^3}{8\beta_0}$$

Differentiating Equation [51.7] with respect to ρ, one has

$$\phi_\rho(\rho,M) = a_1 + 2a_2'\gamma_0 M\rho + 3a_3\epsilon_0 M\rho^2 \quad [51.8]$$

The discussion of Equations [51.7] and [51.8] yields the qualitative features of the phenomena.

A. CONDITION FOR A SOFT SELF-EXCITATION

If $a_1 > 0$, $\gamma_0 < 0$, and $\epsilon_0 = 0$, Equation [51.7] becomes

$$\phi(\rho,M) = (a_1 - a_2'|\gamma_0|M\rho)\rho = (\alpha_0 M - RC - a_2|\gamma_0|M\rho)\rho = 0 \quad [51.9]$$

where a_2 absorbs the constant factor V_s which is of no further interest. In the (ρ,M)-plane this equation represents a straight line $\rho = 0$ and a hyperbola

$$M\alpha_0 - RC - a_2|\gamma_0|M\rho = 0 \quad [51.10]$$

The point of intersection of $\rho = 0$ and the curve [51.10] is given by the equation $M_1\alpha_0 - RC = 0$. The value

$$M_1 = \frac{RC}{\alpha_0} \quad [51.11]$$

is a critical value of the parameter M. Following the method of Poincaré, Section 13, one obtains the diagram of Figure 29.1b. By increasing the parameter M from small values, one has a locus ($\rho = 0$) of stable focal points. The point $M = M_0$ is a branch point of equilibrium; here the focal point undergoes a transition from stability ($M < M_0$) to instability ($M > M_0$), and a stable limit cycle appears; the square of the amplitude of the latter increases with M, following the hyperbolic branch. The asymptotic value of ρ for $M \to \infty$ is clearly $\alpha_0/\alpha_2\overline{\gamma}_0$, which represents the square of the amplitude for infinitely strong coupling M.

The curves representing this case are shown in Figure 29.1a and b. The former represents the condition for $M > M_0$, the latter, the condition for $M < M_0$. The phenomenon is reversible, as shown in Figure 51.1.

B. CONDITION FOR A HARD SELF-EXCITATION

If $\alpha_0 > 0$, $\gamma_0 > 0$ and $\epsilon_0 < 0$, and if we designate by $\overline{\epsilon}_0$ the absolute value of ϵ_0, and put for abbreviation $a_2'\gamma_0 = m$ and $a_3\overline{\epsilon}_0 = n$, Equation [51.6] becomes

$$\phi(\rho,M) = (a_1 + mM\rho - nM\rho^2)\rho = 0 \qquad [51.12]$$

In the (ρ,M)-plane, this equation represents a straight line $\rho = 0$ and a curve

$$(M\alpha_0 - RC) + mM\rho - nM\rho^2 = 0 \qquad [51.13]$$

The intersection of this curve with the line $\rho = 0$ is clearly $a_1 = 0$, which gives the value M_1 previously found, see Equation [51.11].

The tangent to the curve [51.13] is given by the equation

$$\frac{d\rho}{dM} = -\frac{\dfrac{\partial\phi}{\partial M}}{\dfrac{\partial\phi}{\partial\rho}} = -\frac{\alpha_0 + m\rho - n\rho^2}{M(m - 2n\rho)} \qquad [51.14]$$

The curve has a vertical tangent for $\rho_0 = m/2n$; for this value

$$M_0 = \frac{RC}{\alpha_0 + \dfrac{m^2}{4n}} \qquad [51.15]$$

When M_0 is compared with M_1 from Equation [51.11], it is seen that $M_0 < M_1$. Furthermore, it can be shown that the curve does not go to the left of the value $M = M_0$ and has a horizontal asymptote for $\rho = m/n$. This defines the curve shown in Figure 51.4. It is easily seen that $\phi(\rho,M) > 0$ in the shaded area; whence, applying the criteria of Poincaré, Section 13,

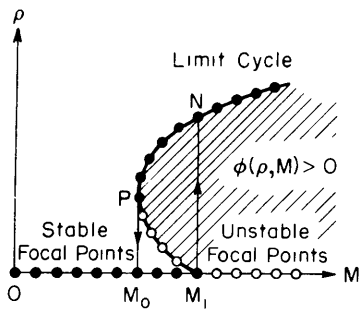

Figure 51.4

one finds that the branch of the curve above point P is stable and below this point unstable. The axis $\rho = 0$ is stable for $M < M_1$ and unstable for $M > M_1$. If M is increased gradually below the value $M = M_1$, the focal points are stable. For $M = M_1$ there is a finite stable limit cycle and a discontinuous jump M_1N, shown in Figure 51.4. If $M > M_1$ and M increases, there is a continuous variation of the amplitude of the limit cycle. If, however, $M > M_1$ and M decreases, when $M = M_1$ is reached, the limit cycle is still stable although there is a transition of the point singularity from instability to stability. This corresponds to Figures 24.6b and c. It follows, therefore, that when M decreases from M_1 to M_0, the stable limit cycle is still being followed. When the point $M = M_0$ is reached, the stable and the unstable limit cycles coalesce, and no limit cycle exists for $M < M_0$, which accounts for the jump PM_0, shown in Figure 51.4.

By comparing the results of this section with those of Section 29 it is seen that if the non-linear characteristic can be approximated by an expression of the form $F_1(x) = a_1 x - a_3 x^3$, where a_1 and a_3 are positive, one has a soft type of self-excitation. If, however, it can be approximated by an expression $F_2(x) = a_1 x + a_3 x^3 - a_5 x^5$, where a_1, a_3, and a_5 are positive, the self-excitation is of a hard type.

In the first case the characteristic has no inflection point (except the point $x = 0$, which is of no interest); in the second case, there is an additional inflection point for $x_1 = \sqrt{\frac{3a_3}{a_5}}$. Since an electon tube exhibits characteristics of both types of self-excitation, depending on the point at which it is biased, each of these cases can be obtained in practice by a suitable adjustment of the circuit.

CHAPTER IX

METHOD OF VAN DER POL

52. ROTATING SYSTEM OF AXES; EQUATIONS OF THE FIRST APPROXIMATION

Consider again the quasi-linear equation

$$\ddot{x} + x = \mu f(x,\dot{x})$$

Its equivalent system is

$$\dot{x} = y; \quad \dot{y} = - x + \mu f(x,y) \qquad [52.1]$$

the notations being the same as in the preceding chapter. For $\mu = 0$, one has the linear equation $\ddot{x} + x = 0$, having a harmonic solution

$$x = a\cos t + b\sin t, \quad \text{with} \quad \dot{x} = - a\sin t + b\cos t \qquad [52.2]$$

where a and b are constants of integration.

The phase trajectories in this case are circles of radii $K = \sqrt{a^2 + b^2}$. If, instead of considering a coordinate system $(x,\dot{x})$ in a fixed phase plane, one introduces a system rotating with angular velocity $\omega = 1$ about the common origin of both systems, in this rotating phase plane Equation [52.2] will represent a fixed point A at a distance OA = $K = \sqrt{a^2 + b^2}$ from origin O. The inclination of OA to a reference line in the rotating plane is given by the angle $\theta = \tan^{-1}\frac{b}{a}$. It is to be noted that this transformation is nothing but the usual method of representing sinusoidal quantities by vectors, used in the theory of alternating currents. We shall call the rotating plane of the variables (a,b) the Van der Pol plane.

Consider now Equations [52.2] as a transformation defining x and $\dot{x}$ in terms of the new variables a and b. This implies that

$$\frac{da}{dt}\cos t + \frac{db}{dt}\sin t = 0$$
$$- \frac{da}{dt}\sin t + \frac{db}{dt}\cos t = \mu f(a\cos t + b\sin t, - a\sin t + b\cos t) \qquad [52.3]$$

whence

$$\frac{da}{dt} = - \mu f(a\cos t + b\sin t, - a\sin t + b\cos t)\sin t$$
$$\frac{db}{dt} = \mu f(a\cos t + b\sin t, - a\sin t + b\cos t)\cos t \qquad [52.4]$$

Since μ is small, by assumption, da/dt and db/dt are also small because $f(x,y)$ is bounded. In other words, a and b are slowly varying quantities in comparison with the rapidly varying trigonometric terms of frequency $\omega = 1$.

For the first approximation it is sufficient therefore to consider a and b as constants on the right side of Equations [52.4]. However, if x and y are replaced by their expressions [52.2], the right sides of [52.4] are periodic and can be expanded in a Fourier series so that [52.4] becomes

$$\frac{da}{dt} = \mu\left[\frac{\phi_0(a,b)}{2} + \phi_1(a,b)\cos t + \bar{\phi}_1(a,b)\sin t + \phi_2(a,b)\cos 2t + \cdots\right]$$

$$\frac{db}{dt} = \mu\left[\frac{\psi_0(a,b)}{2} + \psi_1(a,b)\cos t + \bar{\psi}_1(a,b)\sin t + \psi_2(a,b)\cos 2t + \cdots\right] \qquad [52.5]$$

It must be noted that the system [52.5] now contains t explicitly, whereas the original system [52.1] does not. Van der Pol considers the following "abbreviated" equations as equations of the first approximation

$$\frac{da}{dt} = \mu\frac{\phi_0(a,b)}{2}; \quad \frac{db}{dt} = \mu\frac{\psi_0(a,b)}{2} \qquad [52.6]$$

They are obtained from [52.5] by dropping the terms containing the trigonometric functions. On the other hand,

$$\frac{\phi_0(a,b)}{2} = -\frac{1}{2\pi}\int_0^{2\pi} f(a\cos\xi + b\sin\xi, -a\sin\xi + b\cos\xi)\sin\xi\, d\xi$$

$$\frac{\psi_0(a,b)}{2} = +\frac{1}{2\pi}\int_0^{2\pi} f(a\cos\xi + b\sin\xi, -a\sin\xi + b\cos\xi)\cos\xi\, d\xi \qquad [52.7]$$

Multiplying the first equation [52.6] by a, the second by b, adding and putting $K^2 = a^2 + b^2$, one obtains

$$\frac{1}{2}\frac{dK^2}{dt} = K\frac{dK}{dt} = \frac{\mu}{2\pi}\int_0^{2\pi} f(a\cos\xi + b\sin\xi, -a\sin\xi + b\cos\xi)(-a\sin\xi + b\cos\xi)\, d\xi$$

Putting

$$a\cos\xi + b\sin\xi = K\cos(\xi - \theta)$$

and

$$-a\sin\xi + b\cos\xi = -K\sin(\xi - \theta)$$

where $\theta = \tan^{-1}\frac{b}{a}$, and introducing the variable $u = \xi - \theta$, one has

$$\frac{dK}{dt} = \mu\,\phi(K) \text{ and, by a similar transformation, } \frac{d\theta}{dt} = \mu\,\psi(K) \qquad [52.8]$$

where

$$\phi(K) = -\frac{1}{2\pi}\int_0^{2\pi} f(K\cos u,\ -K\sin u)\sin u\,du$$

$$\psi(K) = +\frac{1}{2\pi K}\int_0^{2\pi} f(K\cos u,\ -K\sin u)\cos u\,du \qquad [52.9]$$

One notes that the definite integrals in these equations coincide with the functions $C(2\pi)$ and $\dot{C}(2\pi)$ of Poincaré, Equations [46.10].

53. TOPOLOGY OF THE PLANE OF THE VARIABLES OF VAN DER POL

In the (x,y) phase plane the limit cycle is reached when the phase trajectory is a circle; in the Van der Pol plane (a,b) the condition for the existence of a stable limit cycle is satisfied when the end of the vector K is a point of stable equilibrium, that is, at this point $\frac{dK}{dt} = \mu\,\phi(K) = 0$. Hence, limit cycles exist for radii K corresponding to the roots of

$$\phi(K) = \frac{1}{2\pi}\int_0^{2\pi} f(K\cos u,\ -K\sin u)\sin u\,du = 0$$

It is to be noted again that this equation coincides with Equation [46.22] of Poincaré's theory.

A root K_i of $\phi(K) = 0$ will correspond to a stable limit cycle if $\phi'(K_i) < 0$. By a method similar to that applied in connection with Equation [50.5], one finds that the condition $\phi'(K_i) < 0$, written explicitly, gives

$$\frac{1}{2\pi}\int_0^{2\pi} f_y(K_i\cos u,\ -K_i\sin u)\,du < 0 \qquad [53.1]$$

The limit cycle is unstable if $\phi'(K_i) > 0$.

Consider now the second equation [52.8]. Here two cases are of interest.

Case 1.

$$\psi(K) = \frac{1}{2\pi K}\int_0^{2\pi} f(K\cos u,\ -K\sin u)\cos u\,du = 0 \qquad [53.2]$$

In this case $\theta = \theta_0$ = constant. The topological picture of trajectories in the plane of variables (a,b) of Van der Pol, in this case, is shown in Figure 53.1. The equilibrium on a limit cycle at a point K_i for $\theta = \theta_0$ is stable if

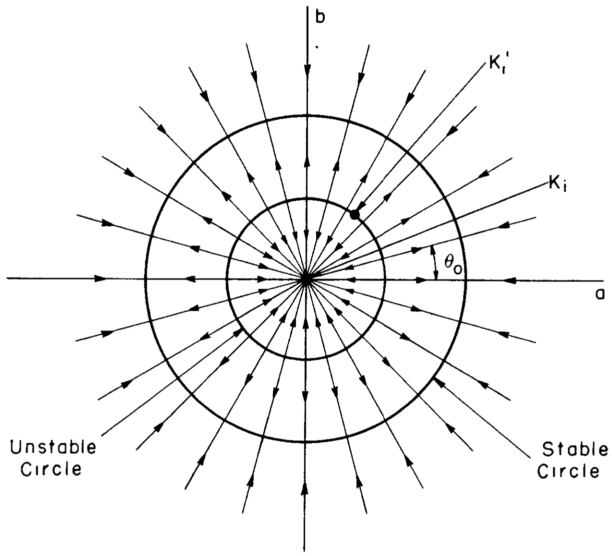

Figure 53.1

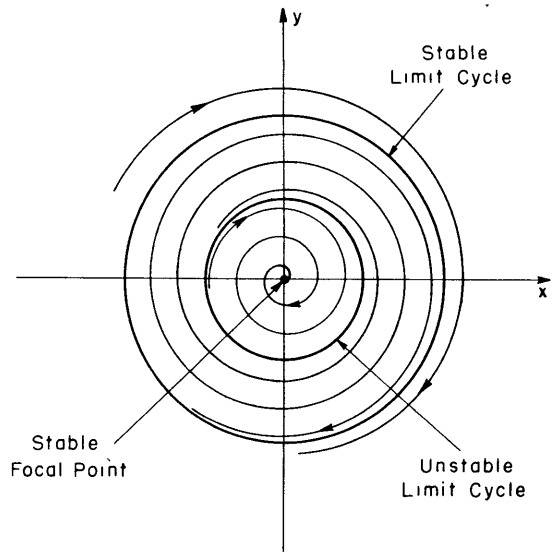

Figure 53.2

the representative point being displaced along the radius returns to K_i. If this point does not return to K_i, as for example Point K_i', such a limit cycle is unstable. The loci of limit cycles in this case are concentric circles corresponding to all possible values of the constant θ_0.

If one returns to the original variables (x,y) of the phase plane, one must apply the equations of transformation [52.2], where the variables a and b are respectively $K_i \cos \theta_0$ and $K_i \sin \theta_0$. This gives

$$x = a\cos t + b\sin t = K_i \cos\theta_0 \cos t + K_i \sin\theta_0 \sin t = K_i \cos(t - \theta_0)$$

[53.3]

$$y = -a\sin t + b\cos t = -K_i\cos\theta_0 \sin t + K_i \sin\theta_0 \cos t = -K_i \sin(t-\theta_0)$$

where θ_0 is arbitrary. This arbitrariness of θ_0 for the (x,y) phase plane is due to the fact that a point of equilibrium in the plane of variables (a,b) of Van der Pol corresponds to a circle in the phase plane of the variables (x,y). The general form of trajectories in the (x,y)-plane is shown in Figure 53.2.

Case 2.

$$\psi(K) = \frac{1}{2\pi K}\int_0^{2\pi} f(K\cos u, -K\sin u)\cos u\,du \neq 0$$

Let $\overline{K}_1$, $\overline{K}_2$, $\cdots$ be the roots of $\psi(K) = 0$, and assume that these roots are different from the roots K_1, K_2, $\cdots$ of $\phi(K) = 0$.

Consider now Equations [52.8]. The motion on a limit cycle is represented in the plane of variables (a,b) by points of equilibrium given by equations

$$a = K_i \cos\left[\mu\psi(K_i)\, t + \theta_0\right]; \quad b = K_i \sin\left[\mu\psi(K_i)t + \theta_0\right] \qquad [53.4]$$

The stability (or instability) of a limit cycle is determined again by the sign of $\phi'(K_i)$ and the direction of rotation by the sign of $\psi(K_i)$.

The topological picture of trajectories in the (a,b)-plane is shown in Figure 53.3. The trajectories "turn back" at points corresponding to the roots $\overline{K}_1$, $\overline{K}_2$, $\cdots$ of $\psi(K_i)$, approaching the stable limit cycles which are again the points of equilibrium in the Van der Pol plane. The topological picture of trajectories in the fixed (x,y)-plane has the same appearance as that shown in Figure 53.2. The only difference between the two cases is that, in the second case, Equations [53.3] become

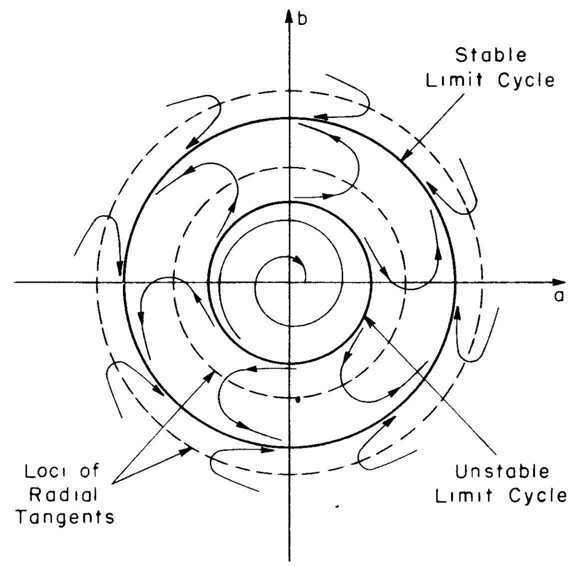

Figure 53.3

$$x = a \cos t + b \sin t = K_i \cos\Big(\left[1 - \mu\psi(K_i)\right] t - \theta_0\Big)$$

$$y = -a \sin t + b \cos t = -K_i \sin\Big(\left[1 - \mu\psi(K_i)\right]t - \theta_0\Big) \qquad [53.5]$$

It is clear that in this case a correction for frequency exists expressed by $\mu\psi(K_i)$. In other words, the velocity along the spiral trajectories is not uniform.

By a further analysis it can be shown that when $\psi(K_i) = 0$ the correction for frequency (and hence for the period) is of the order of μ^2 and consequently can be neglected in the theory of the first approximation. If, however, $\psi(K_i) \neq 0$, this frequency correction appears to the first order of μ.

To sum up the results of this section, it can be said that the use of the variables (a,b) of Van der Pol enables us, if the system is isochronous, to represent a limit cycle by a point in the plane (a,b), that is, by a constant vector. Such representation of a limit cycle is similar to the mode of representing alternating currents by vectors. For transient conditions the representative point moves toward, or away from, the limit cycle point along the length of the radius vector, since the phase angle θ_0 remains constant. The phase angle in this case has no particular physical significance if one single oscillatory phenomenon is considered. If, however, $\psi(K) \neq 0$, that is, if the motion is not isochronous, in the plane of variables (a,b) of Van der Pol the vector K_i undergoes oscillations depending on the roots of $\psi(K) = 0$. This peculiarity resembles the representation of phase-modulated vectors in radio technique. Fixed points in the (a,b)-plane correspond to circles in the (x,y)-plane, and a radial motion in the (a,b)-plane corresponds to a spiral motion in the (x,y)-plane.

54. EXAMPLE: "SOFT" AND "HARD" SELF-EXCITATION OF THERMIONIC CIRCUITS

We now propose to apply the Van der Pol method to the problem previously treated by the method of Poincaré in Section 49. Consider again Equation [30.9]. Here we shall let $\beta = \mu\beta_1$, $\gamma = \mu\gamma_1$, and $\delta = \mu\delta_1$, where μ is a small parameter.

$$\ddot{v} + v = \mu(\beta_1 + 2\gamma_1 v - 3\delta_1 v^2)\dot{v} \qquad [54.1]$$

where β_1, γ_1, and δ_1 are positive.

In this case $f(v,\dot{v}) = (\beta + 2\gamma v - 3\delta v^2)\dot{v}$. Using the first equation [52.9], we have

$$\phi(K) = +\frac{1}{2\pi}\int_0^{2\pi}(\beta + 2\gamma K\cos u - 3\delta K^2\cos^2 u)\,K\sin u\sin u\,du \qquad [54.2]$$

in which the generating solutions $v = K\cos u$, $\dot{v} = -K\sin u$ are substituted. We obtain

$$\phi(K) = \frac{1}{2\pi}\left[\beta K\int_0^{2\pi}\sin^2 u\,du + 2\gamma K^2\int_0^{2\pi}\cos u\sin^2 u\,du - 3\delta K^3\int_0^{2\pi}\cos^2 u\sin^2 u\,du\right] \quad [54.3]$$

Since

$$\int_0^{2\pi}\sin^2 u\,du = \pi, \quad \int_0^{2\pi}\sin^2 u\cos u\,du = \left[\frac{\sin^3 u}{3}\right]_0^{2\pi} = 0$$

and

$$\int_0^{2\pi}\cos^2 u\sin^2 u\,du = \int_0^{2\pi}\sin^2 u\,du - \int_0^{2\pi}\sin^4 u\,du = \pi - \frac{3}{4}\pi = \frac{\pi}{4}$$

we have

$$\phi(K) = \frac{1}{2}\left(\beta K - \frac{3\delta K^3}{4}\right) = \frac{K}{2}\left(\beta - \frac{3\delta K^2}{4}\right) \qquad [54.4]$$

By the first equation [52.8], the condition for the existence of a limit cycle is $\phi(K) = 0$. From Equation [54.4] this takes place for $K_1 = 0$ and $K_2 = \sqrt{\frac{4\beta}{3\delta}}$.

For $K_1 = 0$, the limit cycle reduces to one point, the singular point. The radius of a finite limit cycle is thus

$$K_2 = \sqrt{\frac{4\beta}{3\delta}} \qquad [54.5]$$

In order to ascertain that there is actually a condition of self-excitation, one has to prove that the singularity is unstable and the limit cycle is stable, see Section 29. For the proof of the first point, equations of the first approximation in the sense of Liapounoff must be formed.

The system equivalent to Equation [54.1], upon dropping the non-linear terms, is $\dot{v} = y$ and $\dot{y} + v = \mu\beta y$, that is

$$\dot{v} = y; \quad \dot{y} = -v + \mu\beta y \qquad [54.6]$$

The characteristic equation is $S^2 - \mu\beta S + 1 = 0$ whose roots are

$$S_{1,2} = \frac{\mu\beta \pm \mu\beta\sqrt{1 - \frac{4}{\mu^2\beta^2}}}{2}$$

Since μ is small, it is seen that the roots are complex, with a positive real part. Therefore the singularity is an unstable focal point from which the spiral trajectories unwind themselves approaching the limit cycle $K_2 = \sqrt{\frac{4\beta}{3\delta}}$, provided it is stable.

In order to establish the stability of the limit cycle $K_2 = \sqrt{\frac{4\beta}{3\delta}}$, it is necessary to ascertain the sign of $\phi_K(K_2)$. One finds

$$\phi_K(K_2) = \frac{1}{2}(\beta - 3\beta) < 0$$

The limit cycle is thus stable.

In this discussion it has been assumed that the coefficients β, γ, and δ entering into Equation [54.2] are positive. By waiving this assumption one could analyze additional cases following the same procedure.

It is of interest also to investigate the second equation [52.8], which concerns the frequency of oscillation, $\frac{d\theta}{dt}$.

The function $\psi(K)$ in this case is

$$\psi(K) = \frac{1}{2\pi K}\int_0^{2\pi}(\beta + 2\gamma K\cos u - 3\delta K^2\cos^2 u)\,K\sin u\,\cos u\,du \qquad [54.7]$$

Developing it, we find

$$\psi(K) = \frac{1}{2\pi K}\left[\beta K\int_0^{2\pi}\sin u\cos u\,du + 2\gamma K^2\int_0^{2\pi}\sin u\cos^2 u\,du - 3\delta K^3\int_0^{2\pi}\cos^3 u\sin u\,du\right] \qquad [54.8]$$

Each of the integrals appearing in the above expression is zero so that by the second equation [52.8] $\frac{d\theta}{dt} = 0$. This means that in the phase plane the radius vector of the representative point rotates around the origin with a constant angular velocity, and the frequency correction is zero to the first order.

In order to investigate the variation of K as a function of time, the first equation [52.8] must be integrated upon substituting for $\phi(K)$ its value given by [54.4]:

$$\phi(K) = mK - nK^3$$

where $m = \frac{\beta}{2}$ and $n = \frac{3\delta}{8}$. This gives

$$\frac{dK}{mK - nK^3} = \frac{1}{m}\,\frac{dK}{K(1-pK^2)} = \mu dt$$

where $p = \frac{n}{m} = \frac{3\delta}{4\beta}$.

$$\frac{dK}{K(1-pK^2)} = \frac{dK}{K} + \frac{pKdK}{1-pK^2} = m\mu dt$$

That is, $d\,\log K - d\left[\frac{1}{2}\log(1 - pK^2)\right] = m\mu dt$, or

$$d\left(\log\frac{K}{\sqrt{1-pK^2}}\right) = m\mu dt$$

which, upon integration, gives

$$\log\frac{K}{\sqrt{1-pK^2}} - \log\frac{K_0}{\sqrt{1-pK_0^2}} = m\mu t$$

or

$$\frac{K}{\sqrt{1-pK^2}}\,e^{-m\mu t} = \frac{K_0}{\sqrt{1-pK_0^2}} \qquad [54.9]$$

Finally,

$$K = \frac{K_0}{\sqrt{(1-pK_0^2)e^{-2m\mu t} + pK_0^2}} \qquad [54.10]$$

For $t = 0$, $K = K_0 = \sqrt{\frac{4\beta}{3\delta}}$ and for $t \to -\infty$, $K \to 0$, which means that, for increasing t, the radius vector K increases from zero and approaches the value $K_0 = \sqrt{\frac{4\beta}{3\delta}}$ on the limit cycle. Furthermore, for $t \to +\infty$, $K = K_0$, which shows that the limit cycle is stable.

In order to eliminate the operation with infinities inherent in the asymptotic nature of the process, one can select instead of K and K_0 in Equation [54.10] certain initial and final values K' and K_0' slightly removed from the unstable focal point and the stable limit cycle respectively. In such a case Equation [54.10] can be used for numerical calculations with a view to ascertaining how rapidly the self-excited oscillatory process builds up as a function of time.

It is interesting to note that the terms with γ, ϵ, $\cdots$, corresponding to the even powers in the approximation of the characteristic by a series expansion, disappear from Expression [54.10] for the radius of the limit cycle in the first approximation, as was noted in connection with the vanishing of the term $2\gamma K^2\int_0^{2\pi}\sin^2 u \cos u \, du$ in Equation [54.8].

The inverse passage from the phase plane (a,b) of Van der Pol to the ordinary phase plane (x,y) yields the expressions $x = K\cos t$ and $y = \dot{x} = -K\sin t$, where K is given by Equation [54.10].

The conditions of self-excitation considered above represent the so-called "soft" type of self-excitation. Topologically, it corresponds to the existence either of an unstable singularity surrounded by a stable limit cycle or of a stable singularity within an unstable limit cycle, see Section 29. The first case represents the building up of oscillations asymptotically approaching a stable limit cycle; the second, an asymptotic disappearance of the oscillatory process. If, however, between an unstable singularity and a stable limit cycle an unstable limit cycle exists, one is then concerned with the so-called "hard" self-excitation of oscillations. This subject has already been studied in Section 51 in connection with the theory of Poincaré.

It was shown that the characteristic in this case has an inflection point; in its approximation by expansion in a power series, one has to retain a term ϵv^5 with a negative sign. Since even terms do not have any effect on the determination of limit cycles, one can drop them from the equation. Under these conditions, Van der Pol's equation becomes

$$\ddot{v} + v = \mu(\beta + 3\delta v^2 - 5\eta v^4)\dot{v} \qquad [54.11]$$

In this case $f(v,\dot{v}) = (\beta + 3\delta v^2 - 5\eta v^4)\dot{v}$ and

$$f(K\cos u, -K\sin u) = -(\beta + 3\delta K^2\cos^2 u - 5\eta K^4\cos^4 u)K\sin u$$

From this, by the first equation [52.9], we get

$$\phi(K) = \frac{1}{2\pi}\left[\int_0^{2\pi}\beta K\sin^2 u\,du + \int_0^{2\pi}3\delta K^3\cos^2 u\sin^2 u\,du - \int_0^{2\pi}5\eta K^5\cos^4 u\sin^2 u\,du\right]$$

$$= \frac{K}{2\pi}\left[\beta\int_0^{2\pi}\sin^2 u\,du + 3\delta K^2\int_0^{2\pi}\cos^2 u\sin^2 u\,du - 5\eta K^4\int_0^{2\pi}\cos^4 u\sin^2 u\,du\right]$$

The values of these definite integrals are

$$\int_0^{2\pi}\sin^2 u\,du = \pi;\quad \int_0^{2\pi}\cos^2 u\sin^2 u\,du = \int_0^{2\pi}\sin^2 u\,du - \int_0^{2\pi}\sin^4 u\,du = \pi - \frac{3}{4}\pi = \frac{\pi}{4}$$

$$\int_0^{2\pi}\cos^4 u\sin^2 u\,du = \int_0^{2\pi}\cos^4 u\,du - \int_0^{2\pi}\cos^6 u\,du = \frac{3}{4}\pi - \frac{3\cdot 5}{4\cdot 6}\pi = \frac{\pi}{8}$$

This gives

$$\phi(K) = \frac{K}{2}\left(\beta + \frac{3}{4}\delta K^2 - \frac{5}{8}\eta K^4\right) = 0 \qquad [54.12]$$

as the condition for limit cycles. It is assumed that β, γ, and η are positive. One root is clearly $K = 0$. Following the same procedure as before and making use of Liapounoff's equations of the first approximation, one finds that the singularity is an unstable focal point.

The limit cycles proper are given by the quadratic equation

$$\frac{5}{8}\eta S^2 - \frac{3}{4}\delta S - \beta = 0 \qquad [54.13]$$

where $S = K^2$. Only positive roots are to be considered because $S = K^2$ is essentially positive. Equation [54.13] can be written as

$$\beta = \frac{5}{8}\eta S^2 - \frac{3}{4}\delta S = p^2S^2 - qS = p^2S^2 - qS + \frac{q^2}{4p^2} - \frac{q^2}{4p^2} = \left(pS - \frac{q}{2p}\right)^2 - \frac{q^2}{4p^2}$$

where $p^2 = \frac{5}{8}\eta$ and $q = \frac{3}{4}\delta$.

If this equation is rearranged,

$$\left(\beta + \frac{q^2}{4p^2}\right) = \left[p\left(S - \frac{q}{2p^2}\right)\right]^2 \qquad [54.14]$$

Equation [54.14] represents the parabola $(\beta - \beta_0) = p^2(S - S_0)^2$, as shown in Figure 54.1. If we change the (S,β)-axes to a new system of axes (S_1,β_1) with a new origin at ($\beta_0 = -\frac{q^2}{4p^2}$, $S_0 = \frac{q}{2p}$), Equation [54.14] becomes

$$\beta_1 = p^2S_1^2$$

The second root corresponds to the limit cycle $K_2 = \sqrt{\frac{3\delta}{5\eta}}$ and the first one to an unstable focal point which can be ascertained as explained in the beginning of this section.

For $\beta < 0$, that is, to the left of the origin O, the roots of the quadratic equation $p^2S^2 - qS - \beta = 0$ are

$$S_{1,2} = \frac{q \pm \sqrt{q^2 - 4|\beta|p^2}}{2p^2}$$

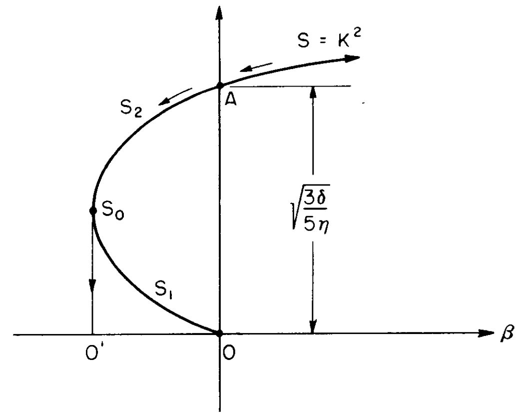

Figure 54.1

They are real and positive only as long as $q^2 - 4|\beta|p^2 \geqq 0$. They correspond to the points S_1 and S_2 of intersection of the parabola with the straight line β = constant. The condition for a double root is $|\beta_0| = \frac{q^2}{4p^2}$; at this value of the negative β, the value of the root is $S_0 = \frac{q}{2p^2}$. The tangent to the parabola at this point is vertical.

In the region O'O, where two roots exist, there are two limit cycles, $K_1 = \sqrt{S_1}$ and $K_2 = \sqrt{S_2}$. The first is unstable and the second stable.

To illustrate this point, differentiate Equation [54.12] with respect to K.

$$\phi'(K) = \frac{\beta}{2} + \frac{3q}{2}K^2 - \frac{5p^2}{2}K^4 = \frac{\beta}{2} + \frac{3q}{2}S - \frac{5p^2}{2}S^2 \qquad [54.15]$$

It is sufficient to substitute into this equation the values of the roots S_1 and S_2 for β in the interval $(0, -q^2/4p^2)$, since the curve does not extend to the left of $\beta = -\frac{q^2}{4p^2}$ and has only one root to the right of $\beta = 0$. Consider, for example, the middle value in this interval, $\beta_1 = -\frac{q^2}{8p^2}$. The corresponding roots are

$$S_{1,2} = \frac{q}{2p^2}\left(1 \pm \sqrt{1 - \frac{1}{2}}\right)$$

that is, $S_1 = 0.293q/2p^2$ and $S_2 = 1.707q/2p^2$.

Substituting these values in Equation [54.15], in which $\beta = -\frac{q}{8p^2}$, one finds, after a reduction, that $\phi'(K_1) > 0$ and $\phi'(K_2) < 0$. Hence the limit cycles on the lower branch S_1 of the parabola are unstable and on the upper branch S_2 they are stable. If β varies now from negative values and reaches the point O', there is no self-excitation throughout the range O'O since the unstable limit cycles S_1, interposed between the stable focal points situated on the β-axis and the stable limit cycles S_2, act as a *barrier* preventing the self-excitation from developing, see Chapter IV. As soon as Point 0 at which the unstable limit cycle disappears is reached, the stable limit cycle $K_2 = \sqrt{\frac{3\delta}{5\eta}}$ is reached abruptly; if β is still further increased, the amplitude of the limit cycle increases continuously, the representative point S following the upper branch of the parabola to the right of Point A. If, however, β is decreased, the amplitude of the stable limit cycle follows the upper branch S_2 until the point $S_0\left(K_0 = \frac{1}{4}\sqrt{\frac{6\delta}{5\eta}}\right)$ is reached. Here the self-excited oscillation disappears abruptly. As previously mentioned, see Section 51, this type of self-excitation is called hard.

It is apparent that the theory of Van der Pol gives exactly the same results as the theory of Poincaré.

55. EXAMPLE: EQUATION OF LORD RAYLEIGH; FROUDE'S PENDULUM

In his researches on the maintenance of vibrations, Lord Rayleigh (14) came across the following equation

$$m\ddot{x} - (\alpha - \beta\dot{x}^2)\,\dot{x} + Kx = 0 \qquad [55.1]$$

in which there is a predominance of negative damping for small values of the velocity $\dot{x}$; for larger velocities the damping becomes positive. By introducing "dimensionless time," as was explained in Section 30, Rayleigh's equation can be put in the form

$$\ddot{x} + x + \mu f(x,\dot{x}) = 0 \qquad [55.2]$$

Assume that the damping terms are small enough to justify the introduction of the small parameter μ. In Rayleigh's equation $f(x,\dot{x}) = f(\dot{x}) = m\dot{x}^3 - n\dot{x}$, where m and n are constants appearing instead of α and β as the result of the transformation of the independent variable. One has finally

$$\ddot{x} + x + \mu(m\dot{x}^3 - n\dot{x}) = 0 \qquad [55.3]$$

where $m > 0$ and $n > 0$.

By Equations [52.9], the functions $\phi(K)$ and $\psi(K)$ are

$$\phi(K) = \frac{1}{2\pi}\int_0^{2\pi} mK\sin^2 u\,du - \frac{1}{2\pi}\int_0^{2\pi} nK^3\sin^4 u\,du = \frac{1}{2}K\left(m - \frac{3}{4}nK^2\right) \qquad [55.4]$$

and

$$\psi(K) = \frac{1}{2\pi K}\int_0^{2\pi} nK^3\sin^3 u\cos u\,du - \frac{1}{2\pi K}\int_0^{2\pi} mK\sin u\cos u\,du \qquad [55.5]$$

From Equation [55.4] we see that the conditions for limit cycles are

$$K = 0; \quad K = \sqrt{\frac{4m}{3n}} \qquad [55.6]$$

The first value $K = 0$ is clearly a point singularity. If the non-linear term $m\dot{x}^3$ is omitted, Liapounoff's equations are $\dot{x} = y$ and $\dot{y} - ny + x = 0$. From this we obtain the characteristic equation $S^2 - nS + 1 = 0$, whose roots are

$$S_{1,2} = \frac{n \pm \sqrt{n^2 - 4}}{2}$$

For $n < 2$, one has an unstable focal point, and for $n > 2$, an unstable nodal point. In both cases the point singularity $K = 0$ is unstable.

In order to ascertain the stability of the limit cycle $K = \sqrt{\frac{4m}{3n}}$, one must differentiate Equation [55.4] with respect to K

$$\phi'(K) = \frac{m}{2} - \frac{9}{8}nK^2 \qquad [55.7]$$

Substituting $K^2 = \frac{4m}{3n}$ in this expression, one finds

$$\phi'\left(\sqrt{\frac{4m}{3n}}\right) = \frac{m}{2} - \frac{9}{8}n\frac{4m}{3n} = \frac{m}{2} - \frac{3}{2}m < 0$$

The limit cycle is therefore stable. The oscillating system governed by Rayleigh's equations, [55.1] or [55.3], thus exhibits the property of being self-excited in a steady state. If the dissipative terms were of the same sign, that is, of the form $\pm(m\dot{x} + n\dot{x}^3)$, there would be no limit cycle in this case, as is easily ascertained, although the oscillation would still be governed by a non-linear differential equation of a dissipative type. The example of Section 31 belongs to the case considered in this section.

It can be shown that the oscillation of Froude's pendulum, see Section 8, is also governed by Rayleigh's equation. In the elementary theory of this phenomenon it was established that the damping is *initially* negative under certain specified conditions. A linear equation does not represent the actual phenomenon because when the coefficient of $\dot{\phi}$ is negative it indicates that the amplitudes of oscillation increase indefinitely, which is clearly impossible from physical considerations. The reason for this inconsistency is the fact that the equation was overlinearized by dropping the non-linear terms.

By conserving at least the first two non-linear terms, we shall be able to establish the existence of a finite limit cycle, characterizing the steady state of oscillation of Froude's pendulum which is actually observed. Expanding the function $M(\omega - \dot{\phi})$ of Section 8 in a Taylor series, we have

$$M(\omega - \dot{\phi}) = M(\omega) - \dot{\phi}M'(\omega) + \frac{\dot{\phi}^2}{2!}M''(\omega) - \frac{\dot{\phi}^3}{3!}M'''(\omega) + \cdots \qquad [55.8]$$

Here we assume that the function M is analytic, that is, that it admits derivatives at least as an idealized feature of the observed phenomenon.

Dividing Equation [8.1] by I, introducing dimensionless time, and keeping only the first two non-linear terms in the expansion, one obtains an equation of the form

$$\ddot{\phi} + \phi = -c\dot{\phi} + k\dot{\phi}^2 - n\dot{\phi}^3 \qquad [55.9]$$

where $c = b + M'(\omega)$, and k and n are suitable constant coefficients obtained by substituting the expansion [55.8] into Equation [8.1]. It must be noted that c is negative according to what has been stated in Section 8. Putting $c = -m$, where $m > 0$, one has

$$\ddot{\phi} + \phi = m\dot{\phi} + k\dot{\phi}^2 - n\dot{\phi}^3 \equiv \mu(m_1\dot{\phi} + k_1\dot{\phi}^2 - n_1\dot{\phi}^3)$$

whence

$$f(\dot{\phi}) = \gamma_1\dot{\phi} + k_1\dot{\phi}^2 - n_1\dot{\phi}^3$$

and

$$f(K\sin u) = m_1 K\sin u + k_1 K^2\sin^2 u - n_1 K^3 \sin^3 u$$

Hence

$$\phi(K) = -\frac{1}{2\pi}\left[m_1 K\int_0^{2\pi}\sin^2 u\,du + k_1 K^2\int_0^{2\pi}\sin^3 u\,du - n_1 K^3\int_0^{2\pi}\sin^4 u\,du\right]$$

that is,

$$\phi(K) = -\frac{1}{2}\left[m_1 K - \frac{3}{4}n_1 K^3\right]$$

The equation for limit cycles is

$$K\left(m_1 - \frac{3}{4}n_1 K^2\right) = 0 \qquad [55.10]$$

The solution $K = 0$ corresponds clearly to a point singularity, and the amplitude of the limit cycle is $K = \sqrt{\frac{4m_1}{3n_1}}$. One can calculate m_1 and n_1 explicitly from the equations of transformation of the original equation to its form involving dimensionless time. It is to be noted that the quadratic term disappears from Expression [55.10] for the limit cycle because $\int_0^{2\pi} \sin^3 u \, du = 0$, a fact which has already been noted in Section 54.

56. MORE GENERAL FORMS OF NON-LINEAR EQUATIONS

The Van der Pol method is applicable also to non-linear equations involving, in addition to variable damping, a *variable "spring constant."* It must be noted, however, that the existence of self-excited oscillations, that is, of limit cycles, depends only on the conditions of the variable damping, as was shown by Van der Pol (15) and generalized later by E. and H. Cartan (16) and Liénard (17). In fact, the initial damping, for small x in the Van der Pol equation or for small $\dot{x}$ in the Rayleigh equation, is first negative then positive for larger values of the corresponding variable (x or $\dot{x}$). Physically this means that there exists, initially, an energy input into the system which, in later stages of motion, becomes an energy drain from the system, which then becomes dissipative. The existence of a steady oscillation, that is, of a limit cycle, depends thus on the average equality between the input and the drain per cycle. We shall come back to this important point later in connection with the Principle of Equivalent Balance of Energy formulated by Kryloff and Bogoliuboff. Consider, for example, a more general type of equation

$$m\ddot{x} + (nx^2 - \alpha)\dot{x} + \beta x + \gamma x^2 = 0 \qquad [56.1]$$

which has variable damping, the term $nx^2\dot{x}$, as well as a variable spring constant, the term $(\beta + \gamma x)x$. As long as the terms $(nx^2 - \alpha)\dot{x}$ and γx^2 are small, we can write this equation in the form

$$m\ddot{x} + \beta x = \mu\left[(\alpha_1 - n_1 x^2)\dot{x} - \gamma_1 x^2\right] \equiv \mu\phi(x,\dot{x}) \qquad [56.2]$$

We can easily reduce this equation to a form previously considered. It is sufficient to divide it by m, putting $\beta/m = \omega_0{}^2$, and pass to a dimensionless independent variable* in order to obtain the equation in the usual form

$$\ddot{x} + x = \mu\left[(\alpha_0 - n_0 x^2)\dot{x} - \gamma_0 x^2\right] \equiv \mu f(x,\dot{x}) \qquad [56.3]$$

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* The last operation, although convenient for practical calculations, is not altogether necessary. In case one does not use it, the generating solutions should be taken in the form $K\cos\omega t$, $-K\omega\sin\omega t$ instead of $K\cos t$, $-K\sin t$.

and to substitute the generating solutions $K \cos u$ and $- K \sin u$ into the first equation [52.9] giving the limit cycles. The same procedure applies to the generalized Rayleigh equation with the variable spring constant.

In all cases the existence of limit cycles requires that the coefficient of $\dot{x}$ satisfy the conditions of Cartan-Liénard. If this condition is not fulfilled, no self-excited oscillations can exist in a steady state, and the system behaves as purely dissipative, while still non-linear.

CHAPTER X

THEORY OF THE FIRST APPROXIMATION OF KRYLOFF AND BOGOLIUBOFF*

57. INTRODUCTORY REMARKS

The method of Kryloff and Bogoliuboff is very similar to that of Van der Pol and is related to it in the following way. While Van der Pol applies the method of variation of constants to the basic solution $x = a \cos \omega t + b \sin \omega t$ of $\ddot{x} + \omega^2 x = 0$, Kryloff and Bogoliuboff apply the same method to the basic solution $x = a \cos (\omega t + \phi)$ of the same equation. Thus in Kryloff and Bogoliuboff's method, the "varied" constants are a and ϕ (polar coordinates), while in Van der Pol's method they are a and b (rectangular coordinates). The method of Kryloff and Bogoliuboff seems more interesting from the point of view of applications, since it deals directly with the amplitude and phase of the quasi-harmonic oscillation.

Before proceeding with a review of the method of Kryloff and Bogoliuboff, a few additional remarks concerning the effect of secular terms may be helpful.

58. EFFECT OF SECULAR TERMS IN SOLUTIONS BY EXPANSIONS IN SERIES

The difficulty arising from the appearance of secular terms has already been mentioned in Sections 44 and 46. In the example given in Section 46 that difficulty was avoided by a rather delicate change from the "old" periods to the "new," or corrected, ones. This change requires a knowledge of the correction for the period, which is not always obtainable as has been demonstrated. Unfortunately, in a great majority of cases in which the approximation consists in abbreviating an infinite series by a few terms, the situation is still more difficult.

In order to see this point, consider again a quasi-linear differential equation of the form

$$\ddot{x} + \omega^2 x + \mu f(x,\dot{x}) = 0 \qquad [58.1]$$

where μ is a small parameter, i.e., $\mu \ll 1$. Since the non-linear term appears with a small coefficient μ, Poisson suggested as a solution an expression of the form

$$x = x_0 + \mu x_1 + \mu^2 x_2 + \cdots \qquad [58.2]$$

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* The subject matter of this and of the following two chapters is taken from the treatise of Kryloff and Bogoliuboff, Reference (5). This subject is also treated in the free translation by S. Lefschetz of the Kryloff-Bogoliuboff text, Princeton University Press, 1943.

For $\mu = 0$, $x = x_0 = a \cos \omega t$ is the known generating solution. For $0 < \mu \ll 1$, it seems logical to consider the effect of the non-linear term $\mu f(x,\dot{x})$ as a small *perturbation* and to assume that this perturbation will also be felt in the modification of the initial generating solution represented as a power series of μ, given by Equation [58.2]. The method of Poincaré, Chapter VIII, is, in fact, a further generalization of Poisson's method.

If one substitutes the solution [58.2] into the differential equation [58.1], limiting the expansion up to the power μ^K of the small parameter, one obtains the following series of differential equations by equating terms containing the same powers of μ:

$$\begin{aligned} &\ddot{x}_0 + \omega^2 x_0 = 0 \\ &\ddot{x}_1 + \omega^2 x_1 = -f_1(x_0, \dot{x}_0) \\ &\ddot{x}_2 + \omega^2 x_2 = \left[-f_x(x_0,\dot{x}_0)\,x_1 + f_{\dot{x}}(x_0,\dot{x}_0)\,\dot{x}_1\right] \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad [58.3]$$

This procedure has already been outlined in connection with Equations [46.5] of the theory of Poincaré. It is easy to show, however, that a direct application of this method is handicapped by the following difficulty.

Consider, for example, the simplest case, that in which $f(x,\dot{x}) \equiv + \dot{x}\omega$. In this case Equation [58.1] is

$$\ddot{x} + \omega^2 x + \mu\omega\dot{x} = 0$$

The exact solution of this equation is

$$x = Ae^{-\frac{\mu\omega t}{2}} \cos\left[\left(\omega\sqrt{1 - \frac{\mu^2}{4}}\right)t + \phi\right] \qquad [58.4]$$

where A and ϕ are the constants of integration determined by the initial conditions. If, however, one proceeds by substituting the expansion [58.2] into Equations [58.3], the first equation gives

$$x_0 = A\cos(\omega t + \phi)$$

Substituting this solution into the second equation [58.3], one has

$$\ddot{x}_1 + \omega^2 x_1 = -\dot{x}_0\omega = A\omega^2 \sin(\omega t + \phi) \qquad [58.5]$$

This equation is satisfied by

$$x_1 = -\frac{A\omega t}{2}\cos(\omega t + \phi) \qquad [58.6]$$

Substituting these values for x_0 and x_1 into Equation [58.2], one has

$$x = A\left(1 - \frac{\mu\omega t}{2}\right)\cos(\omega t + \phi) \qquad [58.7]$$

It is clear that the amplitude of the approximate solution [58.7] increases with time t indefinitely, whereas, according to the exact solution [58.4], it approaches zero, owing to the presence of the exponential term $e^{-\frac{\mu\omega t}{2}}$

As a second example, consider the differential equation

$$\ddot{x} + \omega^2 x(1 + \mu x^2) = 0 \qquad [58.8]$$

which may be considered as the equation of motion of a mechanical mass attracted to the position of equilibrium by a force proportional to the distance, with a perturbation term proportional to the cube of the distance.

Proceeding as before and seeking a solution of the form $x = x_0 + \mu x$, one has

$$\ddot{x}_0 + \omega^2 x_0 = 0$$
$$\ddot{x}_1 + \omega^2 x_1 = -\omega^2 x_0^3 \qquad [58.9]$$

From the first equation [58.9], $x_0 = A\sin(\omega t + \phi)$. Substituting this value for x_0 into the second equation, one has

$$\ddot{x}_1 + \omega^2 x_1 = -\omega^2 A^3 \sin^3(\omega t + \phi) = -\frac{3}{4}\omega^2 A^3 \sin(\omega t + \phi) + \frac{1}{4}\omega^2 A^3 \sin 3(\omega t + \phi)$$

This equation is satisfied by

$$x_1 = \frac{3}{8}\omega t A^3 \cos(\omega t + \phi) - \frac{A^3}{32}\sin 3(\omega t + \phi)$$

whence

$$x = A\sin(\omega t + \phi) + \frac{3\mu}{8}\omega t A^3 \cos(\omega t + \phi) - \frac{A^3\mu}{32}\sin 3(\omega t + \phi) \qquad [58.10]$$

The second term of this expression is a secular term. Thus, this expression for the displacement has no physical meaning. Unfortunately, in this case the exact solution is not known, as Equation [58.8] is not linear. One can, however, affirm the correctness of the above statement by invoking the law of conservation of energy, which holds in this case since the system is conservative. In fact, multiplying Equation [58.8] by $\dot{x}$, one can write

$$\dot{x}\left[\ddot{x} + (1 + \mu x^2)\omega^2 x\right] = \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \frac{\mu\omega^2}{4}x^4\right) = 0$$

From this we can obtain the law of conservation of energy

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \frac{\mu\omega^2}{4}x^4 = \text{constant} = E \qquad [58.11]$$

From this equation it follows that, for $\mu > 0$, the square of the amplitude x^2 has an upper bound $2E/\omega^2$. This result is to be expected, as the system has no sources of energy and is conservative. Hence there is a definite contradiction of Equation [58.10].

From these two examples, we see that the direct application of Poisson's method to problems of dynamics encounters a serious difficulty because of the presence of secular terms.

59. EQUATIONS OF THE FIRST APPROXIMATION

In this section we propose to establish the fundamental points of the theory of the first approximation of Kryloff and Bogoliuboff, which will play an important role in subsequent chapters.

For $\mu = 0$, Equation [58.1] reduces to a simple linear equation whose solution is

$$x = a\sin(\omega t + \phi); \qquad \dot{x} = a\omega\cos(\omega t + \phi) \qquad [59.1]$$

where a and ϕ are constants, the amplitude and the phase respectively. For a quasi-linear equation when $\mu \neq 0$ but is small, it appears logical to retain the form of solutions [59.1], provided that we consider the quantities a and ϕ not as constants but as certain functions of time to be determined.

Differentiating the first equation [59.1], one obtains

$$\dot{x} = \dot{a}\sin(\omega t + \phi) + a\omega\cos(\omega t + \phi) + a\dot{\phi}\cos(\omega t + \phi) \qquad [59.2]$$

Making use of the second equation [59.1], one has

$$\dot{a}\sin(\omega t + \phi) + a\dot{\phi}\cos(\omega t + \phi) = 0 \qquad [59.3]$$

Differentiating the second equation [59.1], one gets

$$\ddot{x} = \dot{a}\omega\cos(\omega t + \phi) - a\omega^2\sin(\omega t + \phi) - a\omega\dot{\phi}\sin(\omega t + \phi) \qquad [59.4]$$

Substituting these values for x, $\dot{x}$, and $\ddot{x}$ into the original quasi-linear equation [58.1], one has

$$\dot{a}\omega\cos(\omega t + \phi) - a\omega\dot{\phi}\sin(\omega t + \phi) + \mu f\left[a\sin(\omega t + \phi), a\omega\cos(\omega t + \phi)\right] \qquad [59.5]$$

Solving Equations [59.3] and [59.5] for $\dot{a}$ and $\dot{\phi}$, one gets

$$\dot{a} = -\frac{\mu}{\omega}f\left[a\sin(\omega t + \phi), a\omega\cos(\omega t + \phi)\right]\cos(\omega t + \phi) \qquad [59.6]$$

$$\dot{\phi} = \frac{\mu}{a\omega} f\left[a\sin(\omega t + \phi), a\omega\cos(\omega t + \phi)\right]\sin(\omega t + \phi) \tag{59.7}$$

It can be seen that the original equation [58.1] of the second order has been reduced to a system of two equations, [59.6] and [59.7], of the first order. The interesting feature of this transformation lies in the fact that these first-order equations are now written in terms of the amplitude and phase as dependent variables. One notes a formal analogy with Equations [52.8] of Van der Pol and also with Equations [46.22] and [46.23] of Poincaré.

From the form of the right side of Equations [59.6] and [59.7], it is seen that both $\dot{a}$ and $\dot{\phi}$ are periodic functions of time. From the fact that the right-hand terms of these equations contain a small parameter μ, one can conclude that both a and ϕ, while being periodic, are functions which vary slowly during one period $T = 2\pi/\omega$ of the trigonometric functions involved.

It is reasonable, therefore, to consider a and ϕ as constant during one period T. It is possible to transform Equations [59.6] and [59.7] into a more convenient form. For this purpose, consider the Fourier expansions of the functions

$$\begin{aligned} f(a\sin\phi, a\omega\cos\phi)\cos\phi &= K_0(a) + \sum_{n=1}^{\infty}\left[K_n(a)\cos n\phi + L_n(a)\sin n\phi\right] \\ f(a\sin\phi, a\omega\cos\phi)\sin\phi &= P_0(a) + \sum_{n=1}^{\infty}\left[P_n(a)\cos n\phi + Q_n(a)\sin n\phi\right] \end{aligned} \tag{59.8}$$

where

$$\begin{aligned} K_0(a) &= \frac{1}{2\pi}\int_0^{2\pi} f(a\sin\phi, a\omega\cos\phi)\cos\phi\, d\phi \\ P_0(a) &= \frac{1}{2\pi}\int_0^{2\pi} f(a\sin\phi, a\omega\cos\phi)\sin\phi\, d\phi \\ K_n(a) &= \frac{1}{\pi}\int_0^{2\pi} f(a\sin\phi, a\omega\cos\phi)\cos\phi\cos n\phi\, d\phi \\ L_n(a) &= \frac{1}{\pi}\int_0^{2\pi} f(a\sin\phi, a\omega\cos\phi)\cos\phi\sin n\phi\, d\phi \\ P_n(a) &= \frac{1}{\pi}\int_0^{2\pi} f(a\sin\phi, a\omega\cos\phi)\sin\phi\cos n\phi\, d\phi \\ Q_n(a) &= \frac{1}{\pi}\int_0^{2\pi} f(a\sin\phi, a\omega\cos\phi)\sin\phi\sin n\phi\, d\phi \end{aligned} \tag{59.9}$$

Equations [59.6] and [59.7] can then be written in the form

$$\frac{da}{dt} = -\frac{\mu}{\omega}K_0(a) - \frac{\mu}{\omega}\sum_{n=1}^{\infty}\Big[K_n(a)\cos n(\omega t + \phi) + L_n(a)\sin n(\omega t + \phi)\Big]$$

$$\frac{d\phi}{dt} = \frac{\mu}{a\omega}P_0(a) + \frac{\mu}{a\omega}\sum_{n=1}^{\infty}\Big[P_n(a)\cos n(\omega t + \phi) + Q_n(a)\sin n(\omega t + \phi)\Big] \qquad [59.10]$$

Integrating these equations between the limits t and $t + T$, and considering $a(t)$ and $\phi(t)$ as remaining approximately constant in this interval, one has as the first approximation

$$\frac{a(t + T) - a(t)}{T} = -\frac{\mu}{\omega}K_0\big[a(t)\big]; \quad \frac{\phi(t + T) - \phi(t)}{T} = \frac{\mu}{a\omega}P_0\big[a(t)\big] \qquad [59.11]$$

since

$$\int_t^{t+T}\cos n(\omega t + \phi)\,dt = \int_t^{t+T}\sin n(\omega t + \phi)\,dt = 0$$

Furthermore, since, by assumption, the variations Δa and $\Delta\phi$ of amplitude and phase are small during the interval $(t, t + T)$, one can write Equations [59.11] to the first approximation

$$\frac{da}{dt} = -\frac{\mu}{\omega}K_0(a); \quad \frac{d\phi}{dt} = \frac{\mu}{a\omega}P_0(a) \qquad [59.12]$$

If these equations are compared with the exact equations [59.10], it is seen that the equations of the first approximation are obtained from the exact equations by *averaging the latter equations over the period*, thus eliminating the rest of the Fourier series under the summation sign. The analogy between Equations [59.12] and the "abbreviated" equations [52.6] of Van der Pol should be noted.

Letting $\psi = \omega t + \phi$, the *total phase* of the motion, we have $d\psi/dt = \omega + d\phi/dt$. Making use of these relations and the relation for $K_0(a)$ and $P_0(a)$ in [59.9], we obtain for the equations of the first approximation

$$\frac{da}{dt} = -\frac{\mu}{\omega}\frac{1}{2\pi}\int_0^{2\pi}f(a\sin\phi, a\omega\cos\phi)\cos\phi\,d\phi \equiv \Phi(a) \qquad [59.13]$$

$$\frac{d\psi}{dt} = \omega + \frac{\mu}{a\omega}\frac{1}{2\pi}\int_0^{2\pi}f(a\sin\phi, a\omega\cos\phi)\sin\phi\,d\phi \equiv \Omega(a) \qquad [59.14]$$

The first approximation will then be $x = a\sin\psi$, where the amplitude a and the phase ψ are obtained from Equations [59.13] and [59.14].

60. NON-LINEAR CONSERVATIVE SYSTEMS

Consider again a quasi-linear differential equation

$$\ddot{x} + \omega^2 x + \mu f(x) = 0$$

in which $f(x)$ does not contain the velocity $\dot{x}$. Equation [59.13] gives

$$\frac{da}{dt} = -\frac{\mu}{2\pi\omega}\int_0^{2\pi} f(a\sin\phi)\cos\phi\, d\phi \qquad [60.1]$$

and Equation [59.14] gives

$$\frac{d\psi}{dt} = \omega + \frac{\mu}{2\pi a\omega}\int_0^{2\pi} f(a\sin\phi)\sin\phi\, d\phi \qquad [60.2]$$

Noting that

$$\int_0^{2\pi} f(a\sin\phi)\cos\phi\, d\phi = \frac{1}{a}\,\psi(a\sin\phi)\Big|_0^{2\pi} = 0$$

where $\psi(x) = \int_0^x f(\xi)d\xi$, Equation [60.1] gives a = constant. Hence, from the first approximation, it follows that the amplitude does not change in the course of time; the system is thus conservative. This can be seen from *a priori* considerations, because the function f does not depend on the velocity $\dot{x}$, whereas dissipative forces generally do depend on $\dot{x}$.

From Equation [59.14] it follows that

$$\psi = \Omega(a)t + \psi_0 \qquad [60.3]$$

since $\Omega(a)$ does not depend on t in this case; ψ_0 is a constant of integration.

Thus, the oscillations will be of the form

$$x = a\sin\left[\Omega(a)t + \psi_0\right] \qquad [60.4]$$

Therefore, to the first approximation, the effect of a non-linearity of this type will be felt only in that the frequency of oscillation will depend on the amplitude a, that is, the oscillations are not isochronous, but the decrement of oscillation is zero since the system is conservative.

Squaring Equation [59.14] and neglecting the term of the second order in μ, one has

$$\Omega^2(a) = \omega^2 + \frac{\mu}{\pi a}\int_0^{2\pi} f(a\sin\phi)\sin\phi\, d\phi = \frac{1}{\pi a}\left[\omega^2 a\int_0^{2\pi}\sin^2\phi\, d\phi + \mu\int_0^{2\pi} f(a\sin\phi)\sin\phi\, d\phi\right]$$

$$= \frac{1}{\pi a}\int_0^{2\pi}\left[\omega^2 a\sin\phi + \mu f(a\sin\phi)\right]\sin\phi\, d\phi = \frac{1}{\pi a}\int_0^{2\pi} F(a\sin\phi)\sin\phi\, d\phi \qquad [60.5]$$

where

$$F(a\sin\phi) = \omega^2 a\sin\phi + \mu f(a\sin\phi) \qquad [60.6]$$

On the other hand, the general form of a non-linear equation, in which the non-linearity is only in the spring constant, is of the form $\ddot{x} + F(x) = 0$. If the term $F(x)$ is not far from linearity, it can be written as $F(x) = \omega^2 x + \mu f(x)$, where $f(x)$ is the non-linear component of $F(x)$. Comparing this expression with Equation [60.6], one finds exactly the same result provided one substitutes the generating solution $a \sin\phi$ for x. From this we obtain the following theorem:

When a system is conservative but not linear, the amplitude a remains constant and the frequency $\Omega(a)$ is given by Equation [60.5], in which $F(x)$ is the term entering into the equation $\ddot{x} + F(x) = 0$, without the necessity of splitting it into a linear component $\omega^2 x$ and a non-linear one, $\mu f(x)$.

In the following section examples are given illustrating the application of the theory of the first approximation.

61. EXAMPLES OF NON-LINEAR CONSERVATIVE SYSTEMS

A. PENDULUM

The differential equation for a pendulum is $\ddot{\theta} + \frac{g}{L}\sin\theta = 0$. In elementary theory, which we may designate as an *approximation of zero order*, it is assumed that for small angles $\sin\theta \approx \theta$. The well-known solution for the period, $T = 2\pi\sqrt{L/g}$, is obtained. It is to be noted that oscillations are isochronous under this assumption.

For the first approximation we can take $\sin\theta \approx \theta - \frac{\theta^3}{6}$; the differential equation then becomes $\ddot{\theta} + \omega^2(\theta - \frac{\theta^3}{6}) = 0$, where $\omega^2 = g/L$. Using Equation [60.5], one obtains

$$\Omega^2(a) = \frac{\omega^2}{\pi a}\int_0^{2\pi}\left(a\sin\phi - \frac{a^3\sin^3\phi}{6}\right)\sin\phi\, d\phi$$

$$= \frac{\omega^2}{\pi a}\left[a\int_0^{2\pi}\sin^2\phi\, d\phi - \frac{a^3}{6}\int_0^{2\pi}\sin^4\phi\, d\phi\right] = \omega^2\left(1 - \frac{a^2}{8}\right)$$

that is,

$$\Omega(a) = \omega\sqrt{1 - \frac{a^2}{8}} \approx \omega\left(1 - \frac{a^2}{16}\right) \qquad [61.1]$$

or, in terms of the period

$$T(a) = T\left(1 + \frac{a^2}{16}\right) \qquad [61.2]$$

It is thus seen that the oscillation is not isochronous; the period increases slightly with increasing amplitudes of oscillation.

Thus, for example,

for amplitudes of the order of 10 degrees, $T(a) = T \times 1.001$;

for amplitudes of the order of 20 degrees, $T(a) = T \times 1.006$;

for amplitudes of the order of 30 degrees, $T(a) = T \times 1.014$.

It should be observed that, although the expression for the period can be established in this case by means of elliptic functions, the theory of the first approximation leads to this result by a more general procedure. Furthermore, the latter method easily leads to correct results in more complicated cases for which exact methods are not available.

B. TORSIONAL OSCILLATIONS OF A SHAFT

Let J_1 and J_2 be the moments of inertia of rotating masses placed at the ends of a shaft S, see Figure 61.1. If θ_1 and θ_2 are the two angles determining the angular position of the masses J_1 and J_2 with respect to a fixed reference angle, the torsional moment M is a certain function of the difference $(\theta_1 - \theta_2)$, say $C(\theta_1 - \theta_2) = M(\theta)$.*

The differential equations of the coupled system are

$$J_1\ddot{\theta}_1 + C(\theta_1 - \theta_2) = 0 \quad \text{and} \quad J_2\ddot{\theta}_2 - C(\theta_1 - \theta_2) = 0$$

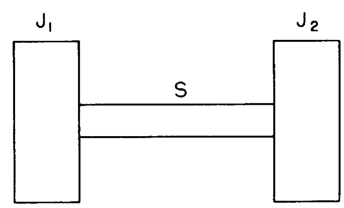

Figure 61.1

Subtracting the second equation from the first and letting $\theta = \theta_1 - \theta_2$, we obtain

$$J_1 J_2 \ddot{\theta} + (J_1 + J_2)C(\theta) = 0 \qquad [61.3]$$

which is the non-linear differential equation of the torsional oscillation of the system.

Equation [61.3] can be written as $\ddot{\theta} + K^2 C(\theta) = 0$, where $K^2 = (J_1 + J_2)/J_1 J_2$. Assume that $C(\theta)$ is of the form $C(\theta) = C_0\theta \pm C_1\theta^3$, where $C_1\theta^3$ is a small non-linear term. From this,

$$F(\theta) = K^2 C_0 \theta \pm K^2 C_1 \theta^3 = m\theta \pm n\theta^3$$

and the frequency is given by

$$\Omega^2(a) = \frac{1}{\pi a}\int_0^{2\pi}(ma\sin\phi \pm na^3\sin^3\phi)\sin\phi\, d\phi = \omega_0^2 \pm \frac{3}{4}K^2 C_1 a^2$$

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* $C(\theta_1 - \theta_2)$ in these notations should be read: C is a function of $(\theta_1 - \theta_2)$, and not C times $(\theta_1 - \theta_2)$.

or

$$\Omega(a) \approx \omega_0\left(1 \pm \frac{3}{8}\,\frac{K^2C_1a^2}{\omega_0^2}\right)$$

The non-linear frequency (and therefore the period) will thus depend on the amplitude. The amplitude of the vibration is constant, but the vibration is not isochronous.

C. ELECTRICAL OSCILLATIONS OF A CIRCUIT CONTAINING AN IRON CORE

Let a circuit be composed of an inductance L and a capacity C with negligible resistance. The inductance coil is wound on an iron circuit A, see Figure 61.2, subject to magnetic saturation. The condition of equilibrium of electromotive forces in the circuit gives

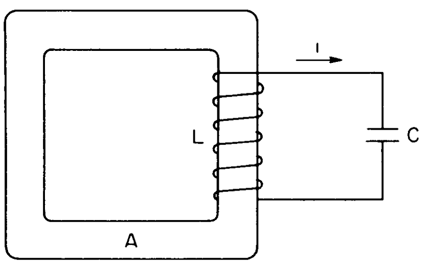

Figure 61.2

$$\frac{d\Phi}{dt} + \frac{1}{C}\int_0^t i\,dt = 0$$

where Φ is the totalized flux through the coil. The condition of saturation can be approximated by the equation $i = A\Phi + B\Phi^3$, whence from the preceding equation one has

$$\ddot{\Phi} + \frac{A\Phi + B\Phi^3}{C} = 0 \qquad [61.4]$$

Equation [61.4] is the non-linear equation of oscillation. Reducing it to the standard form of a quasi-linear equation, where $A/C = \omega^2$ = constant, and assuming that the ratio $\frac{B}{A}\Phi^2 << 1$, one can apply Formula [60.5] and obtain

$$\Omega^2(a) = \frac{1}{\pi a C}\int_0^{2\pi}(Aa\sin\phi + Ba^3\sin^3\phi)\sin\phi\,d\phi = \frac{A}{C}\left(1 + \frac{3Ba^2}{4A}\right)$$

that is,

$$\Omega(a) = \omega\sqrt{1 + \frac{3Ba^2}{4A}} \approx \omega\left(1 + \frac{3}{8}\,\frac{Ba^2}{A}\right) \qquad [61.5]$$

The actual frequency $\Omega(a)$ is thus increased in comparison with the frequency ω for small amplitudes, owing to a decrease of L with the amplitude a of the oscillation. Thus the oscillation is non-ioschronous.

62. SYSTEMS WITH NON-LINEAR DAMPING OF A DISSIPATIVE TYPE

Consider the differential equation

$$m\ddot{x} + Kx + f(\dot{x}) = 0$$

Dividing it by m and putting $K/m = \omega^2$, we get

$$\ddot{x} + \omega^2 x + \frac{1}{m} f(\dot{x}) = 0$$

We shall keep within the limits of the quasi-harmonic theory. In this case $\frac{1}{m} f(\dot{x})$ plays the role taken by $\mu f(x,\dot{x})$ in the general theory. The expressions

$$\int_0^{2\pi} f(a\sin\phi,\, a\omega\cos\phi)\cos\phi\, d\phi \quad \text{and} \quad \int_0^{2\pi} f(a\sin\phi,\, a\omega\cos\phi)\sin\phi\, d\phi$$

entering into Equations [59.13] and [59.14] of the first approximation, in this case are

$$\int_0^{2\pi} f(a\omega\cos\phi)\cos\phi\, d\phi \quad \text{and} \quad \int_0^{2\pi} f(a\omega\cos\phi)\sin\phi\, d\phi$$

respectively, since $f(x,\dot{x})$ reduces to $f(\dot{x})$.

We note that

$$\int_0^{2\pi} f(a\omega\cos\phi)\sin\phi\, d\phi = -\frac{1}{a\omega}\int_0^{2\pi} f(a\omega\cos\phi)\, d(a\omega\cos\phi) = -\frac{1}{a\omega}\,\Phi(a\omega\cos\phi)\Big|_0^{2\pi} = 0$$

so that, by Equation [59.14],

$$\frac{d\psi}{dt} = \omega \qquad [62.1]$$

Furthermore, Equation [59.13] can be written as

$$\frac{da}{dt} = \dot{a} = -\frac{1}{2\pi m\omega}\int_0^{2\pi} f(a\omega\cos\phi)\cos\phi\, d\phi \qquad [62.2]$$

It is clear that the instantaneous frequency $d\psi/dt$ is equal to the constant "linear" frequency ω, and the amplitude a varies according to Equation [62.2]. Thus, the oscillation is generally of the form $x = a\sin(\omega t + \psi_0)$, where ψ_0 is a constant. The frequency is not changed since the frequency correction is of the second order and therefore does not appear in equations of the first approximation. A few examples given below illustrate the application of Equation [62.2] to various types of non-linear damping $f(\dot{x})$.

A. LINEAR DAMPING: $f(\dot{x}) = \lambda\dot{x}$

In this case

$$\int_0^{2\pi} f(a\omega\cos\phi)\cos\phi\, d\phi = a\omega\lambda\int_0^{2\pi}\cos^2\phi\, d\phi = a\omega\lambda\pi$$

and Equation [62.2] gives

$$\dot{a} = -\frac{1}{2\pi m\omega} \cdot a\omega\lambda\pi = -\frac{\lambda a}{2m}$$

whence

$$a = a_0 e^{-\frac{\lambda t}{2m}} \qquad [62.3]$$

Comparing this with the exact solution

$$x = a_0 e^{-\frac{\lambda t}{2m}} \sin(\omega_1 t + \psi_0)$$

where

$$\omega_1 = \omega\sqrt{1 - \frac{1}{4}\left(\frac{\lambda}{\sqrt{Km}}\right)^2}$$

one notes that the first approximation gives the same expression for the amplitude as the exact solution; the difference between the expressions for frequency ω and ω_1 is of the order of $\frac{1}{8}(\lambda/\sqrt{Km})^2$, that is, of the second order, if λ is of the first order, as previously set forth.

B. QUADRATIC DAMPING: $f(\dot{x}) = b\dot{x}^2$

Since $f(\dot{x})$ is an even function of $\dot{x}$, and from physical considerations it should be an odd function of $\dot{x}$, we should write the above expression as $f(\dot{x}) = b|\dot{x}|\dot{x}$. In this case

$$\int_0^{2\pi} f(a\omega\cos\phi)\cos\phi\, d\phi = ba^2\omega^2\int_0^{2\pi}|\cos\phi|\cos^2\phi\, d\phi = ba^2\omega^2\left[\int_0^{\frac{\pi}{2}}\cos^3\phi\, d\phi + \int_{\frac{3\pi}{2}}^{2\pi}\cos^3\phi\, d\phi - \right.$$

$$\left. - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos^3\phi\, d\phi\right] = ba^2\omega^2\left[\frac{2}{3} + \frac{2}{3} + \frac{4}{3}\right] = \frac{8}{3}ba^2\omega^2$$

From Equation [62.2] we get

$$\frac{da}{dt} = -\frac{1}{2\pi m\omega}\int_0^{2\pi} f(a\omega\cos\phi)\cos\phi\, d\phi = -\frac{4b\omega a^2}{3\pi m}$$

that is,

$$-\frac{1}{a^2}\frac{da}{dt} = \frac{d\left(\frac{1}{a}\right)}{dt} = \frac{4b\omega}{3\pi m}$$

From this, on integrating, we obtain

$$\frac{1}{a} - \frac{1}{a_0} = \frac{4b\omega t}{3\pi m} \quad \text{or} \quad a = \frac{a_0}{1 + \frac{4b\omega a_0}{3\pi m} t} \qquad [62.4]$$

It is seen that the law of variation of the amplitude with time in this case is entirely different from that given for Case A.

C. COULOMB DAMPING: $f(\dot{x}) = A \operatorname{sgn}(\dot{x})$*

From this equation

$$\int_0^{2\pi} f(a\omega\cos\phi)\cos\phi\, d\phi = A\left[\int_0^{\frac{\pi}{2}} \cos\phi\, d\phi + \int_{\frac{3\pi}{2}}^{2\pi} \cos\phi\, d\phi - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\phi\, d\phi\right] = 4A$$

for $a \neq 0$. Furthermore, $\int_0^{2\pi} f(a\omega\cos\phi)\cos\phi\, d\phi = 0$ for $a = 0$. By Equation [62.2]

$$\frac{da}{dt} = -\frac{2A}{\pi m\omega} \quad \text{if} \quad a \neq 0$$

$$\frac{da}{dt} = 0 \qquad \text{if} \quad a = 0 \qquad [62.5]$$

Integrating the first equation [62.5], for $a \neq 0$, we get

$$a = a_0 - \frac{2A}{\pi m\omega} t \qquad [62.6]$$

The motion will continue as long as $a_0 - \frac{2A}{\pi m\omega} t > 0$, and will cease for t_1 defined by the equation $a_0 = \frac{2A}{\pi m\omega} t_1$. The motion thus lasts a finite time.

D. MIXED CASES: $f(\dot{x}) = \alpha\dot{x} + \beta\dot{x}^2$

In applications one frequently encounters differential equations in which both linear and quadratic damping are present. Thus, for example, Froude's differential equation for the rolling of a ship in still water is $I\ddot{\theta} + K_1\dot{\theta} + K_2\dot{\theta}^2 + Wh\theta = 0$, where I, K_1, K_2, W, and h are well-known constants. Likewise, the so-called "surge chamber" equation** in hydraulic engineering is of the form $\ddot{x} + p\dot{x}^2 + q\dot{x} + \gamma x = 0$. Writing equations of this kind in the form $\ddot{x} + \omega^2 x + \mu f(\dot{x}) = 0$, one has

---

* The symbol $\operatorname{sgn}(\dot{x})$ designates a discontinuous function defined as follows: $\operatorname{sgn}(\dot{x}) = 1$ for $\dot{x} > 0$; $\operatorname{sgn}(\dot{x}) = -1$ for $\dot{x} < 0$; and $\operatorname{sgn}(\dot{x}) = 0$ for $\dot{x} = 0$.

** The writer is indebted to Dr. W.F. Durand for bringing this equation to his attention.

$$\int_0^{2\pi} f(a\omega\cos\phi)\cos\phi\,d\phi = \alpha a\omega\int_0^{2\pi}\cos^2\phi\,d\phi + \beta a^2\omega^2\left[\int_0^{\frac{\pi}{2}}\cos^3\phi\,d\phi + \int_{\frac{3\pi}{2}}^{2\pi}\cos^3\phi\,d\phi - \right.$$

$$\left. - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos^3\phi\,d\phi\right] = \alpha a\omega\pi + \frac{8}{3}\beta a^2\omega^2$$

From this

$$\dot{a} = -\frac{1}{2\pi\omega}\left[\alpha a\omega\pi + \frac{8}{3}\beta a^2\omega^2\right] = -\left[\frac{\alpha}{2}a + Sa^2\right]$$

where $S = 4\beta\omega/3\pi$. Separating the variables

$$\frac{da}{\frac{\alpha}{2}a + Sa^2} = \frac{2}{\alpha}\frac{da}{a} - \frac{2}{\alpha}\frac{d\left(aS + \frac{\alpha}{2}\right)}{aS + \frac{\alpha}{2}}$$

we get

$$d\,\log\left[\frac{a}{aS + \frac{\alpha}{2}}\right] = -\frac{\alpha}{2}\,dt$$

Upon integrating and putting the constant of integration in the form $C = a_0/\left(a_0 S + \frac{\alpha}{2}\right)$, we obtain

$$\frac{a}{aS + \frac{\alpha}{2}} = \frac{a_0}{a_0 S + \frac{\alpha}{2}}\,e^{-\frac{\alpha}{2}t}$$

or

$$a = \frac{\frac{\alpha}{2}\,\frac{a_0}{a_0 S + \frac{\alpha}{2}}\,e^{-\frac{\alpha}{2}t}}{1 - S\,\frac{a_0}{a_0 S + \frac{\alpha}{2}}\,e^{-\frac{\alpha}{2}t}} \qquad [62.7]$$

For $t = 0$, one finds $a = a_0$; for $S = 0$, one finds $a = a_0 e^{-\frac{\alpha}{2}t}$, as in the case of linear damping, Case A. It is seen that the presence of quadratic damping causes a somewhat more rapid decay of amplitudes owing to the presence of the term $S\frac{a_0}{a_0 S + \frac{\alpha}{2}}\,e^{-\frac{\alpha}{2}t}$ in the denominator than is found with a pure linear damping. This fact is to be expected on physical grounds.

63. SYSTEMS WITH NON-LINEAR VARIABLE DAMPING

By the expression "variable" damping occurring in this section, we shall understand a damping which, for small values of the determining variable (either x or $\dot{x}$ as the case may be), is negative and becomes positive above a certain critical value of the variable. Since negative damping means supply to the system, and positive damping means withdrawal of energy from it, the system considered here is non-conservative. Furthermore, stationary states, or limit cycles, are possible when the average supply of energy per cycle becomes equal to its average dissipation.

According to the mode of production of the non-linear variable damping, there exist two principal types of non-linear self-excited oscillations governed by the following equations, which are solved here by the method of the first approximation.

A. VAN DER POL'S EQUATION

The Van der Pol equation is

$$\ddot{x} + x - \mu(1 - x^2)\dot{x} = 0 \qquad [63.1]$$

We have $f(x,\dot{x}) = \mu(x^2\dot{x} - \dot{x})$; furthermore, $\omega = 1$, hence $f(a \sin\phi,\ a\omega \cos\phi)$ is $f(a \sin\phi,\ a \cos\phi)$. Since $x = a \sin\phi$ and $\dot{x} = a \cos\phi$, it follows that

$$f(x,\dot{x}) = \mu(a^3 \sin^2\phi \cos\phi - a \cos\phi)$$

and

$$f(a \sin\phi,\ a \cos\phi) \cos\phi = \mu(a^3 \sin^2\phi \cos^2\phi - a \cos^2\phi)$$

Whence, by Equation [59.13], we have

$$\frac{da}{dt} = -\frac{\mu}{2\pi}\int_0^{2\pi} f(a \sin\phi,\ a \cos\phi) \cos\phi \, d\phi = -\frac{\mu}{2\pi}\left[a^3\int_0^{2\pi}\sin^2\phi \cos^2\phi \, d\phi - a\int_0^{2\pi}\cos^2\phi \, d\phi\right]$$

which reduces to

$$\frac{da}{dt} = \frac{\mu a}{2}\left(1 - \frac{a^2}{4}\right) \qquad [63.2]$$

Forming the expression $\int_0^{2\pi} f(a \sin\phi,\ a \cos\phi) \sin\phi \, d\phi$, one finds that it is zero. From this, by Equation [59.14], $d\psi/dt = \omega = 1$, that is, $\psi = t + \psi_0$, where ψ_0 is arbitrary.

The solution of Van der Pol's equation to the first order is then of the form

$$x = a \sin(t + \psi_0) \qquad [63.3]$$

where a is given by Equation [63.2].

As we shall mention shortly, the interesting feature of this solution is the variation of the amplitude a as a function of time. From the preceding study of Van der Pol's equation we know that a limit cycle exists in this case. We propose now to establish its existence by use of Equation [63.2].

Multiplying both sides of Equation [63.2] by $2a$, we have

$$\frac{da^2}{dt} = \mu a^2\left(1 - \frac{a^2}{4}\right)$$

that is,

$$\frac{da^2}{a^2\left(1 - \frac{a^2}{4}\right)} = d\left(\log\frac{a^2}{4 - a^2}\right) = \mu\, dt$$

Upon integration, we obtain

$$\log\frac{a^2}{4 - a^2} = \log\frac{a_0^{\,2}}{4 - a_0^{\,2}} + \mu t \qquad [63.4]$$

Expressing this relation in the equivalent form $\frac{a^2}{4 - a^2} = \frac{a_0^2}{4 - a_0^2}\, e^{\mu t}$ and solving for a^2, we get

$$a^2 = \frac{a_0^{\,2} e^{\mu t}}{1 + \frac{1}{4}a_0^2(e^{\mu t} - 1)}; \qquad a = \frac{a_0 e^{\frac{\mu t}{2}}}{\sqrt{1 + \frac{1}{4}a_0^2(e^{\mu t} - 1)}} \qquad [63.5]$$

The fundamental equation $x = a \sin\psi$ of the theory of the first approximation is then

$$x = \frac{a_0 e^{\frac{\mu t}{2}}}{\sqrt{1 + \frac{1}{4}a_0^2(e^{\mu t} - 1)}} \sin(t + \psi_0) = a \sin(t + \psi_0) \qquad [63.6]$$

It is apparent that Equation [63.6] describes the general nature of motion previously investigated in connection with limit cycles. In fact, if for $t = 0$, $a_0 = 0$, $x \equiv 0$. This trivial solution of the Van der Pol equation is, however, unstable. For any finite a_0, however small, the amplitudes increase, approaching the value $a(t) = 2$ as a limit. In the phase plane as $t \to +\infty$, the trajectory spirals toward the circle of radius $a = 2$ from the inside.

B. RAYLEIGH'S EQUATION

Rayleigh's equation* is

$$\ddot{x} + x + \mu(-\alpha + \beta\dot{x}^2)\dot{x} = 0 \qquad [63.7]$$

In this case $\omega = 1$ and $f(x,\dot{x}) = -\alpha\dot{x} + \beta\dot{x}^3$. Since $f(x,\dot{x}) \equiv f(\dot{x})$ and $\omega = 1$, the generating solution $a \cos\phi$ must be substituted into $f(\dot{x})$ resulting in

$$\int_0^{2\pi} f(a\cos\phi)\cos\phi\, d\phi = -\alpha a \int_0^{2\pi} \cos^2\phi\, d\phi + \beta a^3 \int_0^{2\pi} \cos^4\phi\, d\phi = -\alpha a \pi + \frac{3}{4}\pi\beta a^3$$

Hence, by Equation [59.13], with $\omega = 1$, one has

$$\dot{a} = \frac{\mu a}{2}\left(\alpha - \frac{3}{4}\beta a^2\right) \qquad [63.8]$$

The system will reach the limit cycle when $\alpha = \frac{3}{4}\beta a^2$, from which the amplitude of the limit cycle is $a = \sqrt{4\alpha/3\beta}$. The radius of the limit cycle increases as the ratio β/α decreases, which is physically obvious since for $\beta = 0$ the system becomes linear and the amplitude, at least theoretically, increases indefinitely, the damping then being negative.

To find the mode of approach to the limit cycle, one must integrate Equation [63.8]. Following the procedure explained in Section 63A, one obtains**

$$a(t) = \frac{a_0 e^{\frac{\mu\alpha}{2}t}}{\sqrt{1 + \frac{3}{4}\frac{\beta}{\alpha}a_0^2(e^{\mu\alpha t} - 1)}} \qquad [63.9]$$

This gives $a(t)_{t\to\infty} = \sqrt{4\alpha/3\beta}$, which is independent of the initial amplitude a_0, as required by the condition for a limit cycle.

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* It is supposed that the original Rayleigh equation, $m\ddot{x} + Kx + (-A + B\dot{x}^2)\dot{x} = 0$, has first been divided by m and written as

$$\ddot{x} + \omega^2 x + \left(-\frac{A}{m} + \frac{B}{m}\dot{x}^2\right)\dot{x} = 0$$

where $\omega^2 = K/m$, after which a change of the independent variable brings it to the above form, with $\omega = 1$.

** In case Equation [63.7] is not reduced to a unit frequency and has the form

$$x^2 + \omega^2 x + (-\alpha + \beta\dot{x}^2)\dot{x} = 0$$

the generating solution should be taken in the form $a\omega\cos\phi$ (instead of $a\cos\phi$), which finally results in the following equation for $a(t)$

$$a(t) = \frac{a_0 e^{\frac{\mu\alpha}{2}t}}{\sqrt{1 + \frac{3}{4}\frac{\beta}{\alpha}\omega^2 a_0^2(e^{\mu\alpha t} - 1)}} \qquad [63.9a]$$

64. EXISTENCE OF LIMIT CYCLES; SYSTEMS WITH SEVERAL LIMIT CYCLES

Although the scope of the method of the first approximation has been sufficiently ascertained from the previous sections of this chapter, we now propose to introduce certain additional transformations of the form of the fundamental equations [59.13] and [59.14] of this theory. The object of these transformations is to introduce functions similar to those appearing in Equations [52.8] of the Van der Pol theory. By this procedure we will prepare the groundwork for the investigation of an important subject, namely, the existence and stability of limit cycles.

It is useful for this purpose to recapitulate briefly the principal results of the theory of the first approximation.

The solution of a quasi-linear equation is considered in the form $x = a \sin \psi$, where a and ψ are given by the equations

$$\dot{a} = -\frac{\mu}{2\pi\omega}\int_0^{2\pi} f(a\sin\phi,\ a\omega\cos\phi)\cos\phi\, d\phi \equiv \Phi(a) \qquad [64.1]$$

$$\dot{\psi} = \omega + \frac{\mu}{2\pi\omega a}\int_0^{2\pi} f(a\sin\phi,\ a\omega\cos\phi)\sin\phi\, d\phi \equiv \Omega(a) \qquad [64.2]$$

The condition for a stationary oscillation on a limit cycle is

$$\Phi(a) = 0 \qquad [64.3]$$

This is exactly the condition obtained from Equation [46.22] of Poincaré's theory and the first equation [52.9] of Van der Pol's theory.

Equation [64.2] can be transformed by taking account of Equation [60.5]

$$\Omega^2(a) = \omega^2 + \frac{\mu}{\pi a}\int_0^{2\pi} f(a\sin\phi,\ a\omega\cos\phi)\sin\phi\, d\phi$$

and the identity

$$\omega^2 \equiv \frac{1}{\pi a}\int_0^{2\pi} \omega^2 a \sin^2\phi\, d\phi$$

Thus

$$\Omega^2(a) = \frac{1}{\pi a}\int_0^{2\pi} F(a\sin\phi,\ a\omega\cos\phi)\sin\phi\, d\phi \qquad [64.4]$$

where $F(x,\dot{x}) = \omega^2 x + \mu f(x,\dot{x})$ is the non-linear force appearing in the equation $\ddot{x} + F(x,\dot{x}) = 0$. Likewise, Equation [64.1] can be transformed by means of the identity $\int_0^{2\pi} a\omega^2 \sin\phi\cos\phi\, d\phi \equiv 0$, which gives

$$\dot{a} = \Phi(a) = -\frac{\mu}{2\pi\omega}\int_0^{2\pi} F(a\sin\phi,\, a\omega\cos\phi)\cos\phi\, d\phi \qquad [64.5]$$

Equations [59.13] and [59.14] appear now in the form

$$\dot{a} = \Phi(a) \qquad [64.6]$$

$$\dot{\psi} = \Omega(a) \qquad [64.7]$$

where $\Phi(a)$ and $\Omega(a)$ are given by Equations [64.5] and [64.4] in terms of the total non-linear force $F(x,\dot{x})$.

Considering the question of limit cycles generated by a harmonic $x = K\cos t$ when $\mu = 0$, we examine Equation [64.6].

If $\Phi(a) > 0$, the amplitude a increases indefinitely, and hence no such limit cycle exists.

If $\Phi(a) < 0$, the amplitude decreases, and again no such limit cycle exists. This condition characterizes dissipative systems.

If $\Phi(a_1) = 0$, we obtain the condition for a limit cycle with amplitude a_1.

The question of the existence of limit cycles which are not generated by a harmonic $x = K\cos t$ when $\mu = 0$, is not considered.

We now propose to investigate a practical case in which a limit cycle exists, as shown by experiment, and to show how this existence can be ascertained analytically on the basis of this theory. For this purpose consider the differential equation of an oscillating circuit containing a non-linear conductor characterized by a non-linear equation of the form $v = G(i)$. Putting $i = x$, the differential equation is

$$L\dot{x} + G(x) + \frac{1}{C}\int x\, dt = 0 \qquad [64.8]$$

where the constant parameters L and C designate the inductance and capacity of the circuit respectively.

Differentiating this equation with respect to t, dividing it by L, and putting for abbreviation $1/LC = \omega^2$, we obtain

$$\ddot{x} + \omega^2 x + \frac{1}{L}G'(x)\dot{x} = 0 \qquad [64.9]$$

Identifying the term $\frac{1}{L}G'(x)\dot{x}$ with $\mu f(x,\dot{x})$ of the general theory, which incidentally imposes a requirement that it be small in comparison with the first two terms, we obtain on the basis of this theory

$$\dot{a} = -\frac{1}{2\pi\omega L}\int_0^{2\pi} G'(a\sin\phi)a\omega\cos\phi \cdot \cos\phi\, d\phi \equiv \Phi(a) \qquad [64.10]$$

$$\dot{\psi} = \omega + \frac{1}{2\pi L}\int_0^{2\pi} G'(a\sin\phi)\cos\phi \cdot \sin\phi\, d\phi \equiv \Omega(a) \qquad [64.11]$$

Integrating by parts, we see that the second term on the right side of Equation [64.11] is zero, hence,

$$\dot{\psi} = \Omega(a) = \omega \qquad [64.12]$$

and $\psi = \omega t + \psi_0$ where ψ_0 is arbitrary. The oscillation is thus isochronous at least to the first order.

If we let $\frac{1}{\pi}\int_0^{2\pi} G'(a \sin\phi) \cos^2\phi \; d\phi = R(a)$, Equation [64.10] can be written as

$$\dot{a} = -\frac{R(a)a}{2L} = \Phi(a) \qquad [64.13]$$

From the definition of $R(a)$ it follows that $R(a) > 0$, if $G'(x) > 0$, that is, if the voltage across the non-linear conductor increases with the current. Thus, $\dot{a} < 0$, and the final state of the system is $x = 0$, as is obvious from physical considerations since a "positive resistance" characterizes a dissipation of energy. Therefore, the final state of equilibrium is stable, and the point $x = 0$ is either a stable focal, or a stable nodal, point.

If $G'(x) < 0$ (negative resistance), that is, the voltage across the non-linear conductor decreases when the current through it increases, $\Phi(a)$ is positive, and from Equation [64.13] it follows that the amplitude increases. From physical considerations it is apparent that the amplitude cannot increase indefinitely. Analytically this is expressed by the condition $\Phi(a_1) = 0$ for a certain amplitude a_1 which is the amplitude of the limit cycle.

Figures 64.1a, b, c, and d illustrate the various possible cases. Figure 64.1a represents the case of an ohmic conductor (Curve a represents an ideal, and Curve b a real, ohmic conductor). Since $G'(x) \geqq 0$ in this case, $R(a) \geqq 0$; hence $\dot{a} < 0$. The amplitudes always decrease since the system is dissipative.

Figure 64.1b corresponds to the case when $G'(x)$, and hence $R(a)$, are negative for small amplitudes and become positive for larger ones. The root a_1 of the equation $R(a_1) = 0$ corresponds to a stable amplitude $a = a_1$. If the oscillations are started from values $a < a_1$, they will increase until the amplitude $a = a_1$ is reached; if, however, they are started from a value $a > a_1$, they will decrease down to the value $a = a_1$. This condition is indicated by

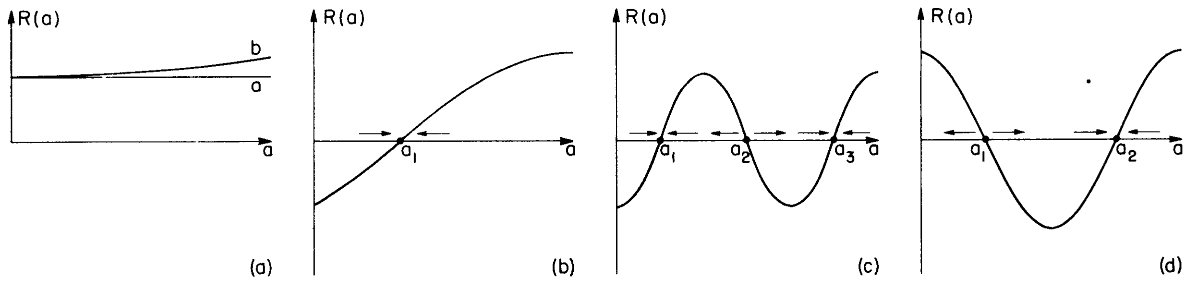

Figure 64.1

arrows in Figure 64.1b. The amplitude $a = a_1$ thus corresponds to a *stable* limit cycle.

Figure 64.1c shows a more complicated form of non-linear characteristic. By a similar argument, one finds that the amplitudes $a = a_1$ and $a = a_3$ correspond to stable limit cycles and $a = a_2$ to an unstable one.

Finally, Figure 64.1d shows a condition sometimes encountered in applications. For small amplitudes the "resistance" of a non-linear conductor is positive, that is, the energy is dissipated. Beginning with a certain critical value $a = a_1$, the resistance becomes "negative," that is, energy is conveyed into the system, and this state exists until an amplitude $a = a_2$ is reached. In such a case if the initial amplitude $a_0 < a_1$, the only stable amplitude is $a = 0$; the system cannot acquire self-excitation. If, however, $a_0 > a_1$, the amplitudes begin to grow and eventually become stabilized at a stable limit cycle a_2. A condition of this kind is designated as a "hard" type of self-excitation, which we have previously investigated.

It is thus seen that the unstable limit cycle $a = a_1$ acts as a kind of "barrier" preventing the amplitudes from building up if the initial amplitude is below the value corresponding to this barrier.

We thus find by the theory of the first approximation a situation exactly the same as that found previously by the topological methods of Poincaré.

65. STABILITY OF LIMIT CYCLES; CRITICAL VALUES OF A PARAMETER; SYSTEMS WITH SEVERAL LIMIT CYCLES

Let us first consider the question of the stability of limit cycles. Let a_1 be a root of the equation $\Phi(a_1) = 0$. For a slightly varied amplitude, $a_1 + \delta a$,

$$\Phi(a_1 + \delta a) = \Phi_a(a_1)\, \delta a$$

to the first order.

From Equation [59.13] it follows that

$$\frac{d(\delta a)}{dt} = \Phi_a(a_1)\,\delta a \qquad [65.1]$$

We shall take δa as the absolute value of the departure. If the initial departure has a tendency to disappear, that is, if $d(\delta a)/dt < 0$, $\Phi_a(a_1) < 0$, which is the condition for a stable limit cycle. If $\Phi_a(a_1) > 0$, by a similar argument one concludes that the limit cycle is unstable.

If self-excitation starts from rest ($a = 0$), the condition for its occurrence is

$$\Phi(0) > 0 \qquad [65.2]$$

which is equivalent to the existence of an unstable singularity in the presence of a stable limit cycle, see Part I, Chapter IV. Similarly, the condition for a critical value of some parameter can easily be established by this method. In fact, assume that $\Phi(a)$, in addition to a, also depends on a parameter λ, that is, it is a function $\Phi(a,\lambda)$. Consider the value of $\Phi(a,\lambda)$ for $a = 0$ and varying λ. When a value $\lambda = \lambda_0$ is reached for which $\Phi_a(0,\lambda) > 0$, the amplitudes begin to grow from zero and the subsequent increase of amplitudes from that moment will be determined by Equation [64.13], which now has the form

$$\dot{a} = \Phi(a,\lambda) \qquad [65.3]$$

The limit cycle is reached for a value a_1 of the amplitude for which

$$\Phi(a_1,\lambda) = 0 \qquad [65.4]$$

For a given value $\lambda = \lambda_1$, the limit cycle will be determined from the equation $\Phi(a_1, \lambda_1) = 0$, and for some other value $\lambda = \lambda_2$, from the equation $\Phi(a_2, \lambda_2) = 0$. Hence, the amplitude of the limit cycle in general is a certain function of the parameter λ.

In some cases the function $\Phi(a,\lambda)$ can be put in the form

$$\Phi(a,\lambda) = \left[\phi(a) - \frac{a}{\lambda}\right]\Phi_1(a,\lambda) \qquad [65.5]$$

where $\Phi_1(a,\lambda) > 0$ for all values of a and λ, and $\phi(0) = 0$. Differentiating Equation [65.5] with respect to a, one has

$$\Phi_a(a,\lambda) = \left[\phi_a(a) - \frac{1}{\lambda}\right]\Phi_1(a,\lambda) + \left[\phi(a) - \frac{a}{\lambda}\right]\frac{d[\Phi_1(a,\lambda)]}{da} \qquad [65.6]$$

Putting $a = 0$, one has

$$\Phi_a(0,\lambda) = \left[\Phi_a(0) - \frac{1}{\lambda}\right]\Phi_1(0,\lambda) \qquad [65.7]$$

Self-excitation will start from rest if

$$\Phi_a(0) > \frac{1}{\lambda} \qquad [65.8]$$

since $\Phi_1(0,\lambda) > 0$.

Consider a curve $y_1 = \Phi(a)$ and a straight line $y_2 = \frac{1}{\lambda}a$ shown in Figure 65.1. $\Phi_a(0)$ is the slope of the tangent to the curve $\Phi(a)$ at the origin, and $\frac{1}{\lambda}$ the slope of the straight line y_2. Condition [65.8] states that self-excitation occurs starting from rest ($a = 0$) only if the initial slope of the tangent to the curve $\Phi(a)$ is greater than the slope of the line a/λ.

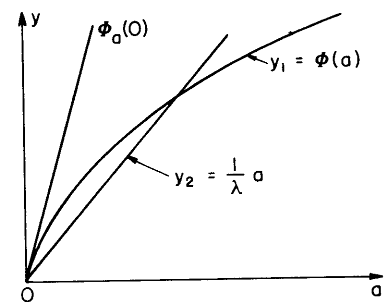

Figure 65.1

As an example, one may mention the self-excitation of a shunt generator. In this case the frequency is zero but the amplitude, Equation [59.13], is still applicable. The function $\Phi(a)$ is the voltage induced in the armature, a is the exciting current, and a/λ is the ohmic drop across the field winding; whence $1/\lambda$ is the resistance of the field winding plus the field resistor. In Figure 65.1, $1/\lambda = \tan\alpha$ is the slope of the straight line $\frac{1}{\lambda}a$. If $\Phi_a(0) - 1/\lambda < 0$, there is no self-excitation. For $\frac{1}{\lambda} < \Phi_a(0)$, self-excitation is possible since $\Phi_a(0,\lambda) > 0$. The equilibrium condition is $\Phi(a_1) = a_1/\lambda$, which corresponds to the intersection of curves $y_1 = \Phi(a)$ and $y_2 = \frac{1}{\lambda}a$. It is interesting to note that the amplitude, Equation [64.10], holds in this case in spite of the fact that the frequency, Equation [64.11], is absent.

From the general considerations explained in connection with Figure 64.1, the following theorems result:*

THEOREM 1. If a system possesses several stable limit cycles forming a sequence a_1, a_3, a_5, $\cdots$, between each pair of consecutive stable limit cycles there is always one unstable limit cycle; these unstable cycles form another sequence a_2, a_4, a_6, $\cdots$.

THEOREM 2. The limit cycle reached spontaneously by a system starting from rest is always the one which corresponds to the smallest root a_1 of the sequence.

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* It should be borne in mind that we are considering only limit cycles that are generated by circles.

THEOREM 3. The stable limit cycles corresponding to larger roots $a_3, a_5, \cdots$ of the stable sequence can be reached only if the system is given a shock excitation carrying it beyond the corresponding unstable limit cycles $a_2, a_4, \cdots$.

66. LIMIT CYCLES IN THE CASE OF POLYNOMIAL CHARACTERISTICS

Let the characteristic of a non-linear conductor be approximated by a polynomial

$$f(i) = A + Bi + Ci^2 + Di^3 + Ei^4 + Fi^5 \qquad [66.1]$$

Differentiating and setting $i = a \sin\phi$, we have

$$f'(a \sin\phi) \cos^2\phi = B\cos^2\phi + 2Ca\sin\phi\cos^2\phi + 3Da^2\sin^2\phi\cos^2\phi + $$
$$+ 4Ea^3\sin^3\phi\cos^2\phi + 5Fa^4\sin^4\phi\cos^2\phi$$

Forming the expression for $R(a) = \frac{1}{\pi}\int_0^{2\pi} f'(a \sin\phi) \cos^2\phi \, d\phi$, we have

$$R(a) = \frac{1}{\pi}\int_0^{2\pi} B\cos^2\phi\, d\phi + \frac{1}{\pi}\int_0^{2\pi} 2Ca\sin\phi\cos^2\phi\, d\phi + \frac{1}{\pi}\int_0^{2\pi} 3Da^2\sin^2\phi\cos^2\phi\, d\phi + $$
$$+ \frac{1}{\pi}\int_0^{2\pi} 4Ea^3\sin^3\phi\cos^2\phi\, d\phi + \frac{1}{\pi}\int_0^{2\pi} 5Fa^4\sin^4\phi\cos^2\phi\, d\phi \qquad [66.2]$$

The second and the fourth terms on the right side of this equation are zero. The remaining terms are

$$\frac{B}{\pi}\int_0^{2\pi}\cos^2\phi\, d\phi = B$$

[66.3]

$$\frac{1}{\pi}\int_0^{2\pi} 3Da^2\sin^2\phi\cos^2\phi\, d\phi = \frac{3Da^2}{\pi}\left[\int_0^{2\pi}\sin^2\phi\, d\phi - \int_0^{2\pi}\sin^4\phi\, d\phi\right] = \frac{3Da^2}{\pi}\pi\left(1 - \frac{3}{4}\right) = \frac{3}{4}Da^2$$

$$\frac{1}{\pi}\int_0^{2\pi} 5Fa^4\sin^4\phi\cos^2\phi\, d\phi = \frac{5Fa^4}{\pi}\left[\int_0^{2\pi}\sin^4\phi\, d\phi - \int_0^{2\pi}\sin^6\phi\, d\phi\right] = \frac{5Fa^4}{\pi}\left(\frac{3}{4} - \frac{3\cdot 5}{4\cdot 6}\right) = \frac{5}{8}Fa^4$$

From this

$$R(a) = B + \frac{3}{4}Da^2 + \frac{5}{8}Fa^4 \qquad [66.4]$$

It must be noted that the coefficient F in this equation must be positive; otherwise, beginning with a certain value $a = a_1$, $R(a)$ would become negative and remain negative for increasing a. In such a case, by Equation [62.2], da/dt would be positive. This is impossible in a physical problem.

Setting $\frac{5}{8}F = m > 0$ and $\frac{3}{4}D = n$, the condition for the existence of a limit cycle is

$$R(a) = ma^4 + na^2 + B = 0 \qquad [66.5]$$

The roots of this biquadratic equation are

$$a_{1,2}^2 = \frac{-n \pm \sqrt{n^2 - 4mB}}{2m} \qquad [66.6]$$

For the existence of limit cycles at least one of the roots a_1^2 or a_2^2 must be positive. Since $m > 0$, the necessary and sufficient condition is that $B < 0$, which expresses the fact that the slope of the characteristic $f(i)$ is negative, that is, "negative" resistance.

When B is positive, two cases are possible:

1. $B > 0$, $n > 0$, that is, $D > 0$. In this case both roots a_1^2 and a_2^2 are negative; hence, the amplitudes a_1 and a_2 of the limit cycles are imaginary. In other words, no limit cycles exist, the system being dissipative. The amplitudes decrease indefinitely, and the only stable solution is $a = 0$.

2. $B > 0$, $n < 0$, that is, $D < 0$. In this case limit cycles are possible as long as $n^2 - 4Bm > 0$, which expresses the condition of the reality of the roots. If we substitute for n and m their values, this gives

$$B < \frac{n^2}{4m} = \frac{9}{40}\frac{D^2}{F} \qquad [66.7]$$

Summing up this discussion, one can state that with a non-linear voltage, approximated by the polynomial [66.1], of the fifth degree, the following conditions exist:

1. On physical grounds the coefficient F must always be positive.

2. If $B < 0$, there is always one stable limit cycle.

3. If $B > 0$ and $D > 0$, no limit cycles exist and the system is dissipative.

4. If $B > 0$, $D < 0$, and $B < \frac{9}{40}\frac{D^2}{F}$, limit cycles are possible with the amplitude a_1 (positive root).

5. If $B > 0$ and $D < 0$, but $B > \frac{9}{40}\frac{D^2}{F}$, the system behaves again as a .ssipative one, and no limit cycles exist.

Since any experimental characteristic can be approximated by a polynomial, the coefficients $A, \cdots F$ in Equation [66.1] are known quantities, and the above procedure permits ascertaining from the form of the characteristic the behavior of the system into which the non-linear element with this particular characteristic is introduced.

CHAPTER XI

APPROXIMATIONS OF HIGHER ORDERS

67. INTRODUCTORY REMARKS

In this chapter we review the extension of the method of Kryloff and Bogoliuboff as applied to approximations of orders higher than the first. Although for practical applications the theory of the first approximation gives a satisfactory degree of accuracy, it is also interesting to consider the possibility of a further refinement of approximate solutions in case greater accuracy is required. The procedure of Kryloff and Bogoliuboff is derived from the classical methods of Gylden and Lindstedt used in celestial mechanics. As explained in Section 44, the object of these methods is to eliminate the secular terms resulting from the resonance effect of subsequent harmonics in the recurrence procedure by which the higher-order terms are determined. The method can be summarized as follows.

Assume that we wish to find a periodic solution of the differential equation

$$\ddot{x} + \omega^2 x + \mu f(x) = 0 \qquad [67.1]$$

with a certain unknown period T. Introducing a new independent variable $\tau = 2\pi t/T = \Omega t$, where $\Omega = 2\pi/T$, we shall look for a solution $x(t) = z(\tau)$, where $z(\tau)$ is a periodic function with period 2π. We shall try to represent the periodic solution in the form

$$z(\tau) = \sum_{n=0}^{\infty} \mu^n z_n(\tau) \qquad [67.2]$$

where $z_n(\tau)$ with $n = 1, 2, \cdots$, are periodic functions with period 2π. We further assume that

$$\Omega^2 = \sum_{n=0}^{\infty} \alpha_n \mu^n \qquad [67.3]$$

where α_n is constant. The transformed equation [67.1] then becomes

$$\Omega^2 \frac{d^2 z}{d\tau^2} + \omega^2 z + \mu f(z) = 0 \qquad [67.4]$$

If one substitutes into this equation the series expansions [67.2] and [67.3], one obtains a series of recurrent differential equations resulting from equating to zero the coefficients of equal powers of μ. For the subsequent approximations one thus obtains a series of differential equations

$$\alpha_0 \frac{d^2 z_0}{d\tau^2} + \omega^2 z_0 = 0$$

$$\alpha_0 \frac{d^2 z_1}{d\tau^2} + \omega^2 z_1 = -f(z_0) - \alpha_1 \frac{d^2 z_0}{d\tau^2}$$

$$\alpha_0 \frac{d^2 z_2}{d\tau^2} + \omega^2 z_2 = -f'(z_0) z_1 - \alpha_2 \frac{d^2 z_0}{d\tau^2} - \alpha_1 \frac{d^2 z_1}{d\tau^2} \qquad [67.5]$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_0 \frac{d^2 z_{n+1}}{d\tau^2} + \omega^2 z_{n+1} = F(z_0, z_1, \cdots, z_n) - \alpha_{n+1} \frac{d^2 z_0}{d\tau^2} - \alpha_n \frac{d^2 z_1}{d\tau^2} - \cdots - \alpha_1 \frac{d^2 z_n}{d\tau^2}$$

where $F(z_0, z_1, \cdots, z_n)$ is a certain polynomial in $z_0, z_1, \cdots, z_n$. It is apparent that if $z_0, z_1, \cdots, z_N$ and $\alpha_0, \alpha_1, \cdots, \alpha_N$ satisfy the first $(N+1)$ equations of the system [67.5], then the expressions

$$x = \sum_{n=0}^{N} \mu^n z_n(\tau) \quad \text{and} \quad \Omega^2 = \sum_{n=0}^{N} \mu^n \alpha_n \qquad [67.6]$$

also satisfy [67.1] up to the order μ^{N+1} and, hence, may be considered as the $(N+1)^{\text{th}}$ approximation.

The method of Lindstedt, which Kryloff and Bogoliuboff follow, consists in determining the coefficients α_i in the subsequent stages of the recurrence procedure so as to eliminate terms with the fundamental period 2π. In fact, if these terms were left on the right side of Equations [67.5], they would account for the "resonance terms," which are of secular form, as previously defined. The determination of α_i by this procedure at the same time leads to the expression for frequency given by Equation [67.3]. By this method the difficulties encountered in the theory of Poincaré (see Chapter VIII) in connection with the appearance of secular terms in the expansions are eliminated, and solutions without secular terms can be obtained.

Poincaré has shown by an example that the approximations generally do not converge. However, nothing better is available, and, in practice, the second or third approximation (and usually, in fact, the first) gives entirely satisfactory results.

The subject matter of this chapter is considerably abbreviated, compared with Kryloff and Bogoliuboff's text (5), and for that reason the reader should refer to the original text for additional details.

68. IMPROVED FIRST APPROXIMATION

Equations [59.13] and [59.14] of the first approximation were obtained by dropping the higher harmonics in the Fourier series on the right side of Equations [59.10]. In reality, owing to the presence of these terms, the slowly varying quantities a and ϕ undergo oscillations of a relatively high frequency.

In order to take this into account, it is convenienent to consider the quantities a and ϕ in Equations [59.10] as practically constant in comparison with the rapidly varying trigonometric terms cos $n(\omega t + \phi)$ and sin $n(\omega t + \phi)$.

Designating the left sides of Equations [59.10] by $d\bar{a}/dt$ and $d\bar{\phi}/dt$, and noting that $-\frac{\mu}{\omega} K_0(a) = \frac{da}{dt}$ and $\frac{\mu}{a\omega} P_0(a) = \frac{d\phi}{dt}$, upon integration of Equations [59.10], one has

$$\bar{a} = a - \frac{\mu}{\omega} \sum_{n=1}^{\infty} \frac{K_n(a) \sin n(\omega t + \phi) - L_n(a) \cos n(\omega t + \phi)}{n\omega}$$

$$\bar{\phi} = \phi + \frac{\mu}{a\omega} \sum_{n=1}^{\infty} \frac{P_n(a) \sin n(\omega t + \phi) - Q_n(a) \cos n(\omega t + \phi)}{n\omega} \qquad [68.1]$$

where a designates the first approximation for the amplitude given by Equation [59.13].

Substituting these values into the equation $x = \bar{a} \sin(\omega t + \bar{\phi})$, one has

$$x = \left[a - \frac{\mu}{\omega^2} \sum_{n=1}^{\infty} \frac{K_n \sin n(\omega t + \phi) - L_n \cos n(\omega t + \phi)}{n}\right] \sin\left[\omega t + \phi + \frac{\mu}{a\omega^2} \sum_{n=1}^{\infty} \frac{P_n \sin n(\omega t + \phi) - Q_n \cos n(\omega t + \phi)}{n}\right] \qquad [68.2]$$

If we let

$$S = \frac{1}{a\omega^2} \sum_{n=1}^{\infty} \frac{P_n \sin n(\omega t + \phi) - Q_n \cos n(\omega t + \phi)}{n}$$

the sine term of this expression can be written as

$$\sin(\omega t + \phi + \mu S) = \sin(\omega t + \phi) \cos \mu S + \cos(\omega t + \phi) \sin \mu S$$

$$\approx \sin(\omega t + \phi) + \mu S \cos(\omega t + \phi)$$

since μ is small. Substituting this into Equation [68.2], one has

$$x = \left[a - \frac{\mu}{\omega^2} \sum_{n=1}^{\infty} \frac{K_n \sin n(\omega t + \phi) - L_n \cos n(\omega t + \phi)}{n}\right]\left[\sin(\omega t + \phi) + \mu S \cos(\omega t + \phi)\right] \qquad [68.3]$$

whence, neglecting the terms with μ^2, μ^3, $\cdots$, one has

$$x = a\sin(\omega t + \phi) - \frac{\mu}{\omega^2}\sum_{n=1}^{\infty}\frac{K_n \sin n(\omega t + \phi) - L_n \cos n(\omega t + \phi)}{n}\sin(\omega t + \phi) +$$

$$+ \frac{a\mu}{a\omega^2}\sum_{n=1}^{\infty}\frac{P_n \sin n(\omega t + \phi) - Q_n \cos n(\omega t + \phi)}{n}\cos(\omega t + \phi) \qquad [68.4]$$

If we put $\omega t + \phi = \tau$,

$$\sum_{n=1}^{\infty}\frac{K_n \sin n\tau - L_n \cos n\tau}{n} = u(\tau)$$

$$\sum_{n=1}^{\infty}\frac{P_n \sin n\tau - Q_n \cos n\tau}{n} = v(\tau) \qquad [68.5]$$

and

$$u(\tau)\sin\tau - v(\tau)\cos\tau = w(\tau) \qquad [68.6]$$

We now rewrite [68.4] as

$$x = a\sin\tau - \frac{\mu}{\omega^2}w(\tau) \qquad [68.7]$$

From Equations [68.5], in view of [59.8], one has further

$$u'(\tau) = f(a\sin\tau, a\omega\cos\tau)\cos\tau - K_0(a)$$

$$v'(\tau) = f(a\sin\tau, a\omega\cos\tau)\sin\tau - P_0(a) \qquad [68.8]$$

On the other hand, differentiating Equation [68.6], one has

$$u'(\tau)\sin\tau - v'(\tau)\cos\tau + u(\tau)\cos\tau + v(\tau)\sin\tau = w'(\tau) \qquad [68.9]$$

If the values [68.8] are substituted into Equation [68.9], one finds

$$w'(\tau) = P_0(a)\cos\tau - K_0(a)\sin\tau + u\cos\tau + v\sin\tau \qquad [68.10]$$

$$w''(\tau) = - P_0(a)\sin\tau - K_0(a)\cos\tau - u\sin\tau + v\cos\tau + u'\cos\tau + v'\sin\tau \qquad [68.11]$$

Thus

$$w''(\tau) + w(\tau) = - P_0(a)\sin\tau - K_0(a)\cos\tau - u\sin\tau + v\cos\tau + u'\cos\tau + v'\sin\tau$$

$$+ u\sin\tau - v\cos\tau = - P_0(a)\sin\tau - K_0(a)\cos\tau + u'\cos\tau + v'\sin\tau \qquad [68.12]$$

Noting that

$$u'(\tau)\cos\tau + v'(\tau)\sin\tau = f(a\sin\tau, a\omega\cos\tau) - K_0(a)\cos\tau - P_0(a)\sin\tau$$

one obtains from Equations [68.8]

$$w''(\tau) + w(\tau) = f(a\sin\tau, a\omega\cos\tau) - 2K_0(a)\cos\tau - 2P_0(a)\sin\tau \quad [68.13]$$

In order to determine the corrective term $w(\tau)$ of the improved first approximation, it is necessary to transform the right-hand term of [68.13] into the known Fourier series.

For this purpose consider the Fourier expansion

$$f(a\sin\tau, a\omega\cos\tau) = f_0(a) + \sum_{n=1}^{\infty}\left[f_n(a)\cos n\tau + g_n(a)\sin n\tau\right] \quad [68.14]$$

in which the coefficients f and g are given by the equations

$$f_0(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a\sin\tau, a\omega\cos\tau)\,d\tau$$

$$f_n(a) = \frac{1}{\pi}\int_0^{2\pi} f(a\sin\tau, a\omega\cos\tau)\cos n\tau\,d\tau \quad [68.15]$$

$$g_n(a) = \frac{1}{\pi}\int_0^{2\pi} f(a\sin\tau, a\omega\cos\tau)\sin n\tau\,d\tau$$

On the other hand, by [59.9], $K_0(a) = \frac{1}{2}f_1$ and $P_0(a) = \frac{1}{2}g_1$. From this

$$f(a\sin\tau, a\omega\cos\tau) - 2K_0(a)\cos\tau - 2P_0(a)\sin\tau$$

$$= f_0 + \sum_{n=2}^{\infty}(f_n\cos n\tau + g_n\sin n\tau) \quad [68.16]$$

Substituting the right-hand term of Equation [68.16] into Equation [68.13], one gets

$$w''(\tau) + w(\tau) = f_0 + \sum_{n=2}^{\infty}(f_n\cos n\tau + g_n\sin n\tau) \quad [68.17]$$

Looking for a solution of Equation [68.17] of the form

$$w(\tau) = a_0 + \sum_{n=2}^{\infty}(a_n\cos n\tau + b_n\sin n\tau)$$

one obtains by identification of the coefficients after the substitution of this expression into the differential equation [68.17]

$$w(\tau) = f_0 - \sum_{n=2}^{\infty} \frac{f_n \cos n\tau + g_n \sin n\tau}{n^2 - 1} \qquad [68.18]$$

Substituting this expression into Equation [68.7], one finally obtains the following expression for the improved first approximation

$$x = a \sin(\omega t + \phi) - \frac{\mu}{\omega^2} f_0(a) + \frac{\mu}{\omega^2} \sum_{n=2}^{\infty} \frac{f_n \cos n(\omega t + \phi) + g_n \sin n(\omega t + \phi)}{n^2 - 1} \qquad [68.19]$$

where a and ϕ satisfy the equations of the first approximation, Equations [59.12], that is, $\dot{a} = -\frac{\mu}{2\omega} f_1(a)$ and $\dot{\phi} = \frac{\mu}{2\omega a} g_1(a)$.

In order to see whether the solution [68.19] satisfies the original quasi-linear equation $\ddot{x} + \omega^2 x + \mu f(x,\dot{x}) = 0$, substitute the solution [68.19] in that equation. This gives, on the other hand,

$$\ddot{x} + \omega^2 x = -\left[\sum_{n=1}^{\infty} f_n \cos n(\omega t + \phi) + g_n \sin n(\omega t + \phi) + f_0\right] + O_1(\mu^2) \qquad [68.20]$$

where O_1 is of the order of μ^2.

On the other hand, in view of Equation [68.19], one has

$$\mu f(x,\dot{x}) = \mu f\left[a \sin(\omega t + \phi),\, a\omega \cos(\omega t + \phi)\right] + O_2(\mu^2)$$

whence, by [68.14],

$$\ddot{x} + \omega^2 x + \mu f(x,\dot{x}) = O_1(\mu^2) - O_2(\mu^2)$$

Thus the expression [68.19] satisfies the original differential equation with accuracy of the order of μ^2.

Furthermore, it can be shown, upon developing the expressions for $O_1(\mu^2)$ and $O_2(\mu^2)$, that the error in the approximate solution [68.19] is uniform in the interval $0 \leqq t < \infty$.

69. APPLICATIONS OF THE THEORY OF THE IMPROVED FIRST APPROXIMATION

We shall consider first oscillations of a conservative system acted on by a non-linear force represented by an odd function of the dependent variable. In this case, $f(x) + f(-x) \equiv 0$. Since the function is odd, no cosine terms are present in the Fourier expansion. Hence

$$f(a \sin\tau) = \sum_{n=1}^{\infty} g_n(a) \sin n\tau$$

and

$$g_n(a) = \frac{2}{\pi}\int_0^{\pi} f(a\sin\tau)\sin n\tau\, d\tau$$

By Equation [68.19], the improved first approximation is

$$x = a\sin(\omega t + \phi) + \frac{\mu}{\omega^2}\sum_{n=2}^{\infty}\frac{g_n(a)\sin n(\omega t + \phi)}{n^2 - 1} \qquad [69.1]$$

The frequency is given by the equation

$$\Omega(a) = \omega + \frac{\mu}{2\omega a}\, g_1(a)$$

that is,

$$\Omega^2(a) = \omega^2 + \frac{\mu}{a}\, g_1(a)$$

to the first order.

We shall now discuss three applications of the theory of the improved first approximation.

A. VARIABLE "SPRING CONSTANT"

Consider the differential equation

$$\ddot{x} + \omega^2 x = 0$$

where $\omega^2 = \omega_0^2 + \mu x^2$, which gives

$$\ddot{x} + \omega_0^2 x + \mu x^3 = 0 \qquad [69.2]$$

so that

$$f(x) = x^3; \qquad f(a\sin\tau) = a^3\sin^3\tau = \frac{3}{4}a^3\sin\tau - \frac{1}{4}a^3\sin 3\tau$$

whence

$$g_1(a) = \frac{2}{\pi}\int_0^{\pi}\left(\frac{3}{4}a^3\sin\tau - \frac{1}{4}a^3\sin 3\tau\right)\sin\tau\, d\tau = \frac{3}{4}a^3$$

and

$$\Omega(a) = \omega_0 + \frac{\mu}{2\omega_0 a}\, g_1(a) = \omega_0 + \frac{3}{8}\omega_0\mu a^2$$

For the next harmonic

$$g_3(a) = \frac{2}{\pi}\int_0^{\pi}\left(\frac{3}{4}a^3\sin\tau - \frac{1}{4}a^3\sin 3\tau\right)\sin 3\tau\,d\tau = -\frac{a^3}{2\pi}\int_0^{\pi}\sin^2 3\tau\,d\tau$$

$$= -\frac{a^3}{6\pi}\int_0^{3\pi}\sin^2 x\,dx = -\frac{3\pi a^3}{12\pi} = -\frac{a^3}{4}$$

From this, by Equation [69.1]

$$x = a\sin(\omega_0 t + \phi) - \frac{\mu a^3}{4\omega_0^2}\left(\frac{1}{3^2 - 1}\right)\sin 3(\omega_0 t + \phi)$$

$$= a\sin(\omega_0 t + \phi) - \frac{\mu a^3}{32\omega_0^2}\sin 3(\omega_0 t + \phi) \qquad [69.3]$$

Thus the improved first approximation introduces a small corrective term $\frac{\mu a^3}{32}\sin 3(\omega_0 t + \phi)$ in the form of a third harmonic.

B. VARIABLE DAMPING

Consider a differential equation of the form

$$\ddot{x} + \omega^2 x + \mu f(x)\dot{x} = 0$$

The non-linear term in this case is

$$f(x)\dot{x} = f(a\sin\tau)\,a\omega\cos\tau$$

Consider the function $F(\lambda) = \int_0^{\lambda} f(\xi)\,d\xi$. For $\lambda = a\cos\phi$, the development of $F(a\cos\phi)$ in a Fourier series gives

$$F(a\cos\phi) = \sum_{n=0}^{\infty} F_n(a)\cos n\phi \qquad [69.4]$$

Differentiating this equation, we have

$$af(a\cos\phi)\sin\phi = \sum_{n=1}^{\infty} nF_n(a)\sin n\phi$$

Putting $\phi = \tau + \frac{3\pi}{2}$, one obtains

$$f(a\sin\tau)\,a\omega\cos\tau = -\omega\sum_{n=0}^{\infty} nF_n(a)\sin n\left(\omega t + \frac{3\pi}{2} + \phi_0\right)$$

where ϕ_0 is an arbitrary phase angle. Hence, by Equation [69.1], the solution is

$$x = a\sin(\Omega t + \phi_0) - \frac{\mu}{\omega}\sum_{n=2}^{\infty}\frac{n}{n^2 - 1}F_n(a)\sin n\left(\Omega t + \frac{3\pi}{2} + \phi_0\right) \qquad [69.5]$$

assuming that $\omega = \Omega$, since in this case the correction for frequency is of a higher order.

If one introduces $\theta_0 = \phi_0 - \pi/2$, Equation [69.5] has the form

$$x = a\cos(\Omega t + \theta_0) - \frac{\mu}{\omega}\sum_{n=2}^{\infty}\frac{n}{n^2 - 1}F_n(a)\sin n(\Omega t + \theta_0) \qquad [69.6]$$

For example, consider the Van der Pol equation

$$\ddot{x} + x - \mu(1 - x^2)\dot{x} = 0$$

In this case $f(x) = x^2 - 1$, $F(x) = \frac{x^3}{3} - x$, and $\omega = 1$, whence

$$F(a\cos\phi) = \frac{a^3\cos^3\phi}{3} - a\cos\phi = a\left(\frac{a^2}{4} - 1\right)\cos\phi + \frac{a^3}{12}\cos 3\phi$$

It follows that $F_1(a) = a\left(\frac{a^2}{4} - 1\right)$ and $F_3(a) = \frac{a^3}{12}$, the other $F_n(a)$ being zero. By Equation [69.6], the oscillation is

$$x = a\cos(t + \theta_0) - \frac{\mu a^3}{32}\sin 3(t + \theta_0) \qquad [69.7]$$

The differential equation for the amplitude is obtained from the equations of the first approximation

$$\frac{da}{dt} = \frac{\mu a}{2}\left(1 - \frac{a^2}{4}\right)$$

For a steady-state condition, $a = 2$; Equation [69.7] in this case becomes

$$x = 2\cos(t + \theta_0) - \frac{\mu}{4}\sin 3(t + \theta_0) \qquad [69.8]$$

C. CORRECTION FOR FREQUENCY

In the preceding notations ω is the linear frequency when $\mu = 0$, and Ω is the frequency of the quasi-linear oscillation when $\mu \neq 0$. For quasi-isochronous motions to the first order, $\omega \approx \Omega$.

It is to be noted first that the exact solution for stationary oscillations can always be developed in a Fourier series

$$x = a\cos(\Omega t + \theta_0) + \sum_{n=2}^{\infty}\left[A_n\cos n(\Omega t + \theta_0) + B_n\sin n(\Omega t + \theta_0)\right] \qquad [69.9]$$

One has identically

$$\int_0^T[\ddot{x}x + \omega^2x^2 + \mu f(x)x\dot{x}]\,dt = 0 \qquad [69.10]$$

since

$$\ddot{x} + \omega^2 x + \mu f(x)\dot{x} = 0 .$$

On the other hand,

$$\ddot{x}x + \dot{x}^2 = \frac{d}{dt}(x\dot{x})$$

and

$$\int_0^T \ddot{x}x\,dt + \int_0^T \dot{x}^2\,dt = x\dot{x}\Big|_0^T = 0$$

in view of the periodicity. Hence

$$\int_0^T \ddot{x}x\,dt = -\int_0^T \dot{x}^2\,dt$$

Likewise

$$\int_0^T f(x)\,x\dot{x}\,dt = 0$$

for the same reason. Hence,

$$\int_0^T \dot{x}^2\,dt = \omega^2\int_0^T x^2\,dt \qquad [69.11]$$

If one replaces x by its expression [69.9] and $\dot{x}$ by its expression obtained by differentiating Equation [69.9], one obtains finally

$$\Omega^2\left[a^2 + \sum_{n=2}^{\infty} n^2(A_n^2 + B_n^2)\right] = \omega^2\left[a^2 + \sum_{n=2}^{\infty}(A_n^2 + B_n^2)\right] \qquad [69.12]$$

since the terms of the form $A_pB_q \cos p(\Omega t + \theta_0) \sin q(\Omega t + \theta_0)$ disappear when integrated over the period It follows that

$$\frac{\Omega^2}{\omega^2} = \frac{a^2 + \sum_{n=2}^{\infty}(A_n^2 + B_n^2)}{a^2 + \sum_{n=2}^{\infty} n^2(A_n^2 + B_n^2)} \qquad [69.13]$$

On the other hand, under the assumption that $f(x,\dot{x}) = f(x)\dot{x}$, by Equation [69.6], $A_n = 0$ and $B_n = -\frac{\mu}{\omega}\frac{n}{n^2 - 1} F_n(a)$.

Substituting these values for A_n and B_n into Equation [69.13], one has

$$\frac{\Omega^2}{\omega^2} = \frac{1 + \dfrac{\mu^2}{\omega^2 a^2}\displaystyle\sum_{n=2}^{\infty}\left[\frac{n}{n^2-1}F_n(a)\right]^2}{1 + \dfrac{\mu^2}{\omega^2 a^2}\displaystyle\sum_{n=2}^{\infty}n^2\left[\frac{n}{n^2-1}F_n(a)\right]^2} \qquad [69.14]$$

If we let $\frac{\mu^2}{\omega^2} = K^2$ and $\left[\frac{F_n(a)}{a}\right] = b_n$, the preceding equation can be written

$$\frac{\Omega^2}{\omega^2} = \frac{1 + K^2\displaystyle\sum_{n=2}^{\infty}\left(\frac{n}{n^2-1}b_n\right)^2}{1 + K^2\displaystyle\sum_{n=2}^{\infty}n^2\left(\frac{n}{n^2-1}b_n\right)^2}$$

$$= \left[1 + K^2\sum_{n=2}^{\infty}\left(\frac{n}{n^2-1}b_n\right)^2\right]\left[1 - K^2\sum_{n=2}^{\infty}n^2\left(\frac{n}{n^2-1}b_n\right)^2\right] \qquad [69.15]$$

Hence, to the second order of the small quantity K, one has

$$\Omega = \omega\left[1 - \frac{\mu^2}{2\omega^2}\sum_{n=2}^{\infty}\left(\frac{n^2}{n^2-1}\right)\frac{F_n^2(a)}{a^2}\right] \qquad [69.16]$$

Applying this formula to the Van der Pol equation with $a = 2$, $F_3(a) = \frac{a^3}{12} = \frac{2}{3}$, and $F_n(a) = 0$ for $n \neq 3$, one has

$$\Omega = 1 - \frac{\mu^2}{16}$$

70. APPROXIMATIONS OF HIGHER ORDERS

Consider, first, the quasi-linear equation

$$\ddot{x} + \omega^2 x + \mu f(x) = 0$$

and assume that it has one or several periodic solutions. Let $x = x(t)$ be such a solution, with an unknown period T and frequency $\Omega = 2\pi/T$. Let $x = z(\Omega t + \phi) \equiv z(\tau)$, with $\tau = \Omega t + \phi$. Then $z(\tau)$ has the period 2π.

Changing the independent variable in the original equation $\ddot{x} + \omega^2 x + \mu f(x) = 0$, one obtains

$$\Omega^2\ddot{z} + \omega^2 z + \mu f(z) = 0 \qquad [70.1]$$

From now on the differential notation $\ddot{z}$ will designate differentiation with respect to τ. By hypothesis, Equation [70.1] has a periodic solution with period 2π. We shall look for solutions $z(\tau)$ and Ω^2 in the form of a power series in μ:

$$z(\tau) = \sum_{n=0}^{\infty}\mu^n z_n(\tau) \quad \text{and} \quad \Omega^2 = \sum_{n=0}^{\infty}\mu^n \alpha_n \qquad [70.2]$$

with $\alpha_0 = \omega^2$, since the existence of a periodic solution of period 2π for [70.1] with $\mu = 0$ requires that $\Omega^2 = \omega^2$.

Furthermore,

$$f(z) = f(z_0 + \mu z_1 + \mu^2 z_2 + \cdots)$$

$$= f(z_0) + \mu z_1 f'(z_0) + \mu^2\left[z_2 f'(z_0) + \frac{z_1^2 f''(z_0)}{2}\right] + \cdots$$

The substitution of these values into Equation [70.1] gives rise to the following recurrent differential equations obtained by equating to zero the coefficients of μ^0, μ^1, μ^2, $\cdots$.

$$\omega^2 \ddot{z}_0 + \omega^2 z_0 = 0$$

$$\omega^2 \ddot{z}_1 + \omega^2 z_1 = - f(z_0) - \alpha_1 \ddot{z}_0$$

$$\omega^2 \ddot{z}_2 + \omega^2 z_2 = - f'(z_0) z_1 - \alpha_2 \ddot{z}_0 - \alpha_1 \ddot{z}_1 \qquad [70.3]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\omega^2 \ddot{z}_{n+1} + \omega^2 z_{n+1} = F(z_0, z_1, \cdots, z_n) - \alpha_{n+1} \ddot{z}_0 - \alpha_n \ddot{z}_1 - \cdots - \alpha_1 \ddot{z}_n$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

where $F(z_0, z_1, \cdots, z_n)$ is a polynomial in $z_1, z_2, \cdots, z_n$.

The solution $z(\tau)$ under consideration is determined up to a translation on τ. We can choose the origin, that is, ϕ, in the substitution $\tau = \Omega t + \phi$ so that

$$\dot{z}(0) = 0 \qquad [70.4]$$

The condition [70.4] then yields the following conditions for $z_n(\tau)$, which are periodic functions of period 2π and which we take as being represented by their Fourier series,

$$\dot{z}_n(0) = 0 \quad \text{for} \quad n = 1, 2; 3 \cdots \qquad [70.5]$$

Let $z_0, z_1, \cdots, z_n$ and $\alpha_1, \cdots, \alpha_n$ be the solutions of the first $(N+1)$ equations of the system [70.3]; it is then clear that

$$x = \sum_{n=0}^{N} \mu^n z_n(\tau) \quad \text{and} \quad \Omega^2 = \sum_{n=0}^{N} \mu^n \alpha_n$$

will satisfy the equation $\ddot{x} + \omega^2 x + \mu f(x) = 0$ to the order of μ^{N+1}, and hence can be considered as the $(N+1)^{\text{th}}$ approximation of x.

The first equation [70.3] gives, in view of [70.5], $z_0 = a \cos \tau$, where a is an arbitrary constant. There exists, however, a certain arbitrariness in the following steps, that is, in the successive determination of z_1, z_2, $\cdots$. We can remove this arbitrariness by requiring that no fundamental harmonic shall appear on the right side of Equations [70.3], for otherwise some z_n would contain secular terms. We wish, however, to obtain a function $z(\tau) = x(t)$ representing the periodic solution of our quasi-linear equation in the whole interval $0 \leqq t < \infty$.

Consider now the second equation of System [70.3]

$$\omega^2(\ddot{z}_1 + z_1) = - f(a \cos \tau) + \alpha_1 a \cos \tau$$

The function $f(a \cos \tau)$ can be developed in a Fourier series which contains only cosine terms, that is,

$$f(a \cos \tau) = \sum_{n=0}^{\infty} f_n(a) \cos n\tau = f_0(a) + f_1(a) \cos \tau + \sum_{n=2}^{\infty} f_n(a) \cos n\tau$$

$$\omega^2(\ddot{z}_1 + z_1) = - \sum_{n=2}^{\infty} f_n(a) \cos n\tau + \left[\alpha_1 a - f_1(a)\right] \cos \tau - f_0(a) \qquad [70.6]$$

It is noted that the secular term is bound to appear in this case unless the coefficient of $\cos \tau$ on the right side of Equation [70.6] is zero. Hence, the condition for the elimination of the secular term is

$$\alpha_1 = \frac{f_1(a)}{a} \qquad [70.7]$$

which determines the approximation (z_1, α_1). One has

$$\ddot{z}_1 + z_1 = - \frac{1}{\omega^2}\left[f_0(a) + \sum_{n=2}^{\infty} f_n(a) \cos n\tau\right] \qquad [70.8]$$

In view of [70.5], the solution of [70.8] is

$$z_1 = A \cos \tau - \frac{1}{\omega^2} f_0(a) + \frac{1}{\omega^2} \sum_{n=2}^{\infty} \frac{f_n \cos n\tau}{n^2 - 1} \qquad [70.9]$$

with $f_n = f_n(a)$ where A is a constant. Substituting z_0 and z_1 into the right side of Equation [70.3] for z_2 and annulling the fundamental harmonic, we obtain an equation involving α_2 and A, so that A remains arbitrary. In order to simplify our solution, we take $A = 0$ so that

$$z_1 = - \frac{1}{\omega^2} f_0 + \frac{1}{\omega^2} \sum_{n=2}^{\infty} \frac{f_n \cos n\tau}{n^2 - 1} \qquad [70.10]$$

and the equation linking α_0 and A, where we set $A = 0$, now yields the value of α_2. By proceeding in this manner the following terms $z_2, z_3, \cdots$ and $\alpha_2, \alpha_3, \cdots$ can be determined. We require that none of $z_n (n \geqq 3)$ should contain the fundamental harmonic, by analogy with z_2. The condition [70.5] is then automatically satisfied. We proceed to show by induction that any z_n can be determined in this manner from Equation [70.3].

Assume that $z_1, z_2, \cdots, z_n$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$, satisfying the first n equations of the system [70.3], have been determined.

The $(n+1)^{\text{th}}$ equation is

$$\omega^2(\ddot{z}_{n+1} + z_{n+1}) = F(z_0, z_1, \cdots, z_n) - \alpha_n \ddot{z}_1 - \cdots - \alpha_1 \ddot{z}_n + \alpha_{n+1} a \cos\tau \quad [70.11]$$

where $z_0 = a \cos\tau$, as before.

Since $z_1, \cdots, z_n$ and $\ddot{z}_1, \cdots, \ddot{z}_n$ contain only cosine terms, $F(z_0, z_1, \cdots, z_n)$ also contains terms of this kind only.

Putting $F(z_0, z_1, \cdots, z_n) - \alpha_n \ddot{z}_1 - \alpha_{n-1}\ddot{z}_2 - \cdots - \alpha_1 \ddot{z}_n = \sum_{m=0}^{\infty} b_m \cos m\tau$, one can write Equation [70.11] as

$$\ddot{z}_{n+1} + z_{n+1} = \frac{1}{\omega^2}\left[b_0 + \sum_{m=2}^{\infty} b_m \cos m\tau\right] + \frac{1}{\omega^2}(\alpha_{n+1} a + b_1)\cos\tau \quad [70.12]$$

The condition for the absence of secular terms is again

$$\alpha_{n+1} = -\frac{b_1}{a} \quad [70.13]$$

Equation [70.12] now becomes

$$\ddot{z}_{n+1} + z_{n+1} = \frac{1}{\omega^2}\left[b_0 + \sum_{m=2}^{\infty} b_m \cos m\tau\right] \quad [70.14]$$

The solution of this differential equation is

$$z_{n+1} = \frac{1}{\omega^2}\left[b_0 - \sum_{m=2}^{\infty} b_m \frac{\cos m\tau}{m^2 - 1}\right] \quad [70.15]$$

the secular term having been removed again from the solution.

71. MOTION OF A CONSERVATIVE NON-LINEAR SYSTEM WITH A CUBIC TERM

As an example of an application of this method, consider the differential equation

$$\ddot{x} + x + \mu x^3 = 0 \quad [71.1]$$

The integration of this equation by the method of the first approximation was given in Chapter X.

We propose now to determine the approximations of higher orders, following the method explained in the preceding section.

Taking as generating solutions $z_0(\tau) = a \cos\tau$, $\omega^2 = 1$, and $\alpha_0 = 1$, we have

$$\ddot{z}_1 + z_1 = -z_0^3 - \alpha_1\ddot{z}_0 = -a^3\cos^3\tau + \alpha_1 a\cos\tau = \left(\alpha_1 a - \frac{3}{4}a^3\right)\cos\tau - \frac{a^3}{4}\cos 3\tau$$

The elimination of the secular term gives

$$\alpha_1 = \frac{3}{4}a^2 \qquad [71.2]$$

and the solution of this equation gives

$$z_1 = \frac{3}{32}a^3\cos 3\tau \qquad [71.3]$$

If we substitute for α_1 and z_1 their values, the third equation [70.3] becomes

$$\ddot{z}_2 + z_2 = -\frac{3}{32}a^5\cos^2\tau\cos 3\tau + \alpha_2 a\cos\tau + \frac{27}{128}\cos 3\tau$$

$$= \left(\alpha_2 a - \frac{3}{128}a^5\right)\cos\tau + \frac{21}{128}a^5\cos 3\tau - \frac{3}{128}a^5\cos 5\tau \qquad [71.4]$$

The condition for the absence of the secular term gives $\alpha_2 = \frac{3}{128}a^4$. Thus the solution of Equation [71.4] is

$$z_2 = -\frac{21}{1024}a^5\cos 3\tau + \frac{a^5}{1024}\cos 5\tau$$

Consequently the approximate solution satisfying Equation [71.1] to the order of μ^2 is

$$x = a\cos(\omega t + \phi) + \mu\frac{a^3}{32}\left(1 - \mu\frac{21}{32}\right)\cos(3\omega t + \phi) + \mu^2\frac{a^5}{1024}\cos(5\omega t + \phi) \qquad [71.5]$$

where a and ϕ are constants of integration.

The frequency Ω is given by the second expression of [70.2], in which α_1, α_2, $\cdots$ have already been determined

$$\Omega^2 = 1 + \frac{3}{4}\mu a^2 + \frac{3}{128}\mu^2 a^4 \qquad [71.6]$$

72. HIGHER APPROXIMATIONS FOR NON-LINEAR, NON-CONSERVATIVE SYSTEMS

We now consider the general form of a quasi-linear equation

$$\ddot{x} + \omega^2 x + \mu f(x, \dot{x}) = 0 \qquad [72.1]$$

We shall attempt to write the general solution for higher approximations in the form of an improved first approximation by replacing $\omega t + \phi$ by $\omega t + \phi - \frac{\pi}{2} = \psi$. One then obtains from Equation [68.19]

$$x = a\cos\psi - \frac{\mu}{\omega^2} F_0(a) + \frac{\mu}{\omega^2} \sum_{n=2}^{\infty} \frac{F_n(a)\cos n\psi + G_n(a)\sin n\psi}{n^2 - 1} \qquad [72.2]$$

with equations of the first approximation

$$\frac{da}{dt} = \frac{\mu}{2\omega} G_1(a); \qquad \frac{d\psi}{dt} = \Omega(a) \qquad [72.3]$$

$$\Omega(a) = \omega + \frac{\mu}{2\omega a} F_1(a) \qquad [72.4]$$

The F_n and G_n in Equation [72.2] are the Fourier coefficients in the development of

$$f(a\cos\tau, -a\omega\sin\tau) = \sum_{n=0}^{\infty} \left[F_n(a)\cos n\tau + G_n(a)\sin n\tau\right] \qquad [72.4a]$$

The condition for a steady state is

$$G_1(a) = 0; \qquad \psi = \Omega(a)t + \psi_0 \qquad [72.5]$$

where ψ_0 is an arbitrary constant.

Equation [72.2] for a steady state becomes

$$x = a\cos\left[\Omega(a)t + \psi_0\right] - \frac{\mu}{\omega^2} F_0(a) +$$

$$+ \frac{\mu}{\omega^2} \sum_{n=2}^{\infty} \frac{F_n(a)\cos n\left[\Omega(a)t + \psi_0\right] + G_n(a)\sin n\left[\Omega(a)t + \psi_0\right]}{n^2 - 1} \qquad [72.6]$$

It was seen that for conservative systems $G_1(a) \equiv 0$. In such a case, Equation [72.6] has two integration constants a and ψ_0, as is to be expected for an equation of the second order. If, however, $G_1(a) = 0$ has only simple roots without being equal to zero identically, the solution [72.6] has only one integration constant ψ_0 since a is determined from the equation $G_1(a) = 0$. This case corresponds, therefore, to the existence of limit cycles corresponding to the roots of $G_1(a) = 0$. In fact, as was shown previously, the stationary oscillation in this case does not depend on the initial amplitude.

We now propose to establish the existence of periodic solutions, that is, of limit cycles in non-conservative systems.

If $x = z(\Omega t + \phi)$ is such a solution, it must clearly satisfy the differential equation

$$\Omega^2 \ddot{z} + \omega^2 z + \mu f(z, \Omega \dot{z}) = 0 \qquad [72.7]$$

We can follow the same procedure as before, assuming solutions of the form

$$\begin{aligned} z &= z_0 + \mu z_1 + \mu^2 z_2 + \cdots \\ \Omega &= \Omega_0 + \mu \Omega_1 + \mu^2 \Omega_2 + \cdots \end{aligned} \qquad [72.8]$$

where z_n are periodic with period 2π. Here we take z_n as represented by their Fourier series.

Forming $\dot{z}$ and $\ddot{z}$ and substituting the values of z, $\dot{z}$, $\ddot{z}$, and Ω into Equation [72.7], one obtains again a series of recurrent differential equations by equating to zero the coefficients of various powers of μ. One has

$$\begin{aligned} &\Omega_0^2 \ddot{z}_0 + \omega^2 z_0 = 0 \\ &\Omega_0^2 \ddot{z}_1 + \omega^2 z_1 = -f(z_0, \Omega_0 \dot{z}_0) - 2\Omega_0 \Omega_1 \ddot{z}_0 \\ &\Omega_0^2 \ddot{z}_2 + \omega^2 z_2 = -f_z(z_0, \Omega_0 \dot{z}_0) z_1 - f_{\dot{z}}(z_0, \Omega_0 \dot{z}_0)\Omega_0 \dot{z}_1 - 2\Omega_0 \Omega_2 \ddot{z}_0 - \\ &\qquad - 2\Omega_0 \Omega_1 \ddot{z}_1 - \Omega_1^2 \ddot{z}_0 - f_{\dot{z}}(z_0, \Omega_0 \dot{z}_0)\Omega_1 \dot{z}_0 \\ &\qquad \cdots\cdots\cdots\cdots\cdots \\ &\Omega_0^2 \ddot{z}_{n+1} + \omega^2 z_{n+1} = -f_z(z_0, \Omega_0 \dot{z}_0) z_n - f_{\dot{z}}(z_0, \Omega_0 \dot{z}_0)\Omega_0 \dot{z}_n - 2\Omega_0 \Omega_{n+1} \ddot{z}_0 - \\ &\qquad - 2\Omega_0 \Omega_1 \ddot{z}_n - F_n(z_0 \cdots z_{n-1}; \dot{z}_0 \cdots \dot{z}_{n-1}; \ddot{z}_0 \cdots \ddot{z}_{n-1}; \Omega_0 \cdots \Omega_{n-1}) \end{aligned} \qquad [72.9]$$

where F_n is a known function of the indicated arguments. As before, $\Omega_0^2 = \omega^2$; we require that $\dot{z}(0) = 0$. Hence $\dot{z}_n(0) = 0$, where $n = 0, 1, 2, \cdots$, so that $z_0 = a \cos \tau$, where a is a constant to be determined, and z_1, z_2, $\cdots$ do not contain sine terms.

Substituting these values for z_0 and Ω_0 into the second equation [72.9], one gets

$$\begin{aligned} \omega^2(\ddot{z}_1 + z_1) &= -f(a\cos\tau, -a\omega\sin\tau) + 2\omega\Omega_1 a\cos\tau \\ &= -\sum_{n=0}^{\infty}\Big[F_n(a)\cos n\tau + G_n(a)\sin n\tau\Big] + 2\omega\Omega_1 a\cos\tau \end{aligned} \qquad [72.10]$$

The condition for absence of a secular term gives

$$G_1(a) = 0\,; \qquad \Omega_1 = \frac{F_1(a)}{2\omega a} \qquad [72.11]$$

From Equations [72.11], a and Ω_1 can be determined. Equation [72.10] then becomes

$$\omega^2(\ddot{z}_1 + z_1) = -F_0(a) - \sum_{n=2}^{\infty}\left[F_n(a)\cos n\tau + G_n(a)\sin n\tau\right] \qquad [72.12]$$

The solution of Equations [72.12] is

$$z_1 = a_1\cos\tau - \frac{F_0(a)}{\omega^2} + \frac{1}{\omega^2}\sum_{n=2}^{\infty}\frac{[F_n(a)\cos n\tau + G_n(a)\sin n\tau]}{n^2-1} \qquad [72.13]$$

in which the amplitude a_1 is to be determined by the condition for the elimination of secular terms on the right side of Equation [72.9] for z_2.

Writing Equation [72.13] as $z_1 = a_1 \cos\tau + u$ where

$$u = -\frac{F_0(a)}{\omega^2} + \frac{1}{\omega^2}\sum_{n=2}^{\infty}\frac{F_n(a)\cos n\tau + G_n(a)\sin n\tau}{n^2-1} \qquad [72.14]$$

and substituting it into the third equation [72.9], one has

$$\omega^2(\ddot{z}_2 + z_2) = -f_z(a\cos\tau, -a\omega\sin\tau)\,a_1\cos\tau +$$

$$+ f_{\dot{z}}(a\cos\tau, -a\omega\sin\tau)\,a_1\omega\sin\tau + 2\omega\Omega_1 a_1\cos\tau + 2\omega\Omega_2 a\cos\tau + v(\tau)$$

where $v(\tau)$ is a periodic function containing the remaining terms of the third equation [72.9]. Since $v(\tau)$ is periodic, one can represent it by a Fourier series

$$v(\tau) = \sum_{n=0}^{\infty}\left[p_n\cos n\tau + q_n\sin n\tau\right]$$

Furthermore, one has the identity

$$-f_z a_1\cos\tau + f_{\dot{z}} a_1\omega\sin\tau = -\frac{\partial}{\partial\mu} f\left[(a+\mu a_1)\cos\tau, -(a+\mu a_1)\omega\sin\tau\right]_{\mu=0}$$

whence, by Equation [72.4a],

$$-f_z a_1\cos\tau + f_{\dot{z}} a_1\omega\sin\tau = -a_1\sum_{n=0}^{\infty}\left[F_n'(a)\cos n\tau + G_n'(a)\sin n\tau\right] \qquad [72.15]$$

where $F_n'(a)$ and $G_n'(a)$ designate the derivatives of $F_n(a)$ and $G_n(a)$. Substituting f_z and $f_{\dot{z}}$ into the third equation [72.9], one gets

$$\omega^2(\ddot{z}_2 + z_2) = -a_1\sum_{n=0}^{\infty}\left[F_n'(a)\cos n\tau + G_n'(a)\sin n\tau\right] +$$

$$+ 2\omega(\Omega_1 a_1 + \Omega_2 a)\cos\tau + \sum_{n=0}^{\infty}\left[p_n\cos n\tau + q_n\sin n\tau\right] \quad [72.16]$$

In view of the periodicity of z_2, the secular terms must be eliminated again. Their elimination gives the conditions

$$a_1 G_1'(a) = q_1 \quad \text{and} \quad \Omega_1 a_1 + \Omega_2 a = -\frac{p_1}{2\omega} + \frac{a_1 F_1'}{2\omega} \quad [72.17]$$

It follows that

$$a_1 = \frac{q_1}{G_1'(a)} \quad [72.18]$$

Since we are in search of the condition for the existence of limit cycles, $G_1(a) = 0$ by the first equation [72.3], we have to add now a second condition $G_1'(a) \neq 0$, since only in this case does Equation [72.18] give the determination of a_1. The equation $G_1'(a) \neq 0$ shows that the root of the equation $G(a) = 0$, which gives the limit cycle, is a simple root.

From the second equation [72.17]

$$\Omega_2 = -\frac{1}{a}\left(\frac{p_1}{2\omega} + \Omega_1 a_1 - \frac{a_1 F_1'}{2\omega}\right) \quad [72.19]$$

Hence, Equation [72.16] can now be written as

$$\omega^2(\ddot{z}_2 + z_2) = (p_0 - a_1 F_0') +$$

$$+\sum_{n=2}^{\infty}\left\{\left[p_n - a_1 F_n'(a)\right]\cos n\tau + \left[q_n - a_1 G_n'(a)\right]\sin n\tau\right\} \quad [72.20]$$

The solution of this equation is

$$z_2 = a_2\cos\tau + \frac{1}{\omega^2}\left(p_0 - a_1 F_0'\right) +$$

$$+\frac{1}{\omega^2}\sum_{n=2}^{\infty}\left[(p_n - a_1 F_n')\cos n\tau + (q_n - a_1 G_n')\sin n\tau\right]\frac{1}{1-n^2} \quad [72.21]$$

where a_2 is again an undetermined coefficient, which, together with Ω_3, is determined by the condition for the absence of secular terms on the right sides of Equations [72.9] for z_3, and so on.

Following this recurrence procedure, it is apparent that subsequent equations [72.9] permit the determination of a_n and Ω_{n+1} from the equations for the elimination of secular terms.

$$a_n G_n'(a) = \xi_n \quad \text{and} \quad 2\omega(\Omega_{n+1} a + \Omega_1 a_n) = \eta_n \qquad [72.22]$$

in which ξ_n and η_n are known from the preceding recurrent operations.

As long as the equation $G_1(a) = 0$ has only simple roots, that is, $G_1'(a) \neq 0$, the process can be continued indefinitely up to any value of the index n.

The expressions for the non-linear oscillation x and its frequency Ω, up to the order of μ^N inclusive, are of the following form

$$x = z_0(\Omega t + \phi) + \mu z_1(\Omega t + \phi) + \cdots + \mu^N z_N(\Omega t + \phi)$$
$$\Omega = \omega + \mu\Omega_1 + \cdots + \mu^N \Omega_N \qquad [72.23]$$

Following this procedure, one finds, as an example, for the second approximation

$$x = (a + \mu a_1)\cos(\Omega t + \phi) - \frac{\mu}{\omega^2} F_0(a) + $$
$$+ \frac{\mu}{\omega^2}\sum_{n=2}^{\infty} \frac{F_n(a)\cos n(\Omega t + \phi) + G_n(a)\sin n(\Omega t + \phi)}{n^2 - 1} \qquad [72.24]$$

where

$$\Omega = \omega + \mu\frac{F_1(a)}{2\omega a}$$

If we compare this expression with the earlier formula [72.6], it is observed that the only difference between these expressions is in the amplitude of the first harmonic, which is now $a + \mu a_1$ instead of a, where a is the root of $G_1(a) = 0$.

This difference is due to the fact that for higher approximations, as can be shown, the amplitude equation is

$$\frac{da}{dt} = \frac{\mu}{2\omega} G_1(a) + \mu^2 S_1(a) + \mu^3 S_2(a) + \cdots \qquad [72.25]$$

If the limit cycle is reached,

$$\frac{\mu}{2\omega} G_1(a) + \mu^2 S_1(a) + \cdots = 0$$

Hence, in view of the factor μ before $G_1(a)$, it is seen that by stopping the approximation for a certain value $n = N$ of the index, the error in the determination of the first harmonic of the limit cycle is only of the order $(N - 1)$, and not of the N^{th} order.

Hence, the N^{th} approximation determines the amplitude to the order $(N + 1)$ and the frequency to the order $(N + 2)$.

For $N = 0$, one has

$$x = a \cos(\Omega t + \phi); \quad \Omega = \omega + \frac{\mu F_1(a)}{2\omega a} \qquad [72.26]$$

which represents the first approximation obtained by a different method in Chapter VIII.

For $N = 1$, x is given by Equation [72.23] and

$$\Omega = \omega + \mu \frac{F_1(a)}{2\omega} + \mu^2 \Omega_2 \qquad [72.27]$$

In order to establish the explicit expression for this approximation, one has to determine a_1 and Ω_2 entering into these formulas.

73. GENERAL FORM OF EQUATIONS OF HIGHER APPROXIMATIONS

We shall now review a generalization of the preceding theory applicable to steady oscillations as well as to the transient conditions of a quasi-linear system

$$\ddot{x} + \omega^2 x + \mu f(x, \dot{x}) = 0 \qquad [73.1]$$

One may attempt to find a periodic solution of the form

$$x = z(\psi, a) \qquad [73.2]$$

Furthermore, by analogy with equations of the first approximation, one can postulate that

$$\frac{da}{dt} = A(a)$$
$$\frac{d\psi}{dt} = \Omega(a) \qquad [73.3]$$

For the time being, the functions $A(a)$ and $\Omega(a)$ remain unknown. In fact, their determination constitutes the object of this procedure.

Proceeding formally, we obtain

$$\dot{x} = \frac{\partial z}{\partial \psi} \Omega + \frac{\partial z}{\partial a} A \qquad [73.4]$$

Differentiating the second time, we find

$$\ddot{x} = \frac{\partial^2 z}{\partial \psi^2}\,\Omega^2 + 2\,\frac{\partial^2 z}{\partial \psi \partial a}\,\Omega A + \frac{\partial^2 z}{\partial a^2}\,A^2 + \frac{\partial z}{\partial \psi}\,\frac{\partial \Omega}{\partial a}\,A + \frac{\partial z}{\partial a}\,\frac{\partial A}{\partial a}\,A \quad [73.5]$$

Replacing x, $\dot{x}$, and $\ddot{x}$ in the quasi-linear equation [73.1] by their expressions [73.2], [73.4], and [73.5], one gets

$$\frac{\partial^2 z}{\partial \psi^2}\,\Omega^2 + 2\,\frac{\partial^2 z}{\partial \psi \partial a}\,\Omega A + \frac{\partial^2 z}{\partial a^2}\,A^2 + \frac{\partial z}{\partial \psi}\,\frac{\partial \Omega}{\partial a}\,A +$$

$$+ \frac{\partial z}{\partial a}\,\frac{\partial A}{\partial a}\,A + \omega^2 z + \mu f\left(z, \frac{\partial z}{\partial \psi}\,\Omega + \frac{\partial z}{\partial a}\,A\right) = 0 \qquad [73.6]$$

It is apparent that if one finds expressions for z, A, and Ω satisfying this equation to a certain degree μ^N of accuracy, these solutions will satisfy the original quasi-linear equation [73.1] to the same degree of approximation, conditions [73.3] being satisfied.

In order to apply the method of successive approximations, let us represent the solutions in the form

$$z(\psi,a) = z_0(\psi,a) + \mu z_1(\psi,a) + \mu^2 z_2(\psi,a) + \cdots$$

$$A(a) = \mu A_1(a) + \mu^2 A_2(a) + \cdots \qquad [73.7]$$

$$\Omega(a) = \omega + \mu\Omega_1(a) + \mu^2\Omega_2(a) + \cdots$$

The method then consists in substituting the series expressions [73.7] for $z(\psi,a)$, $A(a)$, and $\Omega(a)$ into Equations [73.3] and [73.2] and equating to zero the coefficients of μ, μ^2, $\cdots$.

One obtains in this manner the following series of recurrent differential equations (compare with the analogous method of Poincaré, Chapter VIII).

$$\frac{\partial^2 z_0}{\partial \psi^2} + z_0 = 0$$

$$\left(\frac{\partial^2 z_1}{\partial \psi^2} + z_1\right)\omega^2 = -f\left(z_0, \omega\,\frac{\partial z_0}{\partial \psi}\right) - 2\,\omega\,\Omega_1\,\frac{\partial^2 z_0}{\partial \psi^2} - 2\,\omega A_1\,\frac{\partial^2 z_0}{\partial \psi \partial a}$$

$$\left(\frac{\partial^2 z_2}{\partial \psi^2} + z_2\right)\omega^2 = -f_z\left(z_0, \omega\,\frac{\partial z_0}{\partial \psi}\right)z_1 - f_{\dot{z}}\left(z_0, \omega\,\frac{\partial z_0}{\partial \psi}\right)\left(\omega\,\frac{\partial z_1}{\partial \psi} + \Omega_1\,\frac{\partial z_0}{\partial \psi} + A_1\,\frac{\partial z_0}{\partial a}\right) -$$

$$- 2\,\omega\,\Omega_1\,\frac{\partial^2 z_1}{\partial \psi^2} - 2\,\omega A_1\,\frac{\partial^2 z_1}{\partial \psi \partial a} - \Omega_1^2\,\frac{\partial^2 z_0}{\partial \psi^2} - 2\Omega_1 A_1\,\frac{\partial^2 z_0}{\partial \psi \partial a} - A_1^2\,\frac{\partial^2 z_0}{\partial a^2} -$$

$$- \frac{\partial z_0}{\partial \psi}\,\frac{\partial \Omega_1}{\partial a}\,A_1 - \frac{\partial z_0}{\partial a}\,A_1\,\frac{\partial A_1}{\partial a} - 2\,\omega\,\Omega_2\,\frac{\partial^2 z_0}{\partial \psi^2} - 2\,\omega A_2\,\frac{\partial^2 z_0}{\partial \psi \partial a}$$

$$= -E_1 - 2\,\omega\,\Omega_2\,\frac{\partial^2 z_0}{\partial \psi^2} - 2\,\omega A_2\,\frac{\partial^2 z_0}{\partial \psi \partial a} \qquad [73.8]$$

. . . .

$$\left(\frac{\partial^2 z_{n+1}}{\partial \psi^2} + z_{n+1}\right) \omega^2 = - E_n - 2\omega \Omega_{n+1} \frac{\partial^2 z_0}{\partial \psi^2} - 2\omega A_{n+1} \frac{\partial^2 z_0}{\partial \psi \partial a} \qquad [73.8]$$

where E_n is a function of $z_1, \cdots, z_n; A_1, \cdots, A_n; \Omega_1, \cdots, \Omega_n$; and their partial derivatives, which can be considered as known from the solution of the first n equations [73.8] by a recurrence procedure.

As before, the first equation of System [73.8] is solved by putting $z_0 = a \cos \psi$. Substituting this solution into the second equation [73.8], one obtains

$$\left(\frac{\partial^2 z_1}{\partial \psi^2} + z_1\right) \omega^2 = - f(a\cos\psi, - a\omega\sin\psi) + 2\omega\Omega_1 a\cos\psi + 2\omega A_1 \sin\psi$$

$$= -\sum_{n=0}^{\infty} \left[F_n(a)\cos n\psi + G_n(a)\sin n\psi\right] + 2\omega\Omega_1 a\cos\psi + 2\omega A_1 \sin\psi \qquad [73.9]$$

Since we wish to have z_1 periodic, the secular terms on the right side of Equation [73.9] must be eliminated. The conditions for this are

$$2\omega\Omega_1 a - F_1(a) = 0\,; \qquad 2\omega A_1 - G_1(a) = 0 \qquad [73.10]$$

From these conditions A_1 and Ω_1 are determined. Substituting their values into Equation [73.9], one has

$$\left(\frac{\partial^2 z_1}{\partial \psi^2} + z_1\right) \omega^2 = - F_0(a) - \sum_{n=2}^{\infty} \left[F_n(a)\cos n\psi + G_n(a)\sin n\psi\right] \qquad [73.11]$$

The solution of this equation is

$$z_1 = - \frac{1}{\omega^2} F_0(a) + \frac{1}{\omega^2} \sum_{n=2}^{\infty} \frac{F_n(a)\cos n\psi + G_n(a)\sin n\psi}{n^2 - 1} \qquad [73.12]$$

Substituting the values of A_1 and Ω_1 from Equations [73.10] and that of z_1 from Equation [73.12] into the third equation [73.8], one has

$$\left(\frac{\partial^2 z_2}{\partial \psi^2} + z_2\right) \omega^2 = -\sum_{n=0}^{\infty} \left[F_n^{(\prime)}(a)\cos n\psi + G_n^{(\prime)}(a)\sin n\psi\right] +$$

$$+ 2\omega\Omega_2 a\cos\psi + 2\omega A_2 \sin\psi \qquad [73.13]$$

where $F^{(\prime)}$ and $G^{(\prime)}$ are certain functions of a.

The elimination of secular terms again permits determining A_2 and Ω_2 from the equations.

$$2\omega\Omega_2 a - F_1^{(\prime)}(a) = 0\,; \qquad 2\omega A_2 - G_1^{(\prime)}(a) = 0 \qquad [73.14]$$

and the substitution of these values into Equation [73.13] reduces it to

$$\left(\frac{\partial^2 z_2}{\partial \psi^2} + z_2\right)\omega^2 = -\ F_0^{(\prime)}(a) - \sum_{n=2}^{\infty} \frac{F_n^{(\prime)}(a)\cos n\psi + G_n^{(\prime)}(a)\sin n\psi}{n^2-1} \qquad [73.15]$$

The recurrence procedure is now apparent. It is thus seen that the solution so obtained is of the form

$$x = a\cos\psi + \mu z_1(\psi,a) + \mu^2 z_2(\psi,a) + \cdots \qquad [73.16]$$

where a and ψ are given by the equations

$$\frac{da}{dt} = \mu A_1(a) + \mu^2 A_2(a) + \cdots + \mu^N A_N(a) \qquad [73.17]$$

$$\frac{d\psi}{dt} = \omega + \mu \Omega_1(a) + \mu^2 \Omega_2(a) + \cdots + \mu^N \Omega_N(a) \qquad [73.18]$$

On the other hand, by Equations [73.10], $A_1 = \frac{G_1(a)}{2\omega}$ and $\Omega_1 = \frac{F_1(a)}{2\omega a}$, whence

$$\frac{da}{dt} = \mu\,\frac{G_1(a)}{2\omega} - \mu^2 A_2(a) + \mu^3 A_3(a) + \cdots + \mu^N A_N(a)$$

$$\frac{d\psi}{dt} = \omega + \mu\,\frac{F_1(a)}{2\omega a} + \mu^2\Omega_2(a) + \mu^3\Omega_3(a) + \cdots + \mu^N\Omega_N(a) \qquad [73.19]$$

For $N = 1$, Equations [73.19] give the improved first approximation, Equation [68.19].

Furthermore, since by the method of elimination of secular terms the quantities $A_1, A_2, \cdots, A_n$ are expressed in terms of the subsequent first harmonics which are eliminated from expressions $z_1(\psi,a), \cdots, z_n(\psi,a)$, it is apparent that the first equation [73.19] relates to the amplitude of the fundamental harmonic.

The second equation [73.19], viz.,

$$\Omega(a) = \frac{d\psi}{dt} = \omega + \mu\,\frac{F_1(a)}{2\omega a} + \mu^2\Omega_2(a) + \cdots \qquad [73.20]$$

may be designated as the equation of the instantaneous frequency $\Omega(a)$ of the non-linear oscillation.

CHAPTER XII

METHOD OF EQUIVALENT LINEARIZATION OF KRYLOFF AND BOGOLIUBOFF

74. INTRODUCTORY REMARKS

The method of Kryloff and Bogoliuboff outlined in Chapter X was established by assuming a sinusoidal solution $x = a \sin\psi$ for a quasi-linear equation [58.1] and by determining the functions $a(t)$, the amplitude, and $\psi(t)$, the total phase, so as to satisfy the differential equation [58.1] with accuracy of the order of μ^2. As was mentioned, from the standpoint of formal procedure the method resembles that of the variation of constants of Lagrange.

The method of the first approximation stated in Chapter X gives approximate expressions for the frequency and the amplitude of a non-linear oscillation for small values of μ. It is plausible to think that these same approximate relations may be obtained from a linear equation in which the coefficients have been suitably chosen. This is essentially what Kryloff and Bogoliuboff have done and which is designated by them as the *method of equivalent linearization*. The essence of the method is the determination of the *equivalent parameters*, as is indicated in Section 75.

On the basis of formal procedure it is not clear why this particular determination of parameters leads to the possibility of approximating the solutions of a quasi-linear equation by those of a corresponding linear one in which equivalent parameters appear. In order to justify the procedure, Kryloff and Bogoliuboff observe that a non-linear oscillatory process is generally characterized by a certain Fourier spectrum of the component frequencies resulting from the non-linearity of the system. If, however, one limits oneself to the theory of the first approximation, it is logical to assume that the fundamental harmonic of the spectrum should be considered. Hence it is sufficient to determine the equivalent parameters so as to obtain in the linearized problem the same oscillation which appears as the fundamental harmonic of the quasi-linear system. In fact, if one assumes this to be an *a priori* proposition, the Principle of Harmonic Balance, it can be shown that the formulas giving the equivalent parameters follow directly from this principle, see Section 77. One can also justify the introduction of equivalent parameters by postulating that the work per cycle done by a non-linear force F_g and by a corresponding linear one is the same. In fact, if one assumes a Principle of Equivalent Balance of Energy of this kind, one likewise obtains the same formulas for the equivalent parameters, see Section 76.

Viewed from this standpoint, the Principle of Harmonic Balance enables us to determine the equivalent parameters without actually writing the non-linear differential equation. Kryloff and Bogoliuboff show that the solutions so obtained do not differ much from those of the corresponding linear equation. It is to be noted, however, that this argument should not be considered as a proof. In spite of this, the method of equivalent linearization, as we shall see particularly in Part III, plays an important role in the quasi-linear theory and leads to results consistent with experimental data. Thus, for instance, the generalization of the concept of equivalent parameters for several variables makes it possible to absorb the effect of an external periodic excitation by the equivalent parameter and thus to explain a number of phenomena such as asynchronous quenching and excitation, and similar phenomena. Moreover, when an equivalent parameter is a function of the amplitude, the approach of the phenomenon to a limit cycle in this representation amounts to the approach of the equivalent parameter to a critical value at which the linearized decrement vanishes and the oscillation becomes stationary.

We shall encounter numerous applications of the method of equivalent linearization in Part III. In this chapter we shall establish the principal definitions of equivalent parameters and give a few applications of this method.

75. METHOD OF EQUIVALENT LINEARIZATION

It was shown in Chapter X that the solution of a quasi-linear equation

$$m\ddot{x} + Kx + \mu f(x,\dot{x}) = 0 \qquad [75.1]$$

can be written $x = a \cos \psi$,* where the amplitude a and the total phase ψ are given by two differential equations of the first order.

Applying Equations [59.13] and [59.14] to Equation [75.1] with this form of solution, one obtains

$$\frac{da}{dt} = \frac{\mu}{2\pi\omega m}\int_0^{2\pi} f(a\cos\phi, -a\omega\sin\phi)\sin\phi\, d\phi \equiv \Phi(a) \qquad [75.2]$$

$$\frac{d\psi}{dt} = \Omega(a) \qquad [75.3]$$

where

$$\Omega^2(a) = \omega^2 + \frac{\mu}{\pi m a}\int_0^{2\pi} f(a\cos\phi, -a\omega\sin\phi)\cos\phi\, d\phi \qquad [75.4]$$

* As a matter of fact, this solution in Chapter X was taken as $x = a \sin \psi$, which merely reverses the $\sin \phi$ and $\cos \phi$ under the integral sign in the amplitude and phase equations. The notation in the present chapter complies with that used in the text of Kryloff and Bogoliuboff.

The second term on the right side of Equation [75.4] is the frequency correction between the linear, $\omega^2 = \frac{K}{m}$, and the non-linear, $\Omega^2(a)$, frequencies.

If one defines two constants $\overline{\lambda}$ and $\overline{K}$, the equivalent parameters, by the equations

$$\overline{\lambda} = -\frac{\mu}{\pi a \omega}\int_0^{2\pi} f(a\cos\phi, -a\omega\sin\phi)\sin\phi\, d\phi \qquad [75.5]$$

$$\overline{K} = K + \frac{\mu}{\pi a}\int_0^{2\pi} f(a\cos\phi, -a\omega\sin\phi)\cos\phi\, d\phi \qquad [75.6]$$

it can be shown that an "equivalent" linear equation, with coefficients $\overline{\lambda}$ and $\overline{K}$, approximates the solution of the quasi-linear equation [75.1] to an accuracy of the order of μ^2.

In fact, with values [75.5] and [75.6], the amplitude equation [75.2] becomes

$$\dot{a} = -\frac{\overline{\lambda}}{2m} a \qquad [75.7]$$

and the phase equation

$$\dot{\psi} = \Omega(a) = \sqrt{\frac{\overline{K}}{m}} \qquad [75.8]$$

One recognizes these expressions as the usual ones for the decrement and frequency of an ordinary linear equation of the second order.

In order to make sure that the solution $x = a \cos \psi$, with a and ψ given by Equations [75.7] and [75.8], actually satisfies the equivalent linear equation, with accuracy of the order of μ^2, substitute the values x and $\ddot{x}$ into the equivalent linear equation

$$m\ddot{x} + \overline{\lambda}\dot{x} + \overline{K}x = 0 \qquad [75.9]$$

We have

$$\dot{x} = \dot{a}\cos\psi - a\sin\psi\cdot\dot{\psi} = -\frac{\overline{\lambda}}{2m} a\cos\psi - a\Omega\sin\psi$$

$$\ddot{x} = -a\frac{\overline{K}}{m}\cos\psi + \frac{\overline{\lambda}}{m} a\Omega\sin\psi + \frac{\overline{\lambda}}{2m} a^2\frac{\partial\Omega}{\partial a}\sin\psi + \frac{1}{2m}\frac{\partial\overline{\lambda}}{\partial a}\frac{\overline{\lambda}}{2m} a^2\cos\psi \qquad [75.10]$$

$$= -\frac{\overline{K}}{m}x - \frac{\overline{\lambda}}{m}\dot{x} + \frac{\overline{\lambda}}{2m} a^2\frac{\partial\Omega}{\partial a}\sin\psi + \frac{1}{2m}\frac{\partial\overline{\lambda}}{\partial a}\frac{\overline{\lambda}}{2m} ax - \left(\frac{\overline{\lambda}}{m}\right)^2\frac{a}{2}\cos\psi ,$$

Substituting x, $\dot{x}$, and $\ddot{x}$ from these equations into the equivalent linear equation [75.9], one sees that the quasi-linear differential equation [75.1] reduces to a residue of the form $R(\mu^2)$, which proves that the equivalent linear equation is satisfied with accuracy of the order of μ^2.

The transformation of the original quasi-linear equation

$$m\ddot{x} + Kx + \mu f(x,\dot{x}) = 0$$

into an equivalent linear one is accomplished by replacing the term $\mu f(x,\dot{x})$ of the quasi-linear equation by $K_1 x + \overline{\lambda}\dot{x}$, where $K_1 = \overline{K} - K$.

It is apparent that the quantity $\overline{\delta} = \overline{\lambda}/2m = -\dot{a}/a$, as determined by Equation [75.7], is the decrement, and $\Omega = \sqrt{\overline{K}/m}$ is the frequency of the equivalent linear equation. If one substitutes for $\overline{\lambda}$ and $\overline{K}$ their expressions [75.5] and [75.6], one finds equations of the first approximation.

In this manner one obtains a purely formal connection between the equations of the first approximation and the equivalent linear equation with parameters $\overline{\lambda}$ and $\overline{K}$, as defined by Equations [75.5] and [75.6].

76. PRINCIPLE OF EQUIVALENT BALANCE OF ENERGY

From the preceding section it appears that the method of equivalent linearization consists in replacing a quasi-linear force, $F_g = \mu f(x,\dot{x})$, by a linear one, $F_L = K_1 x + \overline{\lambda}\dot{x}$. Furthermore, it has been shown that if the equivalent parameters K_1 and $\overline{\lambda}$ are defined by Equations [75.5] and [75.6], with $K_1 = \overline{K} - K$, the solution of the equivalent linear equation [75.9] differs by a small quantity of the second order from the solution of the original quasi-linear equation [75.1]. We propose now to show why this particular definition of equivalent parameters K_1 and $\overline{\lambda}$ has been adopted. The physical justification for this definition lies in the Principle of Equivalent Balance of Energy, which requires that *the work per cycle of F_g and F_L be the same*, that is

$$\mu\int_0^T f(x,\dot{x})\dot{x}\,dt = \overline{\lambda}\int_0^T \dot{x}^2\,dt \qquad [76.1]$$

The term with K_1 does not enter into this expression because the work of a conservative force per cycle is always zero.

It is to be noted, in view of the fact that the integral in Equation [75.5] is finite, that $\overline{\lambda}$ is of the same order as μ, that is, it is small. On the other hand, to the first approximation, $x = a\cos\psi$ and $\dot{x} = -a\omega\sin\psi$, where $\psi = \omega t + \psi_0$, and a and ψ_0 can be considered as approximately constant during the time interval $2\pi/\omega$. The left side of Equation [76.1], upon changing the limit of integration and substituting the generating solutions $x = a\cos\psi$ and $\dot{x} = -a\omega\sin\psi$, becomes

$$- \mu \int_0^{2\pi} f(a_1 \cos \psi, - a\omega \sin \psi)\, a \sin \psi \, d\psi \qquad [76.2]$$

and the right side is clearly

$$\bar{\lambda} \int_0^{\frac{2\pi}{\omega}} a^2 \omega^2 \sin^2 \psi \, dt = \bar{\lambda} a^2 \omega \int_0^{2\pi} \sin^2 \psi \, d\psi = \bar{\lambda} a^2 \omega \pi \qquad [76.3]$$

Hence, by Equation [76.1],

$$\bar{\lambda} = - \frac{\mu}{\pi a \omega} \int_0^{2\pi} f(a \cos \phi, - a\omega \sin \phi) \sin \phi \, d\phi \qquad [76.4]$$

This is precisely the first equation [75.5], by which the parameter $\bar{\lambda}$ was originally defined.

It is thus seen that the introduction of the equivalent parameter $\bar{\lambda}$ is dictated by the equivalence of work per cycle in both the quasi-linear and the equivalent linear systems.

We next give a suitable physical interpretation to the other equivalent factor, K_1, which does not appear in the energy equation [76.1]. For this purpose it is helpful to utilize the definition of "wattless" or reactive power commonly used in the theory of alternating currents. In this theory the energy (or active) component of power W_a and its wattless (or reactive) counterpart W_r are defined as

$$W_a = \frac{1}{T} \int_0^T e i \cos \phi \, dt \qquad \text{and} \qquad W_r = \frac{1}{T} \int_0^T e i \sin \phi \, dt$$

where e, i, and ϕ are voltage, current, and phase angle respectively.

Defining the active W_a and the reactive W_r components of power for a mechanical system in a similar manner, we have

$$W_a = \frac{1}{T} \int_0^T F(t)\, \dot{x}(t)\, dt\, ; \qquad W_r = \frac{1}{T} \int_0^T F(t)\, \dot{x}\left(t - \frac{T}{4}\right) dt$$

Hence, by equating the expressions for the "reactive powers" for a quasi-linear and for an equivalent linear system, we obtain

$$\mu \frac{1}{T} \int_0^T f\left[x(t), \dot{x}(t)\right] \dot{x}\left(t - \frac{T}{4}\right) dt = \frac{1}{T} \int_0^T \left[K_1 x(t) + \bar{\lambda} \dot{x}(t)\right] \dot{x}\left(t - \frac{T}{4}\right) dt \qquad [76.5]$$

Since in this equation both μ and $\bar{\lambda}$ are of the first order, K_1 is of the same order.

Substituting the generating solutions $x = a\cos(\omega t + \theta)$, $\dot{x} = -a\omega\sin(\omega t + \theta)$, and $T = 2\pi/\omega$, the left side of Equation [76.5] becomes

$$\frac{a\omega\mu}{2\pi}\int_0^{2\pi} f(a\cos\phi,\ -a\omega\sin\phi)\cos\phi\,d\phi$$

and its right side is $a^2\omega K_1/2$. It follows that

$$K_1 = \frac{\mu}{\pi a}\int_0^{2\pi} f(a\cos\phi,\ -a\omega\sin\phi)\cos\phi\,d\phi \qquad [76.6]$$

which is Equation [75.6].

77. PRINCIPLE OF HARMONIC BALANCE

An alternative auxiliary principle serving the same purpose can be described as follows. Consider again the non-linear force $F = \mu f(x,\dot{x})$ and the equivalent linear one, $F_L = K_1 x + \bar{\lambda}\dot{x}$. The harmonic oscillation $x = a\cos(\omega t + \theta)$, where ω is the frequency of the "zero" approximation, is taken again as a generating solution. With this solution, F_L can be written as $F_L = F_{L0}\cos(\omega t + \theta_L)$, where F_{L0} and θ_L are the amplitudes and the phase respectively of F_L. The non-linear force F is represented by a Fourier series of which the fundamental harmonic is $F = F_0\cos(\omega t + \theta)$. If one makes $F = F_L$, which constitutes the *Principle of Harmonic Balance*, it entails two equations, $F_0 = F_{L0}$ and $\theta = \theta_L$, from which again the two parameters $\bar{\lambda}$ and K_1 can be obtained. In fact,

$$F_L = K_1 a\cos(\omega t + \theta_L) - \omega\bar{\lambda} a\sin(\omega t + \theta_L) \qquad [77.1]$$

and the fundamental harmonic of the non-linear force is

$$F = \frac{1}{\pi}\left[\int_0^{2\pi} f(a\cos\tau, -a\sin\tau)\cos\tau\,d\tau\right]\cos(\omega t + \theta) +$$

$$+\frac{1}{\pi}\left[\int_0^{2\pi} f(a\cos\tau,\ -a\sin\tau)\sin\tau\,d\tau\right]\sin(\omega t + \theta) \qquad [77.2]$$

Equating the coefficients of $\cos(\omega t + \theta)$ and $\sin(\omega t + \theta)$ in Equations [77.1] and [77.2], since $\theta = \theta_L$, one obtains the same expressions for K_1 and $\bar{\lambda}$ as before.

It is seen that both principles, that of the Equivalent Balance of Energy and that of Harmonic Balance, are equivalent, because the work of higher harmonics per cycle of the fundamental frequency is zero.

Summing up the results of this and of the preceding sections, one can state:

1. The Principle of Equivalent Linearization consists in defining an equivalent linear system as a system with parameters $\bar{\lambda}$ and K_1 expressing the equality of work per cycle for the non-linear and the equivalent linear systems.

2. The parameter $\bar{\lambda}$ is obtained by equating the active components of power in both cases; K_1, by equating the reactive components.

3. When the equivalent parameters $\bar{\lambda}$ and K_1 are so determined, the equivalent linear differential equation admits a solution differing from that of the quasi-linear equation by a small quantity of the order of μ^2.

4. For practical purposes the formation of the equivalent parameters $\bar{\lambda}$ and K_1 is the only requirement for the solution of the quasi-linear equation, in view of Statement 3.

78. EXAMPLES OF APPLICATION OF THE METHOD OF EQUIVALENT LINEARIZATION

A few examples given below illustrate the application of this method.

A. NON-LINEAR RESTORING FORCE

Consider the differential equation

$$m\ddot{x} + F(x) = 0 \qquad [78.1]$$

where $F(x)$ is of a quasi-linear type. For example, $F(x) = cx + \mu x^3$, where c and μ are constant. The condition for quasi-linearity is that $\frac{\mu x^3}{cx} \ll 1$. Since the system is conservative, the amplitude a remains constant, but the oscillations are not isochronous. Substituting the value of $F(x)$ into Equation [78.1], one has

$$m\ddot{x} + cx + \mu x^3 = 0 \qquad [78.2]$$

Hence $f(x,\dot{x}) = f(x) = \mu x^3$ and, by Equation [76.6],

$$K_1(a) = \frac{\mu}{\pi a}\int_0^{2\pi} f(a\cos\phi)\cos\phi \, d\phi = \frac{\mu a^2}{2}$$

The equivalent spring constant in this case will be $c + \frac{\mu a^2}{2}$, whence

$$\Omega_a = \sqrt{\frac{c + K_1(a)}{m}} = \sqrt{\omega^2 + \frac{\mu a^2}{2m}} = \omega\sqrt{1 + \frac{\mu a^2}{2m\omega^2}} \approx \omega\left(1 + \frac{\mu a^2}{4m\omega^2}\right) = \omega(1 + \alpha a^2) \qquad [78.3]$$

where $\alpha = \frac{\mu}{4m\omega^2}$.

The frequency of non-linear oscillation is here a function of the amplitude a. The oscillation is thus non-isochronous, although the amplitude a is constant. One can easily discover this fact formally by calculating the expression for $\overline{\lambda}$ from Equation [75.5], which in this case gives $\overline{\lambda} = 0$.

B. NON-LINEAR DISSIPATIVE DAMPING

Consider a differential equation of the form

$$m\ddot{x} + Kx + \mu f(\dot{x}) = 0$$

In order to be more specific, assume quadratic damping, that is,

$$\mu f(\dot{x}) = b|\dot{x}|\dot{x}$$

Equation [75.5] for $\overline{\lambda}$ in this case is

$$\overline{\lambda} = -\frac{b}{\pi a\omega}\int_0^{2\pi} f(-a\omega\sin\phi)\sin\phi\, d\phi = \frac{b}{\pi a\omega}\int_0^{2\pi} f(a\omega\cos\phi)\cos\phi\, d\phi$$

where

$$\int_0^{2\pi} f(a\omega\cos\phi)\cos\phi\, d\phi = a^2\omega^2\int_0^{2\pi}|\cos\phi|\cos^2\phi\, d\phi = a^2\omega^2\left[\int_0^{\frac{\pi}{2}}\cos^3\phi\, d\phi - \right.$$

$$\left. - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos^3\phi\, d\phi + \int_{\frac{3\pi}{2}}^{2\pi}\cos^3\phi\, d\phi\right] = \frac{8}{3}a^2\omega^2$$

Hence, $\overline{\lambda} = \frac{8}{3}\frac{ba\omega}{\pi}$. The equivalent decrement $\overline{\delta} = \frac{\overline{\lambda}}{2m} = \frac{4}{3}\frac{ba\omega}{m\pi}$; it is seen that the decrement in this case varies with the amplitude.

Applying Equation [76.6], one finds

$$K_1 = \frac{\mu b a\omega^2}{\pi}\int_0^{2\pi}|\sin\phi|\sin\phi\cos\phi\, d\phi = 0$$

Hence the non-linear correction for the frequency in this case is zero to the first order.

From the fact that for quadratic damping the decrement varies with the amplitude, one concludes that for large amplitudes quadratic damping is more efficient, and for small amplitudes less efficient, than is linear damping, the decrement of which does not depend on the amplitude.

Since the decrement $\overline{\delta} = \frac{4}{3}\frac{b\omega}{m\pi}a \equiv Sa$, where $S = \frac{4}{3}\frac{b\omega}{m\pi}$, the motion under the effect of quadratic damping can be determined. We have

$$\overline{\delta} = Sa = -\frac{\dot{a}}{a} = -\frac{1}{a}\frac{da}{dt}$$

whence

$$S = -\frac{1}{a^2}\frac{da}{dt} = \frac{d}{dt}\left(\frac{1}{a}\right)$$

Hence, on integrating, $\frac{1}{a} - \frac{1}{a_0} = St$, so that finally

$$a = \frac{a_0}{1 + \frac{4}{3}\frac{b\omega}{m\pi}a_0 t} \qquad [78.4]$$

which coincides with Equation [62.4], obtained by the theory of the first approximation.

C. NON-LINEAR RESTORING FORCE AND NON-LINEAR DISSIPATIVE DAMPING

The differential equation is of the form

$$m\ddot{x} + f(\dot{x}) + c(x) = 0$$

It will again be assumed that both $f(\dot{x})$ and $c(x)$ are quasi-linear, that is, they are of the form $f(\dot{x}) = \lambda_0\dot{x} + \mu\phi(\dot{x})$ and $c(x) = c_0 x + \nu\psi(x)$, where λ_0 and c_0 are constant, μ and ν are small parameters, and $\phi(\dot{x})$ and $\psi(x)$ are non-linear terms.

The application of the method of equivalent linearization gives

$$\Omega = \sqrt{\omega^2 + \frac{\overline{K}_1}{m}} \quad \text{and} \quad \overline{\delta} = \frac{\overline{\lambda}}{2m}$$

where $\overline{K}_1$ and $\overline{\lambda}$ are again the equivalent parameters determined by Equations [75.5] and [76.6], applied to the function $f(x,\dot{x}) = \mu\phi(\dot{x}) + \nu\psi(x)$, as explained in connection with the two previous examples.

Equations of this type are of frequent occurrence in practice. For example, Froude's well-known differential equation for the rolling of a ship in still water is of the form

$$I\ddot{\theta} + K_1\dot{\theta} + K_2\dot{\theta}^2 + Wh\sin\theta = 0$$

where I, W, and h are respectively the moment of inertia, the displacement, and the metacentric height of the ship, and K_1 and K_2 are Froude's coefficients of resistance to rolling. If one approximates $\sin\theta$ by $\theta - \frac{\theta^3}{6}$, one has, upon dividing the equation by I, the following equation

$$\ddot{\theta} + b_1\dot{\theta} + b_2\dot{\theta}^2 + \omega^2\theta - \frac{\omega^2\theta^3}{6} = 0$$

which is of the type considered here.

D. ELECTRICAL OSCILLATIONS IN A CIRCUIT CONTAINING A SATURATED CORE REACTOR

Consider an oscillating circuit containing a constant air-core inductance L_0, a variable saturation iron-core inductance L_1, and a fixed capacity C. The non-linearity in this case is due to L_1. In fact, the flux Φ through the coil L_1 is $\Phi = f(i)$, where i is the current. For a sinusoidal current $i = i_0 \cos(\omega t + \theta)$, the fundamental harmonic of magnetic flux is

$$\frac{1}{\pi}\int_0^{2\pi} f(i_0 \cos\phi)\cos\phi\, d\phi \cdot \cos(\omega t + \phi)$$

According to the method of equivalent linearization, the non-linear equation $\Phi = f(i)$ can be replaced by the linear one $\Phi = L_e i$, where L_e is the equivalent linear coefficient of self-inductance,

$$L_e = \frac{1}{\pi i_0}\int_0^{2\pi} f(i_0 \cos\phi)\cos\phi\, d\phi \qquad [78.5]$$

If the constant air-core inductance L_0 is relatively large compared with the non-linear inductance containing iron, the current will be quasi-harmonic and the expression for frequency will be

$$\omega = \frac{1}{\sqrt{(L_0 + L_e)C}} = \frac{1}{\sqrt{L_0 C}}\left(1 - \frac{L_e}{2L_0}\right) \qquad [78.6]$$

E. NON-LINEAR CONDUCTORS

Consider a conductor and let the voltage drop e across its terminal be $e = -f(i)$.

If the current is of the form $i = i_0 \cos(\omega t + \theta)$, the fundamental harmonic of the voltage drop is

$$-\frac{1}{\pi}\int_0^{2\pi} f(i_0 \cos\phi)\cos\phi\, d\phi \cdot \cos(\omega t + \theta) = e_1$$

By putting

$$R_e = \frac{1}{\pi i_0}\int_0^{2\pi} f(i_0 \cos\phi)\cos\phi\, d\phi$$

the non-linear conductor can be replaced by an equivalent linear one having a voltage drop $e_1 = R_e i$. If $R_e > 0$, the non-linear conductor dissipates energy; if $R_e < 0$, energy is brought into the system owing to the non-linearity of the process. Likewise, if the non-linearity appears in the form $i = f(e)$ and the voltage executes a harmonic oscillation $e = e_0 \cos(\omega t + \theta)$, the fundamental harmonic of current will be $\frac{1}{\pi}\int_0^{2\pi} f(e_0 \cos\phi)\cos\phi\, d\phi$. Here again one can replace the non-linear parameter $i = f(e)$ by a linear one, $i = \sigma e$, provided that we define the equivalent conductance as

$$\sigma = \frac{1}{\pi e_0} \int_0^{2\pi} f(e_0 \cos\phi) \cos\phi \, d\phi \qquad [78.7]$$

Depending upon whether σ is greater than or less than zero, one has either absorption or generation of energy.

These considerations are useful in analyzing circuits containing electron tubes. The anode current in this case is of the form $i_a = f(E_0 + e)$, where E_0 is the constant voltage of the "B source," and e is the alternating grid voltage.

If e is sinusoidal, the fundamental harmonic of i_a will be

$$\frac{1}{\pi} \int_0^{2\pi} f(E_0 + e_0 \cos\phi) \cos\phi \, d\phi \cdot \cos(\omega t + \theta)$$

If one defines

$$S_e = \frac{1}{\pi e_0} \int_0^{2\pi} f(E_0 + e_0 \cos\phi) \cos\phi \, d\phi \qquad [78.8]$$

as the average transconductance of the tube, instead of a non-linear relation $i_a = f(E_0 + e)$, one will have an *equivalent* linear relation, $i_a = S_e e$.

F. THERMIONIC GENERATORS

Consider a thermionic circuit arranged according to the diagram shown in Figure 78.1, which is self-explanatory. The resistance R, shown to be in parallel, is supposed to be large so as to obtain only rather small damping in the oscillating circuit LC.

The control voltage is $e = (M - DL)\frac{di}{dt}$, where the term $DL\frac{di}{dt}$ takes care of the anode reaction ($D \ll 1$). The alternating component of the anode current is

$$i_a = S_e\left(M - DL\right)\frac{di}{dt}$$

where S_e is the equivalent transconductance of the linearized problem.

By Kirchoff's law, $i_a = i_L + i_R + i_C$, where $Ri_R = Li_L'$ in the LR-mesh; hence $i_R = \frac{L}{R} i_L'$. In the CL-mesh,

$$\frac{1}{C}\int i_C \, dt = L\frac{di_L}{dt}$$

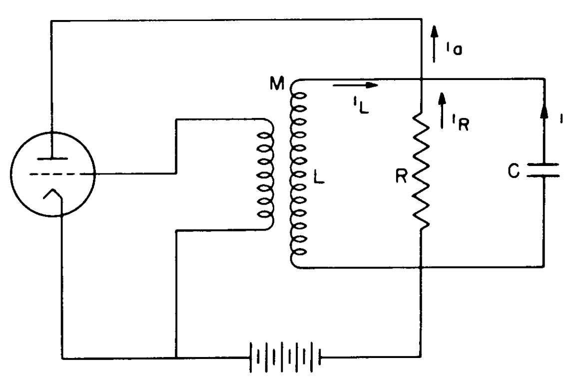

Figure 78.1

whence $i_C = LCi_L''$. Substituting these values into Kirchoff's equation and dropping the subscript L, one obtains

$$LC\frac{d^2i}{dt^2} + \frac{L}{R}\frac{di}{dt} + i = i_a = S_e\left(M - DL\right)\frac{di}{dt}$$

Hence finally

$$LC\frac{d^2i}{dt^2} + \left[\frac{L}{R} - S_e\left(M - DL\right)\right]\frac{di}{dt} + i = 0 \qquad [78.9]$$

This equation is an equivalent linearized equation of the process, since it contains the linearized parameter S_e. From this we get the decrement

$$\delta = \frac{1}{2LC}\left[\frac{L}{R} - S_e\left(M - DL\right)\right] \qquad [78.10]$$

Equation [59.13] of the first approximation is here

$$\frac{di_0}{dt} = \frac{S_e(M - DL) - \frac{L}{R}}{2LC}\cdot i_0 \qquad [78.11]$$

The stable amplitude is reached when

$$S_e(M - DL) = \frac{L}{R} \qquad [78.12]$$

Since, by Equation [78.8], S_e contains e_0, the substitution for S_e of its value from [78.12] determines the equilibrium amplitude e_0 of the grid voltage at which the oscillation reaches a steady state.

As a second example, consider a somewhat modified scheme shown in Figure 78.2, in which the resistance R is supposed to be small so as to be within the range of the quasi-linear theory. If the current in the oscillating circuit is designated by i and the anode current by i_a, the differential equation is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = M\frac{di_a}{dt} = MS_e\frac{de}{dt} \qquad [78.13]$$

where e is the grid voltage given by the equation

$$e = \frac{1}{C}\int i\,dt + D\left(M\frac{di}{dt} - L_a\frac{di_a}{dt}\right)$$

If the anode reaction is neglected, $D \approx 0$, so that

$$e \approx \frac{1}{C}\int i\,dt \qquad [78.14]$$

Introducing the variable e instead of i in Equation [78.13], we obtain

$$LC\frac{d^2e}{dt^2} + \left(RC - MS_e\right)\frac{de}{dt} + e = 0 \qquad [78.15]$$

The stationary condition is reached when

$$S_e = \frac{RC}{M} \qquad [78.16]$$

In the transient state the decrement is

$$\delta = \frac{RC - MS_e}{2LC} \qquad [78.17]$$

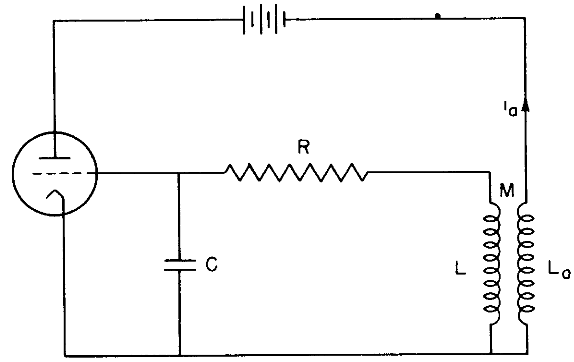

Figure 78.2

Since by [78.8] the equivalent transconductance S_e is a function of the amplitude e_0, the decrement δ varies during the transient state. If the static curve $i_a = f(e)$ is approximated by a polynomial, Equation [78.8] permits calculating the amplitude e_0' at which the decrement δ vanishes and the stationary condition is reached. Assume, for example, that the constant biasing voltage E_0 is such that the characteristic $i_a = f(e)$ of the electron tube can be approximated by the polynomial

$$f(e) = i_a = i_{a0} + \alpha_1 e + \beta_1 e^2 - \gamma_1 e^3 \qquad [78.18]$$

where α_1, β_1, and γ_1 are positive constants. Carrying out the calculation [78.8], one finds

$$S_e = \alpha_1 - \frac{3}{4}\gamma_1 e_0^2$$

and, by Equation [78.16],

$$e_0' = \sqrt{\frac{4}{3\gamma_1}\left(\alpha_1 - \frac{RC}{M}\right)} \qquad [78.19]$$

REFERENCES

(1) "Les méthodes nouvelles de la mécanique céleste" (New Methods in Celestial Mechanics), by H. Poincaré, Gauthier-Villars, Paris, Vol. 1, 1892, Chapters III and IV.

(2) "On Van der Pol's and Related Non-Linear Differential Equations," by J.A. Shohat, Journal of Applied Physics, Vol. 15, No. 7, July 1944, pp. 568-574.

(3) "Differentialgleichungen der Störungstheorie" (Differential Equations of the Theory of Perturbation), by And. Lindstedt, Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg, Vol. XXXI, No. 4, 1883.

(4) "Theory of Oscillations," by A. Andronow and S. Chaikin, Moscow, (Russian), 1937, Chapters VII and VIII.

(5) "Introduction to Non-Linear Mechanics," by N. Kryloff and N. Bogoliuboff, Kieff, (Russian), 1937, Chapters X, XI, and XII.

(6) H. Poincaré, loc. cit., Chapter II.

(7) "Sur la théorie mathématique des auto-oscillations" (On the Mathematical Theory of Self-Excited Oscillations), by A. Andronow and A. Witt, Comptes Rendus, Paris, 1930.

(8) "Sur la théorie mathématique des systèmes auto-oscillatoires à deux dégres de liberté" (On the Mathematical Theory of Self-Excited Systems with Two Degrees of Freedom), by A. Andronow and A. Witt, Journal of Technical Physics, USSR, (Russian), 1934.

(9) H. Poincaré, loc. cit., p. 179.

(10) "Dynamics of a System of Rigid Bodies," by E.J. Routh, London, 1905, Chapter VI.

(11) Über die Bedingungen unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt" (On the Conditions under which an Equation Possesses Only Roots with Negative Real Parts), by A. Hurwitz, Mathematische Annalen, Vol. 46, 1895, p. 273.

(12) "Problème général de la stabilité du mouvement" (General Problemes of Stability of Motion), by M.A. Liapounoff, Annales de la Faculté des Sciences de Toulouse, Paris, Vol. 9, 1907.

(13) "On a Type of Oscillation Hysteresis in a Simple Triode Generator," by E.V. Appleton and B. Van der Pol, Philosophical Magazine, Vol. 42, 1921.

(14) "Theory of Sound," by Lord Rayleigh, London, Vol. 1, 1894.

(15) "On Oscillation Hysteresis in a Triode Generator," by B. Van der Pol, Philosophical Magazine, Vol. 43, 1922.

(16) "Note sur la génération des oscillations entretenues" (Memoir on the Generation of Self-Excited Oscillations), by E. and H. Cartan, Annales des Postes Télégraphes et Téléphones, December 1925.

(17) "Étude des oscillations entretenues" (Study of Self-Excited Oscillations), by A. Liénard, Revue Générale de l'Électricité, Vol. 23, 1928.