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THE AXIALLY SYMMETRIC POTENTIAL FLOW ABOUT ELONGATED BODIES OF REVOLUTION

by

L. Landweber

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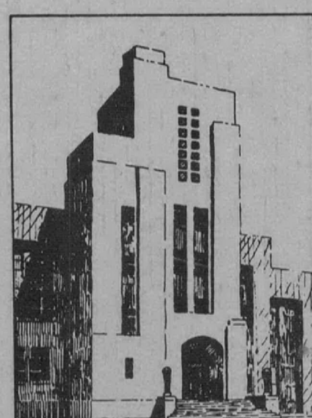

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THE AXIALLY SYMMETRIC POTENTIAL FLOW ABOUT ELONGATED BODIES OF REVOLUTION

by

L. Landweber

ABSTRACT

An iteration formula for Fredholm integral equations of the first kind is applied in two new methods for obtaining the steady, irrotational, axisymmetric flow of an inviscid, incompressible fluid about a body of revolution. In the first method a continuous, axial distribution of doublets is sought as a solution of an integral equation of the first kind. A method of determining the end points and the initial trends of the distribution, and a first approximation to a solution of the integral equation are given. This approximation is then used to obtain a sequence of successive approximations whose successive differences furnish a geometric measure of the accuracy of an approximation. When a doublet distribution has been assumed, the velocity and pressure can be computed by means of formulas which are also given.

In the second method the velocity is given directly as the solution of an integral equation of the first kind. Here also a first approximation is derived and applied to obtain a sequence of successive approximations. In contrast with the first method, which, in general, can give only an approximate solution, the integral equation of the second method has an exact solution.

Both methods are illustrated in detail by an example. The results are compared with those obtained by other well-known methods.

INTRODUCTION

HISTORY

The determination of the flow about elongated bodies of revolution is of great practical and theoretical importance in aero- and hydrodynamics. Such knowledge is required in connection with bodies such as airships, torpedoes, projectiles, airplane fuselages, pitot tubes, etc. Since it is well known that for a streamlined body, moving in the direction of the axis of symmetry, the actual flow is very closely approximated by the potential (inviscid) flow about the body,[1] numerous attempts have been made to find a convenient theoretical method for obtaining numerical solutions of the potential flow problem.

At first the problem was attacked by indirect means. In 1871 Rankine[2] showed how one could obtain families of bodies of revolution of known potential flow, generated by placing several point sources and sinks of various strengths on the axis. This method was extended and used by D.W. Taylor[3] in 1894 and by G. Fuhrmann[4] in 1911. The latter also constructed models of the computed forms and showed that the measured distributions of the pressures over them agreed very well with the computed values except for a small region at the downstream ends. More recently, in 1944, the Rankine method was employed by Munzer and Reichardt[5] to obtain bodies with flat pressure distribution curves, and a further refinement of the technique was published by Riegels and Brandt.[6] Most recently the indirect method has been employed to obtain bodies generated by axisymmetric source-sink distributions on circumferences, rings, disks, and cylinders. This development, which enabled bodies with much blunter noses to be generated, was initiated by Weinstein[7] in 1948 and continued by van Tuyl[8] and by Sadowsky and Sternberg[9] in 1950.

A method of solving the direct problem, i.e., to determine the flow over a given body of revolution, appears to have been first published by von Kármán[10] in 1927. von Kármán reduced the problem to that of solving a Fredholm integral equation of the first kind for the axial source-sink distribution which would generate the given body, and solved the integral equation approximately by replacing it with a set of simultaneous linear equations. Although this method has limited accuracy and becomes very laborious when, for greater refinement, a large number of linear equations are employed, nevertheless it is the best known and most frequently used of the direct methods. A modification of the von Kármán method was published by Wijngaarden[11] in 1948.

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[1]References are listed on page 59.

An interesting attempt to solve the direct problem was made by Weinig[12] in 1928. He also formulated the problem in terms of an integral equation for an axial doublet distribution which would generate the given body, and employed an iteration formula to obtain successive approximations. Since the successive approximations diverged, the recommended procedure was to extrapolate one step backwards to obtain a solution.

In 1935 an entirely different approach, in which a solution for the velocity potential was assumed in the form of an infinite linear sum of orthogonal functions, was made by Kaplan[13] and independently by Smith.[14] The coefficients of this series are given as the solution of a set of linear equations, infinite in number. In practice a finite number of these equations is solved for a finite number of coefficients, and Kaplan has shown that the approximate solution thus obtained is that due to an axial source-sink distribution which is also determined. A simplification of Kaplan's method by means of additional approximations was proposed by Young and Owen[15] in 1943.

It appears to be generally agreed, by those who have tried them, that the aforementioned methods are both laborious and approximate. Thus, according to Young and Owen:[15]

> "In every case, however, the methods proposed are laborious to apply, and the labour and heaviness of the computations increase rapidly with the rigour and accuracy of the process. Inevitably, a compromise is necessary between the accuracy aimed at and the difficulties of computation. All the methods reduce, ultimately, to finding in one way or another the equivalent sink-source distribution, and it is this part of the process which in general involves the heaviest computing."

Furthermore, a fundamental objection is that only a special class of bodies of revolution can be represented by a distribution of sources and sinks on the axis of symmetry. According to von Kármán:[10]

> "This (representability by an axial source-sink distribution) is possible only in the exceptional case when the analytical continuation of the potential function, free from singularities in the space outside the body, can be extended to the axis of symmetry without encountering singular spots."

The dissatisfaction with these methods is reflected by the continuing attempts to devise other procedures.

A new method published by Kaplan[16] in 1943 is free of the assumption of axial singularities and appears to be exact in the sense that the solution can be made as accurate as desired, but the labor required for the same accuracy appears to be much greater than by other methods. The application of the method requires that first the conformal transformation which transforms the given meridian profile into a circle be determined. The velocity potential is then expressed as an infinite series whose terms are universal functions involving the coefficients of the conformal transformation. Kaplan[16] has derived only the first three of these universal functions.

Cummins of the David Taylor Model Basin is developing a method based on a distribution of sources and sinks on the surface of the given body. This method is also exact, but the labor involved in its application has not yet been evaluated.

Another exact method, based on a distribution of vorticity over the surface of the body, is being developed by Dr. Vandry of the Admiralty Research Laboratory, Teddington, England. The methods of both Cummins and Vandry lead to Fredholm integral equations of the second kind, which can be solved by iteration.

The present writer has developed two new methods, an approximate one in which an axial doublet distribution is assumed, and an exact one based on a general application of Green's theorem of potential theory. Both methods lead to Fredholm integral equations of the first kind for which a solution by iteration has been discussed by the author.[17] Indeed, the consideration of this iteration formula was initiated in an attempt to find more satisfactory solutions of the integral equations of von Kármán[10] and Weinig.[12] These new methods will be presented, and, by application to a particular body, compared with other methods from the point of view of accuracy and convenience of application.

FORMULATION OF THE PROBLEM

We will consider the steady, irrotational, axially symmetric flow of a perfect incompressible fluid about a body of revolution. Take the x-axis as the axis of symmetry and let x, y be the coordinates in a meridian plane. Denote the equation of the body profile by

$$y^2 = f(x) \qquad [1]$$

Since the flow is irrotational there exists a velocity potential ϕ which, for axisymmetric flows, depends only on the cylindrical coordinates x, y and satisfies Laplace's equation in cylindrical coordinates

$$\frac{\partial}{\partial x}\left(y\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(y\frac{\partial \phi}{\partial y}\right) = 0 \qquad [2]$$

Also, since the flow is axisymmetric, there exists a Stokes stream function $\psi(x, y)$ which is related to the velocity potential by the equations

$$\frac{\partial \psi}{\partial x} = -y\frac{\partial \phi}{\partial y}, \quad \frac{\partial \psi}{\partial y} = y\frac{\partial \phi}{\partial x} \qquad [3]$$

It is seen that Equation [2] may be interpreted as the necessary and sufficient condition insuring the existence of the function ψ. As is well known, ψ is constant along a streamline and, considering the surface of revolution generated by rotation of a streamline about the axis of symmetry, $2\pi\psi$ may be considered as the flux bounded by this surface. On the surface of the given body and along the axis of symmetry outside the body we have $\psi = 0$. ψ satisfies the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1}{y}\frac{\partial \psi}{\partial y} \qquad [4]$$

which is obtained by eliminating ϕ between Equations [3].

The velocity will be taken as the negative gradient of the velocity potential. Let u, v be the velocity components in the x, y directions, Then by [3], we have

$$u = -\frac{\partial \phi}{\partial x} = -\frac{1}{y}\frac{\partial \psi}{\partial y} \qquad [5]$$

$$v = -\frac{\partial \phi}{\partial y} = \frac{1}{y}\frac{\partial \psi}{\partial x} \qquad [6]$$

For a uniform flow of velocity U parallel to the x-axis we have

$$\phi = -Ux, \quad \psi = -\frac{1}{2}Uy^2 \qquad [7]$$

The boundary condition for the body to be a stream surface may be written in various ways. If the body is stationary the boundary condition is

$$\psi\left(x, \sqrt{f(x)}\right) = 0 \qquad [8a]$$

or, equivalently,

$$\left(\frac{d\phi}{dn}\right)_S = 0 \qquad [8b]$$

where the derivative in [8b] is evaluated on the surface of the body in the direction of the outward normal to the body. If the body is moving with velocity V parallel to the x-axis the boundary condition becomes

$$\left(\frac{d\phi}{dn}\right)_S = -V \cos \beta \qquad [9]$$

where β is the angle between the outward normal to the body and the x-axis.

It is desired to obtain a solution of [2] or [4] which satisfies the boundary conditions [7] at infinity and [8] or [9] on the body.

METHOD OF AXIAL DISTRIBUTIONS

SOURCES AND SINKS

The potential and stream functions for a point source of strength M situated on the x-axis at x = t are

$$\phi = \frac{M}{r}, \qquad \psi = M\left(-1 + \frac{x-t}{r}\right) \qquad [10]$$

where

$$r^2 = (x - t)^2 + y^2 \qquad [11]$$

If the sources are distributed piecewise-continuously along the x-axis between the points a and b (see Figure 1) with a strength $\mu(x)$ per unit length, the potential and stream functions are

$$\phi = \int_a^b \frac{\mu(t)}{r}\, dt \qquad [12]$$

$$\psi = \int_a^b \mu(t)\left(-1 + \frac{x-t}{r}\right) dt \qquad [13]$$

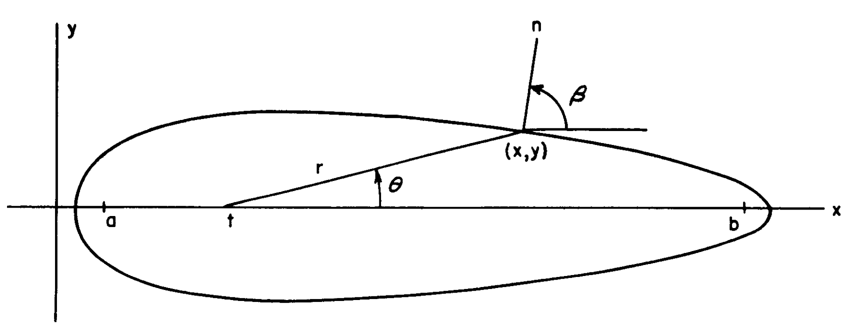

Figure 1 - The Meridian Plane

As is well known, Rankine bodies are obtained by superposition of these flows with a uniform stream so as to obtain a dividing streamline beginning at a stagnation point. Without loss of generality we may suppose this uniform stream to be of unit magnitude. This dividing streamline is the profile of the Rankine body for which, by [7], the stream function is

$$\psi = -\frac{1}{2} y^2 + \int_a^b \mu(t)\left(-1 + \frac{x-t}{r}\right) dt \qquad [14]$$

The boundary condition, Equation [8a], then gives as the implicit equation for the body

$$\int_a^b \mu(t)\left(-1 + \frac{x-t}{r}\right) dt = \frac{1}{2} y^2 \qquad [15]$$

where now $y^2 = f(x)$ and $r^2 = (x-t)^2 + f(x)$. In order to obtain a closed body the total strength of sources and sinks must be zero, i.e.,

$$\int_a^b \mu(t)\, dt = 0$$

In this case [15] becomes

$$\int_a^b \mu(t)\, \frac{x-t}{r}\, dt = \frac{1}{2} y^2 \qquad [15a]$$

In general [15a] cannot be solved explicitly for $f(x)$ when $\mu(t)$ is given. A practical procedure for obtaining $f(x)$ for a given x is to evaluate the integral numerically for various assumed values of $f(x)$ and to determine the value which satisfies [15a] by graphical means.

When $f(x)$ is prescribed [15a] may be considered as a Fredholm integral equation of the first kind for determining the unknown function $\mu(t)$. This equation will not be treated. Indeed it will be shown that, when continuous distributions are considered, it is a special case of the more general equation for doublet distributions which will now be derived.

DOUBLET DISTRIBUTIONS

Let $m(x)$ be the strength per unit length of a continuous distribution of doublets along the x-axis between the points a and b (see Figure 1). The potential and stream functions may be taken as

$$\phi = \int_a^b m(t)\, \frac{t-x}{r^3}\, dt \qquad [16]$$

and

$$\psi = y^2 \int_a^b \frac{m(t)}{r^3} \, dt \qquad [17]$$

The stream function for a Rankine flow now becomes

$$\psi = -\frac{1}{2} y^2 + y^2 \int_a^b \frac{m(t)}{r^3} \, dt \qquad [18]$$

Hence the boundary condition, Equation [8a], gives

$$\int_a^b \frac{m(t)}{r^3} \, dt = \frac{1}{2} \qquad [19]$$

Here again Equation [19] may be considered as an implicit equation for the Rankine body when m(t) is given, or as a Fredholm integral equation of the first kind when the body profile $y^2 = f(x)$ is prescribed.

In order to show the relation between the source and doublet distributions in Equations [15a] and [19], integrate by parts in [19]. We have

$$\int_a^b m(t) \frac{y^2}{r^3} \, dt = m(t) \left. \frac{t-x}{r} \right|_a^b + \int_a^b \frac{dm}{dt} \frac{x-t}{r} \, dt$$

Hence [19] may be written as

$$m(t) \left. \frac{t-x}{r} \right|_a^b + \int_a^b \frac{dm}{dt} \frac{x-t}{r} \, dt = \frac{1}{2} y^2 \qquad [20]$$

The interpretation of Equation [20] is that a doublet distribution of strength m is equivalent to a source-sink distribution of strength dm/dt together with point sources of strength m(a) and -m(b) at the end points. Hence source-sink distributions are completely equivalent only to those doublet distributions which vanish at the end points. This justifies the remark in the previous section that the integral equation for the doublet distributions is more general than that for the source-sink distributions.

MUNK'S APPROXIMATE DISTRIBUTION

Munk[18] has given an approximate solution of Equation [19] for elongated bodies. His formula may be derived as follows. At a great distance from the ends of a very elongated body, the integrand of [19], $m(t)/r^3$, will peak sharply in the neighborhood of $t = x$. In the range of the peak, in which the value of the integral is principally determined, $m(t)$ will vary little from $m(x)$. Also, only a small error will be introduced by replacing the limits of integration by $-\infty$ and $+\infty$. Hence, as a first approximation to a solution of [19], try

$$m_1(x)\int_{-\infty}^{\infty}\frac{dt}{r^3} = \frac{1}{2} \qquad [21]$$

We obtain

$$m_1(x) = \frac{1}{4}\,y^2 \qquad [22]$$

a distribution proportional to the section-area curve of the body. This approximation was independently derived by Weinig[12] who employed it as the first step in a divergent iteration procedure. It has also been rediscovered by Young and Owen[15] and Laitone[19] who have shown the accuracy of the approximation for elongated bodies by several examples.

It is apparent from its derivation that [22] also gives the asymptotic radius of the half-body generated by a constant axial dipole distribution extending from a point on the axis to infinity. It is readily seen that this distribution is equivalent to a point source at the initial point.

As a refinement to Munk's formula, Weinblum[20] has used the approximation

$$m_1(x) = Cy^2 \qquad [23]$$

where C is a factor obtained by comparison of the distributions and section-area curves of several bodies. Weinblum's factor bears an interesting relation to the virtual mass of the body. This is seen by considering the expression for the virtual mass $k_1\Delta$ in terms of the mass of the displaced fluid Δ and the totality of the doublets, $\int_a^b m\,dx$,[21,22,23]

$$k_1\Delta = 4\pi\rho\int_a^b m\,dx - \Delta \qquad [24]$$

where k_1 is designated the longitudinal virtual mass coefficient, and ρ is the density of the fluid. But, from [23],

$$4\pi\rho\int_a^b m_1\,dx = 4\rho C\int_a^b \pi y^2\,dx \cong 4C\Delta$$

since, for elongated bodies, a and b very nearly coincide with the body ends. Hence

$$C = \frac{1}{4}(1 + k_1) \qquad [25]$$

In practice an approximate value of k_1 may be taken as that of the prolate spheroid having the same length-diameter ratio as the given body. The values of k_1 for a prolate spheroid may be computed from the formula[24]

$$k_1 = \frac{\lambda \ln\left(\lambda + \sqrt{\lambda^2 - 1}\right) - \sqrt{\lambda^2 - 1}}{\lambda^2 \sqrt{\lambda^2 - 1} - \lambda \ln\left(\lambda + \sqrt{\lambda^2 - 1}\right)} \qquad [26]$$

where λ is the length-diameter ratio. Hence

$$C = \frac{\frac{1}{4}\left(\lambda^2 - 1\right)^{3/2}}{\lambda^2 \sqrt{\lambda^2 - 1} - \lambda \ln\left(\lambda + \sqrt{\lambda^2 - 1}\right)} \qquad [27]$$

The values of k_1 versus λ have also been tabulated by Lamb[9] and graphed by Munk.[25]

END POINTS OF A DISTRIBUTION

A difficulty in determining the doublet distribution from Equation [19] is that the limits of integration, a and b, are also unknown. In the method of von Kármán[10] the end points are arbitrarily chosen; Kaplan[13] takes the end point of the distribution midway between the end of the body and the center of curvature at that end.

Kaplan based his choice on a consideration of the prolate spheroid. Thus the equation of the spheroid of unit length and length-diameter ratio λ, extending from $x = 0$ to $x = 1$, is

$$y^2 = \frac{1}{\lambda^2}\left(x - x^2\right) \qquad [28]$$

The radius of curvature at $x = 0$ is then $\frac{1}{2\lambda^2}$. The exact doublet distribution, however, extends between the foci of the spheroid which are situated at distances

$$\frac{\lambda - \sqrt{\lambda^2 - 1}}{2\lambda}$$

from the end points. Hence the error in Kaplan's assumption,

$$\frac{\lambda - \sqrt{\lambda^2 - 1}}{2\lambda} - \frac{1}{4\lambda^2} = \frac{1}{16\lambda^4}\left(1 + \frac{1}{2\lambda^2} + \ldots\right)$$

diminishes rapidly with increasing λ.

For the half-body generated by a constant doublet distribution (a point source), Kaplan's assumption gives a poor approximation. Let a^2 be the strength of the distribution. Then it can easily be shown from [19] that the source is at a distance a from the end of the body (stagnation point), and that, if the origin is chosen at the latter point, the equation of the half-body is

$$\left(\frac{y}{a}\right)^2 = \frac{8}{3}\frac{x}{a} - \frac{20}{27}\left(\frac{x}{a}\right)^2 + \frac{16}{243}\left(\frac{x}{a}\right)^3 + \ldots \qquad [29]$$

Hence the radius of curvature at the end is $\frac{4}{3}a$, so that Kaplan's assumption for the start of the distribution gives $\frac{2}{3}a$. This is in error by $\frac{1}{3}a$.

An approximate method for determining the end points of a distribution and its trends at the ends is given in Appendix 1. The given profile is assumed to extend from $x = 0$ to $x = 1$ and to have the equation

$$y^2 = a_1 x + a_2 x^2 + a_3 x^3 + \ldots \qquad [30]$$

The doublet distribution is assumed to extend from $x = a$ to $x = b$, so that $0 < a \ll b < 1$, with a near 0 and b near 1, and to have the equation

$$m(x) = c_0 + c_1 x + c_2 x^2 + \ldots \qquad [31]$$

Only the trends of the distribution near the origin are discussed in Appendix 1. It is clear, however, that by means of a linear transformation the equation of the given profile can be expressed so that the end points of the body exchange their roles. Hence the results in Appendix 1 can be applied to either end of the body.

The method of Appendix 1 consists essentially of expanding the integral in [19] about the origin and equating powers of t on the two sides of the equation to obtain a series of equations in the unknowns a, c_0, c_1, c_2, ... By applying the first four of these equations an approximate solution is obtained in the form

$$a = \frac{a_1}{\alpha} \qquad [32]$$

$$c_0 D = -4a^2\left[3\alpha^3 - 37\alpha^2 + 120\alpha - 96 + 24a_2 + 24a(3\alpha^2 - 15\alpha + 16 - 4a_2)\right] \quad [33]$$

$$c_1 D = a\left[15\alpha^3 - 168\alpha^2 + 512\alpha - 384 + 96a_2 + 48a(5\alpha^2 - 24\alpha + 24 - 6a_2)\right] \quad [34]$$

$$c_2 D = -4\left[(\alpha - 4)^2(\alpha - 1) + 4a_2\right] \quad [35]$$

where

$$\left.\begin{aligned} D = {} & 2(9\alpha^3 - 94\alpha^2 + 272\alpha - 192) + 8\left[(\alpha - 4)^2(\alpha - 1) + 4a_2\right]\ln a \\ & + 96a_2 - 2a(15\alpha^3 - 264\alpha^2 + 944\alpha - 768) - 384aa_2 \\ & - 96a^2(5\alpha^2 - 24\alpha + 24) + 576a^2a_2 \end{aligned}\right\} \quad [36]$$

and α is a root of the seventh-degree polynomial

$$A + a_1B + a_2C + a_1a_2D + a_2^2E + a_1a_2^2F + a_1a_3G + a_1^2a_3H = 0 \quad [37]$$

where

$$\left.\begin{aligned} A(\alpha) &= \alpha(\alpha - 4)^2(5\alpha^4 - 83\alpha^3 + 288\alpha^2 - 368\alpha + 128) \\ B(\alpha) &= 72(\alpha - 4)^2(5\alpha^3 - 25\alpha^2 + 40\alpha - 16) \\ C(\alpha) &= 4\alpha(\alpha - 4)(53\alpha^2 - 148\alpha + 128) \\ D(\alpha) &= -288(\alpha - 4)(5\alpha^2 - 16\alpha + 16) \\ E(\alpha) &= -96\alpha(3\alpha - 4) \\ F(\alpha) &= 1152(2\alpha - 3) \\ G(\alpha) &= 48\alpha(3\alpha - 8) \\ H(\alpha) &= -1152(\alpha - 3) \end{aligned}\right\} \quad [38]$$

The solution gives, for the initial doublet strength at $x = a$,

$$m(a) = -\frac{a^2}{D}\left[(\alpha - 4)(\alpha^2 - 12\alpha + 16) + 48a(\alpha - 4)(\alpha - 2) + 16a_2 - 96aa_2\right] \quad [39]$$

When a_1, a_2, a_3, ... are all small in comparison with unity, an approximate solution for α is

$$\alpha = 4 + a_2 - \frac{1}{2}\sqrt{a_1 a_3}, \quad \text{if } a_3 \geqq 0 \qquad [40]$$

$$\alpha = 4 + a_2, \quad \text{if } a_3 < 0 \qquad [41]$$

and, to the same order of approximation,

$$m(x) = \frac{1}{4}\left(1 + \frac{a_1}{2} + \frac{a_2}{2} \ln \frac{a_1}{4}\right)\left(-\frac{a_1^2}{4} + y^2\right) \qquad [42]$$

and

$$m(a) = \frac{1}{2}\left(1 + \frac{a_1}{2} + \frac{a_2}{2} \ln \frac{a_1}{4}\right)a^2 \sqrt{a_1 a_3}, \quad \text{if } a_3 \geqq 0 \qquad [43]$$

$$m(a) = 0, \quad \text{if } a_3 < 0 \qquad [44]$$

It is seen that Kaplan's assumption that $\alpha = 4$ gives the principal term of the solution in [40] or [41]. The form [42] immediately suggests a modification and refinement of the Munk-Weinblum approximation, Equation [23], which will be considered in the next section.

A graphical procedure for finding the roots α of Equation [35] is also given in the Appendix. For this purpose the functions $A(\alpha)$, $B(\alpha)$, ... $H(\alpha)$ are tabulated in Table 10.

AN IMPROVED FIRST APPROXIMATION

According to its derivation the Munk approximation could be expected to be useful only at a distance from the end points of a distribution. It was seen, however, Equation [42], that under certain circumstances a distribution which was a suitable approximation for the nose and tail of a body also appeared as a generalization of the Munk-Weinblum approximation, [23]. This suggests a procedure for obtaining an improved approximate distribution.

It is desired to obtain a distribution $m(x)$ which satisfies the following conditions:

(a) $m(x)$ assumes known values m_a and m_b at the distribution limits a and b, i.e.,

$$m(a) = m_a, \quad m(b) = m_b \qquad [45]$$

(b) $m(x)$ is nearly equivalent to the Munk-Weinblum approximation [23] at a distance from the distribution limits, i.e.,

$$m(a) \cong Cy^2 \quad \text{for} \quad a \ll x \ll b$$

(c) m(x) satisfies the virtual mass relation [24] which may be written in the convenient form

$$\int_a^b m(x)dx = \frac{1}{4}(1 + k_1)\int_0^1 y^2 dx \qquad [46]$$

It is readily verified that Condition (a) is satisfied by the distribution

$$m(x) = Cy^2 + e_0 + e_1 x \qquad [47]$$

where

$$e_0 = \frac{1}{b-a}\left[bm_a - am_b + C(af_b - bf_a)\right] \qquad [48]$$

and

$$e_1 = \frac{1}{b-a}\left[m_b - m_a + C(f_a - f_b)\right] \qquad [49]$$

If the linear term $e_0 + e_1 x$ in [47] is small in comparison with m(x) at a distance from the ends, then Condition (b) is also satisfied. Finally, Condition (c) can be satisfied by a proper choice of C in [47]. This is accomplished by writing m(x) in the form

$$m(x) = C\left(y^2 - \frac{b-x}{b-a} f_a - \frac{x-a}{b-a} f_b\right) + \frac{b-x}{b-a} m_a + \frac{x-a}{b-a} m_b$$

substituting it into Equation [46], and solving for C. We obtain

$$C = \frac{\frac{1}{4}\ 1+k_1 \int_0^1 y^2 dx - \frac{1}{2}(b-a)(m_a+m_b)}{\int_a^b y^2 dx - \frac{1}{2}(b-a)(f_a+f_b)} \qquad [50]$$

SOLUTION OF INTEGRAL EQUATION BY ITERATION

Now that we have derived a good first approximation to the doublet distribution function in the integral equation [19], it would be very desirable to apply it to obtain a second, closer approximation. This can be accomplished by means of the iteration formula which we will now derive.

Let $m_1(x)$ be a known first approximation and $\psi_1(x)$ the corresponding values of the stream function ψ on the given profile $y^2 = f(x)$. Then, from Equation [18],

$$\psi_1(x) = -\frac{1}{2} f(x) + f(x)\int_a^b \frac{m_1(t)}{r^3} dt \qquad [51]$$

Thus $\psi_1(x)$ is a measure of the error when $m_1(t)$ is tried as a solution of the integral equation [19]. If m(t) is a solution of [19], Equation [51] may be written in the form

$$\psi_1(x) = f(x)\int_a^b \frac{m_1(t)-m(t)}{r^3} dt \qquad [52]$$

But, on the same assumptions as were used to derive Munk's approximate distribution, Equation [22], we obtain as an approximate solution of the integral equation [52]

$$m_1(x) - m(x) = \frac{1}{2}\psi_1(x) \qquad [53]$$

or, denoting the new approximation to m(x) by $m_2(x)$,

$$m_2(x) = m_1(x) - \frac{1}{2}\psi_1(x) \qquad [54]$$

Hence, from [51]

$$m_2(x) = m_1(x) + \frac{1}{2} f(x)\left[\frac{1}{2} - \int_a^b \frac{m_1(t)}{r^3} dt\right] \qquad [55]$$

Since the foregoing procedure can be repeated successively, we obtain the iteration formula

$$m_{i+1}(x) = m_i(x) + \frac{1}{2} f(x)\left[\frac{1}{2} - \int_a^b \frac{m_i(t)}{r^3} dt\right] \qquad [56]$$

and

$$m_{i+1}(x) - m_i(x) = -\frac{1}{2}\psi_i(x) \qquad [57]$$

It is seen that ψ_i is the value of the stream function on the given profile corresponding to the i^{th} approximation $m_i(x)$ and hence serves as a measure of the error when $m_i(t)$ is tried as a solution of the integral equation [19].

Although successive approximations to m(x) may be computed directly from [56], an alternative form, which is both more convenient and more significant, will now be derived. From [56] we may write

$$m_i(x) = m_{i-1}(x) + \frac{1}{2} f(x) \left[\frac{1}{2} - \int_a^b \frac{m_{i-1}(t)}{r^3} \, dt \right] \qquad [56a]$$

Hence, deducting [56a] from [56] and making use of [57], we get

$$\psi_i(x) = \psi_{i-1}(x) - \frac{1}{2} f(x) \int_a^b \frac{\psi_{i-1}(t)}{r^3} \, dt \qquad [58]$$

Also, from [57] we obtain

$$m_{i+1}(x) = m_1(x) - \frac{1}{2} \sum_{j=1}^{i} \psi_j(x) \qquad [59]$$

Thus, in order to obtain $m_{i+1}(x)$, we first assume an $m_1(x)$, then determine $\psi_1(x)$ from [51]. $\psi_2(x)$, $\psi_3(x)$, ... can then be successively obtained from [58], and finally $m_{i+1}(x)$ from [59].

It has been stated that the magnitude of $\psi_i(x)$ is a measure of the proximity of $m_i(x)$. This property of $\psi_i(x)$ can be given a geometrical interpretation. Corresponding to the distribution $m_i(x)$ there is an exact stream surface on which the stream function $\psi_i(x, y) = 0$. Let Δn_i be the distance from a point (x, y) on the given body to this exact stream surface, measured along the normal to the given body, positive outwards. Let u_s be the tangential component of the flow along the body. Then we have

$$u_s = -\frac{1}{y} \frac{\partial \psi_i(x,y)}{\partial n} = -\frac{1}{y} \frac{\Delta \psi_i(x,y)}{\Delta n_i}$$

But $\Delta\psi = -\psi_r(x)$, since $\psi_i(x, y) = 0$ on the exact stream surface. Hence

$$\Delta n_i = \frac{\psi_i(x)}{y u_s} \qquad [60]$$

Since, for an elongated body, $u_s = 1$, except in the neighborhood of the stagnation points, it is seen that $\psi_i(x)$ enables a rapid estimate to be made of the variation from the desired profile of the exact stream surface corresponding to $m_i(x)$. This is an important property because it can be used to monitor the successive approximations. Thus, the sequence $\psi_i(x)$ can be terminated when Δn_i becomes uniformly less than some specified tolerance; or, since there is no assurance that the infinite sequence $\psi_i(x)$ converges, the sequence can conceivably give useful results even without convergence if it is continued as long as Δn_i decreases on the average, and is terminated when the error begins to increase and grows to an unacceptable magnitude at some point along the body. The strong similarity between these remarks and the discussion following Theorem 2 of Reference 17 should be noted.

There is also a strong similarity between the iteration formula of Reference 17 whose convergence was thoroughly discussed, and the present equation [56]. An essential difference between the iteration formulas is that the former employs the iterated kernel of the integral equation, the latter does not, so that the convergence theorems of Reference 17 are not applicable. Nevertheless, it is proposed to use the form in [56] (or the equivalent iteration formula [58]), for the following reasons:

a. The labor of numerical calculations would be greatly increased by iterating the kernel, and even then only convergence in the mean would be guaranteed (Theorem 4 of Reference 17).

b. The physical derivation of Equation [56] indicates that at least the first few approximations should be successively improving.

c. The successive approximations are monitored so that the sequence can be stopped when the error is as small as desired or, in the case of initial convergence and then divergence, when the errors begin to grow.

VELOCITY AND PRESSURE DISTRIBUTION ON THE SURFACE

When an approximate doublet distribution $m_i(x)$ has been obtained, the velocity components u, v can be computed from the corresponding stream function [18]

$$\psi_i(x,y) = y^2 \left[\int_a^b \frac{m_i(t)}{r^3}\, dt - \frac{1}{2} \right] \qquad [61]$$

from which, in accordance with Equations [5] and [6],

$$u = 1 + \int_a^b \left(\frac{3y^2}{r^5} - \frac{2}{r^3} \right) m_i(t)dt \qquad [62]$$

and

$$v = 3y \int_a^b \frac{t-x}{r^5}\, m_i(t)dt \qquad [63]$$

On the given surface we have, from [61],

$$\int_a^b \frac{m_1(t)}{r^3}\, dt = \frac{1}{2} + \frac{\psi_1(x)}{y^2(x)} \qquad [64]$$

where now

$$r^2 = (x-t)^2 + f(x) \qquad [65]$$

Differentiating [64] with respect to x gives

$$3\int_a^b \frac{t-x-yy'}{r^5}\, m_1(t)dt = \frac{\psi_1'(x)}{y^2(x)} - \frac{2\psi_1(x)y'(x)}{y^3(x)} \qquad [66]$$

Hence, from [62] and [64] we obtain

$$u = 3y^2 \int_a^b \frac{m_1(t)}{r^5}\, dt - \frac{2\psi_1(x)}{f(x)} \qquad [67]$$

and, from [63], [66], and [67],

$$v = uy'(x) + \frac{\psi_1'(x)}{y(x)} \qquad [68]$$

where the primes denote differentiation with respect to x. Equations [67] and [68] are the desired expressions for u and v. If the approximation $m_1(t)$ is very good, the contributions of the error function $\psi_1(x)$ should be very small. It is interesting to note that the form of Equation [68] shows the deviation of the resultant velocity from the tangent to the given body.

Bernoulli's equation for steady, incompressible, irrotational flow with zero pressure at infinity now gives the pressure distribution p,

$$\frac{p}{q} = 1 - (u^2 + v^2) \qquad [69]$$

where q is the stagnation pressure.

NUMERICAL EVALUATION OF INTEGRALS

In order to perform the iterations in Equations [56] and [58] and to compute the velocity distribution it will frequently be necessary to evaluate integrals of the form

$$\int_a^b \frac{m(t)}{r^3}\, dt \quad \text{and} \quad \int_a^b \frac{m(t)}{r^5}\, dt$$

where

$$r^2 = (x-t)^2 + f(x)$$

Because in this form these integrals peak sharply in the neighborhood of $t = x$, especially when the body is elongated, they are consequently unsuited for numerical evaluation.

A more suitable form can be obtained by means of the following transformation. Let (x, y) be the coordinates of a point on the body, t the abscissa of a point on the axis, θ the angle between a line joining these two points and the x-axis; see Figure 1. Then

$$x - t = y(x) \cot \theta \qquad [70]$$

We may now transform the integrals so that θ becomes the variable of integration. Then

$$\int_a^b \frac{y^2}{r^3}\, m(t)dt = \int_\alpha^\beta m(t) \sin\theta \, d\theta \qquad [71]$$

and

$$\int_a^b \frac{y^4}{r^5}\, m(t) = \int_\alpha^\beta m(t) \sin^3\theta \, d\theta \qquad [72]$$

where

$$\alpha = \arctan \frac{y}{x-a}, \quad \beta = \arctan \frac{y}{x-b} \qquad [73]$$

An alternate procedure, which eliminates the peak without a transformation of variables, is the following. We have

$$\int_a^b \frac{y^2}{r^3}\, m(t)dt \equiv \int_a^b \frac{y^2}{r^3}\left[m(t)-m(x)\right] dt + m(x) \int_a^b \frac{y^2}{r^3}\, dt$$

and

$$\int_a^b \frac{y^4}{r^5}\, m(t)dt \equiv \int_a^b \frac{y^4}{r^5}\Big[m(t)-m(x)\Big]dt + m(x) \int_a^b \frac{y^4}{r^5}\, dt$$

Hence

$$\int_a^b \frac{y^2}{r^3}\, m(t)dt = \int_a^b \frac{y^2}{r^3}\Big[m(t)-m(x)\Big]dt + m(x)(\cos\alpha - \cos\beta) \qquad [71a]$$

$$\int_a^b \frac{y^4}{r^5}\, m(t)dt = \int_a^b \frac{y^4}{r^5}\Big[m(t)-m(x)\Big]dt$$

$$+ m(x)\Big[\cos\alpha - \cos\beta - \frac{1}{3}(\cos^3\alpha - \cos^3\beta)\Big] \qquad [72a]$$

Gauss' quadrature formula is a convenient and accurate method of evaluating these integrals. The formula may be expressed in the form

$$\int_{-1}^{1} F(\xi)d\xi = \sum_{i=1}^{n} R_{ni}F(\xi_{ni}) \qquad [74]$$

where the ξ_i are the zeros of Legendre's polynomial of degree n and the R_{ni} are weighting factors. These have been tabulated[26] for values of n from 1 to 16. These numbers have the properties

$$R_{ni} = R_{n,n-i+1} \quad \text{and} \quad \xi_{ni} = -\xi_{n,n-i+1} \qquad [75]$$

The value of the integral given by Formula [74] is the same as could be obtained by fitting a polynomial of degree 2n-1 to F(x). The values of R_{ni} and ξ_{ni} are tabulated in Table 1 for n = 7, 11, and 16.

When the limits of integration are α and β , as in Equations [71] and [72], Gauss' formula becomes

$$\int_\alpha^\beta F(\theta)d\theta = \frac{\beta-\alpha}{2}\sum_{i=1}^{n} R_{ni}F(\theta_i) \qquad [76]$$

where

$$\theta_i = \frac{\beta-\alpha}{2}\,\xi_{ni} + \frac{\alpha+\beta}{2} \qquad [77]$$

TABLE 1

ABSCISSAE AND WEIGHTING FACTORS FOR GAUSS' QUADRATURE FORMULA

i	n = 7		n = 11		n = 16	
	ξ_i	R_i	ξ_i	R_i	ξ_i	R_i
1	-0.949108	0.129485	-0.978229	0.055669	-0.989401	0.027152
2	.741531	.279705	.887063	.125580	.944575	.062254
3	-0.405845	.381830	.730152	.186290	.865631	.095159
4	0	0.417959	.519096	.233194	.755404	.124629
5	$\xi_i = -\xi_{n-i+1}$	$R_i = R_{n-i+1}$	-0.269543	.262805	.617876	.149596
6			0	0.272925	.458017	.169157
7			$\xi_i = -\xi_{n-i+1}$	$R_i = R_{n-i+1}$	.281604	.182603
8					-0.095013	0.189451
					$\xi_i = -\xi_{n-i+1}$	$R_i = R_{n-i+1}$

ILLUSTRATIVE EXAMPLE

The foregoing considerations will now be applied to a body of revolution whose meridian profile is given, for $-1 \leqq x \leqq 1$, by

$$y^2 = f(x) = 0.04(1 - x^4) \qquad [78]$$

The body is symmetric fore and aft, has a length-diameter ratio $\lambda = 5$, and a prismatic coefficient

$$\phi = \int_0^1 (1 - x^4)dx = 0.80 \qquad [79]$$

By applying to [78] the transformation

$$x = 2\xi - 1, \quad y = 2\eta \qquad [80]$$

We obtain the equation for the geometrically similar body of unit length, for $0 \leqq \xi \leqq 1$,

$$\eta^2 = 0.08(\xi - 3\xi^2 + 4\xi^3 - 2\xi^4) = 0.08\xi(1 - \xi)(2\xi^2 - 2\xi + 1) \qquad [81]$$

We will also need the slope of the profile which, from [78], is

$$y' = \frac{f'(x)}{2y} = -\frac{0.4x^3}{(1-x^4)^{1/2}} \qquad [82]$$

The profile and f(x) are graphed in Figure 2.

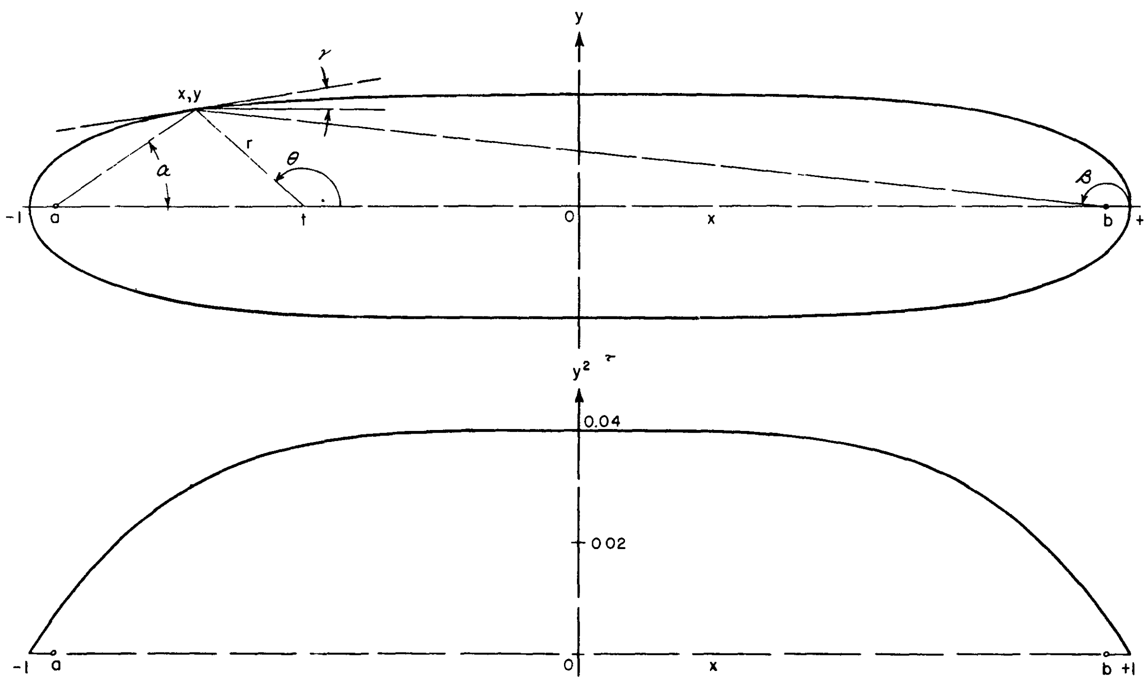

Figure 2 - Graphs of y(x) and $y^2(x)$ for $y^2(x) = 0.04(1 - x^4)$

First let us find the end points of the distribution. We have, from [81], $a_1 = 0.08$, $a_2 = -0.24$, $a_3 = 0.32$. The approximate formula [40] then gives $\alpha = 3.68$ or 3.84, whence $a = \frac{a_1}{\alpha} = 0.0217$ or 0.0208. An examination of the complete polynomial [37] with the aid of Table 10 shows that its zeros occur at $\alpha = 3.65$, 3.85, 12.1. In the application of Table 10 to determine these roots the regions of possible zeros should be determined by inspection, the values of the polynomial in these regions calculated from Equation [37] and Table 10, and then graphed to obtain the zeros. It is seen that in the present case the approximate formula [40] would have been sufficiently accurate for the determination of the roots near $\alpha = 4$. The solution of the complete polynomial equation will always yield an additional large root, corresponding to the large root of Equation [131] of the Appendix; in general, however, this root should be rejected since as will be shown, the initial doublet distribution corresponding to it is less simple than for the roots near $\alpha = 4$.

The initial behavior of the distributions corresponding to each of the three roots, as determined from Equations [33] through [36], and [39], is shown in Table 2. It is seen from the table that the distribution for α = 12.1 begins with practically a zero value for m(a), with a small negative slope and with upward curvature. Since the distribution curve cannot continue very far with upward curvature, there must be an inflection point nearby. In contrast, the distribution corresponding to the other two roots begin with positive slopes and downward curvatures and hence must be considered simpler. Furthermore, the distribution for the first root is considered simpler than for the second since the distribution curves are practically identical except that, for the second root, the curve is extended a distance $\Delta a = 0.0011$, in the course of which m(a) changes from a positive to almost a numerically equal negative value. If we take the point of view that the positive and negative values of this extension counterbalance each other, the curve without the extension, corresponding to the first root, must be considered the simplest.

TABLE 2

Characteristics of Initial Distribution

α	a	m(a)	C_1	C_2
3.65	0.0219	+0.0000216	+0.0375	-0.103
3.85	0.0208	-0.0000191	+0.0376	-0.109
12.1	0.0066	+0.0000008	-0.0064	+0.35

Hence, for the purpose of obtaining a first approximation, we will assume α = 3.65 and, correspondingly, a = 0.022, m(a) = 0.000022. Often, as in this case, the labor of obtaining a and m(a) can be considerably reduced by using the less exact equations [40] through [44] instead of [37] through [39]. Since, as will be seen, the iteration formulas rapidly improve upon the first approximation, great effort should not be expended to determine an initial value for m(a).

The values a = 0.022 and m(a) = 0.000022 have been derived for the profile in the ξ, η-plane. The corresponding values in the x, y-plane are a = -0.956 and m_a = 0.000088. By symmetry we also have b = -a, $m_b = m_a$.

A first approximation can now be obtained from [47], [48], [49], and [50]. Since λ = 5.0, we have k_1 = 0.059. Also, from [78]: f_a = 0.00659, $\int_{-1}^{1} y^2 dx = 0.0640$, $\int_a^b y^2 dx = 0.0637$. Hence from [50], C = 0.328. Then, from [48],

$e_0 = m_a - Cf_a = -0.00207$; from [49], $e_1 = 0$. Finally we obtain from [47]

$$m_1(x) = 0.328y^2 - 0.00207 \qquad [83]$$

We can now apply Equation [51] and the iteration formula [58] to obtain the sequence of functions $\psi_i(x)$. Let us suppose that it is desired to obtain a distribution $m_i(x)$ whose exact stream surface deviates from the given surface by less than one percent of the maximum radius, i.e., $\Delta n < 0.002$. Then, by [60], the sequence $\psi_i(x)$ should be continued until $\psi_i(x) < 0.002\sqrt{f(x)}$ for $a \leqq x \leqq b$, unless the error, as represented by $\psi(x)$, begins to grow before the desired degree of approximation is attained. In the latter case the best approximation attainable would fall short of the specified accuracy.

The integrations in [50] and [51] may be carried out in the form [71] in terms of θ defined in [70]. For a fixed (x, y) on the given profile, α and β are first computed from [73]. Then, to apply Gauss' quadrature formula [76], the interval is subdivided at the points θ_j given by [77] and the integrands evaluated at these points. The corresponding values of t at which $m_1(t)$ in [51] or $\psi_{i-1}(t)$ in [58] is to be read are, from [70],

$$t_j = x - y \cot \theta_j \qquad [70a]$$

Since the values t_j and $\sin \theta_j$ are used repeatedly in the successive iterations at a given (x, y), these should be stored in a form convenient for application.

The calculations for obtaining the integration limits α and β for several values of x are given in Table 3. The values of θ_j from [77], and the corresponding values of $R_j \sin \theta_j$ for application of the Gauss 11 ordinate formula, and the values of t_j from [70a] for each x are entered as the first three columns in Tables 5a through 5h, in which are given the calculations for $\psi_i(x)$.

In order to compute $\psi_1(x)$, $m_1(t)$ is computed from [83], then $m_1 R \sin \theta$ is obtained. These are tabulated in Table 5. Gauss' formula then gives $\int m_1 \sin\theta \, d\theta$. $\psi_1(x)$ is then obtained from [51]; its graph is given in Figure 3. It is important to note that $m_1(t)$ is obtained by calculation, rather than graphically, in this operation. This procedure is recommended since it gives greater accuracy in a critical step. In the subsequent operations on the ψ's considerably less percentage accuracy is required, since the ψ's are of the nature of first differences between the m's, so that graphical operations are sufficiently accurate.

TABLE 3

Calculations for Integration Limits α, β

x	x - a	x - b	y	tan α	tan β	α	β	$\frac{1}{2}(\beta-\alpha)$	$\frac{1}{2}(\alpha+\beta)$
0	0.956	-0.956	0.20000	0.20921	-0.20921	0.2062	2.9354	1.3646	1.5708
-0.20	0.756	-1.156	0.19984	0.26434	-0.17287	0.2584	2.9704	1.3560	1.6144
-0.40	0.556	-1.356	0.19742	0.35507	-0.14559	0.3412	2.9970	1.3279	1.6691
-0.60	0.356	-1.556	0.18659	0.52413	-0.11992	0.4828	3.0222	1.2697	1.7525
-0.70	0.256	-1.656	0.17435	0.68105	-0.10528	0.5979	3.0367	1.2194	1.8173
-0.80	0.156	-1.756	0.15368	0.98513	-0.08752	0.7779	3.0543	1.1382	1.9161
-0.90	0.056	-1.856	0.11729	2.09446	-0.06320	1.1254	3.0785	0.9766	2.1020
-0.956	0	-1.912	0.08117	∞	-0.04245	1.5708	3.0992	0.7642	2.3350

TABLE 4

Calculations for Pressure Distribution p/q

x	y^2	y	y'	ψ'_4	u	uy'	ψ'_4/y	v	$u^2 + v^2$	p/q
0	0.040000	0.20000	0.0000	0.000000	1.02640	0.00000	0.00000	0.00000	1.0535	-0.0535
-0.20	0.039936	0.19984	0.0032	-0.000082	1.03441	0.00331	-0.00041	0.00290	1.0700	-0.0700
-0.40	0.038976	0.19742	0.0259	0.000060	1.05618	0.02739	0.00030	0.02769	1.1163	-0.1163
-0.60	0.034816	0.18659	0.0926	0.000306	1.07907	0.09993	0.00164	0.10157	1.1747	-0.1747
-0.70	0.030396	0.17435	0.1574	0.000317	1.07866	0.16978	0.00182	0.17160	1.1930	-0.1930
-0.80	0.023616	0.15368	0.2665	-0.000129	1.04917	0.27960	-0.00084	0.27876	1.1785	-0.1785
-0.90	0.013756	0.11729	0.4972		0.92425	0.45954		0.4489*	1.0557	-0.0557
-0.956	0.006588	0.08117	0.8611		0.68161	0.58693		0.5768*	0.7973	0.2027

*v obtained from equation $v = \frac{3}{y}\int_{\alpha}^{\beta} m(t)\, \sin^2\theta \cos\theta\, d\theta$.

TABLE 5

Calculations for $\psi_1(x)$ and $u(x)$

(a) $x = 0$: $\frac{1}{2}(\beta - \alpha) = 1.3646$, $y^2 = 0.0400$

θ	t	$R\sin\theta$	$m_1(t)$	$m_1(t)R\sin\theta$	$\psi_1(t)$	$\psi_1(t)R\sin\theta$	$\psi_2(t)$	$\psi_2(t)R\sin\theta$	$\psi_3(t)$	$\psi_3(t)R\sin\theta$	$m_4(t)$	$m_4R\sin^3\theta$
0.2359	-0.8320	0.01301	0.004763	0.0000620	-0.001307	-0.0000170	-0.000428	-0.0000056	-0.000151	-0.0000020	0.005706	0.0000041
.3603	.5309	.04428	.010008	.0004432	-0.000370	-0.0000164	-0.000107	-0.0000047	-0.000019	-0.0000008	.010256	.0000565
.5744	.3090	.10121	.010930	.0011062	+0.000652	+0.0000660	+0.000188	+0.0000190	+0.000063	+0.0000064	.010478	.0003130
.8624	.1713	.17708	.011039	.0019548	.001058	.0001874	.000281	.0000498	.000075	.0000133	.010332	.0010552
1.2030	-0.0771	.24522	.011050	.0027097	.001198	.0002938	.000307	.0000753	.000075	.0000184	.010260	.0021907
1.5708	+0.0000	.27293	.011050	.0030159	.001244	.0003395	.000311	.0000849	.000071	.0000194	.010237	.0027940
1.9386	.0771	.24522	.011050	.0027097	.001198	.0002938	.000307	.0000753	.000075	.0000184	.010260	.0021907
2.2792	.1713	.17708	.011039	.0019548	.001058	.0001874	.000281	.0000498	.000075	.0000133	.010332	.0010552
2.5672	.3090	.10121	.010930	.0011062	+0.000652	+0.0000660	+0.000188	+0.0000190	+0.000063	+0.0000064	.010478	.0003130
2.7813	.5309	.04428	.010008	.0004432	-0.000370	-0.0000164	-0.000107	-0.0000047	-0.000019	-0.0000008	.010256	.0000565
2.9057	+0.8320	0.01301	0.004763	0.0000620	-0.001307	-0.0000170	-0.000428	-0.0000056	-0.000151	-0.0000020	0.005706	0.0000041
			$\Sigma m_1 R\sin\theta = 0.0155677$		$\Sigma\psi_1 R\sin\theta = 0.0013671$		$\Sigma\psi_2 R\sin\theta = 0.0003525$		$\Sigma\psi_3 R\sin\theta = 0.0000900$		$\Sigma R m \sin^3\theta = 0.0100330$	
			$\int m_1 \sin\theta = 0.0212437$		$\int\psi_1 \sin\theta = 0.001866$		$\int\psi_2 \sin\theta d\theta = 0.0004810$		$\int\psi_3 \sin\theta d\theta = 0.0001228$		$\int m \sin^3\theta = 0.013691$	
			$\psi_1 = 0.001244$		$\psi_2 = 0.000311$		$\psi_3 = 0.000071$		$\psi_4 = 0.000010$		$u = 1.0264$	

(b) $x = -0.20$: $\frac{1}{2}(\beta - \alpha) = 1.3560$, $y^2 = 0.039936$

θ	t	$R\sin\theta$	$m_1(t)$	$m_1(t)R\sin\theta$	$\psi_1(t)$	$\psi_1(t)R\sin\theta$	$\psi_2(t)$	$\psi_2(t)R\sin\theta$	$\psi_3(t)$	$\psi_3(t)R\sin\theta$	$m_4(t)$	$m_4R\sin^3\theta$
0.2879	-0.8748	0.01580	0.003366	0.0000532	-0.001189	-0.0000188	-0.000361	-0.0000057	-0.000112	-0.0000018	0.004197	0.0000053
.4115	.6579	.05023	.008592	.0004316	.000997	.0000501	.000329	.0000165	-0.000109	-0.0000055	.009310	.0000748
.6243	.4774	.10889	.010369	.0011291	-0.000098	-0.0000107	-0.000018	-0.0000020	+0.000013	+0.0000014	.010421	.0003877
0.9105	.3552	.18417	.010841	.0019966	+0.000469	+0.0000864	+0.000141	+0.0000260	.000055	.0000101	.010508	.0012072
1.2489	.2666	.24929	.010984	.0027382	.000799	.0001992	.000221	.0000551	.000070	.0000175	.010439	.0023418
1.6144	1913	.27266	.011032	.0030080	.001017	.0002773	.000271	.0000739	.000074	.0000202	.010351	.0028166
1.9799	.1133	.24112	.011048	.0026639	.001152	.0002778	.000300	.0000723	.000076	.0000183	.010284	.0020874
2.3183	-0.0148	.17102	.011050	.0018898	.001240	.0002121	.000310	.0000530	.000072	.0000123	.010239	.0009419
2.6045	+0.1356	.09531	.011046	.0010528	.001121	.0001068	.000296	.0000282	.000076	.0000072	.010299	.0002569
2.8173	.3945	.04001	.010732	.0004294	+0.000300	+0.0000120	+0.000101	+0.0000040	+0.000048	+0.0000019	.010507	.0000427
2.9409	+0.7823	0.01110	0.006136	0.0000681	-0.001336	-0.0000148	-0.000445	-0.0000049	-0.000159	-0.0000018	0.007106	0.0000031
			$\Sigma m_1 R\sin\theta = 0.0154607$		$\Sigma\psi_1 R\sin\theta = 0.0010772$		$\Sigma\psi_2 R\sin\theta = 0.0002834$		$\Sigma\psi_3 R\sin\theta = 0.0000798$		$\Sigma R m_4 \sin^3\theta = 0.0101654$	
			$\int m_1 \sin\theta = 0.0209647$		$\int\psi_1 \sin\theta d\theta = 0.001460$		$\int\psi_2 \sin\theta d\theta = 0.0003843$		$\int\psi_3 \sin\theta d\theta = 0.0001082$		$\int m_4 \sin^3\theta = 0.013784$	
			$\psi_1 = 0.000997$		$\psi_2 = 0.000267$		$\psi_3 = 0.000075$		$\psi_4 = 0.000021$		$u = 1.0344$	

TABLE 5 (Continued)

(c) $x = -0.40$: $\frac{1}{2}(\beta - \alpha) = 1.3279$, $y^2 = 0.038976$

θ	t	R sin θ	$m_1(t)$	$m_1(t)R\sin\theta$	$\psi_1(t)$	$\psi_1(t)R\sin\theta$	$\psi_2(t)$	$\psi_2(t)R\sin\theta$	$\psi_3(t)$	$\psi_3(t)R\sin\theta$	$m_4(t)$	$m_4 R\sin^3\theta$
0.3701	-0.9089	0.02014	0.002096	0.0000422	-0.000982	-0.0000198	-0.000272	-0.0000055	-0.000060	-0.0000012	0.002753	0.0000072
.4912	.7691	.05924	.006460	.0003827	.001328	.0000787	.000440	.0000261	.000158	.0000094	.007423	.0000978
.6995	.6346	.11993	.008922	.0010700	.000895	.0001073	.000292	.0000350	.000092	.0000110	.009562	.0004754
0.9798	.5325	.19364	.009995	.0019354	-0.000376	-0.0000728	-0.000109	-0.0000211	-0.000020	-0.0000039	.010248	.0013684
1.3112	.4524	.25400	.010500	.0026670	+0.000023	+0.0000058	+0.000021	+0.0000053	+0.000027	+0.0000069	.010464	.0024827
1.6691	.3805	.27162	.010775	.0029267	.000360	.0000978	.000115	.0000312	.000050	.0000136	.010512	.0028278
2.0270	.3031	.23592	.010939	.0025807	.000673	.0001588	.000191	.0000451	.000065	.0000153	.010474	.0019912
2.3584	.2017	.16454	.011028	.0018145	.000991	.0001631	.000265	.0000436	.000075	.0000123	.010362	.0008488
2.6387	-0.0411	.08979	.011050	.0009922	.001228	.0001103	.000310	.0000278	.000073	.0000066	.010244	.0002137
2.8470	+0.2506	.03647	.010998	.0004011	+0.000851	+0.0000310	+0.000233	+0.0000085	+0.000071	+0.0000026	.010420	.0000320
2.9681	+0.7264	0.00961	0.007397	0.0000711	-0.001259	-0.0000121	-0.000412	-0.0000040	-0.000147	-0.0000014	0.008306	0.0000024
			$\Sigma m_1 R\sin\theta = 0.0148836$		$\Sigma \psi_1 R\sin\theta = 0.0002761$		$\Sigma \psi_2 R\sin\theta = 0.0000698$		$\Sigma \psi_3 R\sin\theta = 0.0000304$		$\Sigma R m_4 \sin^3\theta = 0.0103474$	
			$\int m_1 \sin\theta = 0.0197639$		$\int \psi_1 \sin\theta = 0.0003666$		$\int \psi_2 \sin\theta = 0.0000926$		$\int \psi_3 \sin\theta = 0.0000404$		$\int m_4 \sin^3\theta = 0.013740$	
			$\psi_1 = 0.000276$		$\psi_2 = 0.000093$		$\psi_3 = 0.000047$		$\psi_4 = 0.000027$		$u = 1.0562$	

(d) $x = -0.60$: $\frac{1}{2}(\beta - \alpha) = 1.2697$, $y^2 = 0.034816$

θ	t	R sin θ	$m_1(t)$	$m_1(t)R\sin\theta$	$\psi_1(t)$	$\psi_1(t)R\sin\theta$	$\psi_2(t)$	$\psi_2(t)R\sin\theta$	$\psi_3(t)$	$\psi_3(t)R\sin\theta$	$m_4(t)$	$m_4 R\sin^3\theta$
0.5104	-0.9333	0.02719	0.001095	0.0000298	-0.000798	-0.0000217	-0.000189	-0.0000051	-0.000011	-0.0000003	0.001594	0.0000104
.6262	.8580	.07360	.003940	.0002900	.001248	.0000919	.000392	.0000289	.000132	.0000097	.004826	.0001221
.8254	.7722	.13689	.006385	.0008740	.001330	.0001821	.000441	.0000604	.000159	.0000218	.007350	.0005432
1.0934	.6965	.20712	.007962	.0016491	.001151	.0002384	.000381	.0000789	.000131	.0000271	.008794	.0014369
1.4103	.6302	.25941	.008981	.0023298	.000873	.0002265	.000287	.0000745	.000090	.0000233	.009606	.0024280
1.7525	.5657	.26843	.009706	.0026054	.000551	.0001479	.000168	.0000451	-0.000042	-0.0000113	.010087	.0026190
2.0947	.4922	.22756	.010280	.0023393	-0.000170	-0.0000387	-0.000041	-0.0000093	+0.000005	+0.0000011	.010383	.0017715
2.4116	.3915	.15551	.010742	.0016705	+0.000311	+0.0000484	+0.000102	+0.0000159	.000049	.0000076	.010511	.0007271
2.6796	-0.2253	.08303	.011016	.0009147	.000928	.0000771	.000251	.0000208	.000072	.0000060	.010390	.0001714
2.8788	+0.0936	.03263	.011049	.0003605	+0.001179	+0.0000385	+0.000302	+0.0000099	+0.000075	+0.0000024	.010271	.0000226
2.9946	+0.6602	0.00816	0.008558	0.0000698	-0.001005	-0.0000082	-0.000332	-0.0000027	-0.000110	-0.0000009	0.009282	0.0000016
			$\Sigma m_1 R\sin\theta = 0.0131329$		$\Sigma \psi_1 R\sin\theta = -0.0007914$		$\Sigma \psi_2 R\sin\theta = -0.0002583$		$\Sigma \psi_3 R\sin\theta = -0.0000773$		$\Sigma R m_4 \sin^3\theta = 0.0098538$	
			$\int m_1 \sin\theta = 0.0166748$		$\int \psi_1 \sin\theta = -0.0010048$		$\int \psi_2 \sin\theta = -0.0003280$		$\int \psi_3 \sin\theta = -0.0000981$		$\int m_4 \sin^3\theta = 0.012511$	
			$\psi_1 = 0.000733$		$\psi_2 = -0.000231$		$\psi_3 = -0.000067$		$\psi_4 = -0.000018$		$u = 1.0791$	

TABLE 5 (Continued)

(e) $x = -0.70$: $\frac{1}{2}(\beta - \alpha) = 1.2194$, $y^2 = 0.030396$

θ	t	R sin θ	$m_1(t)$	$m_1(t)R\sin\theta$	$\psi_1(t)$	$\psi_1(t)R\sin\theta$	$\psi_2(t)$	$\psi_2(t)R\sin\theta$	$\psi_3(t)$	$\psi_3(t)R\sin\theta$	$m_4(t)$	$m_4 R\sin^3\theta$
0.6244	-0.9420	0.03254	0.000719	0.0000234	-0.000711	-0.0000231	-0.000154	-0.0000050	+0.000008	+0.0000003	0.001148	0.0000127
.7356	.8926	.08426	.002722	.0002294	.001093	.0000921	.000318	.0000268	-0.000086	-0.0000072	.003471	.0001317
0.9270	.8308	.14899	.004799	.0007150	.001309	.0001950	.000428	.0000638	.000152	.0000226	.005744	.0005475
1.1843	.7710	.21598	.006414	.0013853	.001329	.0002870	.000441	.0000952	.000159	.0000343	.007379	.0013672
1.4886	.7144	.26191	.007633	.0019992	.001219	.0003193	.000400	.0001048	.000140	.0000367	.008513	.0022144
1.8173	.6561	.26469	.008619	.0022814	.000989	.0002618	.000327	.0000866	.000108	.0000286	.009331	.0023229
2.1460	.5870	.22052	.009492	.0020932	.000663	.0001462	.000206	.0000454	-0.000058	-0.0000128	.009956	.0015458
2.4503	.4893	.14866	.010298	.0015309	-0.000155	-0.0000230	-0.000035	-0.0000052	+0.000008	+0.0000012	.010389	.0006277
2.7076	-0.3238	.07833	.010906	.0008543	+0.000559	+0.0000438	+0.000172	+0.0000135	.000062	.0000049	.010509	.0001455
2.8990	+0.0045	.03016	.011050	.0003333	+0.001242	+0.0000375	+0.000310	+0.0000093	+0.000073	+0.0000022	.010237	.0000178
3.0102	+0.6192	0.00729	0.009121	0.0000665	-0.000821	-0.0000060	-0.000266	-0.0000019	-0.000080	-0.0000006	0.009705	0.0000012
			$\Sigma m_1 R\sin\theta = 0.0115119$		$\Sigma\psi_1 R\sin\theta = -0.0012722$		$\Sigma\psi_2 R\sin\theta = -0.0004119$		$\Sigma\psi_3 R\sin\theta = -0.0001342$		$\Sigma R m_4 \sin^3\theta = 0.0089344$	
			$\int m_1 \sin\theta = 0.0140376$		$\int\psi_1 \sin\theta = -0.0015513$		$\int\psi_2 \sin\theta = -0.0005023$		$\int\psi_3 \sin\theta = -0.0001636$		$\int m_4 \sin^3\theta = 0.010895$	
			$\psi_1 = -0.001160$		$\psi_2 = -0.000384$		$\psi_3 = -0.000133$		$\psi_4 = -0.000051$		$u = 1.0787$	

(f) $x = -0.80$: $\frac{1}{2}(\beta - \alpha) = 1.1382$, $y^2 = 0.023616$

θ	t	R sin θ	$m_1(t)$	$m_1(t)R\sin\theta$	$\psi_1(t)$	$\psi_1(t)R\sin\theta$	$\psi_2(t)$	$\psi_2(t)R\sin\theta$	$\psi_3(t)$	$\psi_3(t)R\sin\theta$	$m_4(t)$	$m_4 R\sin^3\theta$
0.8027	-0.9485	0.04004	0.000431	0.0000173	-0.000655	-0.0000262	-0.000127	-0.0000051	+0.000021	+0.0000008	0.000812	0.0000168
0.9064	.9204	.09887	.001634	.0001616	.000896	.0000886	.000233	.0000230	-0.000037	-0.0000037	.002217	.0001359
1.0850	.8811	.16474	.003143	.0005178	.001158	.0001908	.000347	.0000572	.000102	.0000168	.003947	.0005084
1.3253	.8385	.22619	.004565	.0010326	.001297	.0002934	.000421	.0000952	.000149	.0000337	.005499	.0011704
1.6093	.7941	.26262	.005833	.0015319	.001339	.0003516	.000446	.0001171	.000158	.0000415	.006805	.0017847
1.9161	.7447	.25683	.007015	.0018017	.001299	.0003336	.000427	.0001097	.000151	.0000388	.007954	.0018090
2.2229	.6827	.20887	.008200	.0017127	.001099	.0002295	.000363	.0000760	.000123	.0000257	.008993	.0011865
2.5069	.5913	.13826	.009446	.0013060	-0.000690	-0.0000954	-0.000215	-0.0000297	-0.000061	-0.0000084	.009929	.0004825
2.7472	.4308	.07159	.010598	.0007587	+0.000128	+0.0000092	+0.000051	+0.0000037	+0.000037	+0.0000026	.010490	.0001108
2.9258	-0.0989	.02689	.011049	.0002971	+0.001172	+0.0000315	+0.000302	+0.0000081	+0.000076	+0.0000020	.010274	.0000127
3.0295	+0.5652	0.00623	0.009711	0.0000605	-0.000548	-0.0000034	-0.000169	-0.0000011	-0.000041	-0.0000003	0.010090	0.0000008
			$\Sigma m_1 R\sin\theta = 0.0091979$		$\Sigma\psi_1 R\sin\theta = -0.0015718$		$\Sigma\psi_2 R\sin\theta = -0.0005023$		$\Sigma\psi_3 R\sin\theta = -0.0001635$		$\Sigma R m_4 \sin^3\theta = -.0072185$	
			$\int m_1 \sin\theta = 0.0104690$		$\int\psi_1 \sin\theta = -0.0017890$		$\int\psi_2 \sin\theta = -0.0005717$		$\int\psi_3 \sin\theta = -0.0001861$		$\int m_4 \sin^3\theta = +.008216$	
			$\psi_1 = -0.001339$		$\psi_2 = -0.000444$		$\psi_3 = -0.000158$		$\psi_4 = -0.000065$		$u = 1.0492$	

TABLE 5 (Continued)

(g) x = -0.90: $\frac{1}{2}(\beta - \alpha)$ = 0.9766, y^2 = 0.013756

θ	t	R sin θ	m_1(t)	m_1(t)Rsinθ	ψ_1(t)	ψ_1(t)Rsinθ	ψ_2(t)	ψ_2(t)Rsinθ	ψ_3(t)	ψ_3(t)Rsinθ	m_4(t)	m_4Rsin$^3\theta$
1.1467	-0.9530	0.05074	0.000228	0.0000116	-0.000600	-0.0000304	-0.000104	-0.0000053	+0.000035	+0.0000018	0.000563	0.0000237
1.2357	.9408	.11860	.000772	.0000916	.000720	.0000854	.000154	.0000183	+0.000006	+0.0000007	.001206	.0001276
1.3889	.9216	.18322	.001585	.0002904	.000888	.0001627	.000230	.0000421	-0.000034	-0.0000062	.002161	.0003830
1.5950	.8972	.23312	.002549	.0005942	.001060	.0002471	.000305	.0000711	.000080	.0000186	.003272	.0007623
1.8388	.8678	.25342	.003609	.0009146	.001215	.0003079	.000375	.0000950	.000121	.0000307	.004465	.0010522
2.1020	.8311	.23532	.004790	.0011272	.001307	.0003076	.000428	.0001007	.000151	.0000355	.005733	.0010030
2.3652	.7806	.18414	.006179	.0011378	.001335	.0002458	.000444	.0000818	.000159	.0000293	.007148	.0006462
2.6090	.7010	.11841	.007882	.0009333	.001168	.0001383	.000388	.0000459	.000134	.0000159	.008727	.0002665
2.8151	.5536	.05974	.009818	.0005865	-0.000486	-0.0000290	-0.000145	-0.0000087	-0.000034	-0.0000020	.010151	.0000624
2.9683	-0.2300	.02165	.011013	.0002384	+0.000915	+0.0000198	+0.000248	+0.0000054	+0.000072	+0.0000016	.010395	.0000067
3.0573	+0.4880	0.00469	0.010306	0.0000483	-0.000149	-0.0000007	-0.000033	-0.0000002	+0.000008	+0.0000000	0.010393	0.0000003
			Σm_1 Rsinθ=0.0059739		$\Sigma\psi_1$ Rsinθ=-0.0015351		$\Sigma\psi_2$Rsinθ=-0.0004637		$\Sigma\psi_3$Rsinθ=-0.0001341		ΣRm_4 sin$^3\theta$=0.0043339	
			$\int m_1$ sinθ=0.0058341		$\int\psi_1$ sinθ=-0.001499		$\int\psi_2$ sinθ=-0.0004528		$\int\psi_3$ sinθ=-0.0001309		$\int m_4$ sin$^3\theta$=0.004233	
			ψ_1=-0.001044		ψ_2 =-0.000299		ψ_3 =-0.000073		ψ_4 =-0.000008		u =0.9243	

(h) x = -0.956: $\frac{1}{2}(\beta - \alpha)$ = 0.7642, y^2 = 0.006588

θ	t	R sin θ	m_1(t)	m_1(t)Rsinθ	ψ_1(t)	ψ_1(t)Rsinθ	ψ_2(t)	ψ_2(t)Rsinθ	ψ_3(t)	ψ_3(t)Rsinθ	m_4(t)	m_4Rsin$^3\theta$
1.5872	-0.9547	0.05566	0.000151	0.0000084	-0.000581	-0.0000323	-0.000099	-0.0000055	+0.000039	+0.0000022	0.000472	0.0000263
1.6572	.9490	.12512	.000409	.0000512	.000652	.0000816	.000126	.0000158	.000022	.0000028	.000787	.0000977
1.7773	.9390	.18234	.00850	.0001550	.000745	.0001358	.000167	.0000305	+0.000000	+0.0000000	.001306	.0002282
1.9387	.9247	.21761	.001457	.0003171	.000866	.0001885	.000221	.0000481	-0.000029	-0.0000063	.002015	.0003820
2.1297	.9053	.22291	.002237	.0004986	.001005	.0002240	.000281	.0000626	.000064	.0000143	.002912	.0004670
2.3359	.8782	.19703	.003246	.0006396	.001173	.0002311	.000353	.0000696	.000107	.0000211	.004063	.0004173
2.5421	.8375	.14851	.004595	.0006824	.001300	.0001931	.000421	.0000625	.000150	.0000223	.005531	.0002623
2.7331	.7692	.09293	.006457	.0006000	.001328	.0001234	.000440	.0000409	.000159	.0000148	.007421	.0001096
2.8945	.6362	.04583	.008901	.0004079	-0.000889	-0.0000407	-0.000297	-0.0000136	-0.000094	-0.0000043	.009541	.0000265
3.0146	-0.3288	.01611	.010897	.0001756	+0.000609	+0.0000098	+0.000175	+0.0000028	+0.000061	+0.0000010	.010474	.0000028
3.0843	+0.4182	0.00328	0.010649	0.0000349	+0.000191	+0.0000006	+0.000070	+0.0000002	+0.000040	+0.0000001	0.010498	0.0000001
			Σm_1 Rsinθ=0.0035707		$\Sigma\psi_1$ Rsinθ=-0.0012401		$\Sigma\psi_2$ Rsinθ=-0.0003461		$\Sigma\psi_3$ Rsinθ=-0.0000770		ΣRm_4 sin$^3\theta$=0.0020198	
			$\int m_1$ sinθ=0.0027287		$\int\psi_1$ sinθ=-0.0009477		$\int\psi_2$ sinθ=-0.0002645		$\int\psi_3$ sinθ=-0.0000588		$\int m_4$ sin$^3\theta$=0.001544	
			ψ_1=-0.000565		ψ_2 =-0.000091		ψ_3 = 0.000041		ψ_4 = 0.000070		u =0.6816	

As a check on the accuracy of the integration, $\psi_1(0)$ was also evaluated by two other means, with the following results:

From Gauss 7-ordinate formula $\psi_1(0) = 0.001258$

From Gauss 11-ordinate formula $\psi_1(0) = 0.001243$

From exact integration $\psi_1(0) = 0.001243$

It is seen that the 7-ordinate formula introduces an error in the fifth decimal place.

The first step in the determination of $\psi_2(x)$ is to read the values of $\psi_1(t)$ from the graph, Figure 3. $\psi_1 R \sin\theta$ and $\int \psi_1 \sin\theta \, d\theta$ are then obtained. $\psi_2(x)$ is then given by [58] and graphed in Figure 3. Repeated

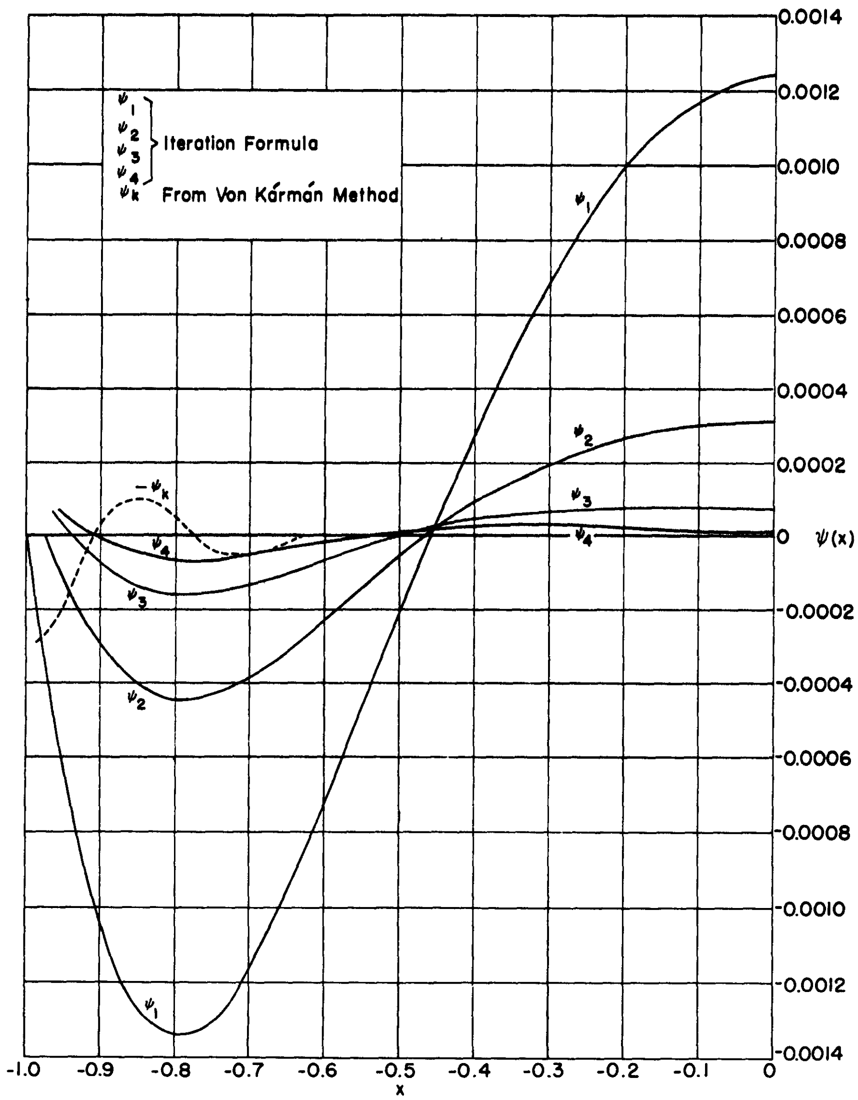

Figure 3 - Comparison of Error Functions $\psi(x)$ from Iteration Formula and von Kármán Method

application of this procedure gives $\psi_3(x)$ and $\psi_4(x)$ which are also graphed in Figure 3. The sequence is stopped at $\psi_4(x)$ since ψ_4 has increased appreciably over ψ_3 at x = -0.956.

Hence, from [59], we have the approximate distribution

$$m_4(x) = m_1(x) - \frac{1}{2}\left[\psi_1(x) + \psi_2(x) + \psi_3(x)\right] \qquad [84]$$

to which $\psi_4(x)$ is the corresponding error function. The distance Δn between the stream surface for $m_4(x)$ and the given profile is seen to be very small; the largest error, ψ_4 = -0.00007 at x = -0.956, gives a Δn of about one percent of the maximum ordinate. A graph of $m_4(x)$ is given in Figure 4. For the sake of comparison the curves for $m_1(x)$ and the original Munk approximation $\frac{1}{4}f(x)$ are also shown.

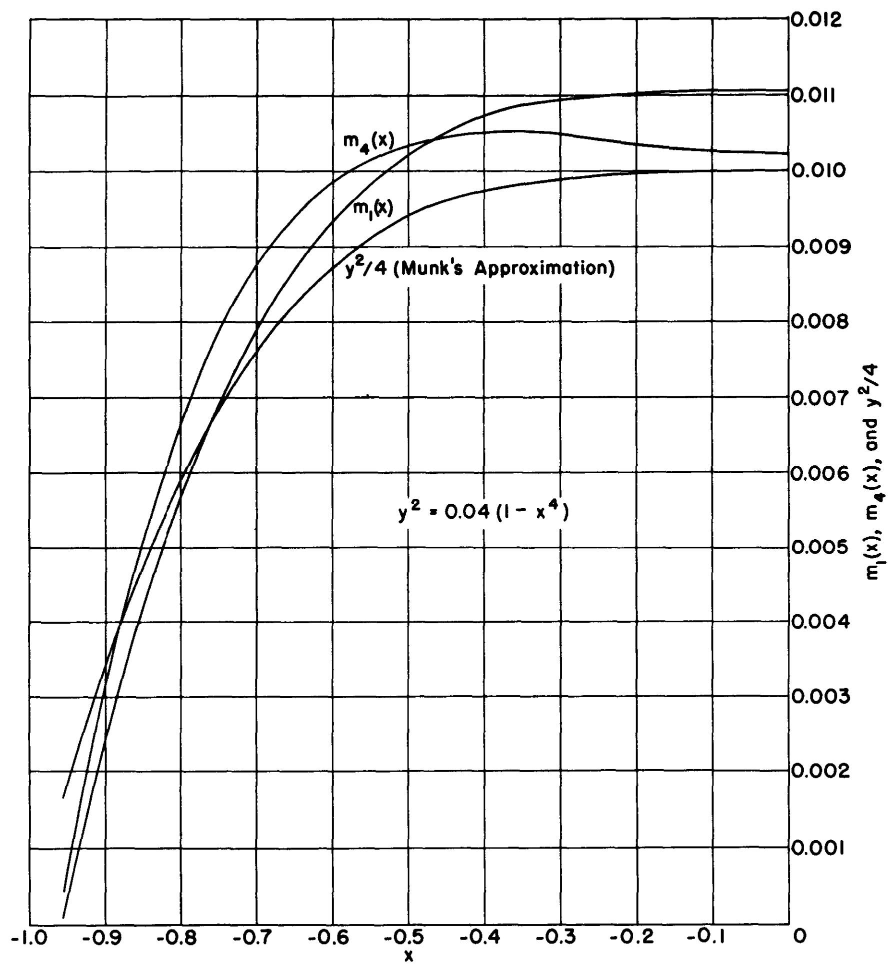

Figure 4 - Comparison of Doublet Distributions $m_1(x)$, $m_4(x)$, and Munk's Approximation $y^2/4$

Table 4 shows the calculations for obtaining the velocity components u, v from [67] and [68], in which the integrals have been evaluated in terms of the polar angle θ, according to Equations [71], [72], and [73]. Here also Gauss' 11-ordinate formula is used. The values of θ and t are again taken from Table 5; the values of $m_4(t)$ are given by [84], in which the ψ's are read from Figure 3 and $m_1(t)$ is given in Table 5.

The pressure distribution can now be obtained from [69]. Graphs of p/q are shown in Figure 5.

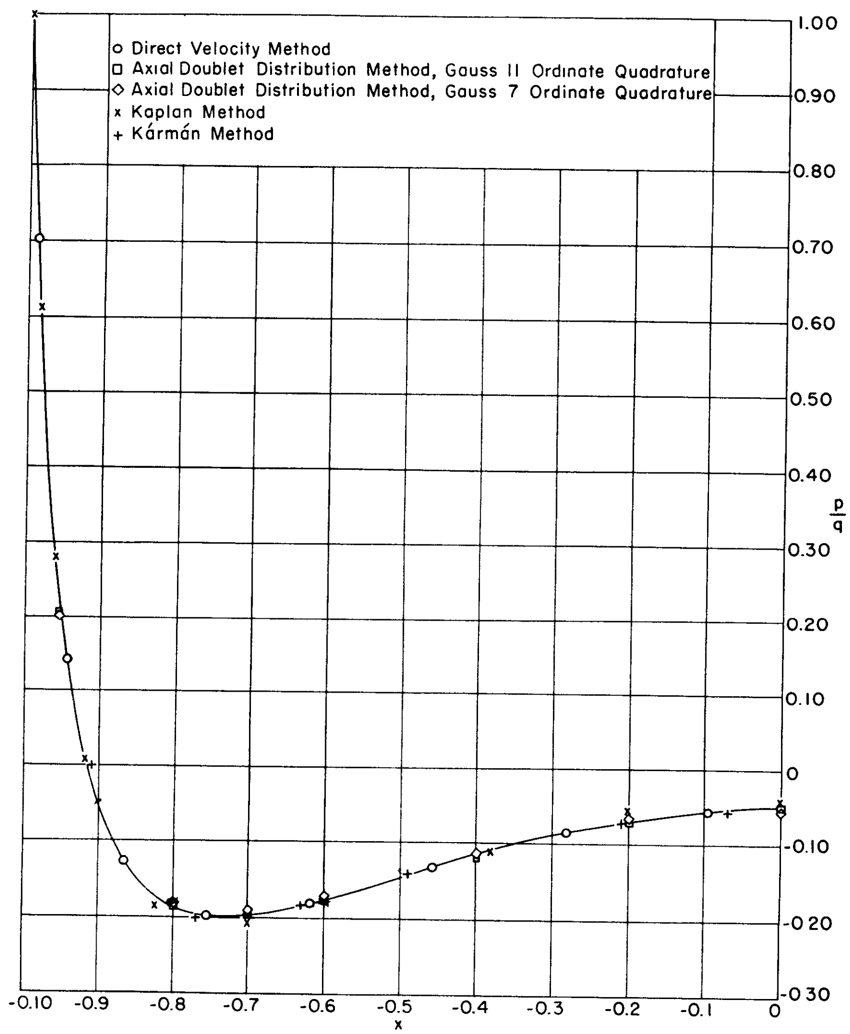

Figure 5 - Comparison of Values of p/q
Obtained by Various Methods

ERROR IN DETERMINATION OF p/q

Let $\Delta(p/q)$, Δu, Δv, and Δm denote errors in p/q, u, v, and m. Then, from [69], we have

$$\Delta \frac{p}{q} = -2(u\,\Delta u + v\,\Delta v)$$

from [68],

$$\Delta v = y'\,\Delta u$$

and from [67] and [72], except near the stagnation points,

$$\Delta u \cong \frac{3\Delta m}{y^2} \int_0^\pi \sin^3 \theta d\theta = \frac{4\Delta m}{y^2}$$

Hence

$$\Delta \frac{p}{q} = -\frac{8u\,\Delta m}{y^2}\left(1 + y'^2\right)$$

If now we assume $u \cong 1$, $y' \cong 0$, $y^2 \cong 4m$ (Munk's approximation), we obtain

$$\Delta \frac{p}{q} \cong -\frac{2\Delta m}{m}$$

Thus an error of one percent in the determination of m would introduce an error of 0.02 in p/q.

In the foregoing example the minimum value of p/q was about -0.20. Hence an error of one percent in m would have produced an error of ten percent in the minimum value of p/q. It was found, in fact, that the results with Gauss' 7-ordinate rule deviated from the values of p/q given by the 11-point rule by less than 0.003 for the entire body. The 7-point rule would have sufficed if an accuracy of only 0.003 in p/q were required; see Figure 5.

If greater accuracy is desired the integrals can be evaluated in the forms [71a] and [72a]. If the latter forms are used in conjunction with the Gauss quadrature formula the values of x should be chosen identical with the t's required by the Gauss formula. This enables the entire calculations, including the iterations and the velocity determinations, to be made arithmetically, without resort to graphical operations, so that the method becomes suitable for processing on an automatic-sequence computing machine. In order to obtain sufficient accuracy in the integrations and to obtain the velocities and pressures at a sufficient number of points along the body a Gauss formula of high order should be used, say $n = 16$. For this reason the procedure that has been illustrated in detail may be less tedious for manual application.

COMPARISON WITH KÁRMÁN AND KAPLAN METHODS

In order to compare the accuracy of the Kármán method with the present one, the error function $\psi_k(x)$ was computed for a distribution derived by the Kármán method, employing 14 intervals extending from $-0.98 \leqq x \leqq 0.98$. $\psi_k(x)$ is graphed in Figure 3. It is seen that the errors are much greater than for the present method, especially near the ends of the body. The oscillatory character of $\psi_k(x)$ is imposed by the condition that the stream function should vanish at the center of each interval. It is conceivable that the amplitude of the oscillations in $\psi_k(x)$ may remain large even when the number of intervals is greatly increased; i.e., the Kármán method may give a poorer approximation when the number of source-sink segments is greatly increased. The pressure distribution obtained by the Kármán method is graphed in Figure 5.

Kaplan's first method[13] was also applied to obtain the pressure distribution. Kaplan expresses the potential function ϕ in the form

$$\phi = \Sigma A_n Q_n(\lambda)\ P_n(\mu)$$

where λ and μ are confocal elliptic coordinates,

$P_n(\mu)$ and $Q_n(\lambda)$ are the nth Legendre and associated Legendre polynomials, and the

A_n's are coefficients to be determined from a set of linear equations which express the condition that the given profile is a stream function.

In the present case it was assumed that ϕ was expressed in terms of the first 9 Legendre functions and the A_n's determined from the conditions that the stream function should vanish at 9 prescribed points (including the stagnation points) on the body. The resulting pressure distribution is also shown in Figure 5.

SOLUTION BY APPLICATION OF GREEN'S THEOREM

GENERAL APPLICATION TO PROBLEMS IN POTENTIAL THEORY

Let ϕ and ω be any two functions harmonic in the region exterior to a given body and vanishing at infinity. Then, a consequence of Green's second identity[27] is

$$\iint \phi \frac{d\omega}{dn}\, dS = \iint \omega \frac{d\phi}{dn}\, dS \qquad [85]$$

where the double integrals are taken over the boundary of the body and dn denotes an element of the outwardly-directed normal to the surface S. Also derivable from Green's formulas is the well-known expression for a potential function in terms of its values and the values of its normal derivatives on the boundary[2]

$$\phi(Q) = \frac{1}{4\pi}\iint\left[-\frac{1}{r}\frac{d\phi}{dn} + \phi\frac{d}{dn}\frac{1}{r}\right]dS \qquad [86]$$

where r is the distance from the element dS on the body to a point Q exterior to the body.

When a distribution of ϕ or $d\phi/dn$ over the surface of the body is given then [85] may be considered as an integral equation of the first kind for finding $d\phi/dn$ or ϕ respectively, on the surface. If the integral equation can be solved, [86] would then give the value of ϕ at any point Q of the region exterior to the body.

AN INTEGRAL EQUATION FOR AXISYMMETRIC FLOW

Equation [85] will now be applied to obtain an integral equation for axisymmetric flow about a body of revolution. Let y be the ordinate of a meridian section of the body and ds an element of arc length along the boundary in a meridian plane. Then we may put

$$dS = 2\pi y\, ds \qquad [87]$$

It will be supposed that the body is moving with unit velocity in the negative x-direction, which is taken to coincide with the axis of symmetry. The condition that the body should be a solid boundary for the flow is that the component of the fluid velocity at the body normal to body is the same as the component of the velocity of the body normal to itself. This gives the boundary condition

$$\frac{d\phi}{dn} = -\sin\,\gamma \qquad [88]$$

where γ is the angle of the tangent to the body with the x-axis. Substitution of Equations [87] and [88] into [85] now gives

$$\int_0^P y\phi\frac{d\omega}{dn}\,ds = -\int_0^P y\omega\sin\gamma\,ds \qquad [89]$$

where 2P is the perimeter of a meridian section and the arc length s is measured from the foremost point of the body.

Now let us choose for ω an axisymmetric potential function and let $\psi(x, y)$ be the corresponding stream function. Then

$$y \frac{d\omega}{dn} = \frac{d\psi}{ds}$$

and

$$\int_0^P y\phi \frac{d\omega}{dn} ds = \phi\psi \Big|_0^P - \int_0^P \psi \frac{d\phi}{ds} ds$$

Also let U be the total velocity along the body when the flow is made steady by superposing a stream of unit velocity in the positive x-direction. Then

$$U = -\frac{d\phi}{ds} + \cos\gamma$$

Furthermore, we have $dx = ds \cos\gamma$, $dy = ds \sin\gamma$. Then [89] may be written

$$\phi\psi \Big|_0^P - \int_0^P \psi(\cos\gamma - U)ds = -\int_0^P y\omega\, dy$$

or

$$\int_0^P U\psi\, ds = \int_0^P (\psi dx - y\omega\, dy) - \phi\psi \Big|_0^P \qquad [90]$$

But, since ω and ψ are corresponding axisymmetric potential and stream functions, we have

$$-y \frac{\partial\omega}{\partial x} = +\frac{\partial\psi}{\partial y}$$

Hence $\psi dx - y\omega dy$ is an exact differential defining a function $\Omega(x, y)$ such that

$$\frac{\partial\Omega}{\partial x} = \psi, \quad \frac{\partial\Omega}{\partial y} = -y\omega \qquad [91]$$

But since also

$$-y \frac{\partial\omega}{\partial y} = \frac{\partial\psi}{\partial x}$$

we obtain from [91]

$$\frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} = \frac{1}{y}\frac{\partial\Omega}{\partial y}$$

which, by comparison with [4], is seen to be the equation satisfied by the Stokes stream function. Conversely, if Ω is a function satisfying [4], it can readily be verified that the functions ω and ψ defined by [92] are corresponding axisymmetric potential and stream functions, i.e., that they satisfy Equations [3]. Written in terms of Ω, [90] now becomes

$$\int_0^P U \frac{\partial \Omega}{\partial x}\, ds = \left(\Omega - \phi \frac{\partial \Omega}{\partial x}\right)\Big|_0^P \qquad [92]$$

If we choose for Ω the stream function of a source of unit strength situated at an arbitrary point of the axis of symmetry within the body, we have, from [10],

$$\Omega = -1 + \frac{x-t}{r}, \qquad r = \left[(x-t)^2 + y^2\right]^{1/2} \qquad [93]$$

Then

$$\frac{\partial \Omega}{\partial x} = \frac{y^2}{r^3}$$

and, since y vanishes at both limits,

$$\left(\Omega - \phi \frac{\partial \Omega}{\partial x}\right)\Big|_0^P = 2$$

Hence [93] becomes

$$\int_0^P \frac{U(x)\; y^2(x)}{2r^3}\, ds = 1 \qquad [94]$$

It is seen that [94] is an integral equation of the first kind in which the unknown function is U(x) and the kernel is $y^2/2r^3$.

In contrast with the integral equations for source-sink or doublet distributions which can be used to obtain the potential flow about bodies of revolution, the integral equation [94] has two important advantages. The first is that a solution exists, a desirable condition which is not in general the case when a solution is attempted in terms of axial source-sink or doublet distributions. The second advantage is that [94] is expressed directly in terms of the velocity along the body so that, when U is determined, the pressure distribution along the body is immediately given by Bernoulli's equation [69]. In the case of the aforementioned distributions, on the other hand, it would first be necessary to evaluate additional integrals, to obtain the velocity along the body, before the pressures could be computed.

KENNARD'S DERIVATION OF THE INTEGRAL EQUATION

A simple, physical derivation of the integral equation [94] has been given by Dr. E.H. Kennard. This will now be presented.

Imagine the body replaced by fluid at rest. Let U be the velocity on the body. Then the field of flow consists of the superposition of the uniform (unit) flow and the flow due to a vortex sheet of density U.

Now subtract the uniform flow. There remains the flow due to the vortex sheet alone, uniform inside the space originally occupied by the body, of unit magnitude.

A vortex ring of strength Uds produces at an axial point distant z from its plane the velocity

$$V = \frac{y^2 U ds}{2(y^2+z^2)^{3/2}}$$

where y is the radius of the ring. Let s be the distance of a point on the body measured along the generator from the forward end, in a meridian plane. The axial and radial coordinates will then be functions x(s), y(s). The velocity due to the sheet at a point t on the axis will then be

$$\int_0^P \frac{U(s)\ y^2(s)}{2r^3}\, ds = 1$$

where $r^2 = [x(s) - t]^2 + y^2(s)$ and P is the total length of a generator. The equivalence of this equation with [94] is evident.

A FIRST APPROXIMATION

If we again make use of the polar transformation $x - t = y(x) \cot \theta$, [94] becomes

$$\int_0^\pi \frac{U(x)\ \sin^2 \theta\, d\theta}{2\ \sin[\theta - \gamma(x)]} = 1 \qquad [95]$$

When $x = t$, $\theta = \frac{\pi}{2}$. For an elongated body the integrand in [94] peaks sharply in the neighborhood of $x = t$, so that a good approximation is obtained when U(x) is replaced by U(t) for the entire range of integration. Also, $\gamma(x)$ will be small except near the ends of the body so that the approximation

$$\sin[\theta - \gamma(x)] \cong \sin\theta \cos\gamma(x) \cong \sin\theta \cos\gamma(t)$$

will also be introduced. We then obtain from [95] the approximation

$$U(t) \cong \cos\gamma(t) \qquad [96]$$

Just as was done in the case of Munk's approximate doublet distribution we can improve upon this approximation in terms of an estimated longitudinal virtual mass coefficient for the body. For this purpose we will first derive a relation between this coefficient and the velocity distribution.

Let T be the kinetic energy of the fluid when the body is moving with unit velocity in the negative x-direction. Then

$$2T = -\rho \iint \phi \frac{d\phi}{dn}\, dS = 2\pi\rho \int_0^P y\,\phi \sin\gamma\, ds$$

by [88]. Integrating by parts and substituting for $d\phi/ds$ from [93] now gives

$$2T = -\pi\rho \int_0^P y^2 \frac{d\phi}{ds}\, ds = \pi\rho \int_0^P U(x)\; y^2(x) ds - \Delta$$

where Δ is the displacement of the body. But also, by definition, $2T = k_1\Delta$. Hence

$$\Delta(1 + k_1) = \pi\rho \int_0^P U(x)\; y^2(x) ds \qquad [97]$$

This is the desired relation between k_1 and $U(x)$.

Now suppose, as a generalization of [96], that an approximate solution of the integral equations [94] is $U(x) = C \cos\gamma$. If this value is substituted into [97], we obtain $C = 1 + k_1$. Hence an improved first approximation to $U(x)$ is

$$U_1(x) = (1 + k_1) \cos\gamma(x) \qquad [98]$$

Equation [98] gives an exact solution for the prolate spheroid.

SOLUTION OF INTEGRAL EQUATION BY ITERATION

In order to solve [94] by means of the iteration formula treated in Reference 17, it would be necessary to work with the iterated kernel of this integral equation. Since this would entail considerable computational labor it is proposed to try a similar iteration formula, but employing the original kernel:

$$U_{n+1}(t) = U_n(t) + \cos\gamma(t)\left[1 - \int_0^P \frac{y^2(x)}{2r^3}\, U_n(x) ds\right] \qquad [99]$$

where $r^2 = (x - t)^2 + y^2(x)$ and $x = x(s)$.

Here also it is convenient to express the iterations in terms of error functions $E_n(t)$ defined by

$$E_n(t) = 1 - \int_0^P \frac{U_n(x)\ y^2(x)}{2r^3}\, ds \qquad [100]$$

or, from [99],

$$E_n(t) \cos \gamma(t) = U_{n+1}(t) - U_n(t) \qquad [101]$$

Hence

$$U_{n+1}(t) = U_1(t) + \cos \gamma(t) \sum_{i=1}^{n} E_i(t) \qquad [102]$$

Also, from [99],

$$E_{n+1}(t) = E_n(t) - \frac{1}{2} \int_{x_0}^{x_1} \frac{E_n(x)\ y^2(x)}{r^3}\, dx \qquad [103]$$

where x_0, x_1 are the nose and tail abscissae. Thus, to obtain $U_{n+1}(t)$, we first obtain $E_1(t)$ from $U_1(t)$ in [100], then E_2, E_3, ... E_n from [103], and finally $U_{n+1}(t)$ from [102].

NUMERICAL EVALUATION OF INTEGRALS

In applying Equations [100] and [103] it will frequently be necessary to evaluate integrals of the form

$$\int_{x_0}^{x_1} \frac{E(x)\ y^2(x)}{r^3}\, dx, \text{ where } r^2 = (t-x)^2 + y^2(x)$$

This form, however, is unsuited for numerical quadrature for elongated bodies, since $y^2(x)$ peaks sharply in the neighborhood of $x = t$. Here, as in the case of the integrals for the doublet distribution, two procedures are available for avoiding this difficulty. The first employs the polar transformation [70], involves several graphical operations, but in general transforms the integrand into a slowly varying function so that the integral can be evaluated by a quadrature formula using relatively few ordinates. The second retains the original variables and eliminates the peak by subtracting from the integrand an integrable function which behaves very much like the original integrand in the neighborhood of the peak. The numerical evaluation of the resulting integral on the second method requires a quadrature formula with more ordinates than the first in order to obtain the same accuracy, but, since all graphical operations are eliminated, the second method is suitable for processing on an automatic-sequence calculating machine.

The result of the polar transformation has effectively been given in [95]. We have

$$\int_{x_0}^{x_1} \frac{E(x)\ y^2(x)}{r^3}\, dx = \int_0^{\pi} \frac{E(x)\ \sin^2\theta \cos\gamma(x)}{\sin[\theta - \gamma(x)]}\, d\theta \qquad [104]$$

where

$$x - t = y(x) \cot\theta \qquad [70]$$

It is desired to evaluate this integral for a series of values of t. In general this can be done with sufficient accuracy by means of the Gauss 7- (or 11-) ordinate quadrature formulas. This gives 7 (or 11) values of θ at which the integrand needs to be determined for a given t. The value of x occurring in the integrand is determined implicitly, for given values of t and θ, by the polar transformation [70]. In practice the 7 (or 11) x's can be obtained graphically from the intersections with a graph of the given profile of the 7 (or 11) rays from the point x = t on the axis at the angles required by the Gauss quadrature formula. If greater accuracy is desired, these graphically determined values of x can be corrected by means of the formula

$$x = x_g + \frac{t - x_g + y(x_g) \cot\theta}{1 - y'(x_g) \cot\theta} \qquad [105]$$

in which x_g is the graphically determined value and y' denotes the derivative of y with respect to x.

Now let us derive an alternate, completely arithmetical procedure for evaluating the integrals. Put

$$k(x,\ t) = \frac{f(x)}{[(x-t)^2 + f(x)]^{3/2}}$$

$$k'(x,\ t) = \frac{g(x,\ t)}{[(x-t)^2 + g(x)]^{3/2}}$$

where $y^2 = f(x)$ is the equation of the given profile and $y^2 = g(x,\ t)$ is the equation of the prolate spheroid whose ends coincide with the ends of the given body, and which intersects the given body at x = t. i.e.,

$$g(x,\ t) = f(t)\ \frac{(x-x_0)(x_1-x)}{(t-x_0)(x_1-t)} \qquad [106]$$

The length-diameter ratio λ of the spheroid is given by

$$\lambda^2 = \frac{(t-x_0)(x_1-t)}{f(t)} \qquad [107]$$

whence the longitudinal virtual mass coefficient $k_1(t)$ can be obtained from [26].

Since $U(x) = (1 + k_1)\cos\gamma(x)$ is an exact solution of [94] for the prolate spheroid, we have

$$\int_{x_0}^{x_1} k'(x,\ t)dx = \frac{2}{1+k_1(t)} \qquad [108]$$

We now obtain, from [98], [100], and [108]

$$E_1(t) = 1 - \frac{1+k_1}{2}\int_{x_0}^{x_1}[k(x,\ t) - k'(x,\ t)]\,dx - \frac{1+k_1}{1+k_1(t)} \qquad [109]$$

Also [103] may be written in the form

$$E_{n+1}(t) = E_n(t) - \frac{1}{2}\int_{x_0}^{x_1} k(x,\ t)[E_n(x) - E_n(t)]dx - E_n(t)\int_{x_0}^{x_1} k(x,\ t)dx$$

But from [98] and [100],

$$\int_{x_0}^{x_1} k(x,\ t)dx = 2\ \frac{1-E_1(t)}{1+k_1}$$

Hence we obtain

$$E_{n+1}(t) = \frac{E_1(t)+k_1}{1+k_1}\ E_n(t) - \frac{1}{2}\int_{x_0}^{x_1} k(x,\ t)\left[E_n(x) - E_n(t)\right]dx \qquad [110]$$

ILLUSTRATIVE EXAMPLE

The present method will now be applied to the same profile [78] as before. By way of contrast with the semi-graphical procedures previously used, a completely arithmetical procedure will be employed.

The velocity U(t) will be determined at the 16 points along the body whose abscissae are $t_i = \xi_i$, the Gaussian values for the 16-point quadrature rule, Table 1. Since the body is symmetrical fore and aft, it is necessary to determine the velocity at only half of these points. Values of y(x), cos γ(x) and k_1(t) for these points are given in Table 6.

In order to apply the Gauss 16-ordinate rule it is necessary to evaluate the integrands in [109] and [110] at the 16 Gaussian abscissae $x_j = \xi_j$ for each of the 8 values of t_i. Thus, there are 16 x 8 = 128 values of k(x, t) and of k'(x, t) to be determined. The matrices $K_{ji} = R_j k(x_j, t_i)$ and $K'_{ji} = R_j k'(x_j, t_i)$ where the R_j's are the Gauss weighting factors, are given in Tables 7 and 8, and applied to evaluate $E_1(t)$ from [109]. E_2, E_3, and E_4 are then obtained from [110]. $U_5(t)$ is then given by [102] and then p/q by [69], in the form $p/q = 1 - U_5^2$. The arrangement of the calculations and the results are given in Table 9. The graph of p/q is included in Figure 5.

TABLE 6

Values of y, cos γ, and k_1(x) for Application of Gauss 16-Point Quadrature Formula

x	y(x)	y'(x)	γ(x)	cos γ(x)	k_1(x)
-0.9894009	0.0408548	1.8965483	1.0856	0.4664	0.096382
9445750	.0903198	0.7464764	0.6412	.8014	.093389
.8656312	.1324422	.3917981	.3734	.9311	.088359
.7554044	.1642411	.2099651	.2070	.9787	.081862
.6178762	.1848527	.1020867	.1017	.9948	.074689
.4580168	.1955501	.0393076	.03932	0.9992	.067885
.2816036	.1993706	.0089607	.008961	1.0000	.062506
-0.0950125	0.1999919	0.0003431	0.0003431	1.0000	0.059509

TABLE 7

Matrix of Values* $K_{ji} = R_j \dfrac{f(x_j)}{\left[(x_j - t_i)^2 + f(x_j)\right]^{3/2}}$

j \ i	1	2	3	4	5	6	7	8
1	0.66460	0.20313	0.02047	0.00338	0.00087	0.00030	0.00013	0.00006
2	.49536	.68926	.29420	.05513	.01304	.00419	.00170	.00081
3	.28022	.45536	.71850	.32626	.07528	.02120	.00777	.00349
4	.14389	.21382	.43441	.75882	.34200	.08574	.02666	.01067
5	.07154	.09665	.17306	.41794	.80929	.35023	.09047	.02997
6	.03563	.04486	.07001	.14347	.40145	.86505	.35410	.09228
7	.01825	.02187	.03008	.05344	.12148	.38470	.91588	.35647
8	.00984	.01140	.01502	.02307	.04319	.10644	.37030	.94729
9	.00565	.00639	.00802	.01137	.01867	.03726	.09772	.36090
10	.00341	.00379	.00460	.00616	.00928	.01615	.03403	.09380
11	.00208	.00228	.00270	.00348	.00495	.00787	.01445	.03205
12	.00121	.00131	.00153	.00192	.00262	.00393	.00660	.01280
13	.00062	.00067	.00078	.00096	.00127	.00183	.00291	.00518
14	.00027	.00028	.00032	.00039	.00051	.00071	.00108	.00183
15	.00007	.00008	.00009	.00010	.00013	.00018	.00027	.00045
16	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00002	0.00004

TABLE 8

Matrix of Values** $K'_{ji} = R_j \dfrac{g(x_j, t_i)}{\left[(x_j - t_i)^2 + g(x_j, t_i)\right]^{3/2}}$

j \ i	1	2	3	4	5	6	7	8
1	0.66458	0.20016	0.01841	0.00271	0.00061	0.00018	0.00007	0.00003
2	.49077	.68925	.29043	.04802	.00980	.00273	.00098	.00044
3	.28639	.45118	.71848	.32161	.06389	.01533	.00493	.00194
4	.16024	.22699	.43201	.75882	.33584	.07174	.01928	.00707
5	.09055	.11637	.19118	.41862	.80927	.34255	.07641	.02291
6	.05223	.06265	.08976	.16366	.40559	.86501	.34633	.08149
7	.03061	.03502	.04556	.07041	.13995	.39080	.91586	.35140
8	.01805	.01998	.02432	.03352	.05520	.11924	.37496	.94729
9	.01058	.01144	.01329	.01694	.02457	.04317	.10217	.36090
10	.00607	.00645	.00725	.00876	.01165	.01788	.03396	.08960
11	.00334	.00351	.00385	.00447	.00561	.00787	.01303	.02749
12	.00171	.00178	.00193	.00218	.00262	.00346	.00523	.00959
13	.00078	.00081	.00086	.00096	.00112	.00142	.00202	.00339
14	.00029	.00030	.00032	.00035	.00040	.00049	.00067	.00107
15	.00007	.00008	.00008	.00009	.00010	.00012	.00016	.00024
16	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00002

*For $i > 8$ use $K_{ji} = K_{17-j,\,17-i}$

**For $i > 8$ use $K'_{ji} = K'_{17-j,\,17-i}$

TABLE 9

Calculations for $E_n(t)$ and $U(t)$

Assume $k_1 = 0.06$: Put $K_{ji} = R_j k_{ji}$, $K'_{ji} = R_j k_{ij}$, $E_n(x_j) = E_n(t_j) = E_{nj}$

(a) $x_1 = -0.989401$; $\cos\gamma = 0.4664$; $\dfrac{1 + k_1}{1 + k_1(x_1)} = 0.966815$

j	K_{j1}	K'_{j1}	$K_{j1}-K'_{j1}$	$E_{1j}-E_{11}$	$K_{j1}(E_{1j}-E_{11})$	$E_{2j}-E_{21}$	$K_{j1}(E_{2j}-E_{21})$	$E_{3j}-E_{31}$	$K_{j1}(E_{3j}-E_{31})$
1	0.66460	0.66458	+0.00002	0	0	0	0	0	0
2	.49536	.49077	+0.00459	-0.00730	-0.00362	-0.00307	-0.00152	-0.00158	-0.00078
3	.28022	.28639	-0.00617	.01799	.00504	.00634	.00178	.00253	.00071
4	.14389	.16024	.01635	.03333	.00480	.01101	.00158	.00403	.00058
5	.07154	.09055	.01901	.05207	.00373	.01651	.00118	.00568	.00041
6	.03563	.05223	.01660	.07213	.00257	.02182	.00078	.00706	.00025
7	.01825	.03061	.01236	.08999	.00164	.02619	.00048	.00802	.00015
8	.00984	.01805	.00821	.10087	.00099	.02870	.00028	.00851	.00008
9	.00565	.01058	.00493	.10087	.00057	.02870	.00016	.00851	.00005
10	.00341	.00607	.00266	.08999	.00031	.02619	.00009	.00802	.00003
11	.00208	.00334	.00126	.07213	.00015	.02182	.00005	.00706	.00001
12	.00121	.00171	.00050	.05207	.00006	.01651	.00002	.00568	-0.00001
13	.00062	.00078	.00016	.03333	-0.00002	.01101	-0.00001	.00403	0
14	.00027	.00029	-0.00002	.01799	0	.00634	0	.00253	0
15	.00007	.00007	0	-0.00730	0	-0.00307	0	-0.00158	0
16	0.00001	0.00001	0	0	0	0	0	0	0
			$\int = -0.08362$ $E_{11} = +0.07750$		$\int = -0.02350$ $E_{21} = +0.02180$		$\int = -0.00793$ $E_{31} = +0.00680$		$\int = -0.00306$ $E_{41} = +0.00241$

$\dfrac{k_1 + E_{11}}{1 + k_1} = 0.12972$ $\quad U_5(x_1) = 0.5450,\quad \dfrac{p}{q} = 0.7030$

(b) $x_2 = -0.944575$; $\cos\gamma = 0.8014$; $\dfrac{1 + k_1}{[1 + k_1(x_2)]} = 0.96947$

j	K_{j2}	K'_{j2}	$K_{j2}-K'_{j2}$	$E_{1j}-E_{12}$	$K_{j2}(E_{1j}-E_{12})$	$E_{2j}-E_{22}$	$K_{j2}(E_{2j}-E_{22})$	$E_{3j}-E_{32}$	$K_{j2}(E_{3j}-E_{32})$
1	0.20313	0.20017	+0.00296	+0.00730	+0.00148	+0.00307	+0.00062	+0.00158	+0.00032
2	.68926	.68927	-0.00001	0	0	0	0	0	0
3	.45536	.45118	+0.00418	-0.01069	-0.00487	-0.00327	-0.00149	-0.00095	-0.00043
4	.21382	.22699	-0.01317	.02603	.00557	.00794	.00170	.00245	.00052
5	.09665	.11638	.01913	.04477	.00433	.01344	.00130	.00410	.00040
6	.04486	.06265	.01779	.06483	.00291	.01875	.00084	.00548	.00025
7	.02187	.03502	.01315	.08269	.00181	.02312	.00051	.00644	.00014
8	.01140	.01998	.00858	.09357	.00107	.02563	.00029	.00693	.00008
9	.00639	.01143	.00504	.09357	.00060	.02563	.00016	.00693	.00004
10	.00379	.00645	.00266	.08269	.00031	.02312	.00009	.00644	.00002
11	.00228	.00350	.00122	.06483	.00015	.01875	.00004	.00548	.00001
12	.00131	.00178	.00047	.04477	.00006	.01344	.00002	.00410	-0.00001
13	.00067	.00081	.00014	.02603	-0.00002	.00794	.00001	.00245	0
14	.00028	.00030	-0.00002	-0.01069	0	-0.00327	0	-0.00095	0
15	.00008	.00008	0	0	0	0	0	0	0
16	0.00001	0.00001	0	+0.00730	0	+0.00307	0	+0.00158	0
			$\int = -0.07484$ $E_{12} = +0.07020$		$\int = -0.02022$ $E_{22} = +0.01873$		$\int = -0.00583$ $E_{32} = +0.00522$		$\int = -0.00158$ $E_{42} = +0.00143$

$\dfrac{k_1 + E_{12}}{1 + k_1} = 0.12283$ $\quad U_5(x_2) = 0.9261;\quad \dfrac{p}{q} = 0.1423$

TABLE 9 (Continued)

(c) $x_3 = -0.865631$; $\cos\gamma = 0.9311$; $\dfrac{1 + k_1}{[1 + k_1(x_3)]} = 0.97394$

j	K_{j3}	K'_{j3}	$K_{j3}-K'_{j3}$	$E_{1j}-E_{13}$	$K_{j3}(E_{1j}-E_{13})$	$E_{2j}-E_{23}$	$K_{j3}(E_{2j}-E_{23})$	$E_{3j}-E_{33}$	$K_{j3}(E_{3j}-E_{33})$
1	0.02047	0.01841	+0.00206	+0.01799	+0.00037	+0.00634	+0.00013	+0.00253	+0.00005
2	.29420	.29039	+0.00381	+0.01069	+0.00314	+0.00327	+0.00096	+0.00095	+0.00028
3	.71850	.71851	-0.00001	0	0	0	0	0	0
4	.43441	.43201	+0.00240	-0.01534	-0.00666	-0.00467	-0.00203	-0.00150	-0.00065
5	.17306	.19118	-0.01812	.03408	.00590	.01017	.00176	.00315	.00055
6	.07001	.08976	.01975	.05414	.00379	.01548	.00108	.00453	.00032
7	.03088	.04556	.01468	.07200	.00222	.01985	.00061	.00549	.00017
8	.01502	.02432	.00930	.08288	.00124	.02236	.00034	.00598	.00009
9	.00802	.01329	.00527	.08288	.00066	.02236	.00018	.00598	.00005
10	.00460	.00725	.00265	.07200	.00033	.01985	.00009	.00549	.00003
11	.00270	.00385	.00115	.05414	.00015	.01548	.00004	.00453	-0.00001
12	.00153	.00193	.00040	.03408	.00005	.01017	-0.00002	.00315	0
13	.00078	.00086	-0.00008	-0.01534	-0.00001	-0.00467	0	-0.00150	0
14	.00032	.00032	0	0	0	0	0	0	0
15	.00009	.00008	+0.00001	+0.01069	0	+0.00327	0	+0.00095	0
16	0.00001	0.00001	0	+0.01799	0	+0.00634	0	+0.00253	0
			$\int = -0.06312$ $E_{13} = +0.05951$		$\int = -0.01750$ $E_{23} = +0.01546$		$\int = -0.00506$ $E_{33} = +0.00427$		$\int = -0.00154$ $E_{43} = +0.00125$

$\dfrac{k_1 + E_{13}}{1 + k_1} = +0.11275$ $U_5(x_3) = 1.0619$; $\dfrac{p}{q} = -0.1276$

(d) $x_4 = -0.755407$; $\cos\gamma = 0.9787$; $\dfrac{1 + k_1}{[1 + k_1(x_4)]} = 0.97980$

j	K_{j4}	K'_{j4}	$K_{j4}-K'_{j4}$	$E_{1j}-E_{14}$	$K_{j4}(E_{1j}-E_{14})$	$E_{2j}-E_{24}$	$K_{j4}(E_{2j}-E_{24})$	$E_{3j}-E_{34}$	$K_{j4}(E_{3j}-E_{34})$
1	0.00338	0.00271	+0.00067	+0.03333	+0.00011	+0.01101	+0.00004	+0.00403	+0.00001
2	.05513	.04802	.00712	.02603	.00144	.00794	.00044	.00245	.00014
3	.32626	.32161	+0.00465	+0.01534	+0.00483	+0.00467	+0.00152	+0.00150	+0.00049
4	.75882	.75882	0	0	0	0	0	0	0
5	.41794	.41862	-0.00068	-0.01874	-0.00783	-0.00550	-0.00230	-0.00165	-0.00069
6	.14347	.16366	.02019	.03880	.00557	.01081	.00155	.00303	.00043
7	.05344	.07041	.01697	.05666	.00303	.01518	.00081	.00399	.00021
8	.02307	.03352	.01045	.06754	.00156	.01769	.00041	.00448	.00010
9	.01137	.01694	.00557	.06754	.00077	.01769	.00020	.00448	.00005
10	.00616	.00876	.00260	.05666	.00035	.01518	.00009	.00399	.00002
11	.00348	.00447	.00099	.03880	.00014	.01081	.00004	.00303	-0.00001
12	.00192	.00218	-0.00026	-0.01874	-0.00004	-0.00550	-0.00001	-0.00165	0
13	.00096	.00096	0	0	0	0	0	0	0
14	.00039	.00035	+0.00004	+0.01534	+0.00001	+0.00467	0	+0.00150	0
15	.00010	.00009	+0.00001	.02603	0	.00794	0	.00245	0
16	0.00001	0.00001	0	+0.03333	0	+0.01101	0	+0.00403	0
			$\int = -0.04523$ $E_{14} = +0.04417$		$\int = -0.01290$ $E_{24} = +0.01079$		$\int = -0.00341$ $E_{34} = +0.00277$		$\int = -0.00087$ $E_{44} = +0.00071$

$\dfrac{k_1 + E_{14}}{1 + k_1} = 0.09827$ $U_5(x_4) = 1.0946$; $\dfrac{p}{q} = -0.1981$

TABLE 9 (Continued)

(e) $x_5 = -0.617876$; $\cos\gamma = 0.9948$; $\dfrac{1 + k_1}{[1 + k_1(x_5)]} = 0.98633$

j	K_{j5}	K'_{j5}	$K_{j5}-K'_{j5}$	$E_{1j}-E_{15}$	$K_{j5}(E_{1j}-E_{15})$	$E_{2j}-E_{25}$	$K_{j5}(E_{2j}-E_{25})$	$E_{3j}-E_{35}$	$K_{j5}(E_{3j}-E_{35})$
1	0.00087	0.00061	+0.00026	+0.05207	+0.00005	+0.01651	+0.00001	+0.00568	0
2	.01304	.00980	.00324	.04477	.00058	.01344	.00018	.00410	+0.00005
3	.07528	.06389	.01139	.03408	.00257	.01017	.00077	.00315	.00024
4	.34200	.33584	.00616	+0.01874	+0.00641	+0.00550	+0.00188	+0.00165	+0.00056
5	.80929	.80927	+0.00002	0	0	0	0	0	0
6	.40145	.40559	-0.00414	-0.02006	-0.00805	-0.00531	-0.00213	-0.00138	-0.00055
7	.12148	.13995	.01847	.03792	.00461	.00968	.00118	.00234	.00028
8	.04319	.05520	.01201	.04880	.00211	.01219	.00053	.00283	.00012
9	.01867	.02457	.00590	.04880	.00091	.01219	.00023	.00283	.00005
10	.00928	.01165	.00237	.03792	.00035	.00968	.00009	.00234	.00002
11	.00495	.00561	-0.00066	-0.02006	-0.00010	-0.00531	-0.00003	-0.00138	-0.00001
12	.00262	.00262	0	0	0	0	0	0	0
13	.00127	.00112	+0.00015	+0.01874	+0.00002	+0.00550	+0.00001	+0.00165	0
14	.00051	.00040	.00011	.03408	.00002	.01017	+0.00001	.00315	0
15	.00013	.00010	+0.00003	.04477	+0.00001	.01344	0	.00410	0
16	0.00001	0.00001	0	+0.05207	0	+0.01651	0	+0.00568	0
			$\int = -0.02219$ $E_{15} = +0.02543$		$\int = -0.00647$ $E_{25} = +0.00529$		$\int = -0.00133$ $E_{35} = +0.00112$		$\int = -0.00018$ $E_{45} = +0.00018$

$\dfrac{k_1 + E_{15}}{1 + k_1} = 0.08059$ $U_5(x_5) = 1.0864$; $\dfrac{p}{q} = -0.1803$

(f) $x_6 = -0.458017$; $\cos\gamma = 0.9992$; $\dfrac{1 + k_1}{[1 + k_1(x_6)]} = 0.99262$

j	K_{j6}	K'_{j6}	$K_{j6}-K'_{j6}$	$E_{1j}-E_{16}$	$K_{j6}(E_{1j}-E_{16})$	$E_{2j}-E_{26}$	$K_{j6}(E_{2j}-E_{26})$	$E_{3j}-E_{36}$	$K_{j6}(E_{3j}-E_{36})$
1	0.00030	0.00018	+0.00012	+0.07213	+0.00002	+0.02182	+0.00001	+0.00706	0
2	.00419	.00273	.00146	.06483	.00027	.01875	.00008	.00548	+0.00002
3	.02120	.01533	.00587	.05414	.00115	.01548	.00033	.00453	.00010
4	.08574	.07174	.01400	.03880	.00333	.01081	.00093	.00303	.00026
5	.35023	.34255	.00768	+0.02006	+0.00703	+0.00531	+0.00186	+0.00138	+0.00048
6	.86505	.86501	+0.00004	0	0	0	0	0	0
7	.38470	.39080	-0.00610	-0.01786	-0.00687	-0.00437	-0.00168	-0.00096	-0.00037
8	.10644	.11924	.01280	.02874	.00306	.00688	.00073	.00145	.00015
9	.03726	.04317	.00591	.02874	.00107	.00688	.00026	.00145	.00005
10	.01615	.01788	-0.00173	-0.01786	-0.00029	-0.00437	-0.00007	-0.00096	-0.00002
11	.00787	.00787	0	0	0	0	0	0	0
12	.00393	.00346	+0.00047	+0.02006	+0.00008	+0.00531	+0.00002	+0.00138	+0.00001
13	.00183	.00142	.00041	.03880	.00007	.01081	.00002	.00303	+0.00001
14	.00071	.00049	.00022	.05414	.00004	.01548	+0.00001	.00453	0
15	.00018	.00012	+0.00006	.06483	+0.00001	.01875	0	.00548	0
16	0.00001	0.00001	0	+0.07213	0	+0.02182	0	+0.00706	0
			$\int = +0.00379$ $E_{16} = +0.00537$		$\int = +0.00071$ $E_{26} = -0.00002$		$\int = +0.00052$ $E_{36} = -0.00026$		$\int = +0.00029$ $E_{46} = -0.00016$

$\dfrac{k_1 + E_{16}}{1 + k_1} = 0.06167$ $U_5(x_6) = 1.0641$; $\dfrac{p}{q} = -0.1323$

TABLE 9 (Continued)

(g) $x_7 = -0.281604$; $\cos \gamma = 1.0000$; $\frac{1 + k_1}{[1 + k_1(x_7)]} = 0.99764$

j	K_{j7}	K'_{j7}	$K_{j7} - K'_{j7}$	$E_{1j} - E_{17}$	$K_{j7}(E_{1j} - E_{17})$	$E_{2j} - E_{27}$	$K_{j7}(E_{2j} - E_{27})$	$E_{3j} - E_{37}$	$K_{j7}(E_{3j} - E_{37})$
1	0.00013	0.00007	+0.00006	+0.08999	+0.00001	+0.02619	+0.00000	+0.00802	+0.00000
2	.00170	.00098	.00072	.08269	.00014	.02312	.00004	.00644	.00001
3	.00777	.00493	.00284	.07200	.00056	.01985	.00015	.00549	.00004
4	.02666	.01928	.00738	.05666	.00151	.01518	.00040	.00399	.00011
5	.09047	.07641	.01406	.03792	.00343	.00968	.00088	.00234	.00021
6	.35410	.34633	.00777	.01786	.00632	.00437	.00155	.00096	.00034
7	.91588	.91586	+0.00002	+0.00000	+0.00000	+0.00000	+0,00000	+0.00000	+0,00000
8	.37030	.37496	-0.00466	-0.01088	-0.00403	-0.00251	-0.00093	-0.00049	-0.00018
9	.09772	.10217	-0.00445	-0.01088	-0.00106	-0.00251	-0.00025	-0.00049	-0.00005
10	.03403	.03396	+0.00007	+0.00000	+0.00000	+0.00000	+0.00000	+0.00000	+0.00000
11	.01445	.01303	.00142	.01786	.00026	.00437	.00006	.00096	.00001
12	.00660	.00523	.00137	.03792	.00025	.00968	.00006	.00234	.00002
13	.00291	.00202	.00089	.05666	.00016	.01518	.00004	.00399	.00001
14	.00108	.00067	.00041	.07200	.00008	.01985	.00002	.00549	.00001
15	.00027	.00016	.00011	.08269	.00002	.02312	.00001	.00644	.00000
16	0.00002	0.00001	+0.00001	+0.08999	+0.00000	+0.02619	+0.00000	+0.00802	+0.00000
			$\int$ = +0.02802 E_{17} = -0.01249		$\int$ = +0.00765 E_{27} = -0.00439		$\int$ = +0.00203 E_{37} = -0.00122		$\int$ = +0.00053 E_{47} = -0.00032

$\frac{k_1 + E_{17}}{1 + k_1} = 0.04482$ $U_5(x_7) = +1.0416$; $\frac{p}{q} = -0.0849$

(h) $x_8 = -0.095013$; $\cos \gamma = 1.0000$; $\frac{1 + k_1}{[1 + k_1(x_8)]} = 1.00046$

j	K_{j8}	K'_{j8}	$K_{j8} - K'_{j8}$	$E_{1j} - E_{18}$	$K_{j8}(E_{1j} - E_{18})$	$E_{2j} - E_{28}$	$K_{j8}(E_{2j} - E_{28})$	$E_{3j} - E_{38}$	$K_{j8}(E_{3j} - E_{38})$
1	0.00006	0.00003	0.00003	0.10087	0.00001	0.02870	0.00000	0.00851	0.00000
2	.00081	.00044	.00037	.09357	.00008	.02563	.00002	.00693	.00001
3	.00349	.00194	.00155	.08288	.00029	.02236	.00008	.00598	.00002
4	.01067	.00707	.00360	.06754	.00072	.01769	.00019	.00448	.00005
5	.02997	.02291	.00706	.04880	.00146	.01219	.00037	.00283	.00008
6	.09228	.08149	.01079	.02874	.00265	.00688	.00063	.00145	.00013
7	.35647	.35140	.00507	.01088	.00388	.00251	.00089	.00049	.00017
8	.94729	.94729	.00000	.00000	.00000	.00000	.00000	.00000	.00000
9	.36090	.36090	.00000	.00000	.00000	.00000	.00000	.00000	.00000
10	.09380	.08960	.00420	.01088	.00102	.00251	.00024	.00049	.00005
11	.03205	.02749	.00456	.02874	.00092	.00688	.00022	.00145	.00005
12	.01280	.00959	.00321	.04880	.00062	.01219	.00016	.00283	.00004
13	.00518	.00339	.00179	.06754	.00035	.01769	.00009	.00448	.00002
14	.00183	.00107	.00076	.08288	.00015	.02236	.00004	.00548	.00001
15	.00045	.00024	.00021	.09357	.00004	.02563	.00001	.00693	.00000
16	0.00004	0.00002	0.00002	0.10087	0.00000	0.02870	0.00000	0.00851	0.00000
			$\int$ = +0.04322 E_{18} = -0.02337		$\int$ = +0.01219 E_{28} = -0.00690		$\int$ = +0.00294 E_{38} = -0.00171		$\int$ = +0.00063 E_{48} = -0.00038

$\frac{k_1 + E_{18}}{1 + k_1} = 0.03456$ $U_5(x_8) = 1.0276$; $\frac{p}{q} = -0.0560$

SUMMARY

Two new methods for computing the steady, irrotational, axisymmetric flow of a perfect, incompressible fluid about a body of revolution are presented.

In the first method a continuous, axial distribution of doublets which generates the prescribed body in a uniform stream is sought as a solution of the integral equation

$$\int_a^b \frac{m(t)}{r^3}\, dt = \frac{1}{2}$$

where r is the distance from a point (t, 0) on the axis to a point (x, y) on the body, $r^2 = (x - t)^2 + y^2(x)$.

A method of determining the end points of the distribution and the values of the distribution at the end points is given. If the equation of the body profile, with the origin of coordinates at one end, is

$$y^2(x) = a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

a very good approximation for the distribution limit a at that end, when the coefficients a_1, a_2, ... are small, is given by

$$\frac{a_1}{a} = 4 + a_2 + \frac{1}{2}\sqrt{a_1 a_3}$$

if $a_3 \geqq 0$. If a_3 is negative, the term containing it is neglected. The corresponding value of the doublet strength at this point is

$$m(a) = \frac{1}{8}\left(1 + \frac{a_1}{2} + \frac{a_2}{2} \log \frac{a_1}{4}\right) a^2 \sqrt{a_1 a_3}$$

Formulas and tables for determining a and m(a), which may be used when the above procedure is insufficiently accurate, are also given. The values a, b, $m_a = m(a)$, $m_b = m(b)$, $f_a = y^2(a)$ and $f_b = y^2(b)$ are then used to obtain the approximate solution of the integral equation

$$m_1(x) = C\left(y^2 - \frac{b-x}{b-a} f_a - \frac{x-a}{b-a} f_b\right) + \frac{b-x}{b-a} m_a + \frac{x-a}{b-a} m_b$$

where

$$C = \frac{\frac{1+k_1}{4}\int_{x_0}^{x_1} y^2 dx - \frac{1}{2}(b-a)\left(m_a+m_b\right)}{\int_a^b y^2 dx - \frac{1}{2}(b-a)\left(f_a+f_b\right)}$$

and k_1 is the longitudinal virtual mass coefficient for the body.

This approximation is used to obtain a sequence of successive approximations by means of the iteration formula

$$m_{i+1}(x) = m_i(x) + \frac{1}{2}\, y^2(x)\left[\frac{1}{2} - \int_a^b \frac{m_i(t)}{r^3}\, dt\right]$$

When a doublet distribution has been assumed, the velocity components at a point (x, y) in a meridian plane are

$$u = 1 + \int_a^b \left(\frac{3y^2}{r^5} - \frac{2}{r^3}\right) m(t)dt$$

$$v = 3y \int_a^b \frac{t-x}{r^5}\, m(t)dt$$

and the pressure is given by

$$\frac{p}{q} = 1 - (u^2 + v^2)$$

where q is the stagnation pressure.

The iterations are most conveniently performed in terms of the differences between successive approximations to m(x), which also furnish, at each iteration, a geometric measure of the accuracy of an approximation. Simpler forms for the velocity components at the surface of the body are given in terms of this difference or error function.

Gauss' quadrature formulas are recommended for the numerical evaluation of the integrals. Two methods of carrying out the iterations are given. The first employs a polar transformation and a graphical operation between successive iterations; the second is completely arithmetical and is suitable for processing on an automatic-sequence computing machine. All of these procedures are illustrated in detail by an example, in which the semigraphical method is employed. The accuracy of the method is analyzed; the results are compared with those obtained by the methods of Kármán and Kaplan.

In the second method the velocity U(x) on the surface of the given body is given directly as the solution of the integral equation

$$\int_0^P \frac{U(x)y^2(x)}{2r^3}\,ds = 1$$

where s is the arc length along the profile,

x is equal to x(s), and

2P is the perimeter of a meridian section.

An approximate solution to this integral equation is

$$U_1(x) = (1 + k_1)\cos\gamma(x)$$

where k_1 is the longitudinal virtual mass coefficient and $\gamma = \arctan\frac{dy}{dx}$. $U_1(x)$ is used to obtain a sequence of successive approximations by means of the iteration formula

$$U_{n+1}(t) = U_n(t) + \cos\gamma(t)\left[1 - \int_0^P \frac{y^2(x)}{r^3}\,U_n(x)ds\right]$$

Here, also, the iterations are most conveniently carried out in terms of the differences between successive approximations to U(x) which also furnish a measure of the error in the integral equation. Two methods of carrying out the iterations are again available, of which one is semi-graphical, the other completely arithmetical. The latter technique is employed on the same example as was used to illustrate the first method.

APPENDIX

END POINTS OF A DISTRIBUTION

An approximate method for determining the end points of a distribution and its trends at the ends will now be described. Let $y^2 = f(x)$ be the equation of the given profile extending from $x = 0$ to $x = 1$; let $m(x)$ be the corresponding doublet distribution, extending from $x = a$ to $x = b$. It will be assumed that $0 < a \ll b < 1$ and that a is near 0, b is near 1. Then $m(x)$ is given by the integral equation

$$\int_a^b \frac{m(t)dt}{[(x-t)^2 + f(x)]^{3/2}} = \frac{1}{2} \qquad [111]$$

Various conditions on $m(x)$ may now be obtained by differentiating [111] repeatedly with respect to x. We get

$$\int_a^b \frac{m(t)}{r^5}[2x - 2t + f'(x)]\, dt = 0 \qquad [112]$$

$$\int_a^b m(t)\left[-\frac{5}{2r^7}(2x - 2t + f')^2 + \frac{1}{r^5}(2 + f'')\right]dt = 0 \qquad [113]$$

$$\int_a^b m(t)\left[\frac{35}{4r^9}(2x-2t+f')^3 - \frac{15}{2r^7}(2+f'')(2x-2t+f') + \frac{f'''(x)}{r^5}\right]dt = 0 \qquad [114]$$

When $x = 0$, $r = t$ and, writing $f(x)$ as a Taylor expansion

$$(x) = a_1x + a_2x^2 + a_3x^3 + \ldots \qquad [115]$$

then also $f'(0) = a_1$, $f''(0) = 2a_2$, $f'''(0) = 6a_3$. Now, setting $x = 0$ in Equations [111] and [113], we obtain

$$\int_a^b \frac{m(t)}{t^3}\, dt = \frac{1}{2} \qquad [116]$$

$$\int_a^b \frac{m(t)}{t^5}(a_1 - 2t)dt = 0 \qquad [117]$$

$$\int_a^b \frac{m(t)}{t^7}\left[5a_1^2 - 20a_1t + 4(4 - a_2)t^2\right]dt = 0 \qquad [118]$$

$$\int_a^b \frac{m(t)}{t^9}\left[35a_1^3 - 210a_1^2 t + 60a_1(6-a_2)t^2 + 40(3a_2-4)t^3 + 24a_3 t^4\right]dt = 0 \qquad [119]$$

Also assume that m(x) may be expressed as a power series

$$m(x) = c_0 + c_1 x + c_2 x^2 + \ldots \qquad [120]$$

Then Equation [116] gives

$$\frac{c_0}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + c_1\left(\frac{1}{a} - \frac{1}{b}\right) + c_2 \log\frac{b}{a} + \ldots = \frac{1}{2}$$

or, neglecting $1/b^2$ in comparison with $1/a^2$ and setting b = 1 in comparison with 1/a,

$$c_0 + 2c_1 a(1-a) + 2c_2 a^2 \log\frac{1}{a} + \ldots = a^2 \qquad [121]$$

Similarly, Equations [117], [118], and [119] give, approximately

$$c_0(3a_1-8a) + 4c_1 a(a_1-3a) + 6c_2 a^2(a_1-4a+4a^2) = 0 \qquad [122]$$

$$2c_0\left[5a_1^2-24a_1a+6(4-a_2)a^2\right] + 4c_1 a\left[3a_1^2-15a_1a+4(4-a_2)a^2\right] + c_2a^2\left[15a_1^2-80a_1a+24(4-a_2)a^2\right] = 0 \qquad [123]$$

$$\begin{aligned} &3c_0\left[35a_1^3-240a_1^2a+80a_1(6-a_2)a^2+64(3a_2-4)a^3+48a_3a^4\right] \\ &+24c_1\left[5a_1^3a-35a_1^2a^2+12a_1(6-a_2)a^3+10(3a_2-4)a^4+8a_3a^5\right] \\ &+4c_2\left[35a_1^3a^2-252a_1^2a^3+90a_1(6-a_2)a^4+80(3a_2-4)a^5+72a_3a^6\right] = 0 \end{aligned} \qquad [124]$$

Equations [121] through [124] are sufficient in number to determine the unknowns a, c_0, c_1, c_2. Since the latter three equations are linear and homogeneous in c_0, c_1, and c_2, a can be determined from the condition that the determinant of their coefficients must vanish. In this way the following equation of the 7th degree in $\alpha = \frac{a_1}{a}$ was obtained:

$$\alpha(\alpha - 4)^2(5\alpha^4 - 83\alpha^3 + 288\alpha^2 - 368\alpha + 128) - 96a_2^2\alpha(3\alpha - 4)$$
$$+ 4a_2\alpha(\alpha - 4)(53\alpha^2 - 148\alpha + 128) + 1152a_1a_2^2(2\alpha - 3)$$
$$+ 72a_1(\alpha - 4)^2(5\alpha^3 - 25\alpha^2 + 40\alpha - 16) + 48a_1a_3\alpha(3\alpha - 8) \qquad [125]$$
$$- 288a_1a_2(\alpha - 4)(5\alpha^2 - 16 + 16) - 1152a_1^2a_3(\alpha - 3) = 0$$

Corresponding to a solution α of [125], c_0, c_1, and c_2 can be obtained from Equations [121], [122], and [123]. The solution of the latter equations gives

$$c_0D = -4a^2\left[3\alpha^3 - 37\alpha^2 + 120\alpha - 96 + 24a_2 + 24a(3\alpha^2 - 15\alpha + 16 - 4a_2)\right] \qquad [126]$$

$$c_1D = a\left[15\alpha^3 - 168\alpha^2 + 512\alpha - 384 + 96a_2 + 48a(5\alpha^2 - 24\alpha + 24 - 6a_2)\right] \qquad [127]$$

$$c_2D = -4\left[(\alpha - 4)^2(\alpha - 1) + 4a_2\right] \qquad [128]$$

where

$$D = 2(9\alpha^3 - 94\alpha^2 + 272\alpha - 192) + 8\left[(\alpha - 4)^2(\alpha - 1) + 4a_2\right]\log a + 96a_2$$
$$-2a(15\alpha^3 - 264\alpha^2 + 944\alpha - 768) - 384aa_2 - 96a^2(5\alpha^2 - 24\alpha + 24)$$
$$+ 576a^2a_2 \qquad [129]$$

The initial doublet strength at x = a is

$$m(a) = c_0 + c_1a + c_2a^2 + \ldots$$

or, from Equations [126] through [129],

$$m(a) = -\frac{a^2}{D}\left[(\alpha - 4)(\alpha^2 - 12\alpha + 16) + 48a(\alpha - 4)(\alpha - 2) + 16a_2 - 96aa_2\right] \qquad [130]$$

Equations [125] through [130] determine the end points of the distribution and its initial trends. In general, Equation [125] will have more than one real root. In this case the initial trends corresponding to each of the roots should be examined, and that root chosen which appears to give the "simplest" trend.

The equations can be solved explicitly in the case of a very elongated body for which a_1, a_2, a_3, ... in [115] are all very small. First let us suppose that they are so small that all the terms in [125] containing them are negligible, so that the first product term alone may be equated to zero, i.e.,

$$\alpha(\alpha - 4)^2(5\alpha^4 - 83\alpha^3 + 288\alpha^2 - 368\alpha + 128) = 0 \qquad [131]$$

whose real roots are α = 0, 0.547, 4.0, 4.0, and 12.429.

Let us consider the solution $\alpha = 4$; i.e., $a = \frac{a_1}{4}$. Since the radius of curvature at x = 0 is $a_1/2$, this solution is seen to be in accord with Kaplan's assumption for the end points of the distribution. Furthermore, substituting $\alpha = 4$ into Equations [129] and [130], we obtain, to the same order of approximation,

$$D = 64, \quad c_0 = -\frac{a_1^2}{16}, \quad c_1 = \frac{a_1}{4}, \quad c_2 = 0$$

whence

$$m(x) = -\frac{a_1^2}{16} + \frac{a_1}{4}x, \quad m(a) = 0 \qquad [132]$$

In order to obtain a second approximation it will be assumed that not only a_1, a_2, a_3, ... but also $(\alpha - 4)$ are small to the first order. Then, neglecting terms of third and higher order, Equation [125] becomes

$$-3072(\alpha - 4)^2 + 6144a_2(\alpha - 4) - 3072a_2^2 + 768a_1a_3 = 0 \qquad [133]$$

whence

$$\alpha = 4 + a_2 \pm \frac{1}{2}\sqrt{a_1a_3}^{*} \qquad [134]$$

provided

$$a_3 \geqq 0$$

Corresponding to this value of α we obtain from Equations [126] through [129], to the same order of approximation,

*The smaller of these two roots has given the preferred solution in all cases tried thus far.

$$\left.\begin{aligned} m(x) &= C\left(-\frac{a_1^2}{4} + a_1x + a_2x^2 + \ldots\right) \\ \text{where}\qquad & \\ C &= \frac{1}{4}\left(1 + \frac{a_1}{2} + \frac{a_2}{2}\log\frac{a_1}{4}\right) \end{aligned}\right\} \qquad [135]$$

and

$$m(a) = \pm\frac{1}{2}\,Ca^2\sqrt{a_1a_3} \qquad [136]$$

The expression for m(x) in [135] may also be written as

$$m(x) = C\left(-\frac{a_1^2}{4} + y^2\right) \qquad [135a]$$

When $a_3 < 0$ the solution for α in [134] indicates that there would be no real roots near $\alpha = 4$. In this case a graph of the complete polynomial in [125] should be examined either for the possibility that more complete calculations would show that there are real roots near $\alpha = 4$ nevertheless, or that the maximum value of the complete polynomial in the neighborhood of $\alpha = 4$ is so nearly zero, that the value of α corresponding to this maximum may be taken as an approximate solution. On this assumption, the second order analysis would give

$$\alpha = 4 + a_2, \quad a_3 < 0 \qquad [137]$$

Since a_3 does not occur explicitly in Equations [135], it is seen that they would also be obtained, to the same order of approximation, if the value of α in [137] were substituted into Equations [126] through [129].

If it is determined that not even an approximate solution can be assumed near $\alpha = 4$ it would be necessary to consider solutions in the neighborhood of the other roots of Equation [131].

In order to facilitate the computations for graphing the polynomial in [125], the functions $A(\alpha)$, $B(\alpha)$, ... $H(\alpha)$, where

$$
\begin{aligned}
A(\alpha) &= \alpha(\alpha - 4)^2(5\alpha^4 - 83\alpha^3 + 288\alpha^2 - 368\alpha + 128) \\
B(\alpha) &= 72(\alpha - 4)^2(5\alpha^3 - 25\alpha^2 + 40\alpha - 16) \\
C(\alpha) &= 4\alpha(\alpha - 4)(53\alpha^2 - 148\alpha + 128) \\
D(\alpha) &= -288(\alpha - 4)(5\alpha^2 - 16\alpha + 16) \\
E(\alpha) &= -96\alpha(3\alpha - 4) \\
F(\alpha) &= 1152(2\alpha - 3) \\
G(\alpha) &= 48\alpha(3\alpha - 8) \\
H(\alpha) &= -1152(\alpha - 3)
\end{aligned} \qquad [138]
$$

have been tabulated in Table 10. In terms of these functions, Equation [125] becomes

$$
A + a_1B + a_2C + a_1a_2D + a_2^2E + a_1a_2^2F + a_1a_3G + a_1^2a_3H = 0 \qquad [139]
$$

It is of interest to compare the approximate value for α from Equation [134] with the exact value for the prolate spheroid $y^2 = \frac{1}{\lambda^2}(x - x^2)$. In this case we have

$$
a_1 = -a_2 = \frac{1}{\lambda^2}, \qquad a_3 = 0
$$

and the exact value of α is

$$
\alpha = 2 + 2\sqrt{1 - \frac{1}{\lambda^2}} = 4 - \frac{1}{\lambda^2} - \frac{1}{4\lambda^4} - \dots
$$

But when the length-diameter ratio λ is large, Equation [134] gives the approximate value $\alpha = 4 - \frac{1}{\lambda^2}$, which is seen to consist of the first two terms of the series expansion of the exact value of α. Table 11 shows that the approximate formula gives excellent agreement with the exact values even for very thick sections. Both the exact and the approximate formulas give $m(a) = 0$. Thus the present approximate methods work very well for the prolate spheroid.

TABLE 10

Functions for Determining Limits of Doublet Distributions

α	$A(\alpha)$	$B(\alpha)$	$C(\alpha)$	$D(\alpha)$	$E(\alpha)$	$F(\alpha)$	$G(\alpha)$	$H(\alpha)$
0	0	-18432.0	0	18432.0	0	-3456.0	0	3456.0
0.1	143.0	-13409.7	-177.4	16230.2	35.5	-3225.6	-37.0	3340.8
.2	188.5	-9315.5	-305.6	14227.2	65.3	-2995.2	-71.0	3225.6
.3	169.7	-6027.4	-392.4	12414.2	89.3	-2764.8	-102.2	3110 4
.4	112.5	-3433.9	-445.1	10782.7	107.5	-2534.4	-130.6	2995.2
.5	36.4	-1433.3	-470.8	9324.0	120.0	-2304.0	-156.0	2880.0
.6	-44.4	66.6	-475.6	8029.4	126.7	-2073.6	-178.6	2764.8
.7	-120.1	1148.7	-465.4	6890.4	127.7	-1843:2	-198.2	2649.6
.8	-184.5	1887.4	-445.6	5898.2	122.9	-1612.8	-215.0	2534.4
0.9	-234.8	2349.1	-421.1	5044.3	112.3	-1382.4	-229.0	2419.2
1.0	-270.0	2592.0	-396.0	4320.0	96.0	-1152.0	-240.0	2304.0
.1	-291.2	2667.3	-374.3	3716.6	73.9	-921.6	-248.2	2188.8
.2	-300.5	2619.2	-359.1	3225.6	46.1	-691.2	-253.4	2073.6
.3	-300.9	2485.3	-353.4	2838.2	12.5	-460.8	-255.8	1958.4
.4	-295.9	2297.3	-359.3	2545.9	-26.9	-230.4	-255.4	1843.2
.5	-288.9	2081.3	-378.8	2340.0	-72.0	0	-252.0	1728.0
.6	-283.1	1857.9	-412.9	2211.8	-122.9	230.4	-245.8	1612.8
.7	-281.5	1643.5	-462.5	2152.8	-179.5	460.8	-236.6	1497.6
.8	-286.2	1449.7	-527.8	2154.2	-241.9	691.2	-224.6	1382.4
1.9	-298.8	1284.4	-608.6	2207.5	-310.1	921.6	-209.8	1267.2
2.0	-320.0	1152.0	-704.0	2304.0	-384.0	1152.0	-192.0	1152.0
.1	-349.8	1054.0	-812.8	2435.0	-463.7	1382.4	-171.4	1036.8
.2	-387.3	989.1	-933.3	2592.0	-549.1	1612.8	-147.8	921.6
.3	-430.9	954.0	-1063.1	2766.2	-640.3	1843.2	-121.4	806.4
.4	-478.2	943.7	-1199.3	2949.1	-737.3	2073.6	-92.2	691.2
.5	-526.3	951.8	-1338.8	3132.0	-840.0	2304.0	-60.0	576.0
.6	-572.0	970.9	-1477.5	3306.2	-948.5	2534.4	-25.0	460.8
.7	-611.7	993.5	-1611.4	3463.2	-1062.7	2764.8	13.0	345.6
.8	-641.8	1011.9	-1735.4	3594.2	-1182.7	2995.2	53.8	230.4
2.9	-658.9	1018.9	-1844.2	3690.7	-1308.5	3225.6	97.4	115.2
3.0	-660.0	1008.0	-1932.0	3744.0	-1440.0	3456.0	144.0	0
.1	-642.8	974.2	-1992.4	3745.4	-1577.3	3686.4	193.6	-115.2
.2	-606.1	914.2	-2018.5	3686.4	-1720.3	3916.8	245.8	-230.4
.3	-549.6	826.8	-2003.0	3558.2	-1869.1	4147.2	301.0	-345.6
.4	-474.9	713.3	-1937.8	3352.3	-2023.7	4377.6	359.0	-460.8
.5	-385.3	578.3	-1814.8	3060.0	-2184.0	4608.0	420.0	-576.0
.6	-286.2	429.5	-1624.8	2672.6	-2350.1	4838.4	483.8	-691.2
.7	-185.8	278.7	-1358.5	2181.6	-2521.9	5068.8	550.6	-806.4
.8	-94.8	142.2	-1006.0	1578.2	-2699.5	5299.2	620.2	-921.6
3.9	-27.0	40.6	-556.8	853.9	-2882.9	5529.6	692.6	-1036.8
4.0	0	0	0	0	-3072.0	5760.0	768.0	-1152.0
.1	-34.7	52.1	675.9	-992.2	-3266.9	5990.4	846.2	-1267.2
.2	-156.4	234.5	1482.8	-2131.2	-3467.5	6220.8	927.4	-1382.4
.3	-394.3	591.5	2433.3	-3425.8	-3673.9	6451.2	1011.4	-1497.6
.4	-782.7	1174.1	3540.3	-4884.5	-3886.1	6681.6	1098.2	-1612.8
.5	-1360.2	2040.8	4817.3	-6516.0	-4104.0	6912.0	1188.0	-1728.0
.6	-2170.8	3257.6	6278.2	-8329.0	-4327.7	7142.4	1280.6	-1843.2
.7	-3263.7	4899.2	7937.7	-10332.0	-4557.1	7372.8	1376.2	-1958.4
.8	-4693.4	7048.4	9810.7	-12533.8	-4792.3	7603.2	1474.6	-2073.6
4.9	-6520.2	9797.5	11912.8	-14942.9	-5033.3	7833.6	1575.8	-2188.8
5.0	-8810.0	13248.0	14260.0	-17568.0	-5280.0	8064.0	1680.0	-2304.0
0	0	-18432	0	18432	0	-3456	0	3456
1	-270	2592	-396	4320	96	-1152	-240	2304
2	-320	1152	-704	2304	-384	1152	-192	1152
3	-660	1008	-1932	3744	-1440	3456	144	0
4	0	0	0	0	-3072	5760	768	-1152
5	-8810	13248	14260	-17568	-5280	8064	1680	-2304
6	-75840	116352	55104	-57600	-8064	10368	2880	-3456
7	-302400	488592	141876	-128736	-11424	12672	4368	-4608
8	-819200	1456128	299008	-239616	-15360	14976	6144	-5760
9	-1700550	3535200	556020	-398880	-19872	17280	8208	-6912
10	-2790720	7475328	947520	-615168	-24960	19584	10560	-8064
11	-3417260	14306040	1513204	-897120	-30624	21888	13200	-9216
12	-1966080	25362432	2297856	-1253376	-36864	24192	16128	-10368
13	4706910	42363648	3351348	-1692576	-43680	26496	19344	-11520
14	22052800	67420800	4728640	-2223360	-51072	28800	22848	-12672
15	58820520	103097808	6489780	-2854368	-59040	31104	26640	-13824

$$A + a_1B + a_2C + a_1a_2D + a_2^2E + a_1a_2^2F + a_1a_3G + a_1^2a_3H = 0$$

$A = \alpha(\alpha - 4)^2(5\alpha^4 - 83\alpha^3 + 288\alpha^2 - 368\alpha + 128)$ $\qquad$ $E = -96\alpha(3\alpha - 4)$

$B = +72(\alpha - 4)^2(5\alpha^3 - 25\alpha^2 + 40\alpha - 16)$ $\qquad$ $F = 1152(2\alpha - 3)$

$C = 4\alpha(\alpha - 4)(53\alpha^2 - 148\alpha + 128)$ $\qquad$ $G = 48(3\alpha - 8)\alpha$

$D = -288(\alpha - 4)(5\alpha^2 - 16\alpha + 16)$ $\qquad$ $H = -1152(\alpha - 3)$

TABLE 11

Comparison of Exact and Computed Values of $\alpha = \frac{a_1}{a}$ for a Prolate Spheroid

λ	2	3	4	5	6
Exact α	3.732	3.886	3.936	3.960	3.972
Approximate α	3.750	3.889	3.937	3.960	3.972

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