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UNITED STATES EXPERIMENTAL MODEL BASIN

NAVY YARD, WASHINGTON, D.C.

ON THE RESISTANCE OF A HEAVY FLEXIBLE CABLE FOR TOWING A SURFACE FLOAT BEHIND A SHIP

BY J. G. THEWS AND L. LANDWEBER

RECLASSIFIED

Per T-M-B

Serial A6-5-(2)

Dated 6/6/47

EXPERIMENTAL MODEL BASIN
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MARCH 1936 REPORT NO. 418

ON THE RESISTANCE OF A HEAVY FLEXIBLE CABLE FOR TOWING A SURFACE FLOAT BEHIND A SHIP

by

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U.S. Experimental Model Basin
Navy Yard, Washington, D.C.

March 1936 Report No. 418

NOTATION

W Weight per unit length of cable

R Drag per unit length of cable when normal to stream

F Drag per unit length of cable when parallel to stream

V Speed of ship

ϕ Angle of cable to horizontal

Y Ordinate of any point of cable from origin 0 at lowest point

S Length of cable from origin 0

T Tension of cable at any point

T_o Tension of cable at origin 0

L Length of cable between points having equal ordinates

p Defined by $p^2 = \sqrt{4r^2 + 1} + 2r$

$f = F/W$

$r = R/W$

$t = T/T_o; \qquad t_1 = T_1/T_o, \quad T_1 \equiv T \text{ for } \phi < 0$

$\qquad\qquad t_2 = T_2/T_o, \quad T_2 \equiv T \text{ for } \phi > 0$

$Z_1 = -\tan \phi/2, \quad \phi < 0$

$Z_2 = \tan \phi/2, \quad \phi > 0$

$C = 2f/p$

$z_1 = pZ_1, \quad z_2 = pZ_2$

$y = \frac{Wp^2}{4T_o} Y$

u, v Defined by $z_1 = \tan u, \quad z_2 = \tanh v$

THE RESISTANCE OF A HEAVY FLEXIBLE CABLE FOR TOWING
A SURFACE FLOAT BEHIND A SHIP

Introduction

In an attempt to predict the behavior of a float towed behind a ship by a long, heavy, flexible cable, two distinct problems arose, for whose solution a knowledge of the forces in the cable was required.

1. What must be the strength of the cable? To answer this, it was necessary to know the tension in the cable at the ship.

2. How does the cable affect the behavior of the float? For it is clear that the cable has a two-fold influence upon the float. The horizontal component of the cable tension at the float balances the resistance of the fluid to the motion of the float. In addition to this, there is the vertical component of the tension which in effect increases the displacement of the float and consequently affects its resistance and behavior. To answer this second question it is necessary, then to know the horizontal and vertical components of the tension, or, what is the same thing, to know the tension at the float and the angle that the cable makes with the horizontal at that point.

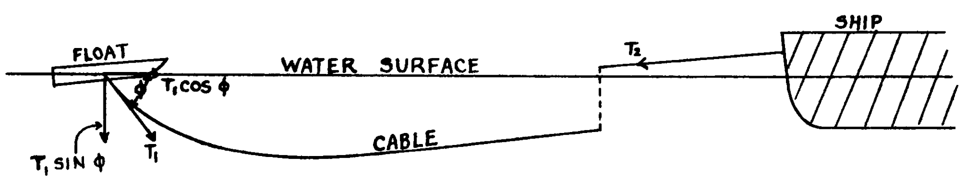

FIG. 1.

Previous work on the form and resistance of a towing cable has furnished no solution of the present problem. Mathematical expressions[1] and numerical solutions[2]

(1) A.R.McLeod. On the action of wind on flexible cables, with applications to cables towed below aeroplanes and balloon cables. R. and M. 554. Oct. 1918.
(2) H.Glauert. The form of a heavy flexible cable used for towing a heavy body below an aeroplane. R. and M. 1592. Feb. 1934.

for the form assumed by a cable used to tow a heavy body below an aeroplane have been developed only for the case where the cable is concave downwards. In addition it has been assumed hitherto that the frictional resistance of the fluid is negligible compared with the resultant forces of lift and drag acting on the cable. Mathematical expressions and numerical solutions have therefore been derived here suitable to the present problem in which the cable is concave upwards and where the angle of the cable with the horizontal is so small over a considerable length of cable that frictional resistance cannot be neglected. Hence in addition to the physical assumptions used in reference (1), an assumption regarding the magnitude of the frictional resistance must be introduced.

General Analysis

Consider a flexible cable of weight W per unit length. Let R* be the drag per unit length of the cable when at right angles to a stream of velocity V; F* the drag per unit length when parallel to the stream. When the cable is inclined at an angle ϕ to the stream, the force per unit length on the cable due to the stream will be assumed to consist of a component $R \sin^2\phi$ at right angles to the length of the cable, and a component F along it. These assumptions are approximations to the actual experimental results. Their experimental basis is reviewed in Appendix I. No other physical assumptions will be made in the analysis of the problem. The general shape of the cable is shown in Fig. 2, the origin O being taken at the point

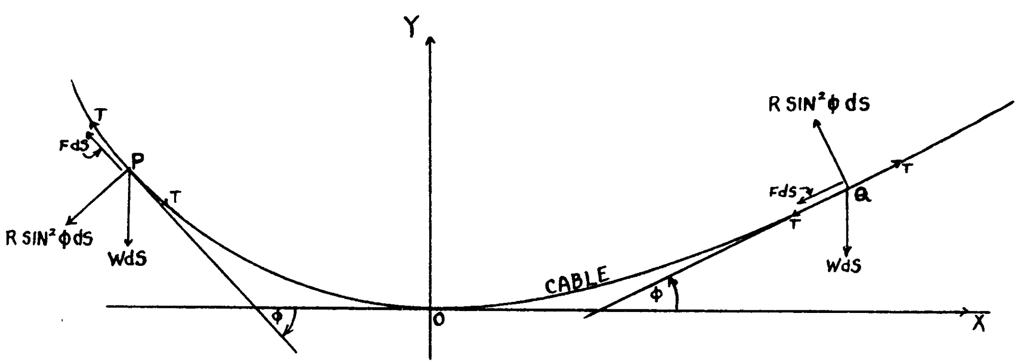

FIG. 2.

of zero slope. Associated with each point P (or Q) of the cable there is the ordinate Y, the arc length S measured from O, and the angle of inclination ϕ of the cable with the horizontal at the point P (or Q). These variables are related by the equation

$$dY/dS = \sin\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

*See Appendix II

The element dS of the cable at P (or Q) is in equilibrium under the action of the system of forces comprising its weight WdS, the force $R \sin^2\phi$ dS normal to the cable, the force FdS along the cable and the tension at its ends. Denoting the tension of the cable at any point by T and resolving along the cable, we obtain the equation

$$dT/dS = F + W \sin\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

which may be combined with (1) to eliminate $\sin\phi$ and then integrated to give

$$T = T_0 + FS + WY \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where T_0 is the tension of the cable at the origin. Resolving also at right angles to the cable we obtain

$$Td\phi/dS = W \cos\phi + R \sin^2\phi \quad , \quad \phi < 0 \quad \text{(at P)} \ldots\ldots\ldots (4a)$$

and
$$Td\phi/dS = W \cos\phi - R \sin^2\phi \quad , \quad \phi > 0 \quad \text{(at Q)} \ldots\ldots\ldots (4b)$$

Equations (2) and (4a) may be combined to eliminate S and then integrated to give a relationship between T and ϕ. For, from (2) and (4a) we obtain

$$dT/Td\phi = \frac{F + W \sin\phi}{W \cos\phi + R \sin^2\phi} = \frac{f + \sin\phi}{\cos\phi + r \sin^2\phi}, \quad \text{where } f = F/W \text{ , } r = R/W \quad \ldots (5)$$

Introducing the substitution $Z_1 = -\tan\frac{\phi}{2}$, (5) may be integrated (see Appendix III) to give

$$\log t_1 = 2f \frac{p - 1/p}{p^2 + 1/p^2} \tan^{-1} pZ_1 + \frac{2}{p^2 + 1/p^2} \log \frac{1 + p^2 Z_1^2}{1 - Z_1^2/p^2} -$$

$$f \frac{p + 1/p}{p^2 + 1/p^2} \log \frac{1 + Z_1/p}{1 - Z_1/p} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

where $p^2 = \sqrt{4r^2 + 1} + 2r$, and $t_1 = T_1/T_0$, $T_1 \equiv T$ for $\phi < 0$

We can now express Y in terms of T and Z_1 by eliminating S between equations (1) and (4a). Thus we obtain

$$dY/T_1 d\phi = \frac{\sin\phi}{W \cos\phi + R \sin^2\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Making the substitution $Z_1 = -\tan\phi/2$, (7) becomes

$$Y = \int_0^{Z_1} \frac{4T_0}{W} \frac{t_1 Z_1 dZ_1}{(p^2 - Z_1^2)(1/p^2 + Z_1^2)} \quad \ldots\ldots\ldots\ldots\ldots (8)$$

Equations (6) and (8) have been derived for the case $\phi < 0$. Correspondingly, for $\phi > 0$, we obtain from equations (2) and (4b)

$$dT/Td\phi = \frac{F + W \sin \phi}{W \cos \phi - R \sin^2\phi} = \frac{f + \sin \phi}{\cos \phi - r \sin^2\phi} \quad \ldots\ldots\ldots (9)$$

Introducing the substitution $Z_2 = \tan \phi/2$, (9) may be integrated (see Appendix III) to give

$$\log t_2 = -2f \frac{p - 1/p}{p^2 + 1/p^2} \tan^{-1} Z_2/p + \frac{2}{p^2 + 1/p^2} \log \frac{1 + Z_2^2/p^2}{1 - p^2 Z_2^2} +$$

$$f \frac{p + 1/p}{p^2 + 1/p^2} \log \frac{1 + pZ_2}{1 + pZ_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

where $t_2 = T_2/T_O$, $T_2 \equiv T$ for $\phi > 0$.
Also from equations (1) and (4b), we obtain

$$dY/Td\phi = \frac{\sin \phi}{W \cos \phi + R \sin^2\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Making the substitution $Z_2 = \tan \phi/2$, (11) becomes

$$Y = \int_0^{Z_2} \frac{4T_O}{W} \frac{t_2 Z_2 dZ_2}{(p^2 + Z_2^2)(1/p^2 - Z_2^2)} \quad \ldots\ldots\ldots\ldots\ldots (12)$$

We conclude the general analysis by deriving a relation between the tensions and the length of the cable L between points determined by a prescribed ordinate Y. For a given ordinate Y let T_1, S_1 for $\phi < 0$ correspond to T_2 and S_2 for $\phi > 0$. Then by (3),

$$T_1 = T_O + FS_1 + WY \quad \text{and} \quad T_2 = T_O + FS_2 + WY. \quad \text{Hence}$$

$$T_2 - T_1 = F(S_2 - S_1) = F \times L \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

noting by (1) that S_1 is always negative.

<u>Application to determination of cable resistance.</u> <u>Approximations.</u>

In the particular problem with which we shall concern ourselves, the ends of the given cable are points having the same value for the ordinates. It is required to find the tensions at these ends as functions of the angle ϕ at the rear end of the cable ($\phi < 0$); or, what is the same thing, as functions of Z_1. But by (13)

$$T_1 \equiv \frac{T_1}{T_O} \frac{T_2 - T_1}{T_2/T_O - T_1/T_O} = F \times L \frac{t_1}{t_2 - t_1} \quad \ldots\ldots\ldots\ldots\ldots (14)$$

Hence a procedure for obtaining numerical solutions of T_1 and T_2 as functions of Z_1 may be outlined as follows:

1. Compute values of t_1 against Z_1 from (6)

2. Enter these values of t_1 in (8) to compute Y/T_0 against Z_1 by graphical or numerical integration. Thus we obtain values of Y/T_0 against t_1.

3. Obtain values of t_2 against Y/T_0 from (10) and (12), in a similar way.

4. Taking values of t_1 and t_2 corresponding to the same Y/T_0, T_1 can be computed from (14) and T_2 from (13).

Thus the problem as stated above is formally solved. Yet it is easily seen that the equations (6), (8), (10) and (12), involving as they do the two independent parameters f and p, do not readily lend themselves to the calculation of a general, dimensionless set of solutions for T_1 and T_2. It will be shown in Appendix IV that a close approximation is obtained by replacing equations (6), (8), (10), and (12) by the following one-parameter set of equations:

$$\log t_1 = -c \tan^{-1} z_1 \quad \text{(6a)}$$

$$y = \int_0^{z_1} \frac{t_1 z dz}{1 + z^2} \quad \text{(8a)}$$

$$\log t_2 = \frac{c}{2} \log \frac{1 + z_2}{1 - z_2} \quad \text{(10a)}$$

$$y = \int_0^{z_2} \frac{t_2 z dz}{1 - z^2} \quad \text{(12a)}$$

where $c = 2f/p$ and $z_1 = pZ_1$, $z_2 = pZ_2$, $y = \frac{wp^2}{4T_0} Y$.

Forms more convenient for calculation are obtained by introducing the substitutions

$$u = \tan^{-1} z_1 \text{ and } v = \tfrac{1}{2} \log \frac{1 + z_2}{1 - z_2}. \text{ Then } z_1 = \tan u \text{ and } z_2 = \tanh v,$$

and the above equations become

$$t_1 = e^{-cu} \quad \text{(6b)}$$

$$y = \int_0^u e^{-cu} \tan u \, du \quad \text{(8b)}$$

$$t_2 = e^{cv} \quad \text{(10b)}$$

$$y = \int_0^v e^{cv} \tanh v \, dv \quad \text{(12b)}$$

Tables 1 and 2 for computing y are based upon equations (8b) and (12b) respectively. For, expanding e^{-cu} into a power series, (8b) becomes

$$y = \log \sec u - c \int_0^u u \tan u \, du + \frac{c^2}{2!} \int_0^u u^2 \tan u \, du - \ldots \qquad \ldots\ldots (8c)$$

and similarly from (12b)

$$y = \log \cosh v + c \int_0^v v \tanh v \, dv + \frac{c^2}{2!} \int_0^v v^2 \tanh v \, dv + \ldots \qquad \ldots\ldots (12c)$$

The coefficients of c^3 and higher powers were not computed because, in general, c will be small.

For values of c = .02, .04, .06, .08, .10, .12 and .14, t_1, t_2 and y were computed from (6b), (10b), (8c) and (12c) respectively. Taking values of t_1 and t_2 corresponding to the same y, T_1/FL was computed from (14). Fig. 3 is a plot of T_1/FL against $\tan \phi/2$. To facilitate the use of Fig. 3 to compute the tensions as functions of ϕ, Table 3 was constructed, giving values of $\tan \phi/2$, $\sin \phi$ and $\cos \phi$ against ϕ. To illustrate the method, the tension T_1 and its vertical and horizontal components are computed against ϕ for a hypothetical cable 1000 feet long, weighing one pound per foot moving at a speed such that F = f = 1 lb. per ft., p = 20, and hence c = 2f/p = .10. The results are shown in table 4. It is seen that the vertical component $T_1 \sin \phi$ is nearly constant for values of ϕ between 60° and 90°.

Conclusion

We can now formulate the following procedure for answering the questions raised in the introduction. Suppose a given cable towing a given float.

(a) Obtain R and F cable-resistance curves against speed either by new tests or by using the data of Wieselsberger for R and of Kempf for F, for a smooth cable (see Appendix II). Recalling that r = R/w and $p^2 = \sqrt{4r^2 + 1} + 2r$, determine f = F/W and c = 2 f/p as functions of speed.

(b) From Fig. 3 and table 3, compute T_1, $T_1 \sin \phi$, $T_1 \cos \phi$ for speeds and f values corresponding to C = .02, .04, ...,.14, as illustrated in table 4. As in table 4, the additional displacement $T_1 \sin \phi$ will vary slowly with the drag of the float, $T_1 \cos \phi$. Choose a mean value of $T_1 \sin \phi$ in the neighborhood of the estimated drag $T_1 \cos \phi$. Plot these mean values of $T_1 \sin \phi$ against speed.

(c) Using these mean values of the corrected displacement corresponding to each speed, obtain the resistance curve of the float. This is equivalent to obtaining $T_1 \cos \phi$ and hence also T_1 for each speed. T_2 is then calculable from (13).

Let us illustrate the procedure in (a), (b), and (c) with the example from which table 4 was computed.

(a') Suppose that the cable-resistance curves have been obtained, as described in (a) above, and that f and c have been plotted against speed. From these curves read value of speed corresponding to $c = .10$, and also value of f at this speed. Suppose we find for the speed $v = 30$ ft./sec. and $f = 1.00$ lb./ft.

(b') From Fig. 3 and table 3 compute T_1, $T_1 \sin \phi$, $T_1 \cos \phi$ for $c = .10$, and hence at 30 ft./sec. with $f = 1.00$ lb./ft. This yields table 4. Suppose now that although the resistance of the float and its displacement when towed at 30 ft./sec. are unknown, we can estimate that its resistance will lie between 500 and 1000 pounds. For values of $T_1 \cos \phi$ in this range we see from table 4 that the additional displacement $T_1 \sin \phi$ varies from 1726 lbs. to 1815 lbs. Take 1770 lbs. as a mean value of the additional displacement. Supposing that the weight of the float itself is 300 lbs., we have 1770 + 300 = 2070 lbs. for the total displacement of the float when towed by the cable at 30 ft./sec.

(c') By model tests, or otherwise, find the resistance of the float for a speed of 30 ft./sec. when its displacement is 2070 lbs. Suppose we find that the resistance at this speed and displacement is 690 lbs., i.e. $T_1 \cos \phi = 690$ lbs., whence, from table 4, we have $T_1 \sin \phi = 1800$ lbs. as a corrected value for the additional displacement. The resistance of the float at 30 ft./sec. could then be corrected using 1800 + 300 = 2100 lbs. for the total displacement. Finally, we see again from table 4 that when $T_1 \cos \phi = 690$ lbs., $T_2 = 2930$ lbs. which is the tension in the cable at the ship at this speed.

TABLE 1.

$$y = \ln \sec u - c \int_0^u u \tan u \, du + \frac{c^2}{2!} \int_0^u u \tan u \, du$$

u	$z_1 = \tan u$	$\ln \sec u$	$\int_0^u u \tan u \, du$	$\int_0^u u^2 \tan u \, du$
.1	.100	.005	0	0
.2	.203	.021	.003	0
.3	.309	.046	.009	.002
.4	.423	.083	.022	.007
.5	.546	.131	.044	.017
.6	.684	.191	.078	.035
.7	.842	.267	.127	.070
.8	1.03	.362	.197	.123
.9	1.26	.475	.294	.205
1.0	1.56	.615	.427	.330
1.1	1.96	.790	.610	.521
1.2	2.57	1.015	.871	.824
1.3	3.60	1.320	1.247	1.297
1.4	5.80	1.771	1.864	2.133
1.45	8.24	2.12	2.349	2.823
1.48	10.98	2.40	2.770	3.442
1.50	14.10	2.65	3.140	3.994
1.52	19.67	2.98	3.64	4.75
1.53	24.5	3.20	3.97	5.26
1.54	32.5	3.48	4.40	5.92
1.55	48.1	3.88	5.01	6.86
1.56	92.6	4.53	6.02	8.43

TABLE 2.

$$y = \ln \cosh v + c \int_0^v v \tanh v \, dv + \frac{c^2}{2!} \int_0^v v^2 \tanh v \, dv$$

v	$\ln \cosh v$	$\int_0^v v \tanh v \, dv$	$\int_0^v v^2 \tanh v \, dv$
.1	.005	0	0
.2	.019	.003	0
.3	.044	.009	.002
.4	.078	.021	.006
.6	.171	.067	.030
.8	.291	.152	.090
1.0	.434	.280	.206
1.2	.595	.456	.328
1.4	.768	.680	.621
1.6	.949	.951	1.028
1.8	1.135	1.269	1.570
2.0	1.327	1.632	2.260
2.2	1.521	2.039	3.115
2.4	1.718	2.489	4.151
2.7	2.02	3.245	6.09
3.0	2.31	4.09	8.51
3.3	2.61	5.04	11.48
3.6	2.91	6.07	15.05
3.9	3.21	7.19	19.27
4.2	3.51	8.41	24.20
4.5	3.81	9.71	29.88
5.0	4.31	12.09	41.16

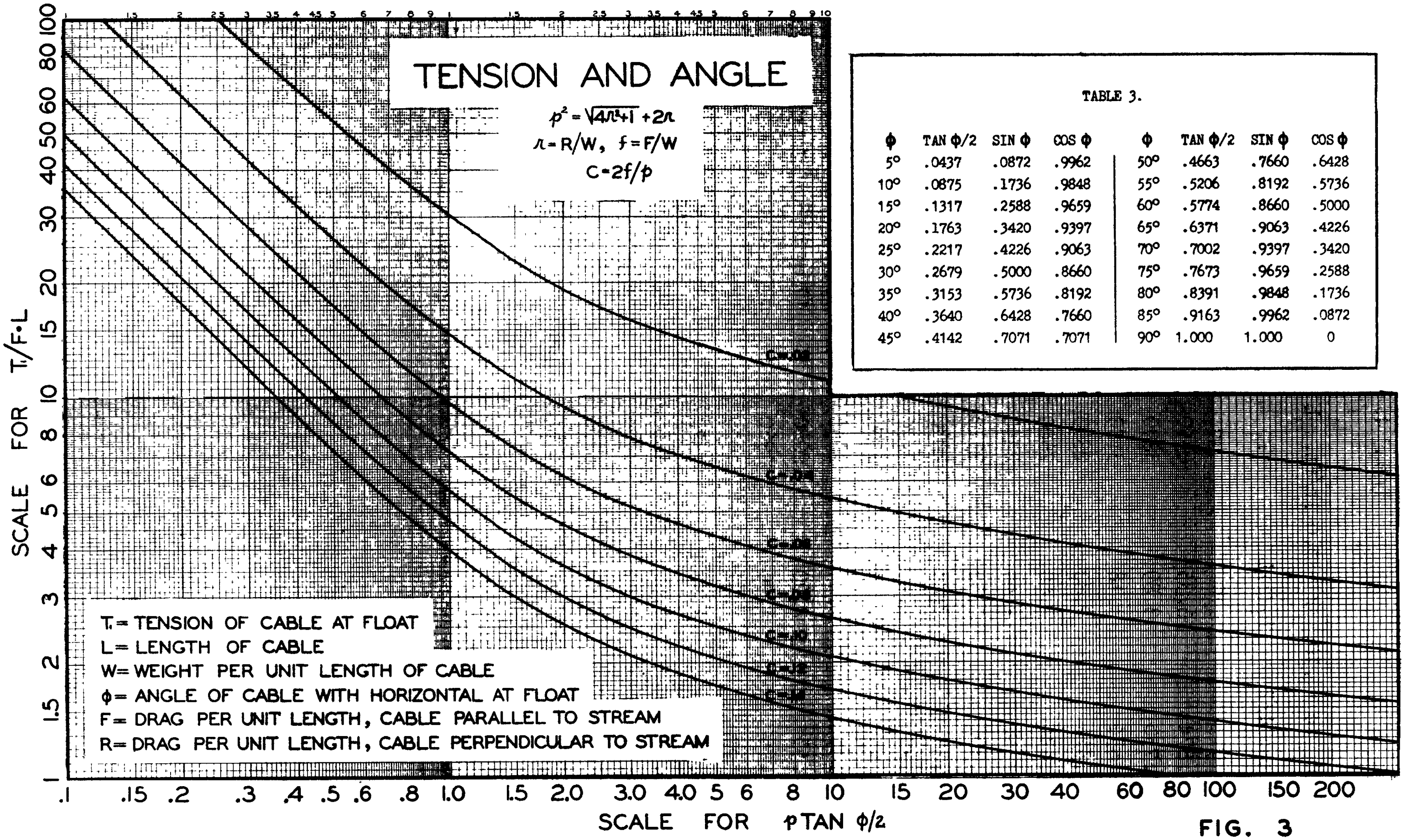

FIG. 3

TABLE 3.

ϕ	TAN $\phi/2$	SIN ϕ	COS ϕ	ϕ	TAN $\phi/2$	SIN ϕ	COS ϕ
5°	.0437	.0872	.9962	50°	.4663	.7660	.6428
10°	.0875	.1736	.9848	55°	.5206	.8192	.5736
15°	.1317	.2588	.9659	60°	.5774	.8660	.5000
20°	.1763	.3420	.9397	65°	.6371	.9063	.4226
25°	.2217	.4226	.9063	70°	.7002	.9397	.3420
30°	.2679	.5000	.8660	75°	.7673	.9659	.2588
35°	.3153	.5736	.8192	80°	.8391	.9848	.1736
40°	.3640	.6428	.7660	85°	.9163	.9962	.0872
45°	.4142	.7071	.7071	90°	1.000	1.000	0

TABLE 4.

$p = 20$, $c = .10$, $F = 1.0$ lb/ft, $L = 1000$ ft., $T_2 = T_1 + F \times L$

ϕ	$p \tan \phi/2$	T_1/FxL	T_1	$T_1 \sin \phi$	$T_1 \cos \phi$	T_2
0	0	∞	∞	500	∞	∞
5°	.874	6.35	6350	554	6320	7350
10°	1.750	3.89	3890	675	3830	4890
15°	2.634	3.18	3180	824	3070	4180
20°	3.526	2.80	2800	958	2630	3800
25°	4.434	2.60	2600	1100	2350	3600
30°	5.358	2.44	2440	1220	2110	3440
35°	6.306	2.32	2320	1330	1900	3320
40°	7.28	2.23	2230	1435	1710	3230
45°	8.28	2.17	2170	1535	1535	3170
50°	9.33	2.10	2100	1610	1350	3100
55°	10.41	2.05	2050	1680	1175	3050
60°	11.55	2.00	2000	1726	1000	3000
65°	12.75	1.96	1960	1775	828	2960
70°	14.00	1.92	1920	1805	657	2920
75°	15.35	1.88	1880	1815	487	2880
80°	16.79	1.85	1850	1823	321	2850
85°	18.34	1.82	1820	1813	159	2820
90°	20.00	1.80	1800	1800	0	2800

Appendix I

The physical assumptions we have made above concerning the components of the forces on the cable normal and parallel to it are based on the experiments and data of Relf and Powell (R. and M. No. 307, January, 1917). They measured the lift and drag of an infinite, straight cable, situated in a uniform stream of air, and inclined at an angle ϕ to the direction of air flow.

Let R be the force per unit length acting on the cable when it is normal to the wind. Let F be the force per unit length acting on the cable when it is parallel to the wind. Our assumptions then are that for a given speed the components of the force per unit length acting on the cable are $R \sin^2 \phi$ normal to the cable and a constant F parallel to the cable. The values of the normal and parallel components were computed from the paper cited above for the case of a stranded wire cable of diameter 0.388 inches. Relf and Powell made their measurements with a wind speed of 40 ft./sec. The experimental results are compared with the assumed laws in table 5 and fig. 4 taking R = .074 and F = .0023.

TABLE 5.

	Normal Component		Tangential Component	
ϕ	Experimental	$.074 \sin^2\phi$	Experimental	F.
0	0	0	.0023	.0023
10°	.0027	.0022	.0028	.0023
20°	.0089	.0087	.0031	.0023
30°	.0174	.0185	.0033	.0023
40°	.0280	.0306	.0033	.0023
50°	.0409	.0434	.0033	.0023
60°	.0544	.0555	.0023	.0023
70°	.0654	.0654	.0018	.0023
80°	.0735	.0719	.0010	.0023
90°	.0740	.0740	0	.0023

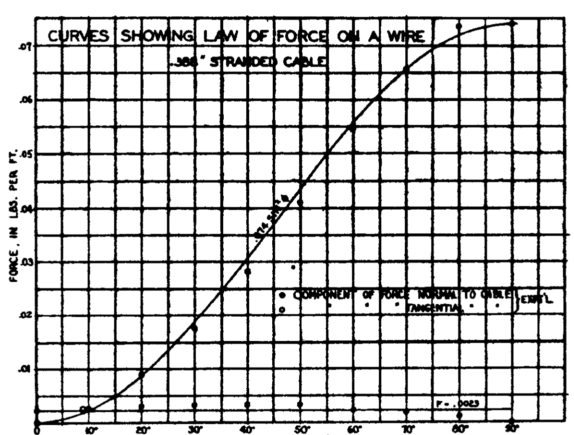

FIG. 4

Appendix II

A. <u>Force on a cable normal to stream.</u>

If the stream velocity is V, the force per unit length will be of the form

$$R = \tfrac{1}{2} k D \rho V^2 , \quad k = k\,(R') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

where D is the diameter of the cable, ρ is the density of the fluid, and k, a non-dimensional coefficient, is a function of the Reynold's number $R' = VD/\nu$. ν is the kinematic viscosity. Table (6) gives values[1] of k corresponding to values of R' in the range $10^5 < R' < 10^6$.

TABLE 6.

$R' \times 10^{-5}$	.10	.20	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0
k	1.10	1.20	1.20	1.20	1.14	.94	.66	.33	.32	.32	.33

B. <u>Force on a cable tangential to stream.</u>

The force per unit length is given by[2]

$$F = \pi D \rho V \lambda \; (2\nu/e + V) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

where $e = 2.62 \times 10^{-5}$ ft. λ depends upon the curvature and roughness of the cylinder. Table (7) gives corresponding values of λ and D for a smooth cylinder.

TABLE 7.

D(inches)	∞	20.0	10.	5.	2.5	1.4	0.7
$\lambda \times 10^3$	1.07	1.14	1.17	1.21	1.26	1.31	1.36

(1) Data due to Wieselsberger, 1922.

(2) Kempf "On the Frictional Resistance of Surfaces of Various Forms" 1924, Proceedings of the 1st International Congress for Applied Mechanics.

Appendix III

A. We shall now solve the differential equation encountered in eq. (5). We had

$dT/T = \frac{f + \sin\phi}{\cos\phi + r\sin^2\phi}\, d\phi$. Put $Z = -\tan\phi/2$. Then

$$\tan\phi = -\frac{2Z}{1 - Z^2}\,,\quad \sin\phi = -\frac{2Z}{1 + Z^2}\,,\quad \cos\phi = \frac{1 - Z^2}{1 + Z^2}\,,\quad d\phi = -\frac{2dZ}{1 + Z^2}$$

Hence

$$dT/T = \frac{f - \frac{2Z}{1 + Z^2}}{\frac{1 - Z^2}{1 + Z^2} + r\,\frac{4Z^2}{(1 + Z^2)^2}} \times -\frac{2dZ}{1 + Z^2} = -2f\,\frac{1 - 2Z/f + Z^2}{1 + 4rZ^2 - Z^4}\, dZ.$$

But $1 + 4rZ^2 - Z^4 \equiv 1 + 4r^2 - (2r - Z^2)^2 = (p^2 - Z^2)(1/p^2 + Z^2)$

where $p^2 = \sqrt{4r^2 + 1} + 2r$. Hence we have

$$dT/T = -2f\,\frac{1 - 2Z/f + Z^2}{(p^2 - Z^2)(1/p^2 + Z^2)}\, dZ = -\frac{2f}{p^2 + 1/p^2}\left[\frac{1 - 1/p^2 - 2Z/p}{1/p^2 + Z^2} + \frac{1 + p^2 + 2p/f}{2p(p + Z)} + \frac{1 + p^2 - 2p/f}{2p(p - Z)}\right] dZ$$

as may be readily verified. Integrating, we obtain

$$\log T/To = -\frac{2f}{p^2 + 1/p^2}\left[p(1 - 1/p^2)\tan^{-1} pZ - 1/f\,\log(1 + p^2Z^2) + \frac{1 + p^2}{2p}\log\frac{1 + Z/p}{1 - Z/p} + \frac{1}{f}\log(1 - Z^2/p^2)\right]$$

where T_o is the tension for $Z = 0$. This last is seen to be equivalent to equation (6).

B. Treating equation (9) in a similar fashion, we have

$$dT/T = \frac{f + \sin\phi}{\cos\phi - r\sin^2\phi}\,.\quad \text{Put } Z = \tan\phi/2. \text{ Then}$$

$$\tan\phi = \frac{2Z}{1 - Z^2}\,,\quad \sin\phi = \frac{2Z}{1 + Z^2}\,,\quad \cos\phi = \frac{1 - Z^2}{1 + Z^2}\,,\quad d\phi = \frac{2dZ}{1 + Z^2}$$

Hence
$$dT/T = \frac{\left(f + \frac{2Z}{1 + Z^2}\right)\frac{2dZ}{1 + Z^2}}{\frac{1 - Z^2}{1 + Z^2} - \frac{4\,r\,Z^2}{(1 + Z^2)^2}} = 2f\,\frac{1 + 2Z/f + Z^2}{1 - 4rZ^2 - Z^4}\,dZ$$

But $1 - 4rZ^2 - Z^4 \equiv 1 + 4r^2 - (2r + Z^2)^2 = (p^2 + Z^2)(1/p^2 - Z^2).$

$$\therefore \quad dT/T = 2f\,\frac{(1 + 2Z/f + Z^2)\,dZ}{(p^2 + Z^2)(1/p^2 - Z^2)} = \frac{2f}{p^2 + 1/p^2}\left[\frac{1 - p^2 + 2Z/f}{p^2 + Z^2} + \frac{1 + p^2 - 2p/f}{2(1 + pZ)} + \frac{1 + p^2 + 2p/f}{2(1 - pZ)}\right]$$

$$\therefore \quad \log T/T_o = \frac{2f}{p^2 + 1/p^2}\left[\frac{1 - p^2}{p}\tan^{-1} Z/p + \frac{1}{f}\log(1 + Z^2/p^2) + \frac{p^2 + 1}{2p}\log\frac{1 + pZ}{1 - pZ} - \frac{1}{f}\log(1 - p^2Z^2)\right]$$

where T_o is the tension for $Z = 0$. This last is seen to be equivalent to equation (10).

Appendix IV

It has been stated that in finding T_1 as a function of ϕ, a close approximation to the set of equations

$$\log t_1 = -2f \frac{p - 1/p}{p^2 + 1/p^2} \tan^{-1} pZ_1 + \frac{2}{p^2 + 1/p^2} \log \frac{1 + p^2 Z_1^2}{1 - Z_1^2/p^2} -$$

$$f \frac{p + 1/p}{p^2 + 1/p^2} \log \frac{1 + Z_1/p}{1 - Z_1/p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

$$\log t_2 = -2f \frac{p - 1/p}{p^2 + 1/p^2} \tan^{-1} Z_2/p + \frac{2}{p^2 + 1/p^2} \log \frac{1 + Z_2^2/p^2}{1 - p^2 Z_2^2} +$$

$$f \frac{p + 1/p}{p^2 + 1/p^2} \log \frac{1 + pZ_2}{1 - pZ_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

where $$\int_0^{Z_1} \frac{t_1 Z_1 dZ_1}{(1 - Z_1^2/p^2)(1 + p^2 Z_1^2)} = \int_0^{Z_2} \frac{t_2 Z_2 dZ_2}{(1 + Z_2^2/p^2)(1 - p^2 Z_2^2)}$$

(from (8) and (12)) . . . . (17)

is given by the set of equations

$$\log t_1^{\circ} = -\frac{2f}{p} \tan^{-1} pZ_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6a)$$

$$\log t_2^{\circ} = \frac{f}{p} \log \frac{1 + pZ_2}{1 - pZ_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10a)$$

where $$\int_0^{Z_1} \frac{t_1^{\circ} Z_1 dZ_1}{1 + p^2 Z_1^2} = \int_0^{Z_2} \frac{t_2^{\circ} Z_2 dZ_2}{1 - p^2 Z_2^2}$$ (from (8a) and (12a)) . . . . (18)

Assume that $p \gg 1$. Since $p^2 = \sqrt{4r^2 + 1} + 2r \doteqdot 4r$, this is equivalent to assuming that $2\sqrt{r} \gg 1$. Compare (6a) with (6). It is easily shown that the term for $\log t_1^{\circ}$ is a good approximation to the sum of the first and third terms for $\log t_1$. In fact, one finds on calculation that the difference is given by

$$\frac{2f}{p} \tan^{-1} pZ_1 - \left[2f \frac{p - 1/p}{p^2 + 1/p^2} \tan^{-1} pZ_1 + f \frac{p + 1/p}{p^2 + 1/p^2} \log \frac{1 + Z_1/p}{1 - Z_1/p} \right] =$$

$$\frac{2f}{p^3} (\tan^{-1} pZ_1 - pZ_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

where $0 < Z_1 < 1$.

Thus the difference is zero when pZ_1 is small, and is of the second order in $\frac{1}{p}$ as Z_1 increases to one.

Similarly, comparing (10a) with (10), one can show that the term for log t_2° is a close approximation to the sum of the first and third terms for log t_2, the difference being given by

$$\frac{f}{p} \log \frac{1 + pZ_2}{1 - pZ_2} - \left[f \frac{p + 1/p}{p^2 + 1/p^2} \log \frac{1 + pZ_2}{1 - pZ_2} - 2f \frac{p - 1/p}{p^2 + 1/p^2} \tan^{-1} Z_2/p \right] =$$

$$\frac{f}{p^3} (2pZ - \log \frac{1 + pZ_2}{1 - pZ_2}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

Again the difference is zero when pZ_2 is small, and of the second order in $1/p$ for larger values of Z_2. The remaining terms,

$$\frac{2}{p^2 + 1/p^2} \log \frac{1 + p^2 Z_1^2}{1 - Z_1^2/p^2} \text{ from (6) and } \frac{2}{p^2 + 1/p^2} \log \frac{1 + Z_2^2/p^2}{1 - p^2 Z_2^2} \text{ from (10),}$$

are, ab initio, of the second order in $1/p$, and further, the following considerations show that they mutually nullify each other in the determination of T_1.

First observe that since log t_1 and log t_2 are both of the first order in $1/p$ and consequently small, both t_1 and t_2 are, to a first approximation, equal to unity. Putting $t_1 = t_2 = 1$, we obtain from (17), on integrating,

$$\frac{2}{(p^2 + 1/p^2)} \log \frac{1 + p^2 Z_1^2}{1 - Z_1^2/p^2} = \frac{2}{(p^2 + 1/p^2)} \log \frac{1 + Z_2^2/p^2}{1 - p_2 Z_2^2} \quad \ldots\ldots \quad (21)$$

as a first approximation to the relation between Z_1 and Z_2. Considering (17) again, we see that if we substitute for t_1 and t_2 the exponentials in functions of Z_1 and Z_2 given by (6) and (10), the factor

$$e^{\left(\frac{2}{p^2 + 1/p^2} \log \frac{1 + p^2 Z_1^2}{1 - Z_1^2/p^2}\right)}$$

appears in the left member of the equation, and

$$e^{\left(\frac{2}{p^2 + 1/p^2} \log \frac{1 + Z_2^2/p^2}{1 - p^2 Z_2^2}\right)}$$

in the right member. Since both exponents are small, and by (21) approximately equal to each other, we obtain a second approximation to the relation between Z_1 and Z_2 by cancelling both factors. Further, since the factors $1 - Z_1^2/p^2$ and $1 + Z_2^2/p^2$ in (17) are very nearly equal to unity, and the uncancelled factors of t_1 and t_2 in (17) are equal to t_1° and t_2° to the second order of approximation, we see that the relation (18) is a close approximation to (17).

Finally let us investigate what approximation is made in computing

$$T_1/FxL = \frac{t_1}{t_2 - t_1}\text{, the quantity of greatest interest.}$$

Writing for the moment $\log t_1 = -U$, and $\log t_2 = V$, we have

$$\frac{t_1}{t_2 - t_1} = \frac{e^{-U}}{e^{V} - e^{-U}} = \frac{1}{e^{(U+V)}-1} .$$

But by (6) and (10) we see that U+V involves terms that are a second order approximation to $U^{\circ} + V^{\circ} \equiv \log t_2{}^{\circ} - \log t_1{}^{\circ}$ (by 19) and (20)) while the remaining terms are not only small but also act to cancel each other, by (21). Hence we conclude that a close approximation to T_1 is obtained by using equations (6a), (8a), (10a) and (12a) instead of the set of equations (6), (8), (10) and (12).