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SUBJECT: TRANSIENTS FOR CRYOTRON TREE SWITCH

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Abstract: A description of a cryotron tree switch, the initial conditions for the various transients, and the circuit parameter value are described. A simple method for calculating the pole zero pattern is found. Finally, an asymptotic expression is found for the slowest and fastest exponentials in a 2^n position switch.

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Introduction:

The cryotron tree switch is one of the possible circuit configurations that may be used in a cryotron computer.† One of the problems is a description of the transient (especially its duration) involved in switching from one switch position to another. The following analysis describes the transient exponentials that will make up the transient. It obtains the number of exponentials and their location in the complex frequency plane with a comparatively simple method for locating the slowest (most important) exponential for any size switch.

Cryotron:

The cryotron is an active circuit element in which a coil of wire can, by its magnetic field (on suitable materials at liquid helium temperatures), change another wire from a zero resistance to a finite resistance and vice versa. For a complete description see Memorandum 6M-3843 by Dudley A. Buck. For our purposes, it is sufficient to consider the cryotron as a four terminal black box, in which a current of Io or greater in the control lead pair causes the gate lead pair to see a resistance. The control leads always have a lossless inductance across them, (Fig. 1).

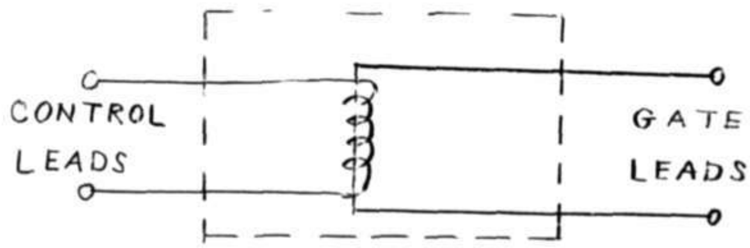

Fig. 1

The cryotron usually is driving a load inductance composed of the control windings of other cryotrons. At the present, the experimental cryotrons have full resistances of 8 milliohms and inductances of .1 microhenry. It is planned to build cryotrons with higher resistances and lower inductances so that circuitry will be faster than one megacycle.

Tree Switch:

The cryotron tree switch is made in a simple fashion so that at each step the current has two paths from which to choose. Each path

† A. Shortell is presently building and measuring an 8 position switch as part of his thesis work.

is controlled by complementary current sources so that only one path is open. Finally each path is connected to a load which is an inductance (Fig. 2). In Figure 2, there is a 1 position switch with no choices, a 2 position switch with one choice and a 4 position switch with two choices. Also there is a load at each output. The complementary current sources are represented as flip-flops to simplify notation. Thus the tree switch is a typical computer binary switch with 2n input lines and 2^n outputs.

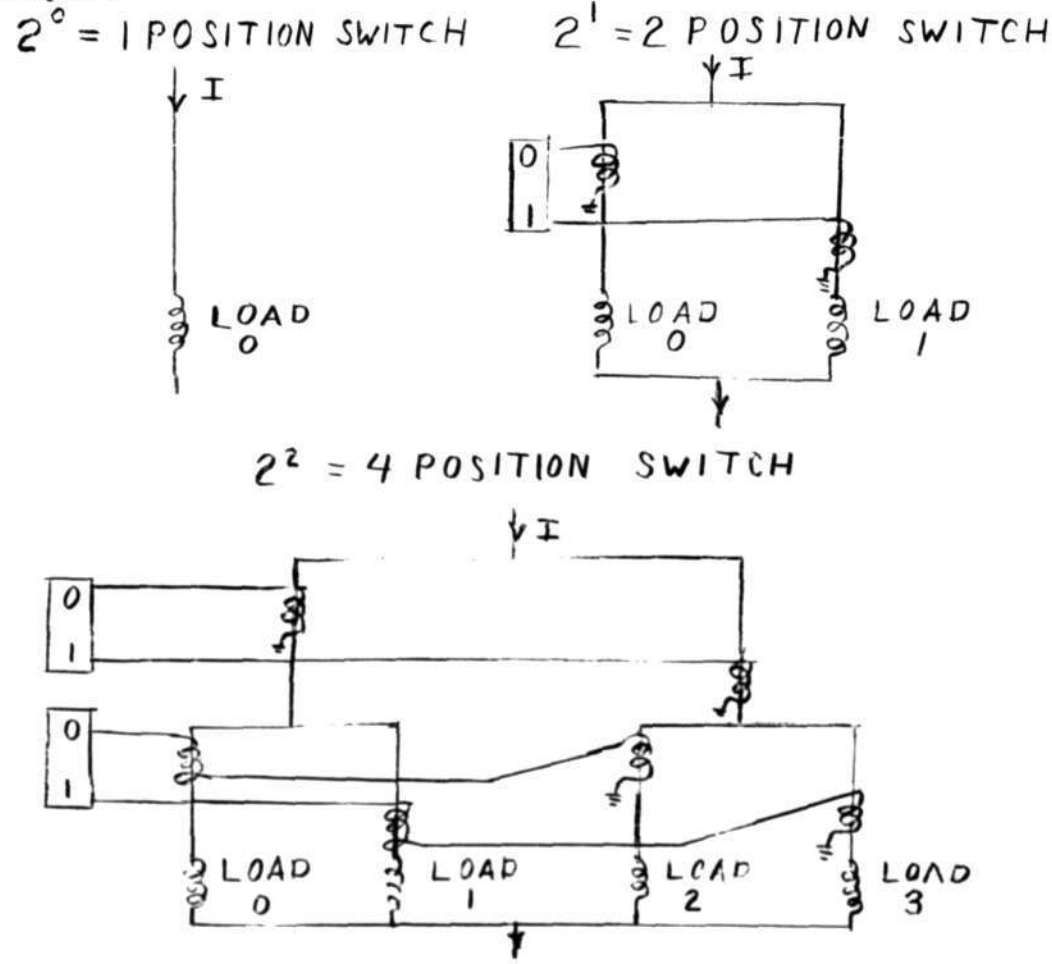

Fig. 2

Assuming all the resistances are equal, all the load inductances are equal, and that all the flip-flops are in a position, where the zero side has current (and so makes the affected cryotrons resistive) then the switches look like the following configurations (Fig. 3). It is likely that similar cryotrons will be used throughout the switch. Since the switch is completely symmetrical, the assumption of a given flip-flop position does not change the circuit.

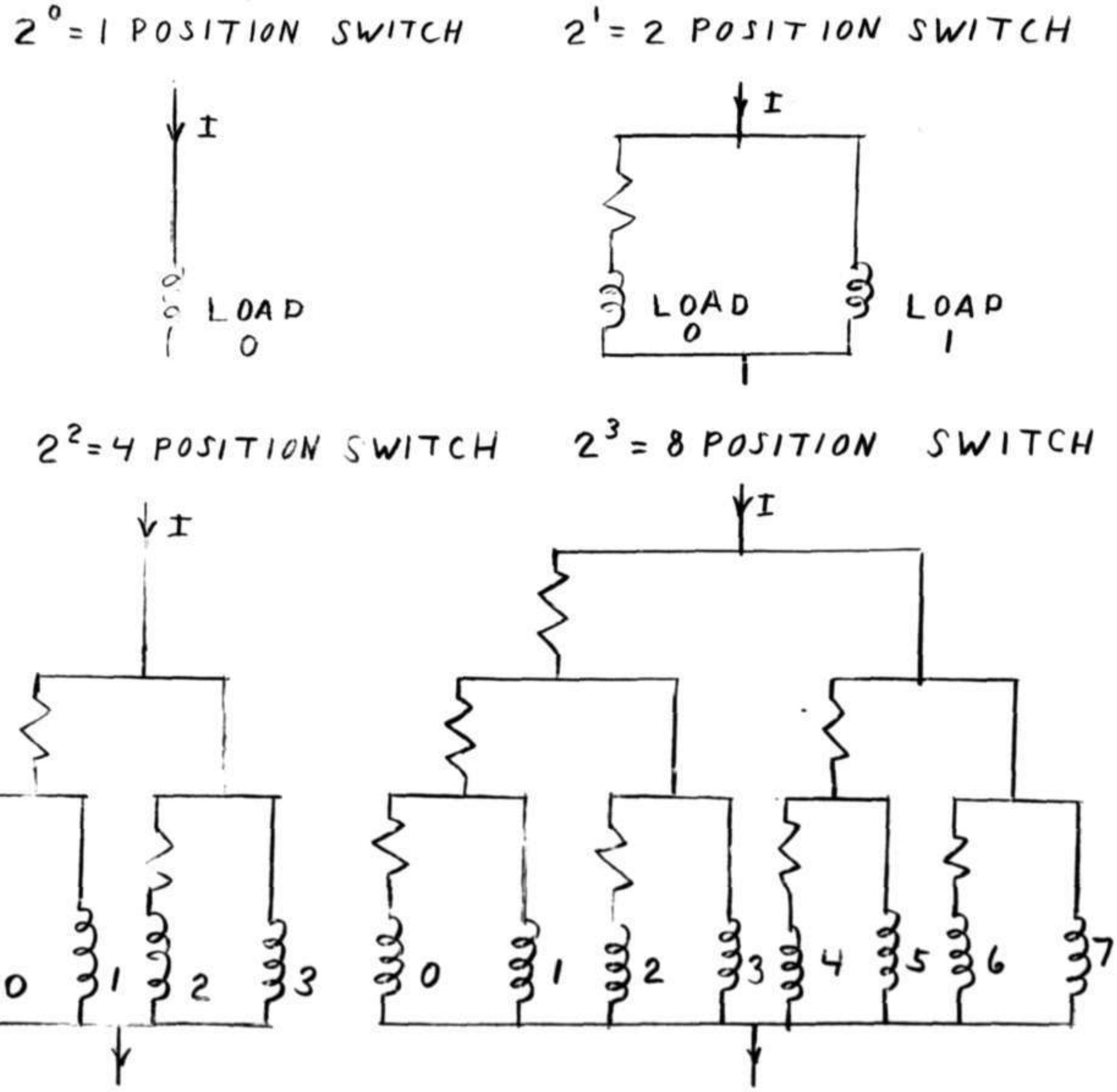

Fig. 3

If we assume that the cryotrons switch instantaneously and that the flip-flops are all triggered on the same pulse, then the initial conditions, immediately after a change of the input flip-flops, are easy to determine. The inductance loads have exactly the same current they had before the flip-flops changed. This is because there is only a finite resistance to any path and any inductive circuit with finite current and finite resistance cannot change the current in the inductance instantaneously. Depending on whether 1, 2 or n flip-flops were complemented in the switch the exact transient will be slightly different. This different transient is represented in a switch by changing the initial conditions. For example, if in a 2^3 position switch no flip-flops have changed, it corresponds to all current initially in load 7 as shown in Fig. 3. One flip-flop changed corresponds to all current initially in 3, 5, or 6. Two flip-flops changed corresponds to all current initially in 1, 2, or 4. Three flip-flops changed corresponds to all current initially in load 0.

Using the results of circuit theory several comments can be made. There will be a finite number of finite natural frequencies or transients (actually $2^n - 1$ in a 2^n position switch) in the circuit which will compose the transients in all cases except that different transients will have varying amounts of the transient frequencies. Because the circuit is composed solely of inductances and resistances all transient frequencies are decoying exponentials. If a connection is made across any two nodes of this network and the admittance is determined, then the zeros of the admittance correspond to the natural frequencies of the network. Since the transient frequencies are so numerous, and the initial conditions are also numerous, the approach will not be to try to determine the exact nature of each transient but only to find the slowest transient frequency in a given size switch as being the slowest response possible.

Analysis:

The most straightforward approach might be to calculate loop equations and then find the natural frequencies. It, however, is enormously difficult. The approach taken here is to get a recurrence relation among the admittances across the selected load. The first step is to redraw the original network in slightly different form with the admittance across the selected load (See Fig. 4) and compare with Fig. 3. This can be done with rigor because the cryotrons gate circuits have exactly zero resistance. Redrawing a matrix of diodes or transistors in this manner would involve the approximation that the resistance in one direction is zero. By then looking at the redrawn diagrams, one can see a pattern: The addmittance Y_n looking into the selected output terminal of a 2^n- position switch is that of a 2^{n-1} switch with a resistor, R, in series plus that of a 2^{n-1} position switch in parallel. In other words, one can regard the next smaller switch as a black box, and a simple picture of a switch in terms of this black box is possible. This Y_1 is represented in terms of Y_0, Y_2 is represented in terms of Y_1, etc. After a little consideration one can see the reason for the relationship of Y_{n+1} to Y_n. By induction, a 2^{n+1} switch is made by taking two 2^n switches and putting a resistance in series with one and a short in series with the other. (See Fig. 6).

A simplification in the succeeding work can be made by using the impedance and frequency transformations of equations (1) and (2) so that all resistances and inductances have unit size.

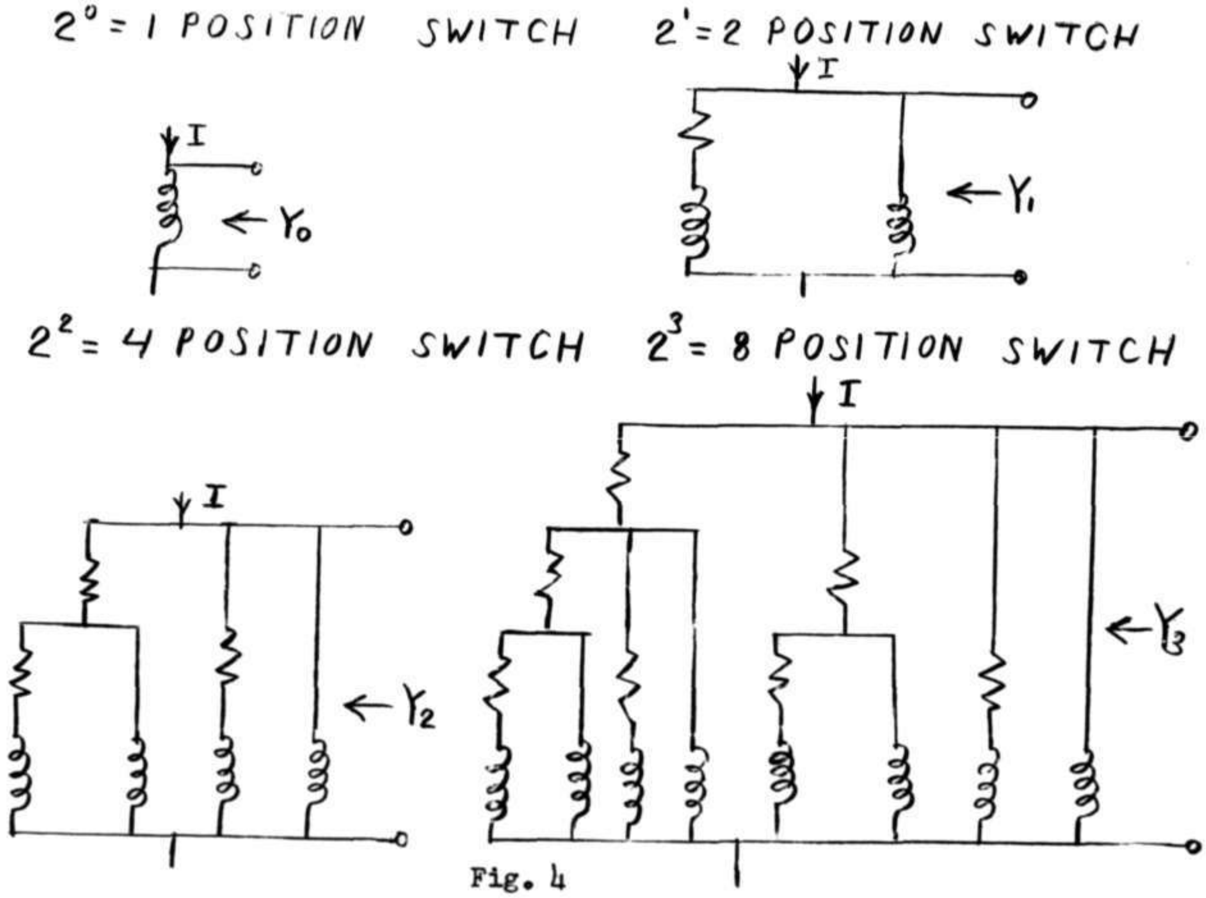

Fig. 4

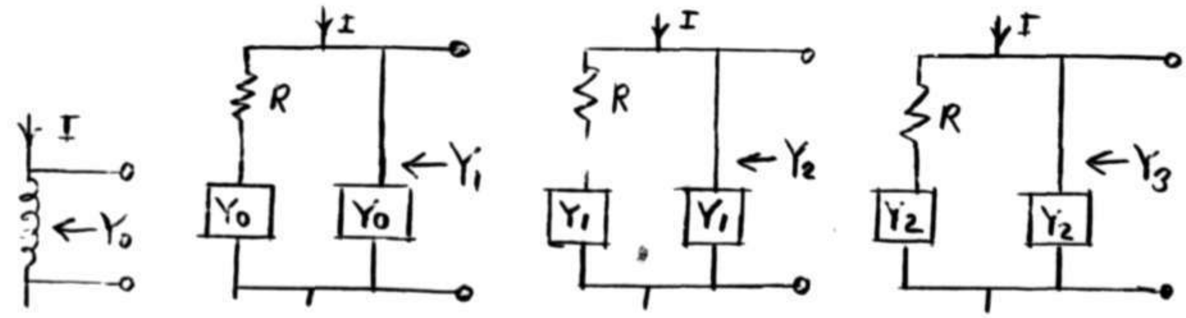

Fig. 5

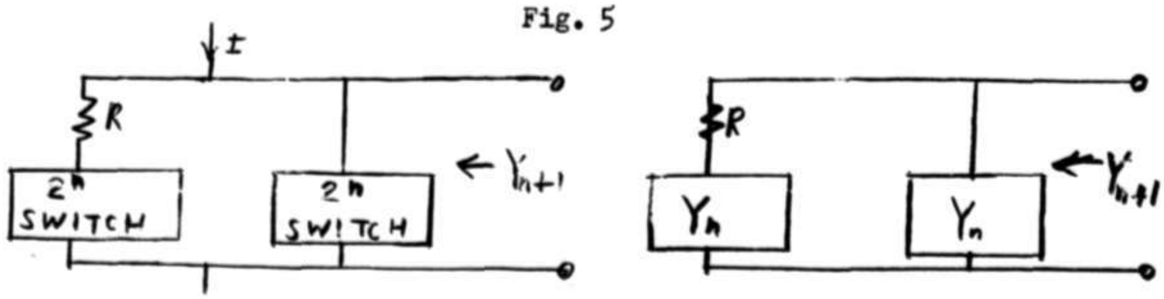

Fig. 6

(1) $$Z \text{ standard} = \frac{Z \text{ old}}{R}$$

(2) $$S \text{ standard} = S \text{ old} \frac{L}{R}$$

Using our standard notation and parallel series rules on Y_{n+1} a complete description of all Y_n is shown in equation (3), and one of equations (4), (5), or (6).

(3) $$Y_o = \frac{1}{S}$$

(4) $$Y_{n+1} = f(Y_n) = Y_n + \frac{1}{1 + \frac{1}{Y_n}} \quad n = 0, 1, 2, \ldots$$

(5) $$Y_{n+1} = Y_n + 1 - \frac{1}{Y_n + 1}$$

(6) $$Y_{n+1} = \frac{Y_n(Y_n + 2)}{Y_n + 1}$$

A graph of f for real values of its argument allows us to see the following important facts. (Fig. 7).

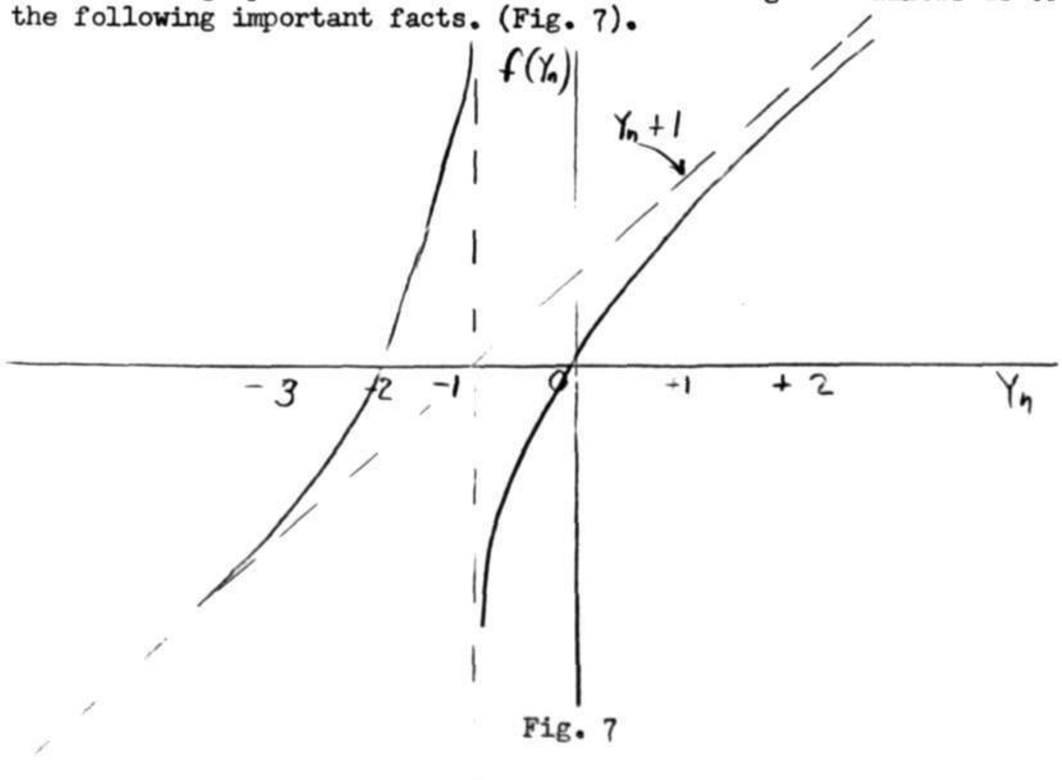

Fig. 7

These can also be seen from the algebra:

(7) $f(\infty)=\infty$; $f(-1)=\infty$ (also can be seen from equation 6)

(8) $f(0)=0$; $f(-2)=0$ (also can be seen from equation 6)

(9) Range of $0 \leq Y_n < \infty$ goes into $0 \leq f(Y_n) \leq \infty$

(10) Range of $-1 < Y_n \leq 0$ goes into $-\infty < f(Y_n) \leq 0$

(11) Range of $-2 \leq Y_n < -1$ goes into $0 \leq f(Y_n) < \infty$

(12) Range of $-\infty < Y_n \leq -2$ goes into $-\infty < f(Y_n) \leq 0$

As the argument moves from $-\infty$ to $+\infty$, the value of f(Yn) moves twice from $-\infty$ to $+\infty$. For any A, the equation $f(x) = A$ has exactly two solutions for x. This is seen in the graph because any horizontal line crosses the graph twice. The derivative of f(Y) ($f'(Y_n) = 1 + 1/(Y_{n+1})^2$) is always positive.

At this point this accumulated background allows us to plot in rough form the shape of Yo (S), Y_1(S) .... Y_n(S) for real values of argument. This is the graph usually used in network theory for RL networks because it places all the zeros and poles of admittance in evidence although one is more interested in the jw axes. (See Fig. 8) $Y_o(S) = 1/S$ is easy to draw. By using equation 7 and equation 8 the zeros and poles of Y_1 are easy to find from Y_o. Where Y_o has a pole or is equal to -1, Y_1 has a pole, Y_1 has a zero where Y_o has a zero or where $Y_o = -2$. The sign of Y_1 in between its poles and zeros can be seen from equations 9-12. Now using the additional knowledge that Y_1(S) has a negative slope a rough sketch of Y_1 is drawn. Again by using same techniques the poles and zeros of Y_2 are located and a rough sketch of Y_1 is drawn. Again by using same techniques the poles and zeros of Y_2 are located and a rough sketch made and so on. This procedure directly shows that Y_n has 2^n-1 zeros and 2^n poles. Also the following rules allow one to get the pole zero pattern of Y_{n+1} from that of Y_n. (This can be seen in two ways. One may know that the derivative of an RL network is always negative on the real axis. Also in this case it can easily be proven that

$$\frac{d(Y_{n+1}(S))}{dS} = f'(Y_n(S)) \frac{d(Y_n(S)}{dS} < 0$$

by knowing $f'(Y_n)$ is always positive and induction on the knowledge that

$$\frac{dYo(S)}{dS} < 0)$$

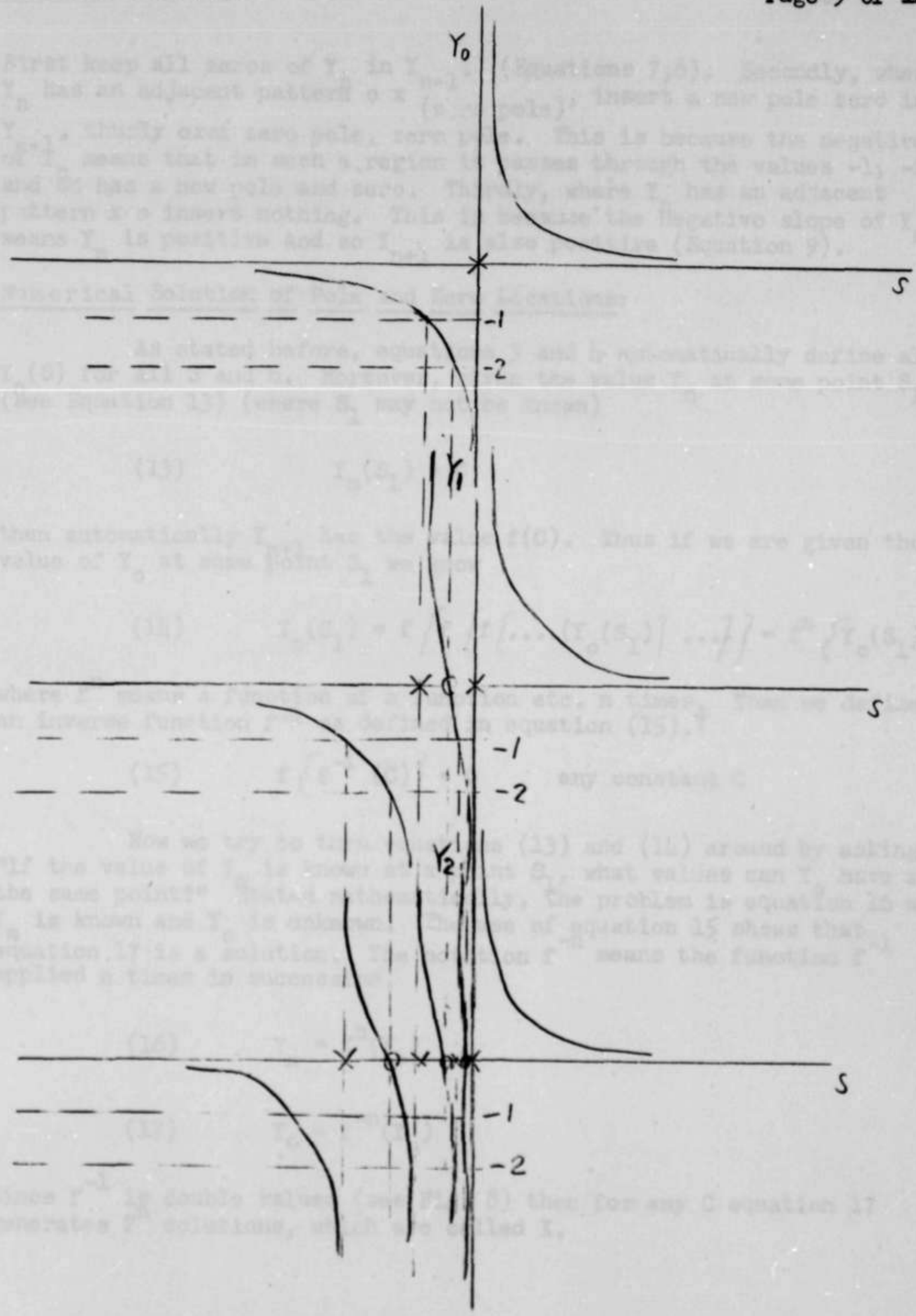

Fig. 8

First keep all zeros of Y_n in Y_{n+1}. (Equations 7,8). Secondly, where Y_n has an adjacent pattern o x $\binom{n+1}{\text{(zero pole)}}$, insert a new pole zero in Y_{n+1}, thusly oxox zero pole, zero pole. This is because the negative slope of Y_n means that in such a region it passes through the values -1, -2 and so has a new pole and zero. Thirdly, where Y_n has an adjacent pattern x o insert nothing. This is because the negative slope of Y_n means Y_n is positive and so Y_{n+1} is also positive (Equation 9).

Numerical Solution of Pole and Zero Locations:

As stated before, equations 3 and 4 automatically define all $Y_n(S)$ for all S and n. Moreover, given the value Y_n at some point S_1 (see Equation 13) (where S_1 may not be known)

$$(13) \qquad Y_n(S_1) = C$$

then automatically Y_{n+1} has the value f(C). Thus if we are given the value of Y_o at some point S_1 we know

$$(14) \qquad Y_n(S_1) = f\left\{f\left\{f\left\{\ldots.(Y_o(S_1)\right\}\ldots\right\}\right\} = f^n\left\{Y_o(S_1)\right\}$$

where f^n means a function of a function etc. n times. Then we define an inverse function f^{-1} as defined in equation (15).†

$$(15) \qquad f\left\{f^{-1}(C)\right\} = C \qquad \text{any constant } C$$

Now we try to turn equations (13) and (14) around by asking, "If the value of Y_n is known at a point S_1, what values can Y_o have at the same point?" Stated mathematically, the problem is equation 16 where Y_n is known and Y_o is unknown. The use of equation 15 shows that equation 17 is a solution. The notation f^{-n} means the function f^{-1} applied n times in succession.

$$(16) \qquad Y_n = f^n(Y_o)$$

$$(17) \qquad Y_o = f^{-n}(Y_n)$$

Since f^{-1} is double valued (see Fig. 8) then for any C equation 17 generates 2^n solutions, which are called **X**.

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† Notice that if a recurrence relation is not point to point such as $Y_{n+1}(S) = Y_n(S) + Y_n(S+A)$ then an inverse function is not meaningful.

(18) $$Y_o(S_1) = \frac{1}{S_1} = X$$

Then if Y_n has the value C, Y_o can be any one of the 2^n solutions X and the frequency at which Y_n has this value can be at any one of the frequencies which is reciprocal to X. Thus all solutions to equation 19 where C is known and S_1 is unknown are given in equation 20. Because of the form of Y_o the large negative values of Y_o correspond to long time constants or small negative S, while small negative values of Y_o correspond to short time constants or large negative S.

(19) $$Y_n(S_1) = C$$

(20) $$S_1 = \frac{1}{f^{-n}(C)}$$

To solve for f^{-1} use equation 6.

(21) $$f(x) = \frac{x^2 + 2x}{x + 1} = A$$

(22) $$x^2 + (2-A)x - A = 0$$

(23) $$x = f^{-1}(A) = \frac{A}{2} - 1 \pm \sqrt{1 + (\frac{A}{2})^2}$$

Thus to find the zeros of Y_o, Y_1, Y_2,.... using equations 20 and 23

(24) $$Y_o = 0 \text{ at } \infty$$

To find the zeros of Y_1 solving $f^{-1}(o)$

(25) $$f^{-1}(o) = o, -2 \quad Y_1 = o \text{ at } \infty, -\frac{1}{2}$$

To find the zeros of Y_2 solving $f^{-2}(o)$

(26) $$f^{-2}(o) = f^{-1}(o), f^{-1}(-2) = o, -2, -2 + \sqrt{2}, -2 - \sqrt{2}$$

$$Y_2 = o \text{ at } \quad , -\frac{1}{2}, -1.7068, -.2929$$

Taking f^{-1} successively n times one can get all the zeros of Y_o, Y_1, ... Y_n. Similarly for finding the poles solve equation 20 and equation 23 for $c = \infty$ (see equation 27).

(27) poles of Y_n at $\frac{1}{f^{-n}(\infty)}$

The same procedure can be used for finding where Y_n has any value, but the usual interest is in zeros and poles.

It seems reasonable that the slowest exponential will be the dominating term in the transient. If this is true, then we want the zero which is closest to the origin as it represents the slowest transient. As stated before, this is represented by the most negative value of Y_o for which $Y_n = o$. By looking at Fig. 8 (or seeing $df(S)/dS > 0$ for all S) it is clear that if for the two numbers, x_1, x_2: $x_1 < x_2$ then $f^{-1}(x_1) < f^{-1}(x_2)$ (taking the most negative value of f^{-1} in each case). Using induction, the most negative solution of $f^{-n}(A)$ is found by taking the negative sign of equation 23 each time. A table below lists the closest zeros for switches up to 2^{12}.

(28) $$f^{-1}(A) = \frac{A}{2} - 1 - \sqrt{1+\left(\frac{A}{2}\right)^2}$$

An asymptotic expansion can be gotten of $f^{-n}(o)$.

(29) Definition $X_n = f^{-n}(o)$ largest negative value

TABLE

Size of switch		Zero
2^0	$f^{-o}(o) = 0$	∞
2^1	$f^{-1}(o) = -2$	- .5000
2^2	$f^{-2}(o) = -3.414$	- .2929
2^3	$f^{-3}(o) = -4.685$	- .2134
2^4	" = -5.890	- .1698
2^5	" = -7.055	- .1417
2^6	" = -8.194	- .1220
2^7	$f^{-7}(o) = -9.314$	- .1074
2^8	" = -10.420	- .0960
2^9	" = -11.515	- .0868
2^{10}	" = -12.601	- .0794
2^{11}	" = -13.680	- .0731
2^{12}	" = -14.753	- .0678

(30) $$X_{n+1} = \frac{X_n}{2} - 1 - \sqrt{1 + \left(\frac{X_n}{2}\right)^2} = \frac{X_n}{2} - 1 - \frac{X_n}{2}\sqrt{1 + \left(\frac{4}{X_n}\right)^2}$$

Using a Taylor expansion

(31) $$X_{n+1} = \frac{X_n}{2} - 1 + \frac{X_n}{2}\left(1 + \frac{2}{X_n^2}\right. \ldots\ldots$$

(32) $$X_{n+1} \approx X_n - 1 + \frac{1}{X_n} - \frac{1}{X_n^3}$$

Since all values of X_n are negative and less than -1 then $\frac{1}{X_n}$ dominates all successive terms and X_n+1 is less than X_n by at least 1. So X_n grows into an infinitely large number. Also using equation 32 we can see that the difference of X_{n+1} and X_n approaches -1. Finally an asymptotic expression for X_n can be calculated.

(33) $$\lim_{n \to \infty} X_n = \infty$$

(34) $$\lim_{n \to \infty} (X_{n+1} - X_n) = -1$$

(35) $$\lim_{n \to \infty} (X_n - n - \log_e n \text{ is finite})$$

The finite number in equation 35 has not been calculated but it is between 0 and + 1/2.

Using equation 35 it is seen that the lowest natural frequency of Y_n approaches $1/n + \log n$.

Similarly the same argument shows that the smallest value of Y_o for which $Y_n = 0$ is gotten by taking the positive sign of equation 23.

(36) $$X_{n+1} = \frac{X_n}{2} - 1 + \sqrt{1 + \left(\frac{X_n}{2}\right)^2}$$

Since X_n is small a Taylor expansion gives the following result.

(37) $$X_{n+1} = \frac{X_n}{2} - 1 + 1 + 1/2\left(\frac{X_n}{2}\right)^2 = \frac{X_n}{2} + \frac{X_n^2}{8}$$

This shows that for large n the smallest value of Y_o for which $Y_n = 0$ increases as 2^{2^n}. Thus the largest finite exponential of Y_n can be approximated as $S = -2^n$.

Summary:

The cryotron switch has 2^n-1 exponentials in the transient. The slowest transient has approximately $n + \log n$ time constants or $S = - 1/ n + \log n$ and the fastest exponential has $S \sim -2^n$. The slowest exponential allows an approximation of the response time.

A recurrence relation and its inverse can be used to find natural frequencies in simple fashion.

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