Digital Computer Laboratory
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SUBJECT: GROUP 63 SEMINAR ON MAGNETISM, XXII

To: Group 63 Staff

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Figure 42 represents a schematic review of the previous lectures.

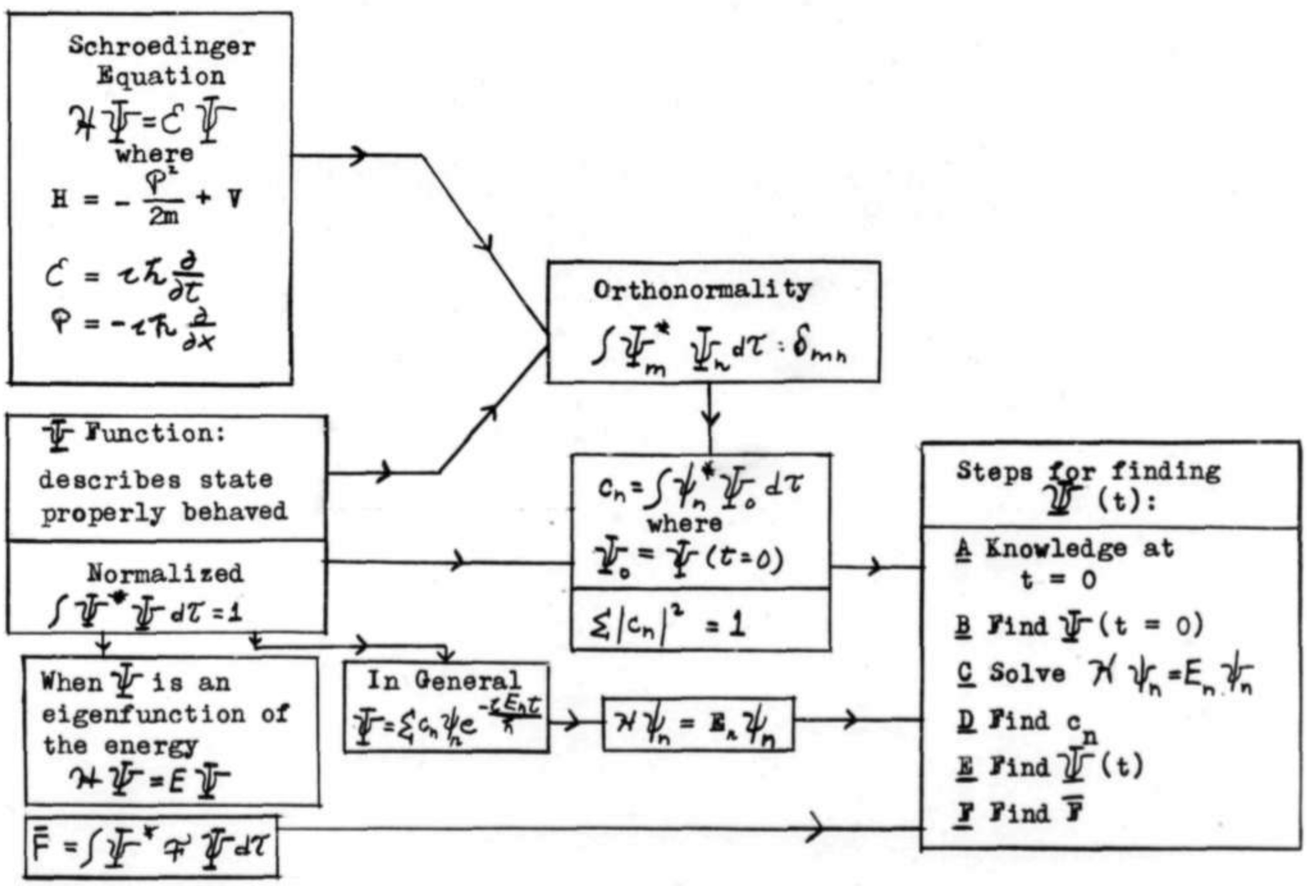


Figure 42

We will now prove that if Ψ is an eigenfunction of the energy at any time, the energy of the system remains constant in time.

If $\mathcal{E}\Psi = E_m \Psi_0$

where we define $\Psi_0 = \Psi(t = 0)$,

then

$$\Psi_0 = c_m \psi_m$$

In general,

$$\Psi = \sum_n c_n \psi_n e^{-iE_n t/\hbar} \qquad \text{XXII - 1}$$

At the previous meeting we found $c_n = \int \psi_n^* \Psi_0 \, d\tau$.

Hence, since $\Psi_0 = c_m \psi_m$.

$$c_n = c_m \int \psi_n^* \psi_m \, d\tau$$

$$= c_m \delta_{mn}$$

Therefore, the only non-zero c_n term is c_m, and from XXII - 1,

$$\Psi = c_m \psi_m e^{-\frac{iE_m t}{\hbar}}$$

This shows that if Ψ_0 is an eigenfunction of energy with eigenvalue E_m, then $\Psi(t)$ is also an eigenfunction, so that the system described by $\Psi(t)$ keeps a constant energy E_m. This is a property peculiar to the $\mathcal{E}$ operator, and does not hold in general for all operators.

Potential Well Problem

As an example of the method of solution of a physical problem using Schroedinger's equation, we will undertake the solution of a particle in a one dimensional potential well. We may consider a bead on a wire, with obstructions on the wire at x = 0 and x = a. The bead is constrained by these obstructions to the region from x = 0 to x = a, and so the stops may be thought of as representing an infinitely high potential barrier. The potential is shown in Figure 43.

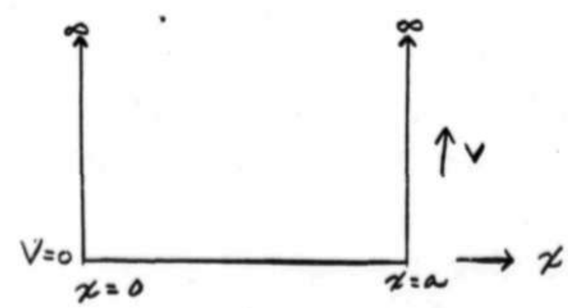


Figure 43

Since the particle cannot penetrate the infinite potential barrier, it will never be found outside the potential well. The function Ψ must therefore be zero outside this region. Furthermore, if Ψ is to be a properly behaved function as previously defined, it must be continuous at the boundaries. Therefore,

$$\Psi\,(x=0) = 0$$

Temporarily dispensing with steps A and B in Figure 42, we first solve $\mathcal{H}\psi_n = E_n\,\psi_n$.

Since V = a constant within the region of interest, and setting the zero of energy such that V = 0 in this range,

$$-\frac{\hbar^2}{2m}\,\frac{d^2\psi_n}{dx^2} = E_n\,\psi_n$$

$$\frac{d^2\psi_n}{dx^2} + \frac{2mE_n}{\hbar^2}\,\psi_n = 0$$

The solution of this equation is

$$\psi_n = A_n e^{i\sqrt{\frac{2mE_n}{\hbar^2}}\,x} + B_n e^{-i\sqrt{\frac{2mE_n}{\hbar^2}}\,x} \qquad \text{XXII - 2}$$

Imposing the condition $\psi_n(x = 0) = 0$

$$0 = A_n + B_n$$

$$\left.\begin{aligned}\psi_n &= A_n\left(e^{i\sqrt{\frac{2mE_n}{\hbar^2}}\,x} - e^{-i\sqrt{\frac{2mE_n}{\hbar^2}}\,x}\right)\\ &= 2iA_n \sin\sqrt{\frac{2mE_n}{\hbar^2}}\,x.\end{aligned}\right\} \qquad \text{XXII - 3}$$

and, since $\psi_n\,(x = a) = 0$

$$0 = 2iA_n \sin\sqrt{\frac{2mE_n}{\hbar^2}}\,a$$

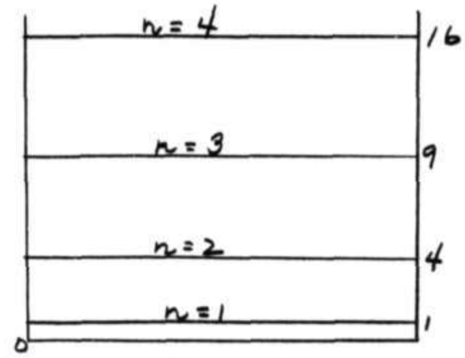
Thus we see that a solution is not generally possible. A solution can exist only when

$$\sqrt{\frac{2mE_n}{\hbar^2}}\,a = n\pi \qquad \text{XXII - 4}$$

where n is an integer. Then,

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad \text{XXII - 5}$$

Since n can only have integer values, only discrete energy levels are permissible. A plot of energy levels as a function of the quantum number n is shown in Figure 44.



E ↑ in units of $\frac{\pi^2\hbar^2}{2ma^2}$

Figure 44

From equation XXII-5 we see that a narrow well width (small a) leads to a large separation of energy levels.

Signed ______________________
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Group 62 (20)