Digital Computer Laboratory
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SUBJECT: GROUP 63 SEMINAR ON MAGNETISM, II

To: Group 63 Staff

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In order not to commit ourselves to a particular system of units, whether Gaussian or MKS, a constant c is used whose value is 1 in the MKS system and which equals the velocity of radiation in vacuo in the Gaussian system. A fuller discussion may be found in "Electromagnetic Theory."*

Maxwell's Equations

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\left(4\pi \gamma \vec{E} + \frac{\partial \vec{D}}{\partial t}\right) \qquad \text{(II - 1)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \text{(II - 2)}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \text{(II - 3)}$$

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho \qquad \text{(II - 4)}$$

Let us look at the physical significance of Maxwell's Equations. To help us in this interpretation, we make use of Stokes' Theorem and Gauss' Theorem, which state:

1. Stokes Theorem:

The line integral of a vector taken about a closed contour equals the surface integral of the curl of that vector through a surface bounded by the contour. It can be shown that

$$\oint \vec{A} \cdot d\vec{l} = \int_s (\vec{\nabla} \times \vec{A}) \cdot d\vec{s} \qquad \text{(II - 5)}$$

2. Gauss Theorem:

The surface integral of a vector through a closed surface equals the volume integral of the divergence of that vector over the volume bounded by that closed surface.

$$\int_s \vec{A} \cdot d\vec{s} = \int_v \vec{\nabla} \cdot \vec{D} \; dv \qquad \text{(II - 6)}$$

* Stratton, J. A., Electromagnetic Theory, McGraw-Hill Co., New York, 1941, p. 238.

Then, from II - 1 and II - 5

$$\int (\vec{\nabla} \times \vec{H}) \cdot d\vec{s} = \oint \vec{H} \cdot d\vec{\ell} = \frac{4\pi}{c} \int_s \vec{i} \cdot d\vec{s} + \frac{1}{c} \frac{\partial}{\partial t} \int_s \vec{D} \cdot d\vec{s}$$

Therefore

$$\oint \vec{H} \cdot d\vec{\ell} = \frac{4\pi I}{c} + \frac{1}{c} \frac{d}{dt} \int_s \vec{D} \cdot d\vec{s} = I + I'$$

where I = current
I' = displacement current

In the steady state, the displacement current is zero, and the first Maxwell equation states that the conduction current through any regular surface is equal to the line integral of the vector $\vec{H}$ about its contour. If the field is variable, then a displacement current I' must be considered as well as the conduction current, where

$$I' = \frac{1}{c} \frac{\partial}{\partial t} \int_s \vec{D} \cdot d\vec{s}$$

Similarly from II - 2 and II - 5

$$\int_s (\vec{\nabla} \times \vec{E}) \cdot ds = \oint \vec{E} \cdot d\ell = -\frac{1}{c} \frac{\partial}{\partial t} \int_s \vec{B} \cdot d\vec{s}$$

By definition, the magnetic flux $\Phi = \int_s B \cdot ds$

$$\therefore \oint E \cdot d\ell = -\frac{1}{c} \frac{\partial \Phi}{\partial t}$$

Thus, Maxwell's second equation equates the line integral of the vector $\vec{E}$ about a closed path to the time rate of decrease of the magnetic flux through any surface spanning that curve. This is just the induced electromotive force.

From equation II - 3 and II - 6

$$\int_v \vec{\nabla} \cdot \vec{B}\, dv = \int_s \vec{B} \cdot d\vec{s} = 0$$

The third Maxwell equation therefore states that the total flux of the vector B crossing any closed surface is equal to zero.

Similarly from II - 4 and II - 6

$$\int_v \vec{\nabla} \cdot \vec{D}\, dv = \int_s \vec{D} \cdot d\vec{s} = 4\pi \int \rho\, dv = 4\pi q$$

Thus the last Maxwell equation states that the number of "flux lines" of the vector $\vec{D}$ through a closed surface is proportional to the total charge $\underline{q}$ in the volume enclosed by the surface.

In addition to the four Maxwell equations, we have the continuity equation, which is nothing more than a statement of the indestructibility of charge, and Ohm's Law. These are respectively:

$$\vec{\nabla}\cdot\vec{i} + \frac{\partial\rho}{\partial t} = 0 \qquad \text{(II - 7)}$$

$$\vec{i} = \gamma\vec{E} \qquad \text{(II - 8)}$$

Wave Equations

From the above, and the equations:

$$\vec{D} = \epsilon\vec{E}$$
$$\vec{B} = \mu\vec{H}$$

we can express Maxwell's equations in the form

$$\vec{\nabla}\times\vec{H} = \frac{1}{c}\left(4\pi\gamma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t} + \vec{E}\frac{\partial\epsilon}{\partial t}\right) \qquad \text{(II - 9)}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\mu}{c}\frac{\partial\vec{H}}{\partial t} - \frac{\vec{H}}{c}\frac{\partial\mu}{\partial t} \qquad \text{(II - 10)}$$

$$\vec{\nabla}\cdot\vec{H} = -\vec{H}\cdot\frac{\nabla\mu}{\mu} \qquad \text{(II - 11)}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{4\pi\rho}{\epsilon} - \vec{E}\cdot\frac{\nabla\epsilon}{\epsilon} \qquad \text{(II - 12)}$$

Measurements taken of the value of $\nabla\times E$ vs. time have been made, and in some cases they lead to a doubly peaked curve. Since the coefficients μ and H in equation II - 10 are varying as well as their time derivatives, a careful analysis of a particular case can prove quite complicated. However, if μ and H have different rates of change, and one term is not dwarfed by the other, the maximum of each of the two terms will occur at different times and be separately visible as maxima.

In most cases, the peak due to the time variation of the permeability occurs considerably later than that due to the magnetic field change, because of the failure of μ to follow sudden changes in H directly.

For the special case where ϵ and μ are constants, equations II - 9 to II - 12 reduce to:

$$\vec{\nabla}\times\vec{H} = \frac{1}{c}\left(4\pi\gamma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t}\right) \qquad \text{(II - 13)}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\mu}{c}\frac{\partial\vec{H}}{\partial t} \qquad \text{(II - 14)}$$

$$\vec{\nabla}\cdot\vec{H} = 0 \qquad \text{(II - 15)}$$

$$\vec{\nabla}\cdot E = \frac{4\pi\rho}{\epsilon} \qquad \text{(II - 16)}$$

A vector identity states:

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{F}] \equiv \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F} \qquad \text{(II - 17)}$$

Therefore:

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{H}] = \vec{\nabla} \times \left[\frac{1}{c}(4\pi\gamma\vec{E} + \mathcal{E}\frac{\partial \vec{E}}{\partial t})\right]$$

$$= \frac{1}{c}\left[4\pi\gamma(\vec{\nabla} \times \vec{E}) + \mathcal{E}\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})\right]$$

$$= \frac{1}{c}\left[4\pi\gamma(-\frac{\mu}{c}\frac{\partial \vec{E}}{\partial t}) + \mathcal{E}\frac{\partial}{\partial t}(-\frac{\mu}{c}\frac{\partial \vec{H}}{\partial t})\right]$$

$$= -\frac{4\pi\gamma\mu}{c^2}\frac{\partial \vec{H}}{\partial t} - \frac{\mathcal{E}\mu}{c^2}\frac{\partial^2 H}{\partial t^2}$$

$$= \vec{\nabla}(\vec{\nabla} \cdot H) - \nabla^2 \vec{H} = -\nabla^2 \vec{H}$$

$$\nabla^2 \vec{H} = \frac{4\pi\gamma\mu}{c^2}\frac{\partial \vec{H}}{\partial t} + \frac{\mathcal{E}\mu}{c^2}\frac{\partial^2 H}{\partial t^2}$$

Finally:

$$\nabla^2 \vec{H} - \frac{4\pi\gamma\mu}{c^2}\frac{\partial \vec{H}}{\partial t} - \frac{\mathcal{E}\mu}{c^2}\frac{\partial^2 H}{\partial t^2} = 0 \qquad \text{(II - 18)}$$

The above is a wave equation in which the third term corresponds to a wave having a velocity $c/\sqrt{\mathcal{E}\mu}$ and the second term represents the damping of the oscillations.

Similarly, by starting with the equation

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot E) - \nabla^2 \vec{E}$$

we would find

$$\nabla^2 \vec{E} - \frac{4\pi\gamma\mu}{c^2}\frac{\partial \vec{E}}{\partial t} - \frac{\mathcal{E}\mu}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \qquad \text{(II - 19)}$$

For the special case of monochromatic radiation

$$\vec{E} = \vec{E}^{\circ} e^{j\omega t} \; ; \; \frac{\partial \vec{E}}{\partial t} = j\omega E^{\circ} e^{j\omega t} \; ; \; \frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 E^{\circ} e^{j\omega t}$$

$$\vec{H} = \vec{H}^{\circ} e^{j\omega t} \; ; \; \frac{\partial \vec{H}}{\partial t} \; j\omega \vec{H}^{\circ} e^{j\omega t} \; ; \; \frac{\partial^2 \vec{H}}{\partial t^2} = -\omega^2 \vec{H}^{\circ} e^{j\omega t}$$

Substituting the above into equations II - 16 and II - 17

$$\nabla^2 \vec{E}^\circ e^{j\omega t} - \frac{4\pi\gamma\mu}{c^2} j\omega \vec{E}^\circ e^{j\omega t} + \frac{\varepsilon\mu}{c^2} \vec{E}^\circ e^{j\omega t} = 0$$

$$\nabla^2 \vec{H}^\circ e^{j\omega t} - \frac{4\pi\gamma\mu}{c^2} j\omega \vec{H}^\circ e^{j\omega t} + \frac{\varepsilon\mu}{c^2} \vec{H}^\circ e^{j\omega t} = 0$$

Therefore

$$\left.\begin{aligned} \nabla^2 \vec{E}^\circ - j\frac{4\pi\gamma\omega\mu}{c^2}\vec{E}^\circ + \frac{\varepsilon\mu}{c^2}\vec{E}^\circ &= 0 \\ \nabla^2 \vec{H}^\circ - j\frac{4\pi\gamma\omega\mu}{c^2}\vec{H}^\circ + \frac{\varepsilon\mu}{c^2}\vec{H}^\circ &= 0 \end{aligned}\right\} \quad (II - 20)$$

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