Digital Computer Laboratory
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SUBJECT: GROUP 63 SEMINAR ON MAGNETISM XI

To: Group 63

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In the previous lecture we discussed the problem of the damped linear harmonic oscillator subject to a sinusoidally varying external force. In this lecture we will show how the harmonic oscillator may be used to explain the properties of a dielectric when subjected to electromagnetic radiation.

Optical Properties of a Dielectric

We have already discussed the microscopic picture of bound charges in a dielectric. When an electric field is applied the negative and positive charges in the dielectric will become slightly displaced, and the restoring force for small amplitudes is assumed to be directly proportional to the displacement from equilibrium. We further assume the damping force is proportional to the velocity, and apply an external field $E = E_o\, e^{2\pi i \nu t}$.

Using the notation of Seitz*, the equation describing the above situation is:

$$m \frac{d^2y}{dt^2} + 2\pi m \gamma \frac{dy}{dt} + ky = -e\ E_o\ e^{2\pi i \nu t} \qquad (XI-1)$$

wherein

$-e$ = charge
E_o = electric field amplitude
k = restoring force coefficient
$2\pi m \gamma$ = damping coefficient
m = mass
y = displacement
ν = frequency of imposed radiation

* Seitz, F. Modern Theory of Solids, McGraw Hill Co., New York, 1940, p. 633ff

Equation XI-1 is the exact form discussed in the previous lecture, with only the coefficient notations changed. The solution of equation XI-1 is then

$$y = -\frac{e}{4\pi^2 m} \frac{E_o e^{2\pi i \nu t - i\varphi}}{\sqrt{(\nu_o^2 - \nu^2)^2 + \gamma^2 \nu^2}} \qquad \text{(XI-2)}$$

wherein: $\nu = \frac{\omega}{2\pi}$

$$\nu_o = \sqrt{\frac{k}{4\pi^2 m}}$$

$$\varphi = \tan^{-1} \frac{\gamma \nu}{(\nu_o^2 - \nu^2)}$$

Then

$\vec{i}$ = current density

$$= -n_o e \frac{dy}{dt} = \frac{n_o e^2}{4\pi^2 m} \frac{2\pi \nu e^{-i(\varphi - \frac{\pi}{2})}}{\sqrt{(\nu_o^2 - \nu^2)^2 + \gamma^2 \nu^2}} E_o e^{2\pi i \nu t} \qquad \text{(XI-3)}$$

wherein n_o = number of oscillators/unit volume.

Since Ohm's law states

$$\vec{i} = \sigma \vec{E} \qquad \text{(XI-4)}$$

Then we may, by analogy, say

$$\sigma_c = \frac{n_o e^2}{4\pi^2 m} \frac{2\pi \nu e^{-i(\varphi - \frac{\pi}{2})}}{\sqrt{(\nu_o^2 - \nu^2)^2 + \gamma^2 \nu^2}} \qquad \text{(XI-5)}$$

Wherein σ_c is a complex conductivity. The real part of σ_c is just the conductivity due to free charge motion, and the imaginary part is related to the displacement current $\frac{\partial D}{\partial t}$, which arises as a result of the polarization of the dielectric.

Further it can be shown that

$$\sigma_c = \sigma + 2\pi i \nu \alpha \qquad \text{(XI-6)}$$

where α = polarizability, and is defined by the equation

$$\vec{P} = \alpha \vec{E}$$

Also,

$$\vec{D} = \epsilon \vec{E} = \vec{E} + 4\pi \vec{P}$$

$$\therefore \vec{P} = \frac{\vec{E}(\epsilon - 1)}{4\pi}$$

Therefore

$$\alpha = \frac{\epsilon - 1}{4\pi} \qquad \text{(XI-7)}$$

From XI-5 and XI-6, since

$$e^{-i(\varphi - \frac{\pi}{2})} = \sin\varphi + i\cos\varphi$$

we have

$$\sigma = \frac{n_0 e^2}{4\pi^2 m} \frac{2\pi \nu \sin\varphi}{\sqrt{(\nu_0^2 - \nu^2)^2 + \gamma^2\nu^2}} = \frac{n_0 e^2}{4\pi^2 m} \frac{2\pi\gamma\nu^2}{[(\nu_0^2 - \nu^2)^2 + \gamma^2\nu^2]} \qquad \text{(XI-8)}$$

$$\alpha = \frac{n_0 e^2}{4\pi^2 m} \frac{\cos\varphi}{\sqrt{(\nu_0^2 - \nu^2)^2 + \gamma^2\nu^2}} = \frac{n_0 e^2}{4\pi^2 m} \frac{(\nu_0^2 - \nu^2)}{[(\nu_0^2 - \nu^2)^2 + \gamma^2\nu^2]} \qquad \text{(XI-9)}$$

Curves of φ, σ, and α are shown in Figure 18, plotted against frequency.

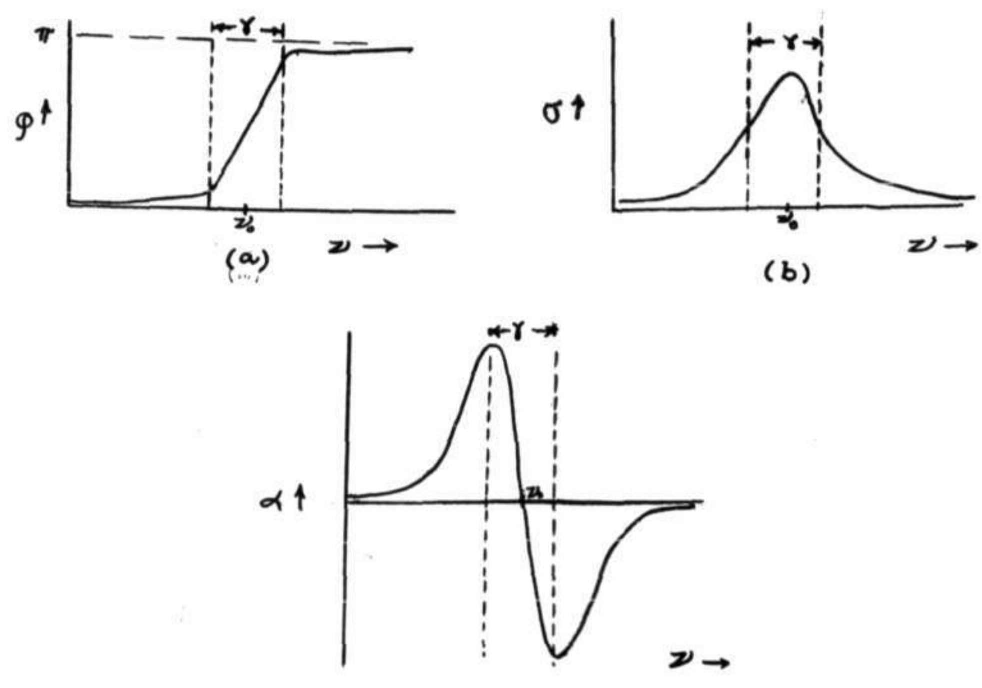

Figure 18

The proportion of electromagnetic radiation reflected at the surface of a semi-infinite piece of material is given by the reflection coefficient R.

$$R = \frac{(n-1)^2 + k^2}{(n+1)^2 + k^2} \qquad \text{(II-10)}$$

where n = index of refraction

k = extinction or absorption coefficient

$$n^2 = \frac{\epsilon + \sqrt{\epsilon^2 + 4\left(\frac{\sigma}{\nu}\right)^2}}{2} \qquad \text{(II-11)}$$

$$k^2 = \frac{-\epsilon + \sqrt{\epsilon^2 + 4\left(\frac{\sigma}{\nu}\right)^2}}{2} \qquad \text{(II-12)}$$

For a fuller discussion, particularly on thin metallic films, see the references given below.*+

In Figure 18 (a) we see the conductivity is appreciable only in the resonance region. When $\sigma = 0$, $k = 0$ when ϵ is positive; $\underline{k}$ is appreciable only when σ is appreciable. And as $k \rightarrow \infty$, $R \rightarrow 1$; hence in the absorption band the reflection is very high, so not much radiation is absorbed in spite of the high absorption coefficient.

On the low frequency side of ν_0, where $|\nu_0 - \nu| \gg \gamma$, γ is small but positive and σ is negligible (≈ 0).

Then, since $\alpha = \frac{\epsilon - 1}{4\pi}$,

$$\epsilon = 4\pi\alpha + 1 > 1$$

Substituting $\epsilon > 1$, and $\sigma = 0$ into XI-11 and XI-12 we obtain

$$n^2 = \epsilon > 1$$

$$k = 0$$

Therefore, in this region the system is transparent and has a refractive index greater than one.

Similarly, in the high frequency side of ν_0, $\sigma \approx 0$. However, since γ is negative in this region, we find that, depending on the value of α, ϵ can be either positive or negative.

For ν sufficiently high, however, α becomes very small, and ϵ is positive. Here the region is again transparent.

Between this high frequency transparent region and the absorption region, is a range wherein ϵ is negative. For negative ϵ, we see from equations XI-11 and XI-12 that

$$n = 0$$

$$k = \text{finite value}$$

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* Stratton, J. A., Electromagnetic Theory, McGraw Hill Book Co., New York, 1941 pp. 321ff, 511ff.

\+ Harris, Beasley and Loeb, Journal of Optical Society of America, 41, 1951, p. 604.

Therefore, from XI-10

$$R = \frac{k^2}{k^2} = 1$$

and the medium is totally reflecting independent of frequency over a range of frequencies which is known as the metallic reflection region.

The regions described fall approximately as shown in Figure 19. The curve of α is also drawn for comparative purposes, and it should be noted that the sign of ϵ changes when the value of α passes through

$$-\frac{1}{4\pi}.$$

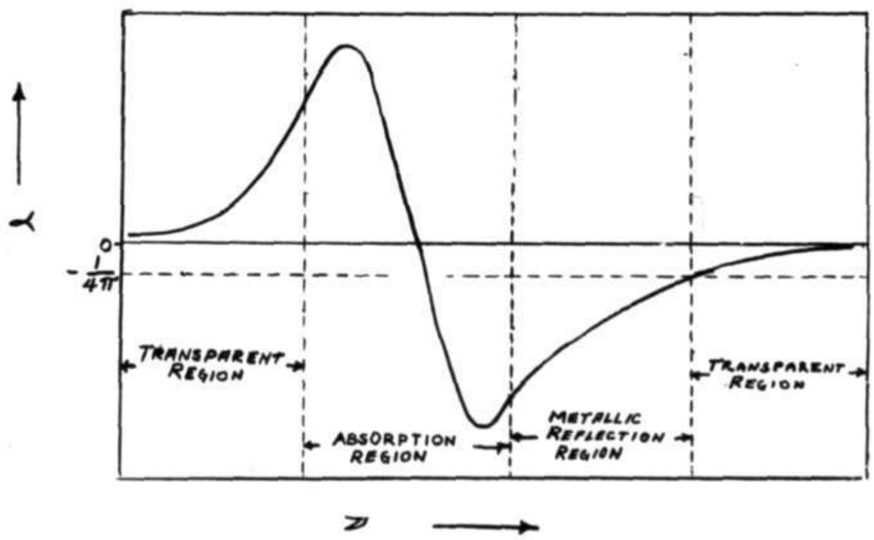

Figure 19

Drude-Zener Theory

For ideal metals, Drude and Zener developed a classical treatment of optical properties based on the assumption of perfectly free electrons. For this case, the equation of motion would be

$$m \frac{d^2y}{dt^2} + 2\pi m \frac{dy}{dt} = -e E_o e^{2\pi i \nu t} \qquad \text{(XI-13)}$$

which is the same as XI-1 except that the restoring force coefficient $\underline{k} = 0$.

Following the procedure used in determining the optical properties of dielectrics leads to the values

$$\sigma = n_o \frac{e^2}{4\pi^2 m} \frac{2\pi\gamma}{\nu^2 + \gamma^2} \qquad \text{(XI-14)}$$

$$\gamma = n_o \frac{e^2}{4\pi^2 m} \frac{1}{\nu^2 + \gamma^2} \qquad \text{(XI-15)}$$

Comparing these equations with XI-8 and XI-9 we see they are identical for the case $\nu_o = 0$. We can, therefore, expect a system of free electrons to have optical properties similar to those of an insulator on the high frequency side of the center of the absorption region; at low frequency (DC and ordinary AC currents), the energy is merely absorbed as heat; for higher frequencies (microwaves, infrared) the energy is largely reflected. Beyond this region the Drude-Zener theory does not generally hold because there are bound electrons with natural frequencies in this region.

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