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SUBJECT: GROUP 63 SEMINAR ON MAGNETISM XIII

To: Group 63

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In the previous lecture a system of two coupled oscillators was described (Figure 23) and we arrived at the equations of motion of the system. They are:

$$m_1 \frac{d^2 y_1}{dt^2} + k_1 y_1 + k(y_1 - y_2) = 0 \qquad \text{(XIII-1)}$$

$$m_2 \frac{d^2 y_2}{dt^2} + k_2 y_2 + k(y_2 - y_1) = 0 \qquad \text{(XIII-2)}$$

The effect of pendulum 1 on pendulum 2 and vice versa may be thought of as perturbation effects. If we removed the perturbation by considering the system as two separate pendulums attached to springs of force constant $\underline{k}$, but with the other end of the springs attached to a fixed point, the equations of motion would then be:

$$m_1 \frac{d^2 y_1}{dt^2} + (k_1 + k)\, y_1 = 0 \qquad \text{(XIII-3)}$$

$$m_2 \frac{d^2 y_2}{dt^2} + (k_2 + k)\, y_2 = 0 \qquad \text{(XIII-4)}$$

We call this system the "unperturbed" system.

From our discussion of the simple harmonic oscillator we know that these lead to natural resonant frequencies

$$\omega_1^2 = \frac{k_1 + k}{m_1}$$

$$\omega_2^2 = \frac{k_2 + k}{m_2} \qquad \text{(XIII-5)}$$

Let us return to the perturbed system described by equations XIII-1 and XIII-2, transforming y_1 and y_2 to x_1 and x_2 coordinates which are related by the equations

$$x_1 = \sqrt{m_1}\, y_1$$

$$x_2 = \sqrt{m_2}\, y_2$$

Then

$$\frac{d^2 x_1}{dt^2} = \sqrt{m_1} \frac{d^2 y_1}{dt^2} \quad \text{and} \quad \frac{d^2 x_2}{dt^2} = \sqrt{m_2} \frac{d^2 y_2}{dt^2}$$

Substituting into XIII-1 and XIII-2

$$\sqrt{m_1} \frac{d^2 x_1}{dt^2} + \frac{(k_1 + k)}{\sqrt{m_1}} x_1 - \frac{k}{\sqrt{m_2}} x_2 = 0$$

$$\sqrt{m_1} \frac{d^2 x_2}{dt^2} + \frac{(k_2 + k)}{\sqrt{m_2}} x_2 - \frac{k}{\sqrt{m_1}} x_1 = 0$$

Therefore,

$$\frac{d^2 x_1}{dt^2} + \omega_1^2 x_1 - \omega_c^2 x_2 = 0 \qquad \text{(XIII-6)}$$

$$\frac{d^2 x_2}{dt^2} + \omega_2^2 x_2 - \omega_c^2 x_1 = 0 \qquad \text{(XIII-7)}$$

where

$$\omega_c^2 = \frac{k}{\sqrt{m_1 m_2}}$$

$$\omega_1^2 = \frac{k_1 + k}{m_1} \qquad \text{(XIII-5)}$$

$$\omega_2^2 = \frac{k_2 + k}{m_2}$$

To solve XIII-6 and XIII-7

Assume

$$x_1 = a_1 e^{i\omega t}$$

$$x_2 = a_2 e^{i\omega t} \qquad \text{(XIII-8)}$$

Then from XIII-6 we have

$$-a_1 \omega^2 e^{i\omega t} + \omega_1^2 a_1 e^{i\omega t} - \omega_c^2 a_2 e^{i\omega t} = 0$$

$$-a_1 \omega^2 + a_1 \omega_1^2 - a_2 \omega_c^2 = 0$$

$$a_1 (\omega_1^2 - \omega^2) - a_2 \omega_c^2 = 0 \qquad \text{(XIII-9)}$$

Similarly from XIII-7

$$-a_1 \omega_c^2 + a_2 (\omega_2^2 - \omega^2) = 0 \qquad \text{(XIII-10)}$$

Then, from XIII-9

$$\frac{a_1}{a_2} = \frac{\omega_c^2}{(\omega_1^2 - \omega^2)}$$

from XIII-10

$$\frac{a_1}{a_2} = \frac{\omega_2^2 - \omega^2}{\omega_c^2}$$

Therefore consistency requires:

$$\frac{\omega_2^2 - \omega^2}{\omega_c^2} = \frac{\omega_c^2}{(\omega_1^2 - \omega^2)}$$

$$\omega_c^4 = (\omega_2^2 - \omega^2)(\omega_1^2 - \omega^2)$$

$$\omega^4 - \omega^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2 - \omega_c^4 = 0$$

$$\omega^2 = \frac{(\omega_1^2 + \omega_2^2)}{2} \pm \frac{1}{2}\sqrt{(\omega_1^2 + \omega_2^2)^2 - 4(\omega_1^2\omega_2^2 - \omega_c^4)}$$

and since

$$(\omega_1^2 + \omega_2^2)^2 - 4\omega_1^2\,\omega_2^2 = (\omega_1^2 - \omega_2^2)^2 \quad ,$$

$$\omega^2 = \frac{\omega_1^2 + \omega_2^2}{2} \pm \frac{1}{2}\sqrt{(\omega_1^2 - \omega_2^2)^2 - 4\omega_c^4}$$

Therefore
$$\omega^2 = \frac{\omega_1^2 + \omega_2^2}{2} \pm \frac{\omega_1^2 - \omega_2^2}{2}\sqrt{1 - \frac{4\omega_c^4}{(\omega_1^2 - \omega_2^2)^2}} \qquad \text{(XIII-11)}$$

For the case

$$4\,\omega_c^4 \ll (\omega_1^2 - \omega_2^2)^2$$

we may use the approximation

$$\sqrt{1 - \frac{4\omega_c^4}{(\omega_1^2 - \omega_2^2)^2}} \approx 1 - \frac{2\omega_c^4}{(\omega_1^2 - \omega_2^2)^2} \qquad \text{(XIII-12)}$$

Substituting the approximation XIII-12 into XIII-11, we find for the two modes of oscillation

$$\left.\begin{aligned} \omega_I^2 &= \omega_1^2 - \frac{\omega_c^4}{(\omega_1^2 - \omega_2^2)} \\ \omega_{II}^2 &= \omega_2^2 + \frac{\omega_c^4}{(\omega_1^2 - \omega_2^2)} \end{aligned}\right\} \qquad \text{(XIII-13)}$$

Thus we have two possible modes of vibration, with angular frequencies ω_I and ω_{II}. In quantum mechanics we would refer to these modes as the two states of the system.

In the above solution we have assumed $\omega_1 \neq \omega_2$. For the case $\omega_1 = \omega_2$ we have degeneracy and this problem will be treated later.

From the equations XIII-8, and the resultant frequencies,

$$\left.\begin{aligned} X_1 &= a_{1I}\, e^{i\omega_I t} + a_{1II}\, e^{i\omega_{II} t} \\ X_2 &= a_{2I}\, e^{i\omega_I t} + a_{2\,II}\, e^{i\omega_{II} t} \end{aligned}\right\} \qquad (XIII\text{-}14)$$

Substituting into XIII-6,we find

$$\left.\begin{aligned} a_{1\,I} &= \frac{\omega_c^2}{\omega_1^2 - \omega_I^2}\; a_{2\,I} \\ a_{2\,II} &= \frac{\omega_c^2}{\omega_2^2 - \omega_{II}^2}\; a_{1\,II} \end{aligned}\right\} \qquad (XIII\text{-}15)$$

Let us assume that at time $t = 0$, $x_1 = x_1^o$ and $x_2 = x_2^o$ where x_1^o and x_2^o are positions of maximum amplitude.

Then, substituting these boundary conditions into equation XIII-14, we find

$$\left.\begin{aligned} x_1^o &= a_{1\,I} + a_{1\,II} \\ x_2^o &= a_{2\,I} + a_{2\,II} \end{aligned}\right\} \qquad (XIII\text{-}16)$$

Then, from XIII-15 and XIII-13

$$a_{1\,I} = \frac{\omega_c^2}{\omega_1^2 - \left[\omega_1^2 - \dfrac{\omega_c^4}{\omega_1^2 - \omega_2^2}\right]}\, a_{2\,I} = \frac{\omega_1^2 - \omega_2^2}{\omega_c^2}\, a_{2\,I}$$

$$a_{2\,II} = \frac{\omega_c^2}{\omega_2^2 - \left[\omega_2^2 + \dfrac{\omega_c^4}{\omega_1^2 - \omega_2^2}\right]} a_{1\,II} = \frac{\omega_2^2 - \omega_1^2}{\omega_c^2} a_{1\,II}$$

Therefore

$$\begin{aligned} a_{2\,I} &= -p\, a_{1\,I} \\ a_{1\,II} &= p\, a_{2\,II} \end{aligned} \qquad \text{(XIII-17)}$$

where $p = \dfrac{\omega_c^2}{\omega_2^2 - \omega_1^2}$ and is called the perturbation parameter.

Substituting XIII-17 into XIII-16

$$\begin{aligned} x_1^o &= a_{1\,I} + p\, a_{2\,II} \\ x_2^o &= p\, a_{1\,I} + a_{2\,II} \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_1^o &= -\frac{1}{p}\, a_{2\,I} + a_{1\,II} \\ x_2^o &= a_{2\,I} + \frac{1}{p}\, a_{1\,II} \end{aligned}$$

These equations lead to the values

$$\left.\begin{aligned} a_{1\,I} &= \frac{x_1^o - p\, x_2^o}{1 + p^2} \\ a_{1\,II} &= p\, \frac{x_1^o + p x_1^o}{1 + p^2} \\ a_{2\,I} &= p\, \frac{x_1^o - p x_2^o}{1 + p^2} \\ a_{2\,II} &= \frac{x_2^o + p x_1^o}{1 + p^2} \end{aligned}\right\} \qquad \text{(XIII-18)}$$

It should be noted that when the coupling is small, as was assumed in deriving XIII-18, $p << 1$. Therefore, $a_{1\,II}$ and $a_{2\,I}$ are of smaller order of magnitude than $a_{1\,I}$ and $a_{2\,II}$, indicating that oscillator "1" vibrates mostly with frequency ω_I, with slight admixture of the mode II. Similarly oscillator "2" moves mostly in mode II. The coefficient $a_{1\,I}$ is determined primarily by x_1^o, for the coefficient of x_2^o is of smaller order of magnitude than that of x_1^o. Thus for oscillator "1" the amount of oscillation in mode I is determined primarily by the initial displacement of oscillator "1" itself. The amount of admixture of mode II in the motion of oscillator "1" is determined by coefficient $a_{1\,II}$, and is seen to depend mostly on the initial displacement of oscillator "2". This is easily understood when we consider that this admixture of the second mode is due to the perturbation, i.e. to the fact that the spring goes from pendulum "1" to pendulum "2" rather to a fixed point; therefore the amount of admixture of the second mode depends primarily on the displacement of the second pendulum. The motion of the second pendulum can be similarly analysed.

If, at time $t = 0$, $x_1^o = -x_2^o = x^o$

Then, from XIII-14 and XIII-18

$$x_1 = \frac{x^o(p+1)}{1+p^2}\, e^{i\omega_I t} + \frac{px^o(p-1)}{1+p^2}\, e^{i\omega_{II} t}$$

$$= \frac{x^o}{1+p^2}\left[(p+1)\, e^{i\omega_I t} + p(p-1)\, e^{i\omega_{II} t}\right]$$

$$x_2 = -\frac{px^o(p+1)}{1+p^2}\, e^{i\omega_I t} + \frac{x^o(p-1)}{1+p^2}\, e^{i\omega_{II} t}$$

$$= \frac{x^o}{1+p^2}\left[-p(p+1)\, e^{i\omega_I t} + (p-1)\, e^{i\omega_{II} t}\right]$$

$$= \frac{x^o}{1+p^2}\left[(-p)(-p-1)\, e^{i\omega_I t} + (1-p)\, e^{i\omega_{II} t}\right]$$

Therefore,

$$x_1 = \frac{x^o}{1+p^2}\left[\left(1+\frac{\omega_c^2}{\omega_2^2-\omega_1^2}\right)e^{i\omega_I t} + \frac{\omega_c^2}{\omega_2^2-\omega_1^2}\left(\frac{\omega_c^2}{\omega_2^2-\omega_1^2}-1\right)e^{i\omega_{II} t}\right]$$

$$x_2 = -\frac{x^o}{1+p^2}\left[\left(1+\frac{\omega_c^2}{\omega_1^2-\omega_2^2}\right)e^{i\omega_{II} t} + \frac{\omega_c^2}{\omega_1^2-\omega_2^2}\left(\frac{\omega_c^2}{\omega_1^2-\omega_2^2}-1\right)e^{i\omega_I t}\right]$$

$$\left(\text{since } \frac{\omega_c^2}{\omega_1^2-\omega_2^2} = -p\right)$$

Thus we see the solutions are symmetrical.

The method of solution shown here for two coupled oscillators may be used for n such oscillators. In that case, there would be n modes of vibration or "states". These n modes will be the solutions of the determinental equation

$$\begin{vmatrix} (\omega_{11}^2-\omega^2) & \omega_{12}^2 & \omega_{13}^2 & \cdots & \omega_{1n}^2 \\ \omega_{21}^2 & \omega_{22}^2-\omega^2 & \omega_{23}^2 & \cdots & \omega_{2n}^2 \\ \omega_{31}^2 & \omega_{32}^2 & \omega_{33}^2-\omega^2 & \cdots & \omega_{3n}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \omega_{n1}^2 & \omega_{n2}^2 & \omega_{n3}^2 & \cdots & \omega_{nn}^2-\omega^2 \end{vmatrix} = 0$$

where

$$\omega_{\ell\ell} = \frac{k_{\ell} + k}{m_{\ell}}$$

$$\omega_{\ell j} = \frac{k_{\ell j}}{\sqrt{m_{\ell} m_j}}$$

and k_{1j} is the restoring force connecting oscillators 1 and j. This will be treated later in greater detail.

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