

Memorandum M-1697

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SUBJECT: GROUP 63 SEMINAR ON MAGNETISM, APPENDIX II  
 To: Group 63  
 From: Arthur Loeb and Norman Menyuk  
 Date: October 28, 1952

The atom consists of a nucleus and electrons. We will usually consider the electrons as spinning around a stationary nucleus. These electrons have an orbital angular momentum due to their motion about the nucleus, and a spin angular momentum due to rotation about their own axis. To find the total angular momentum of an electron, the spin and orbital angular momenta must be added vectorially. In systems containing more than one electron, the method of adding up the individual momentum vectors depends upon the type of coupling existent between these vectors. Most frequently we have Russell-Saunders Coupling, in which case the individual orbital angular momentum vectors of all electrons are added to obtain a total orbital angular momentum vector. Similarly the total spin angular momentum is found, and these two resultants are then added to obtain the total angular momentum of the system.

Thus far nothing has been said about discreteness. However, according to Bohr, the only permissible electron orbits are those for which the angular momentum is  $\frac{nh}{2\pi}$ ,

where  $n = \text{integer} = 1, 2, 3, \dots$

$h = \text{Planck's Constant} = 6.624 \times 10^{-27} \text{ erg-sec}$

We thus have a quantity  $\frac{h}{2\pi}$  which may be considered a fundamental unit of angular momentum.

Spin angular momentum, orbital angular momentum, total angular momentum, and the magnetic field component of angular momentum are all of interest. All these quantities are limited to certain discrete values, and are, therefore, referred to as being quantized. The nature of the quantizations is as follows:

Quantization of the Orbital and Spin Angular Momentum of the Individual Electrons

The orbital angular momentum of an electron is  $\frac{1h}{2\pi}$  and its spin angular momentum is  $\frac{3h}{2\pi}$ ,

where  $\vec{l} = 0$  or a positive integer

and  $\vec{s} = 1/2$

Quantization of the Orbital and Spin Angular Momentum of a System of Electrons

For an entire system of electrons,

orbital angular momentum =  $\frac{\vec{L}h}{2\pi}$ , where  $\vec{L} \equiv \sum \vec{l}$

spin angular momentum =  $\frac{\vec{S}h}{2\pi}$ , where  $\vec{S} \equiv \sum \vec{s}$

with the further condition that

$\vec{L} = 0$  or a positive integer

$\vec{S} = 0$  or a positive integer, or a positive half integer,

depending upon whether there are an even or odd number of electrons in the system.

Quantization of the Total Angular Momentum

Further, when the spin and orbital momenta are added vectorially, they must be so directed with respect to each other that the total angular momentum  $\frac{\vec{J}h}{2\pi}$  has

$\vec{J} = 0$ , positive integer, or positive half integer

(Depending on whether  $\vec{S}$  is a full or half integer)

An example of the vector addition of  $\vec{S}$  and  $\vec{L}$  is shown in Figure III for the case  $\vec{L} = 2$ ,  $\vec{S} = 3/2$

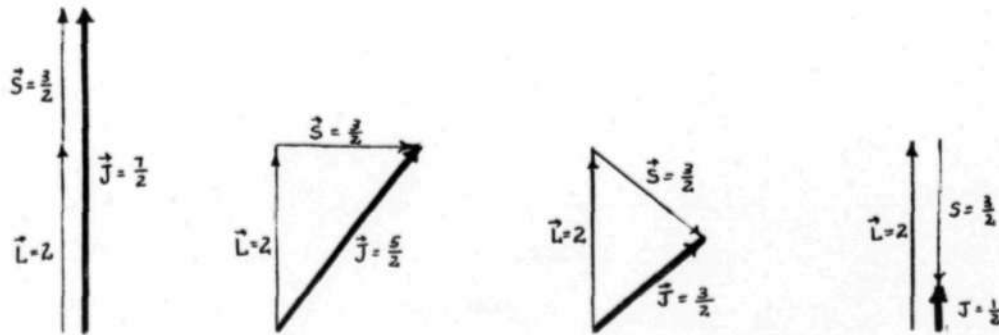


Figure III

From Figure III we see that the total angular momentum vector  $\vec{J}$  may take any of the values  $L - S$ ,  $L - S + 1$ , -----  $L + S$ .

Quantization of the Component of Angular Momentum with respect to External Magnetic field Direction

Another term of interest is the quantity  $\vec{M}$ , where  $\vec{M} \frac{h}{2\pi}$  is the component of the angular momentum  $\frac{\vec{J}h}{2\pi}$  in the direction of an external magnetic field. Further, the angular momentum vector can take only certain discrete directions in a magnetic field. This means that  $\vec{J}$  is space quantized in a magnetic field. It will be so quantized that the component  $\vec{M}$  of angular momentum  $\vec{J}$  will be half integral when  $\vec{J}$  is half integral and integral when  $\vec{J}$  is integral. Examples of this are shown in Figure IV for  $\vec{J} = 2$  and  $\vec{J} = \frac{3}{2}$

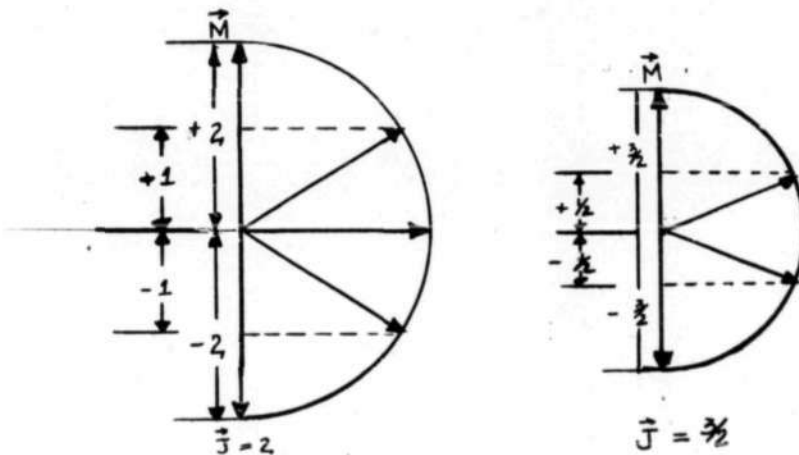


Figure IV

Thus we see that  $\vec{M}$  may take the values

$$\vec{M} = \vec{J}, \vec{J} - 1, \vec{J} - 2, \text{-----}, -\vec{J}.$$

Now let us consider the magnetic moment of an atomic system. According to classical physics, the magnetic moment due to a revolving negative point charge is

$$\vec{\mu} = - \frac{e}{2mc} \vec{p} \tag{B - 1}$$

where  $\vec{\mu}$  = magnetic moment  
 $\vec{p}$  = angular momentum of charged particle  
 $m$  = mass of charged particle

Applying this to our vector model, we find that for atoms, the magnetic moment would be

$$\vec{\mu} = -\frac{e}{2mc} \times \frac{\vec{J}h}{2\pi} = -\frac{eh}{4\pi mc} \vec{J} \quad (B-2)$$

The quantity  $\frac{eh}{4\pi mc}$  is known as the Bohr magneton.

$$\beta = \frac{eh}{4\pi mc} = 9.27 \times 10^{-27} \text{ ergs/oersted}$$

However, the simple relationship given in equation B-2 does not hold. It was found that correct results were obtained by assuming that the spin quantum number has twice as great an effect as that expected.

Thus we obtain

$$\begin{aligned} \vec{\mu}_S &= -\frac{e}{mc} \frac{\vec{S}h}{2\pi} \\ \vec{\mu}_L &= -\frac{e}{2mc} \frac{\vec{L}h}{2\pi} \\ \vec{\mu} &= \vec{\mu}_S + \vec{\mu}_L \end{aligned} \quad (B-3)$$

The resultant magnetic moment, therefore, is no longer in the direction of  $\vec{J}$ , but precesses around  $\vec{J}$ . Since the precession causes an effective cancellation of the magnetic effect perpendicular to  $\vec{J}$ , we need only consider the component of  $\vec{\mu}$  in the  $\vec{J}$  direction,  $\vec{\mu}_J$ , to calculate the magnetic effect.

From Figure V we see that

$$\vec{\mu}_J = \vec{\mu}_L \cos(L, J) + \vec{\mu}_S \cos(S, J) \quad (B-4)$$

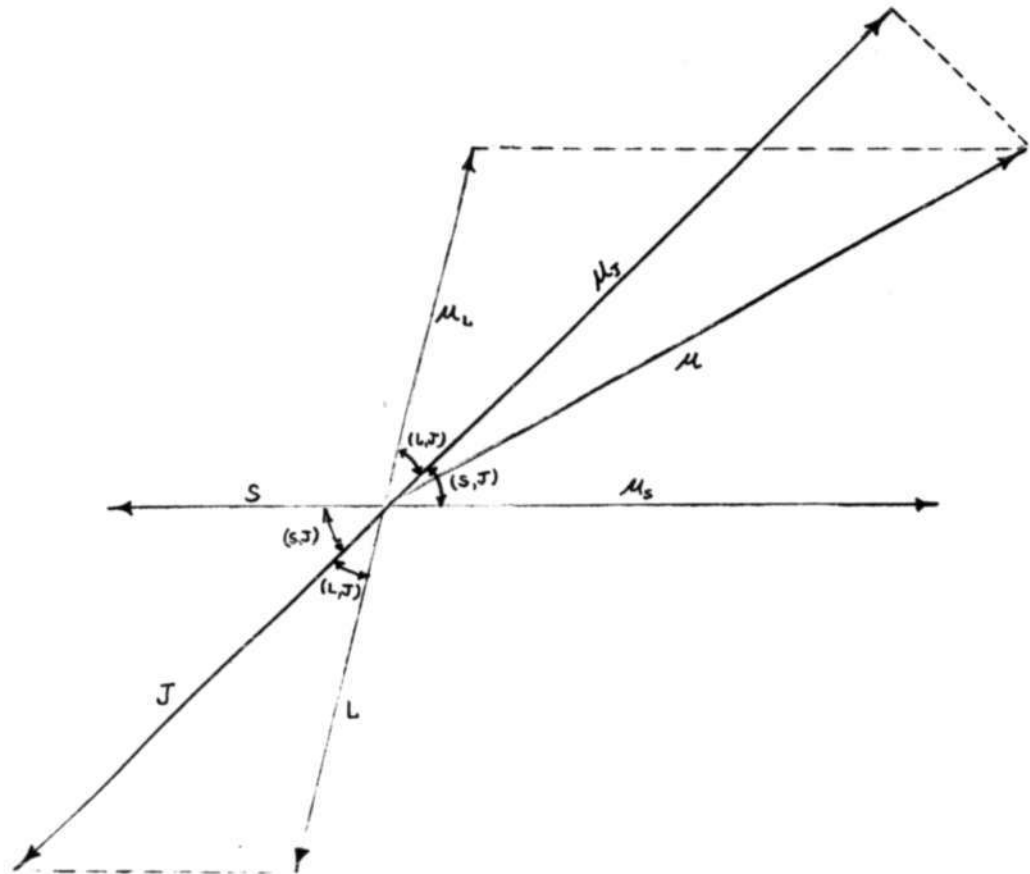


Figure V

In order to maintain the simple form of equation B - 2, in formal notation, one writes

$$\mu_J = -g \frac{eh}{4\pi mc} J \quad (B - 5)$$

where, by equating the right hand side of equations B - 4 and B - 5, we may determine the value of g, known as the Landé g-factor.

To do this, with the aid of Figure V and the law of cosines, we see

$$\begin{aligned} L^2 + J^2 - 2LJ \cos(L, J) &= S^2 \\ \cos(L, J) &= \frac{J^2 + L^2 - S^2}{2LJ} \end{aligned} \quad (B - 6)$$

Similarly,

$$\begin{aligned} S^2 + J^2 - 2SJ \cos(S, J) &= L^2 \\ \cos(S, J) &= \frac{J^2 + S^2 - L^2}{2SJ} \end{aligned} \quad (B - 7)$$

Substituting B-6 and B-7 into equation B-4, we have

$$\begin{aligned} \mu_J &= -\frac{eh}{4\pi mc} L \left( \frac{J^2 + L^2 - S^2}{2LJ} \right) - \frac{eh}{2\pi mc} S \left( \frac{J^2 + S^2 - L^2}{2SJ} \right) \\ &= -\frac{eh}{4\pi mc} \left( \frac{J^2 + L^2 - S^2}{2J} + \frac{2(J^2 + S^2 - L^2)}{2J} \right) \\ &= -\frac{eh}{4\pi mc} \left( \frac{3J^2 + S^2 - L^2}{2J} \right) = -\frac{eh}{4\pi mc} \left( J + \frac{J^2 + S^2 - L^2}{2J} \right) \\ &= -\frac{eh}{4\pi mc} J \left( 1 + \frac{J^2 + S^2 - L^2}{2J^2} \right) \end{aligned} \quad (B - 8)$$

Equating this to equation B - 5

$$-\frac{eh}{4\pi mc} J g = -\frac{eh}{4\pi mc} \left( 1 + \frac{J^2 + S^2 - L^2}{2J^2} \right)$$

$$g = 1 + \frac{J^2 + S^2 - L^2}{2J^2} \quad (B - 9)$$

It should be noted that the advent of quantum mechanics has led to a refinement in the above. Instead of  $L^2$ ,  $S^2$ ,  $J^2$ , the quantities should be  $L(L + 1)$ ,  $S(S + 1)$ , and  $J(J + 1)$  respectively. This leads to the value for  $g$

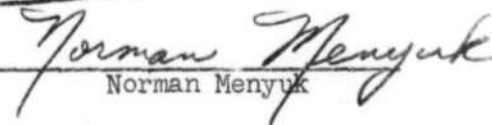
$$g = 1 + \frac{J(J + 1) + S(S + 1) - L(L + 1)}{2J(J + 1)}$$

Signed



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