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The Postulate-Method and the Map-Problem.

One of the great tendencies of modern mathematics is that which is expressed by the increasing use of postulates in mathematical research. Geometry, which used to be regarded by mathematicians as the study of certain eternal verities concerning the world in which we live, involving logic, it is true, but in addition an element over and above logic, is now treated as the science which draws the conclusions that logically follow from a certain set of presuppositions or postulates, ~~concerning as to~~ in the truth or validity of which the geometer has not the least concern. These postulates enable us to form ~~the~~ a precise and complete account of the task which geometry has before it, and furnish in themselves the entire ~~complete~~ basis for the accomplishment of this task.

Ordinary ~~Euclidean~~ metrical Geometry is not the only one of the mathematical sciences in which this method has been found useful. Projective geometry, descriptive geometry, the theory of groups, and many other branches of mathematics have made use of postulates, or have even found their origin in the investigation of the consequences of a set of postulates. Not only branches of mathematics, but even individual problems have been handled by this method. In this paper, we shall discuss its applicability to a certain famous mathematical problem — that of proving — or disproving, as the case may be — that any map on the plane can be colored in four or less colors in such a manner that no two contiguous regions shall be colored in the same color.

This problem is, of course, purely geometrical, and has nothing whatever to do with colors. It can therefore be ~~proved~~ solved — if it can be solved at all — by means of the ordinary postulates of geometry. It is obvious that any solution on such a basis, however, requires a fearfully complicated apparatus of theorems and definitions from the field of the theory of sets of points, for a region, a connected region, the contact of two regions etc. all need to be defined and to have their properties determined before any progress in the solution of the map-problem by this method is possible. Furthermore, a little reflection will convince one that all this work is labor lost so far as the actual solution of the map-problem is concerned. It will readily be seen that the conclusion at which the map-problem aims is entirely concerned with the formal properties of the relation of contact between the regions in a given map: it says that the terms that enter into this relation — that is, the regions in the map — can be divided into four ~~class mutually exclusive~~ classes such that no two members of a single class bear the relation of contact to one another. A map, then, as far as the solution of the map-problem is concerned, may be regarded ~~of~~ as represented by a certain matrix of terms in which each row and each column represents a region, and in which a mark at the intersection of a row and a column indicates that the corresponding regions touch, while ~~the absence~~ a zero indicates the absence of such a contact. All maps, then, determine matrices, and the colorability of a map can be discovered from the matrix. It is obviously false, however, that all matrices can be made to represent maps on the plane, as may be shown by simple examples such as that of a matrix of five rows and columns in which every intersection of a row with a column representing a different region is filled by a mark. This can also be made intuitively obvious by reflecting that all maps on multiply connected surfaces determine matrices, even though these maps may be ~~are~~ essentially different in character from planar maps. If we are able to determine just what limitations are necessary and sufficient to secure that a matrix represents a planar map, we may regard these conditions as a set of postulates for the map-coloring problem. Such a set of postulates will lead us at once to the heart of the matter, without involving us in that endless preliminary work which ~~is~~ would be inevitable if we should start directly from the ordinary postulates of geometry. Of course, if our map-problem is to be solved for an ordinary geometrical plane, it must be ~~proved that~~ solved on the basis of the ordinary postulates of geometry, but it is far more convenient to do this mediately by the specification of the formal properties of the relation of contact by means of a set of postulates than immediately from our geometrical postulates themselves.

Before we go on to specify what the postulates of the map-problem are actually to be, let us consider a few well known preliminary reductions ~~of~~ to which this problem is subject. In the first place, we can limit our discussion of maps to that of those in which no more than three regions meet at a point. Obviously if more than three regions meet at a point we may, by bringing into linear contact, distinct regions at this point which had no such contact previously, transform our map into one in which there is one less point at which more than three regions meet, and so ultimately into a map in which no more than three regions meet at any point. Let us suppose that it is possible to color this latter map: since every contact in our original map is preserved in this, though new contacts are introduced, our original map can be colored. Thus the problem of coloring any map with points where more than three regions meet reduces itself to the

—which, for the purposes of the map-problem, is the only sort of contact—

coloring of a map without such points, so that only maps without such points need be considered in the discussion of the map-problem.

Another kind of map which can be eliminated from discussion in the consideration of the map-problem is that in which a ~~single~~ region of the map contains other regions as enclaves. In such a case, the coloration of the total map reduces itself to the coloration of two distinct maps of a smaller number of regions, one consisting of a region immediately containing a number of enclaves together with these enclaves, while the other consists of the total map with these enclaves removed. For suppose that each of these maps can be colored in four colors. They only come into contact in the single region containing the enclaves, so that if this region is of the same color in the two partial maps, the total map can be colored. If the two partial maps, however disagree in respect of the coloration of their common region, a mere permutation of the colors of one of them which does not affect ~~the~~ its colorability will make the two partial maps agree in this respect. Thus the problem of coloring maps with regions containing enclaves reduces itself to the problem of coloring maps in a smaller number of colors, and ultimately to the problem of coloring maps in which no region contains an enclave.

In a precisely similar manner, we can eliminate from discussion all maps in which a pair of regions conjointly enclose an enclave or in which a ring of three regions, each touching each, divides the map into two parts, for ~~in~~ each map of either of these sorts can be divided into two distinct maps of a smaller number of colors which possess a region in common which must be colorable in each of the partial maps — if both of these be colorable — in manners which differ only by a simple permutation of all of the colors of one of the partial maps.

Maps subject to all the limitations we have so far named which can be drawn on the plane will be called ordinary maps in the remainder of this paper. It is easy to see that if we give a set of postulates which will be satisfied by all ordinary maps, ~~we shall have given~~ and if we prove that all the maps which satisfy our postulates lie on the plane, we shall have given a completely adequate set of postulates for the map problem, for if the ~~no~~ four-color theorem is true, since it is true for all planar maps, it is true for all maps which satisfy our postulates, while if it is false, since it will then be false for some ordinary map, it will be false for some map which satisfy our postulates. A complete scrutiny of the maps that satisfy our postulates would therefore determine the truth or falsity of the four-color theorem.

Our next problem is to define in terms of the relation of contact alone a sufficient apparatus of notions to enable us to state our set of postulates for the map-problem. Our definitions run as follows:

Definition 1. A net in a map M is a collection N of regions in M such that we can pass from any region a of N to any other b through a succession of regions of N which begins with a and ends with b, and which is such that each region of the chain touches the next.

Definition 2. A ring in a map M is a net each region of which touches two and only two others.

We now are ready to go on to the explicit formulation of our postulates. ~~First of all come two propositions~~ All these postulates are to be conditions which limit and specify the formal properties of the relation of contact among regions on a planar map, and which involve no other specific notion, for the four-color theorem itself is a purely formal proposition concerning the properties of the relation of contact. We are at perfect liberty, however, to introduce into our postulates the notions of 'net' and 'ring' which we have just defined for these notions have involved in their definition no other specific notion than that of contact. Our postulates will begin with ~~two~~ the statement of ~~two~~ three very general properties of maps which are independent of the fact that they can be constructed on a plane. These are

Postulate 1. If a region a in a map M touches a region b in the same map, b touches a.

Postulate 2. A map contains a finite number of regions: i.e. only a finite number of regions enter into the relation of contact in a given map.

Postulate 3. A map forms a single net.

The meaning and validity of postulate 1 are obvious. As to postulate 2, it might be that the four-color theorem is true for both finite and infinite maps, but if it should hold for finite, but not for infinite maps, it would still be said to hold good. That is, in its usual acceptation, the map-problem is a problem concerning maps of a finite number of regions. As to ~~Fo~~ postulate 3, it is obvious at once that it puts no real limitation on the map-problem for if a map consists of a number of disconnected nets, the problem of coloring the map reduces itself to that of coloring each of these nets separately. Postulate 3, however makes certain portions of my later work in this paper more simple.

Our two remaining postulates will have to do with properties that belong to the relation of contact on the plane, as distinguished from contact in maps on such multiply-connected regions as the surface of the torus, and with properties which belong not to all maps on the plane, but to all ordinary maps on the plane. Let us consider all rings on the plane. Rings on the plane either may be reduced to mere corners where several regions join or they have an inside and an outside. The first case has already been eliminated except in cases where just three regions form a ring by the restriction of our consideration in this paper to ordinary maps. As to the second contingency, we mean that our map is composed of our ring and two nets which have no member of one adjacent to a member of the other ~~in common~~. We thus get the following postulate:

Postulate 4. If M be an ordinary map and R be a ring in M of more than three regions, the portion of M left after R is removed consists of two nets which do not together form a single net.

Another characteristic feature of ordinary planar maps is that all the regions which touch any given one of our regions form a ring. That this is so may be seen as follows: in the first place, it is not possible for any region touching a given region r to touch more than two other regions touching r. For suppose this were possible. Then we should have five regions, r, s, t, u, and v, such that r touches s, t, u, and v; while s touches r, t, u, and v. In a plane, it is impossible so to arrange t, u, and v so that we do not either have r, s, and t separating u from v and enclosing u or v, or r, s, and u separating t from v, or r, s, and v separating t from u. These contingencies, however, are excluded by our definition of ordinary ~~regions~~ maps. In the second place, it may be shown that lacunae in a plane map have exactly the same representation in a contact-matrix as corners where ~~several~~ more than three regions meet, and can, like the latter, be eliminated in ~~maps~~ ordinary maps. We can consequently regard our plane map as densely packed with regions everywhere. Since our map being ordinary, has no ~~excludes~~ enclaves in any single region or pair of regions, if a region s touches a region r, s touches on either side of r ~~two distin~~ a region which is distinct from that touched on the other side of r. Since all corners on our map are corners where only three regions join, each of these regions touches r. As a consequence, the regions that touch r form one or more chains. This latter contingency is obviously impossible, so they all must form a single chain. We thus obtain the following postulate:

Postulate 5. In any ordinary map, all the regions which touch a given region form a ring.

It is obvious from what we have said that all ordinary maps satisfy our postulates: we now wish to prove that all maps which satisfy our postulates are planar maps. Obviously any contact matrix whatever can be represented by regions on the plane if we permit our regions to touch other regions through tentacles, as it were, extending through space. If these tentacles can be so disposed that they lie in our plane without crossing any other tentacle or any region, our theorem is obviously demonstrated, and the adequacy of our set of postulates is assured. I shall try to demonstrate the truth of this theorem by showing that either we can ~~either~~ lower the tentacles one by one into the plane in such a manner that ~~they~~ each tentacle does not conflict with the map already established on the plane or else we come sooner or later to a configuration that is not compatible with our postulates. We can obviously lower ~~our~~ the tentacles, if we are able to lower them at all, in such an order that at any stage the regions in which the tentacles already lowered terminate form a connected map. When we come to lower any new tentacle into the plane, it either terminates

(1) In some region not in the connected map of planar regions already formed.

or (2) In two adjacent regions of the connected map.

or (3) In two regions of the connected map, which form part of a ring of more than three regions of the connected map, and are only connected by the tentacle in question.

or (4) In two regions of the connected map, which are each in contact with some third region of the connected map, but are not in contact with one another, and do not form part of a ring of ~~three~~ more than three regions on the connected map.

~~All~~ any tentacle~~s~~ of form (1) can be lowered into the plane. That this is so can be seen from the fact that if, in making our connected region, we allow a lacuna to remain wherever three regions meet, it will be possible to move any region so as to touch the edge of any other. As one of our regions here is not part of the connected region already formed, any single tentacle which attaches it to a member of that connected region can be removed, so that the two areas are brought into direct contact.

Any tentacle of type (2) can be removed, for it fulfils no function.

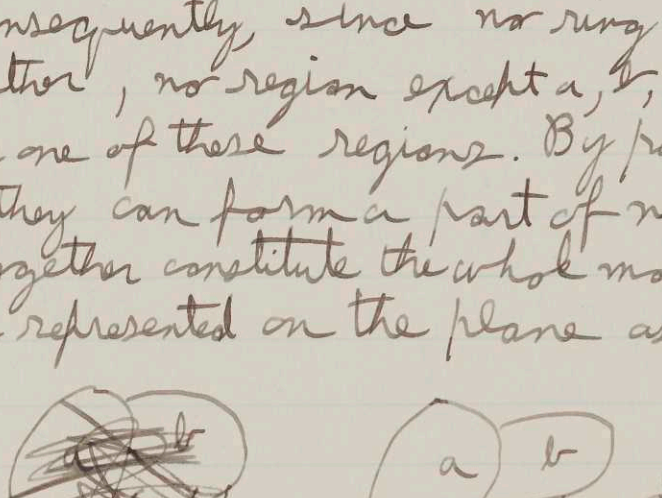
Any tentacle of type (3) can be removed. By postulate 3, if a ring of more than three regions is removed from a map, the map is divided in two. If our map contains such a ring, as it does when it contains a tentacle of type (3), ~~a fortiori the~~ it is divided in two, and it follows *a fortiori* that the connected portion of it already down on the plane is divided up. Let the regions connected by our tentacle be *a* & *b*. One of the ~~divisions~~ 'splits' of our plane map must be such that some regions touching a and some touching b are on one side of it, while other regions touching a and other regions touching *b* are on the other side of it. That this is the case may be shown in the following way. Let the regions of the ring in question, taken in order, be a, b, c, d, e, f, g, . . . m. Let μ and ν be regions on opposite sides of the ring. Since our map is connected, there must be a ~~chain~~ net connecting μ and ν. This ~~chain~~ net, going from μ to ν, must have a first ~~point~~ region where it cuts the ring, for it must cut the ring, since μ and ν are on opposite sides of it. ~~This chain must com~~ Let this first region be k. The chain before it ~~cuts~~ passes through k, must pass through some region on the ~~k~~ μ side of the ring. Let this region be r. Since all the regions in contact with k form a ring, and since a ring

The question of the ~~unicity~~ unicity of this first region is unimportant, and need not be discussed is a net, it is possible to go from r to s, t, and other regions touching k, still remaining in a net from μ. Sooner or later in this net we shall reach l, a region in our ring on one side of k. Let u be the region we hit just before l. Let v be the region touching l on the other side of u from k. We can now follow around the ring surrounding ~~k~~ till we strike the next region m, and so on till we reach a and b. We shall never have this process interfered with by finding ~~some~~ member of our ring in contact with no member of the net we thus produce, for then we should have to have three regions of our ring, each touching one another, which is impossible from the definition of a ring unless, contrary to our hypothesis, our ring consists of three regions only. We have thus proved that there are regions touching a & b on the μ-side of our ring. The same conclusion can be proved concerning the v-side of our ring in the same manner. If our ring is now removed, a and b will be found to lie in a channel which extends on either side of each of them and which encloses at least two islands. If this channel be filled up by reinserting a, b, and the remainder of the regions of the ring, there will still be an open channel around the islands connecting a and b, and in this channel the tentacle connecting a and b can be deposited. We have thus found a method of lowering all tentacles except those of case (4).

In case (4), let a and be be the regions connected by the tentacle in question, and let c be the region with which both are in contact. The

by postulate 5

regions in contact with c form a ring which must, of course, contain a and b. By hypothesis, there is no ring of more than three terms connecting a with b. Consequently, the regions that touch c must be three in number — namely, a, b, and a fourth region d, which touches both a and b. We consequently have a surrounded by the ring bcd, b surrounded by the ring acd, c surrounded by the ring abd, and d surrounded by the ring abc. Consequently, since no ring can form a part of another, no region except a, b, c, or d can touch any one of these regions. By postulate 3, then, since they can form a part of no larger map, they together constitute the whole map. This map can be represented on the plane as follows:



Consequently all maps containing tentacles of the fourth kind and satisfying our postulates — namely, this map alone — can have their tentacles lowered and can be mapped on the plane. This completes our proof that all ~~ma~~ contact-matrices which satisfy all of our postulates represent maps that can be constructed on the plane. As we have already shown that all ordinary maps satisfy our postulates, and that the solution of the four-color problem for all plane maps and for all ~~plane~~ ordinary maps are one and the same ~~problem~~ task, we have proved that a necessary and sufficient condition of the validity of the four-color theorem for all plane maps is that it can be ~~proved~~ shown from the postulates we have given to hold of all maps that satisfy these postulates. That is, the four-color problem reduces itself to the question: given a relation which satisfies the formal conditions the satisfaction of which is demanded of the relation of contact between regions in postulates 1 – 5 inclusive, is it necessarily true that the field of this relation can be divided into four classes, such that no two members of one of these classes bear the relation of contact to one another?

It has become a habit among mathematicians nowadays to affix to any work they do on postulates neat little independence proofs, or examples of systems which satisfy all but any given one of their postulates. Just what ~~math~~ mathematical or logical function proportionate to the effort expended is thereby fulfilled is a trifle obscure, especially when the work is carried on so as to secure the 'complete independence' of one's postulates from one another, and to make each postulate independent for its validity on the affirmation or negation of any of the others. However, in deference to this custom, which it seems, must serve some useful end, I shall give contact-matrices which represent relations of contact which break down only on the first of my postulates, or only on the second, or only on the third, and so on.

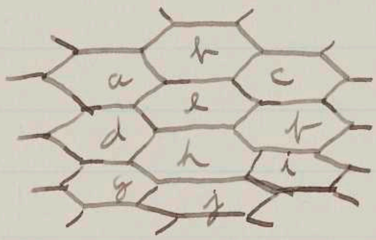
Postulate 1

	a	b	c	d	e
a		o	o	o	o
b	✓		✓	✓	✓
c	✓	✓		✓	✓
d	✓	✓	✓		✓
e	o	✓	✓	✓	

This table satisfies postulates 2 and 3, as may be seen on inspection. Postulate 4 is satisfied vacuously. Postulate 5 is satisfied, since a is surrounded by the ring b c d, b is surrounded by the ring c d e, c is surrounded by the ring b d e, d is surrounded by the ring b c e, while e is surrounded by the ring bcd. Postulate 1, however, is not satisfied, since b touches a while a does not touch b.

Postulate 2.

The example here is a map made up of an infinite number of regular hexagons fitting together and filling all ~~space~~ the plane, as follows:

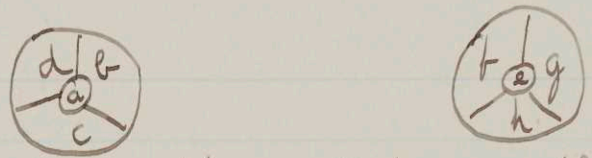


Since we can take the relation of contact as symmetrical in this map, postulate 1 is satisfied. The validity of the others may be seen on inspection.

Postulate 3.

	a	b	c	d	e	f	g	h
a		v	v	v	o	o	o	o
b	v		v	v	o	o	o	o
c	v	v		v	o	o	o	o
d	v	v	v		o	o	o	o
e	o	o	o	o		v	v	v
f	o	o	o	o	v		v	v
g	o	o	o	o	v	v		v
h	o	o	o	o	v	v	v	

Since this matrix represents the following map,



the fact that it satisfies all the postulates but postulate 3, while it fails to satisfy postulate 3 — that our map must be connected — is obvious.

Postulate 4.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
a		v	o	v	v	v	o	o	o	o	o	o	o	v	v	o
b	v		v	o	o	v	v	o	o	o	o	o	o	o	v	v
c	o	v		v	o	o	v	v	o	o	o	o	v	o	o	v
d	v	o	v		v	o	o	v	o	o	o	o	v	v	o	o
e	v	o	o	v		v	o	v	v	v	o	o	o	o	o	o
f	v	v	o	o	v		v	o	o	v	v	o	o	o	o	o
g	o	v	v	o	o	v		v	o	o	v	v	o	o	o	o
h	o	o	v	v	v	o	v		v	o	o	v	o	o	o	o
i	o	o	o	o	v	o	o	v		v	o	v	v	v	o	o
j	o	o	o	o	v	v	o	o	v		v	o	o	v	v	o
k	o	o	o	o	o	v	v	o	o	v		v	o	o	v	v
l	o	o	o	o	o	o	v	v	v	o	v		v	o	o	v
m	o	o	v	v	o	o	o	o	v	o	o	v		v	o	v
n	v	o	o	v	o	o	o	o	v	v	o	o	v		v	o
o	v	v	o	o	o	o	o	o	o	v	v	o	o	v		v
p	o	v	v	o	o	o	o	o	o	o	v	v	v	o	v	

A more or less laborious inspection of this matrix will convince us that it satisfies all of our postulates save postulate 4. As to postulate 4, if you remove the ring a b c d from the map, you will still have a connected map. The map I have given is, if I am not mistaken, the simplest map on the torus that satisfies postulate 5 and cannot be drawn on the plane.

Postulate 5.

A ~~single region sats~~ map consisting of a single region obviously satisfies postulates 2, and 3, while it satisfies postulates 1 and 4 vacuously. It obviously fails to satisfy postulate 5, as in this map there is a region which is such that the regions which touch it, as there are none such, do not form a ring. It is easy to give less trivial examples: such a one as the following will do.

	a	b	c	d
a		v	v	o
b	v		v	v
c	v	v		v
d	o	v	v	

Here the regions touching a do not constitute a ring, while all the postulates but 5 are obviously satisfied.