THE METHOD OF POSTULATES IN MODERN MATHEMATICS.

Mathematics is often looked upon as the most stable of all sciences, and in a sense this is so. The theorems of geometry which were proved by the ancient Greek mathematicians form part and parcel of the mathematics of the present day. The astronomy of the ancient Hindus has long possessed merely an historical interest, but their work in the theory of numbers is still deemed worthy of study. Notwithstanding such facts as these, however, and notwithstanding the general permanence of individual items of mathematical knowledge, in so far as the comprehension of the meaning and purpose of mathematics as a whole is concerned, as great changes have taken place as is the case with any natural science.

The classical conception of mathematics, which is still retained by the layman today, is that mathematics is the science of number, quantity and extension. However, within the last century many sciences of a distinctly mathematical character have come into being which concern themselves at bottom with none of these things. A set of objects may have an arrangement which is neither spatial nor temporal: a group of people, for example, might be arranged in order of wisdom. The study of the possible arrangements of a ~~given~~ set of objects is accordingly neither a study of time nor of space. However, it forms the topic for a great mathematical discipline:- what may be called combinatory analysis, in its widest sense. Many parts of this discipline, such as what is known as the theory of permutation-groups, involve much material of a non-numerical, non-quantitative character. Again, the whole field of logic is now being handled in a strictly mathematical manner, although its subject-matter is far more general than number, quantity, or extension. Accordingly, a new definition of the nature and purpose of mathematics has become necessary, and this new definition has reacted on mathematics itself and its application to the natural sciences.

One of America's greatest mathematicians, Benjamin Pierce, defined mathematics as "the science which draws necessary conclusions". "Conclusions about what?" we may ask. Pierce's reply would be "conclusions about anything whatever". The characteristic feature of such a mathematical formula as 'two and two make four', is that it is true of everything from shoes to kings, by way of ships, sealingwax, and cabbages; two things together with any other two things always give you four things. Geometry may not seem to conform to this definition so readily. It seems at first sight as though geometry had a peculiar subject-matter- space. If one only reflects a little, however, he will see that if one thing is certain of the space in which he lives, it is that no visible entities in it precisely satisfy the laws of geometry. Nobody ever say a line without width, a point without magnitude, nor a plane infinite in extent and without thickness. Yet we draw irrefutable conclusions concerning these things in geometry. How is it then that we draw these conclusions? the answer is, of course, "from the axioms of geometry". It is entirely on these axioms that the certainty of geometry rests. That is if anything whatsoever satisfies these axioms, all the conclusions of geometry become immediately applicable to it. Geometry does not say,"In our every-day space, any two intersecting lines determine a plane"; it merely tells us that this is true in any system obeying the axioms of geometry. Thus geometrical statements apply to the whole universe as fully as arithmetical statements.

We thus see that the purpose of mathematics as 'the science which draws necessary conclusions' is to take certain hypotheses, which may be true or false of things in the world about us, and to deduce their consequences. Once this is recognized to be the true nature of mathematics, a great limitation on the scope of mathematical research is removed. Any set of hypotheses whatever is seen to yield us a mathematical system, if only the hypotheses are not such as to specify some concrete field as their subject-matter. The practical work of the mathematician comes to be the analyis of blank-form hypotheses, so that when these hypothese turn up in the investigations, of the natural scientist, he may know what to expect of them, and so that the scientist may know in advance what hypotheses are likely to prove useful in the investigation he has in hand. We can see that this has actually been the function of mathematics in the history of science. It was the great mathematical development of the first half of the past century ~~which made it~~ and the end of the eighteenth which made it possible for Clerk-Maxwell to formulate his far-reaching hypotheses concerning the relation between light anh electricity. Again, the present use of imaginaries in alternating-current theory would have been impossible but for the mass of deductions which had been made by the mathematicians in a purely theoretical manner on the basis of the assumption that a system existed in which -1 would have a square-root It is not the quantitive aspect of mathematics that is the first source of its fruitfulness; it is its hypothetical aspect.

This being the case, it becomes a task of prime importance to isolate the hypothetical phase of each mathematical discipline, and to exhibit it in its purity. This is the task of the method of postulates. A set of postulates for a mathematical system is a group of propositions valid in that system, from which the remaining propositions of the system can be deduced. To put it another way, a mathematical system is one vast hypothetical proposition, in which the postulates ates are the hypothesis and the remaining formulae of the system form the conclusion. The word postulate is used rather than axiom, because 'axiom' seems to imply a sort of inherent certainty which does not appertain to the postulates of a mathematical system. The function of the method of postulates is to arrange the propositions belonging to already existing branches of mathematics in a deductive system, and to suggest deductive systems which shall lead to future branches of mathematics.

At first sight, it would seem that the deductive order of the propositions in a mathematical system would be obvious on the face of it, and that the method of postulates wold have a definite course marked out for it in each case. This, however, is by no means true. In most if not all mathematical disciplines, there are several alternative deductive orders in which their facts can be arranged. For example, Euclid's fifth postulate, to the effect that if PQ is parallel to LK, and A and B are pohnts on LK and PQ respectively, any line through A in the angle BAK meets PQ, may be replaced by the assumption that through any point in the plane there goes one and only one line parallel to a given line. It consequently happens that we often have a considerable degree of choice between various sets of postulates for a given mathematical system, all of which are perfectly legitimate, so that the choice between these becomes a matter of considerable interest.

The first desideratum of any set of postulates is of course that it be consistent. Now, it is a simple enough matter to prove that a set of postulates is inconsistent, by deducing from it contradictory results. However, the fact that such results are not forthcoming on a casual inspection offers no proof that they will not be found by a more careful study. Accordingly, to prove the consistency of a set of postulates, it is not enough that no contradictory conclusions have been deduced from them. It is a remarkable fact that so far no method of proving the consistency of a set of postulates has been found other than tie actual exhibition of a system which satisfies them all. Concerning the ultimate nature of this exhibition, there is no complete agreement among postulate-workers. It is generally accepted that the consistency of a set of postulates has been sufficiently demonstrated if a system can be exhibited which satisfies them and is made up of various arrangements of the positive integers. Thus fractions are explained as relations between integers, irrational numbers are defiined in terms of sequences of fractions that approach them as a limit, so that the whole number system is brouht back to the integers. 5hen the consistency of the postulates of geometry is demonstrated by showing that if triads of numbers be called points, and pairs of linear equations between these numbers be called lines, the postulates for geometry will be satisfied. The process of demonstrating the consistency of postulates by the exhibition of systems of constructions from the positive integers which satisfy them is known as the arithmetization of mathematics, and has been a prominent feature of all mathematical work done within the last forty years. It is all right as far as it goes, but it is not directly applicable to all known mathematical systems, and there are certain grave difficulties in the demonstration of the consistency of all the formal properties ordinarily attributed to the positive integers themselves.

Of course, all the formal properties of a given system should be deducible from a correct set of postulates for that system. In many cases, our information concerning a given system leaves a certain amount of variation opem in the details of its formal structure. In most cases, however, we are interested in determining the formal properties of our system down to the minutest detail. A set of postulates for such a system should leave none of its formal properties indeterminate. Such a set of postulates is called <u>categorical</u> or <u>sufficient</u>. The sufficiency of a set of postulates is demonstated by showing that any two systems which satisfy it correspond term for term.

Another desideratum of a set of postulates is that it be as simple as possible. One feature of this simplicity is that it should possess no redundancies:- that is, that no postulate should be deducible from any of the rest. This is shown by the actual exhibition of ~~postulates~~ systems which only fail to satisfy one of the postulates, thereby demonstrating that the postulate in question is not redundant. ~~It is little~~. The postulate in question is then said to be *independent* of the rest.

There are other properties which are also desirable in a set of postulates. For one thing, it should concern itself with as simple a set of fundamental notions as possible; it should possess very few postulates which explicitly demand the existence of anything; it should not contain as a single proposition any that can be divided directcy into two others, and so on. As to whether a large or a small set of postulates is preferable, there are two opinions. Some hold that a large set marks a finer and more complete logical analysis, while others prefer the greater simplicity of a small set.

Now as to the technique of the postulate method. In discussing this, I shall use as an example a problem on which I am now at work:- that of forming a simplified set of postulates for ordinary complex algebra. A set of postulates for this algebra has already been developed for this algebra by Professor E. V. Huntington. His set of postulates concerns itself with all complex numbers, the operations of addition and multiplication, the sub-class of all real numbers, and the order of magnitude among these numbers. It was obtained by taking a large group of simple and fundamental propositions in ordinary algebra, and weeding out those which are redundant. It contains such propositions as the associative and commutative laws for addhtion and multiplication, and the distributive law for multiplication with regard to addition. It consequently possesses a high degree of familiaarity and naturalness; however, it has some counterbalancing disadvantages. It is unduly large, containing twenty-seven postulates, and has far too many undefined notions. It has also a large number of postulates demanding the existence of certain special entities, such as 1, 0, and i. It consequently becomes worth while to look for a simpler set to take its place.

The first thing to look for was an operation in terms of which all of Huntington's relations could be defined. This was a long task, and came down in the last analysis to a sheer use of trial and error. Operations in terms of which addition and multiplication can be defined are quite easy to discover - sucd operations as $1 - xy$, or $1 - \frac{x}{y}$

have this property - and I had already seen some work of Dr. H. M. Sheffer, in which he had defined real numbers and the relation of greater and less in terms of the relation between a complex number and its conjugate. My search was confined to operations which combine the conjugate operation with an operation generating addition and multiplication, but that still left many alternatives open. The work thus consisted of a long and tedious elimination of operations which would not yield the result I wanted, and a general confidence in time and good luck. It was like nothing so much as the solution of a chess problem. At last I hit on the combination, and found that the operation 1 - the conjugate of $\frac{x}{y}$ would generate by iteration all Huntington's fundamental notions. I then knew that I could get a set of postulates for algebra in terms of this operation, and it seemed fairly obvious that it was going to be simpler than Huntington's set, but I had the formulation of the postulates still ahead of me. The first thing to do was to collect a large number of true propositions concerning my operation, and to gradually think myself into its manipulation, just as a schoolboy must think himself into ordinary algebra. Having brought these propositions together, it was now necessary to see if Huntington's postulates could be deduced from them. Once this was done, the task became one of elimination, combined with such modifications of the postulates retained as facilitated this work. When further elimination seemed impossible, the next thing was do demonstrate this impossibility by ~~existence~~ independence proofs. In the case of ~~th~~ the twelve postulates that remained, I was able to secure independence-proofs for seven, and toshow that two of the remaining postulates, taken together, were independent of all the rest. I was not able to go further, owing to the unfamilharity of the subject.

Once the postulates had been developed so far, the next thing in order was to see if Huntington's postulates could be obtained from ~~th~~ this reduced set. This would seem of itself to be enough to demonstrate the categorical character of the set, forHuntington's set is known to be categorical; however, this is not so. The set might be categorical for Huntington's operations without being categorical for mine. It was therefore necessary to show that after Huntington's operations had been defined in terms of mine, mine could be defined again by an iteration of Huntington's. This was shown.

The consistency of my set of postulates followed directly from that of Huntington's, for my operation could be defined in terms of his in the manner already indicated.

In rejecting redundant postulates, I was always careful to discard, where possible, those asserting the existence of particular entities, as well as all others of an especially intricate nature. By care in rejecting the undesirable postulates first, a fairly simple and uniform set was obtained.

So much for the technique of postulate-work. It is a field where everyone to a certain extent must develop his own technique. Now, again, what is the whole value of this sort of work? Mathematics, as everyone knows, is not a mass of isolated disciplines; on the contrary, one might say that every branch of mathematics exists for the sakn of other mathematical and non-mathematical disciplines. In order that a mathematical discipline may be applicable, certain conditions must be fulfilled. As we have already seen, the postulates constitute a statement of these conditions. A simple set of postulates is desirable in that it facilitates the recognition of the applicability of the branch of mathematics to which it belongs. Again, a new set of postulates in terms of a new relation may suggest important new theorems in an old branch of mathematics.

The field which has been covered by modern postulate-work is very large. It comprises Euclidean geometry(where Euclid's postulates have been found defective in many ways), the non-Euclidean geometries, projective geometry, descriptive geometry, analysis sius, algebra, the various forms of the theories of groups and fields, the theory of measurement in general, the theory of functions of a real variable, and many other disciplines. It has invaded the sphere of mathematical physics, an a very interesting book on "A Theory of Time and Space", by A. A. Robb, consists of a set of postulates for the theory of relativity in physics. The method has already played a large part in clarifying our mathematical ideas, and althougd the mass of actual new mathematical results to its credit is not yet great, it is only a question of time when it will play its part in the discovery of mathematical principles.