The Shortest Line Dividing an Area in a Given Ratio. By NORBERT WIENER, Ph.D.* (*Communicated by* Mr G. H. HARDY.)

[*Read* 9 November 1914.]

THE question we set out to answer in this paper is: given a connected area on a plane, what can we say, apart from any particular information we may have concerning its form, about the shape of the shortest segment of a curve, lying entirely in the area, and dividing it in a given ratio, provided such a curve exists? To put the problem more concretely, let us suppose a farmer wants to divide an irregular field of his evenly between his two sons, and suppose he wants to use as ~~little barbed wire~~ short a hedge as possible. How shall he shape his fence?

The method by which one would at first thought set out to solve this problem would be that of the calculus of variations. But a little reflection will convince us that the condition that the arc dividing our area in a given ratio must lie entirely within the area, is difficult to express, and next to impossible to handle, by the methods of the calculus of variations.

Our problem is, however, easily amenable to an elementary treatment. It is easy to show that the line of our fence, for example, will be either an arc of a (finite or infinite) circle, or will be a chain of such arcs, such that two successive arcs only meet on the boundary of the area.

To demonstrate this I shall first have to prove the following lemmas.

LEMMA 1. *Given a circle, and any two points on its periphery, then an arc of a circle can always be found passing through these two points, and dividing the circle in any desired ratio.*

No italics

For let the circle be called c and the ~~two~~ points A and B. Draw the chord $\overline{AB}$. Construct its perpendicular bisector, and let the latter meet c in the points C and D. Let E be a point on $\overline{CD}$ between C and D. Let F be a point on $\overline{CE}$ between C and E. Then draw the circles AEB and AFB. The lune $ACBE$ is greater than the lune $ACBF$. For AEB and AFB only intersect at A and B, and E is outside $ACBF$. Moreover, by choosing E and F near enough together, we can make the lune $AEBF$ as small as we ~~will.~~ wish) For we can construct a circle concentric with AEB and passing through F. The ring between this circle and AEB will contain the lune in question, and will ~~be~~ have the area

$$\cancel{\pm \pi(\overline{EF})^2 + 2\pi r(\overline{EF})},$$

$2\pi r(\overline{EF}) \pm \pi(\overline{EF})^2$)

where r is the radius of AEB†. As $\overline{EF}$ decreases without limit, this will also. Therefore the area of the lune $ACBE$ is a monotone continuous function of the length of $\overline{CE}$ within the region ($\overline{CE} = \overline{CD}$) from ~~$E = C$~~ ($\overline{CE} = 0$) to ~~$E = D$~~. Therefore it can easily be shewn by a continuity argument that

$$\frac{\text{area of } ACBE}{\text{area of } ADBE}$$

is a monotone continuous function of $\overline{CE}$, from ~~$E = C$~~ ($\overline{CE} = 0$) to ~~$E = D$~~ ($\overline{CE} = \overline{CD}$) and that in this region it takes every positive value.

LEMMA 2. *The shortest line passing through two given points on the boundary of a given circle, dividing the area of the circle in a given ratio, is an arc of a circle.*

Let our circle be, as before, c, and the two points A and B. By Lemma 1, there is an arc of a circle dividing c in the desired ratio: let it be AEB. ~~Let~~ AFB be any other curve dividing c in the same ratio. Complete the circle AEB, and let AGB be the other arc determined by A and B on this circle. Let ACB be the arc of c within the circle $AEBG$.

If AEB be a straight line our lemma needs no proof. If not let

Then the area of the circle $AEBG$ and that of the figure bounded by AFB and AGB will be identical. For the two have the lune $ACBG$ in common, and, by hypothesis, the area of the lune $ACBE$ equals that of the figure bounded by AFB and ACB. By Steiner's theorem the perimeter of $AEBG$ must be less than that of $AFBG$, for it is a circle. Hence, since the two perimeters

have AGB in common, the length of AEB is less than that of AFB. This proves our lemma.

Our theorem is now easy enough to prove. For let us suppose our area given, and the shortest line dividing it in a given ratio also given. Let us call the latter l. From any point on l at a positive distance from the boundary as a centre, we can describe a circle lying entirely within the area. Except, at the most, ~~a finite number of points~~ about points of a set having l_2, we can make this circle small enough to cut l in two points only. Within the circle, l must be an arc of a circle. For, call our little circle c. Let l divide c in the ratio $\frac{m}{n}$. Construct the arc of a circle cutting c in the same points as l, and dividing c in the ratio $\frac{m}{n}$. Then consider the curve formed by this arc and the portions of l outside c. This must divide the area in the same ratio as l, and, if it is not the same curve as l, must be shorter.

In the same way, it may be shewn that l cannot contain two arcs of distinct circles meeting inside the area. For, as before, around the meeting-point of these arcs describe a circle, c, cutting each arc in one point only, and lying entirely within the area. Then, by the same reasoning as before, the portions of the two arcs lying within c must form a single arc, which is impossible. Thus our theorem is proved*.

* It is almost self-evident that the shortest line to divide a *convex* area in a given ratio is a *single* arc of a circle, but this I have not yet been able to prove.

---

* The following article is on a topic suggested to the author by Dr Otto Szász, *Privatdozent* at Frankfurt am Main. It was the author's original intention to have this article, ~~together~~ with some further work of Dr Szász, appear together under the joint authorship of ~~himself and Dr Szász~~ Dr Szász and himself, but the war has rendered Dr Szász at least temporarily inaccessible, and this plan impossible.

† If AEB is a straight line, then $AEBF$ may be enclosed in a rectangle whose base is constant, and whose altitude may be made as small as you will.

7

THE SHORTEST LINE DIVIDING AN AREA IN A GIVEN RATIO

BY

NORBERT WIENER, Ph.D.

Reprinted from the
PROCEEDINGS OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, Vol. XVIII. Part 2

CAMBRIDGE
AT THE UNIVERSITY PRESS

January 30, 1915

[Extracted from the *Proceedings of the Cambridge Philosophical Society*, Vol. XVIII. Pt. II.]

The Shortest Line Dividing an Area in a Given Ratio. By NORBERT WIENER, Ph.D.* (Communicated by Mr G. H. HARDY.)

[*Received* 27 October 1914. *Read* 23 November 1914.]

The question we set out to answer in this paper is: given a simply connected area on a plane, what can we say, apart from any particular information we may have concerning the area, about the shape of the shortest segment of a curve, lying entirely in it, and dividing it in a given ratio, provided such a curve exists? To put the problem more concretely, let us suppose a farmer wants to divide an irregular field of his evenly between his two sons, and suppose he wants to use as short a hedge as possible. How shall he shape his fence? The conditions of the problem demand that the curve in question must have a length and be continuous. We shall limit our discussion in this paper to curves whose slope, considered as a function of the length of the curve from one end to the point where the curve has the slope in question, possesses only a finite number of discontinuities.

The method by which one would, at first thought, set out to solve this problem, would be that of the calculus of variations. But a little reflection will convince us that the condition that the arc dividing our area in a given ratio must lie entirely within the area, is difficult to express, and next to impossible to handle, by the methods of the calculus of variations.

Our problem is, however, easily amenable to an elementary treatment. It is easy to show that the line of our fence, for example, will be either an arc of a (finite or infinite) circle, or will be a chain of such arcs, such that two successive arcs only meet on the boundary of the area.

To demonstrate this I shall first have to prove the following lemmas.

LEMMA 1. *Given a circle, and any two points on its periphery, then an arc of a circle can always be found passing through these two points, and dividing the circle in any desired ratio.*

For let the circle be called c and the two points A and B. Draw the chord $\overline{AB}$. Construct its perpendicular bisector, and let the latter meet c in the points C and D. Let E be a point on

* The following article is on a topic suggested to the author by Dr Otto Szász, *Privatdozent* at Frankfurt am Main. It was the author's original intention to have this article, with some further work of Dr Szász, appear under the joint authorship of Dr Szász and himself, but the war has rendered Dr Szász at least temporarily inaccessible, and this plan impossible. Dr Szász' work consisted in a rigorous demonstration that the shortest line dividing any *scalene triangle* in a given ratio is a circle with its most acute apex as centre.

$\overline{CD}$ between C and D. Let F be a point on $\overline{CE}$ between C and E. Then draw the circles AEB and AFB. The lune $ACBE$ is greater than the lune $ACBF$. For AEB and AFB only intersect at A and B, and E is outside $ACBF$. Moreover, by choosing E and F near enough together, we can make the lune $AEBF$ as small as we wish. For we can construct a circle concentric with AEB and passing through F. The ring between this circle and AEB will contain the lune in question, and will have the area

$$2\pi r\,(\overline{EF}) \pm \pi\,(\overline{EF})^2,$$

where r is the radius of AEB*. As $\overline{EF}$ decreases without limit, this will also. Therefore the area of the lune $ACBE$ is a monotone continuous function of the length of $\overline{CE}$ within the region from $\overline{CE}=0$ to $\overline{CE}=\overline{CD}$. Therefore it can easily be shown by a continuity argument that

$$\frac{\text{area of } ACBE}{\text{area of } ADBE}$$

is a monotone continuous function of $\overline{CE}$, from $\overline{CE}=0$ to $\overline{CE}=\overline{CD}$. and that in this region it takes every positive value.

LEMMA 2. *The shortest line passing through two given points on the boundary of a given circle, dividing the area of the circle in a given ratio, is an arc of a circle.*

Let our circle be, as before, c, and the two points A and B. By Lemma 1, there is an arc of a circle dividing c in the desired ratio: let it be AEB. If AEB be a segment of a straight line, our lemma needs no proof. If not, let AFB be any other curve dividing c in the same ratio. Complete the circle AEB, and let AGB be the other arc determined by A and B on this circle. Let ACB be the arc of c within the circle $AEBG$.

Then the area of the circle $AEBG$ and that of the figure bounded by AFB and AGB will be identical. For the two have the lune $ACBG$ in common, and, by hypothesis, the area of the lune $ACBE$ equals that of the figure bounded by AFB and ACB. By Steiner's theorem the perimeter of $AEBG$ must be less than that of $AFBG$, for it is a circle. Hence, since the two perimeters have AGB in common, the length of AEB is less than that of AFB. This proves our lemma.

Our theorem is now easy enough to prove. For let us suppose our area given, and the shortest line dividing it in a given ratio also given. Let us call the latter l. From any point on l at a positive distance from the boundary as a centre, we can describe

* If AEB is a straight line, then $AEBF$ may be enclosed in a rectangle whose base is constant, and whose altitude may be made as small as you will.

a circle lying entirely within the area. Except, at the most, in a finite number of points, we can make this circle small enough to cut l in two points only*. Within the circle, l must be an arc of a circle. For, call our little circle c. Let l divide c in the ratio $\frac{m}{n}$. Construct the arc of a circle cutting c in the same points as l, and dividing c in the ratio $\frac{m}{n}$. Then consider the curve formed by this arc and the portions of l outside c. This must divide the area in the same ratio as l, and, if it is not the same curve as l, must be shorter. This contradicts our hypothesis.

In the same way, it may be shown that l cannot contain two arcs of distinct circles meeting inside the area. For, as before, about the meeting-point of these arcs describe a circle, c, cutting each arc in one point only, and lying entirely within the area. Then, by the same reasoning as before, the portions of the two arcs lying within c must form a single arc, which is impossible. Thus our theorem is proved†.

* This is a consequence of the condition which we laid down for all curves considered in this article—that the slope of the curve at a point p, considered as a function of the length of the curve between p and one of the end-points of the curve, possesses only a finite number of discontinuities. This is at once obvious, if we reflect that any curve $y = f(x)$, which intersects a circle c described about some point on it more than twice, must have a maximum or a minimum in y between one of the points where it intersects c and the centre of c.

† It is almost self-evident that the shortest line to divide a *convex* area in a given ratio is a *single* arc of a circle, but this I have not yet been able to prove.

THE SHORTEST LINE DIVIDING AN AREA IN A GIVEN RATIO

BY

NORBERT WIENER, Ph.D.

Reprinted from the

PROCEEDINGS OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, Vol. XVIII. Part 2

CAMBRIDGE
AT THE UNIVERSITY PRESS

January 30, 1915

[Extracted from the *Proceedings of the Cambridge Philosophical Society*, Vol. XVIII. Pt. II.]

The Shortest Line Dividing an Area in a Given Ratio. By NORBERT WIENER, Ph.D.* (Communicated by Mr G. H. HARDY.)

[*Received* 27 October 1914. *Read* 23 November 1914.]

The question we set out to answer in this paper is: given a simply connected area on a plane, what can we say, apart from any particular information we may have concerning the area, about the shape of the shortest segment of a curve, lying entirely in it, and dividing it in a given ratio, provided such a curve exists? To put the problem more concretely, let us suppose a farmer wants to divide an irregular field of his evenly between his two sons, and suppose he wants to use as short a hedge as possible. How shall he shape his fence? The conditions of the problem demand that the curve in question must have a length and be continuous. We shall limit our discussion in this paper to curves whose slope, considered as a function of the length of the curve from one end to the point where the curve has the slope in question, possesses only a finite number of discontinuities.

The method by which one would, at first thought, set out to solve this problem, would be that of the calculus of variations. But a little reflection will convince us that the condition that the arc dividing our area in a given ratio must lie entirely within the area, is difficult to express, and next to impossible to handle, by the methods of the calculus of variations.

Our problem is, however, easily amenable to an elementary treatment. It is easy to show that the line of our fence, for example, will be either an arc of a (finite or infinite) circle, or will be a chain of such arcs, such that two successive arcs only meet on the boundary of the area.

To demonstrate this I shall first have to prove the following lemmas.

LEMMA 1. *Given a circle, and any two points on its periphery, then an arc of a circle can always be found passing through these two points, and dividing the circle in any desired ratio.*

For let the circle be called c and the two points A and B. Draw the chord $\overline{AB}$. Construct its perpendicular bisector, and let the latter meet c in the points C and D. Let E be a point on

* The following article is on a topic suggested to the author by Dr Otto Szász, *Privatdozent* at Frankfurt am Main. It was the author's original intention to have this article, with some further work of Dr Szász, appear under the joint authorship of Dr Szász and himself, but the war has rendered Dr Szász at least temporarily inaccessible, and this plan impossible. Dr Szász' work consisted in a rigorous demonstration that the shortest line dividing any *scalene triangle* in a given ratio is a circle with its most acute apex as centre.

$\overline{CD}$ between C and D. Let F be a point on $\overline{CE}$ between C and E. Then draw the circles AEB and AFB. The lune $ACBE$ is greater than the lune $ACBF$. For AEB and AFB only intersect at A and B, and E is outside $ACBF$. Moreover, by choosing E and F near enough together, we can make the lune $AEBF$ as small as we wish. For we can construct a circle concentric with AEB and passing through F. The ring between this circle and AEB will contain the lune in question, and will have the area

$$2\pi r\,(\overline{EF}) \pm \pi\,(\overline{EF})^2,$$

where r is the radius of AEB*. As $\overline{EF}$ decreases without limit, this will also. Therefore the area of the lune $ACBE$ is a monotone continuous function of the length of $\overline{CE}$ within the region from $\overline{CE}=0$ to $\overline{CE}=\overline{CD}$. Therefore it can easily be shown by a continuity argument that

$$\frac{\text{area of } ACBE}{\text{area of } ADBE}$$

is a monotone continuous function of $\overline{CE}$, from $\overline{CE}=0$ to $\overline{CE}=\overline{CD}$. and that in this region it takes every positive value.

LEMMA 2. *The shortest line passing through two given points on the boundary of a given circle, dividing the area of the circle in a given ratio, is an arc of a circle.*

Let our circle be, as before, c, and the two points A and B. By Lemma 1, there is an arc of a circle dividing c in the desired ratio: let it be AEB. If AEB be a segment of a straight line, our lemma needs no proof. If not, let AFB be any other curve dividing c in the same ratio. Complete the circle AEB, and let AGB be the other arc determined by A and B on this circle. Let ACB be the arc of c within the circle $AEBG$.

Then the area of the circle $AEBG$ and that of the figure bounded by AFB and AGB will be identical. For the two have the lune $ACBG$ in common, and, by hypothesis, the area of the lune $ACBE$ equals that of the figure bounded by AFB and ACB. By Steiner's theorem the perimeter of $AEBG$ must be less than that of $AFBG$, for it is a circle. Hence, since the two perimeters have AGB in common, the length of AEB is less than that of AFB. This proves our lemma.

Our theorem is now easy enough to prove. For let us suppose our area given, and the shortest line dividing it in a given ratio also given. Let us call the latter l. From any point on l at a positive distance from the boundary as a centre, we can describe

* If AEB is a straight line, then $AEBF$ may be enclosed in a rectangle whose base is constant, and whose altitude may be made as small as you will.

a circle lying entirely within the area. Except, at the most, in a finite number of points, we can make this circle small enough to cut l in two points only*. Within the circle, l must be an arc of a circle. For, call our little circle c. Let l divide c in the ratio $\frac{m}{n}$. Construct the arc of a circle cutting c in the same points as l, and dividing c in the ratio $\frac{m}{n}$. Then consider the curve formed by this arc and the portions of l outside c. This must divide the area in the same ratio as l, and, if it is not the same curve as l, must be shorter. This contradicts our hypothesis.

In the same way, it may be shown that l cannot contain two arcs of distinct circles meeting inside the area. For, as before, about the meeting-point of these arcs describe a circle, c, cutting each arc in one point only, and lying entirely within the area. Then, by the same reasoning as before, the portions of the two arcs lying within c must form a single arc, which is impossible. Thus our theorem is proved†.

* This is a consequence of the condition which we laid down for all curves considered in this article—that the slope of the curve at a point p, considered as a function of the length of the curve between p and one of the end-points of the curve, possesses only a finite number of discontinuities. This is at once obvious, if we reflect that any curve $y = f(x)$, which intersects a circle c described about some point on it more than twice, must have a maximum or a minimum in y between one of the points where it intersects c and the centre of c.

† It is almost self-evident that the shortest line to divide a *convex* area in a given ratio is a *single* arc of a circle, but this I have not yet been able to prove.