Definitions of the Fundamental Notions of Projective Geometry in Terms of the Relation of Intersection among Convex Surfaces

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$$\text{pnt}_R = \hat{\alpha}\{\alpha = p`\overrightarrow{R}``\alpha\} \quad \text{Df}$$

$$\text{cont}_R = \dot{-}(\breve{R} \mid \dot{-} R) \quad \text{Df}$$

$$\text{bet}_R(x, y, z) \,.\!=:\, (\exists u) \,.\, uRx \,.\, uRz \,.\, \overrightarrow{\text{cont}_R`u \cap \overrightarrow{R}`x} \cap \overrightarrow{R}`z = \overrightarrow{R}`y \quad \text{Df}$$

$$\text{con}_R = \hat{\alpha}\{(\exists \beta, x) \,.\, \beta \,\varepsilon\, \text{pnt}_R \,.\, \alpha = \hat{y}[z \,\varepsilon\, \beta \,.\supset_z : \text{bet}_R(x, y, z) \,.\vee.\, \text{bet}_R(y, z, x) \,.\vee.\, \text{bet}_R(z, x, y)]\} \quad \text{Df}$$

$$\bar{R} = \hat{\alpha}\hat{\beta}\{\alpha, \beta \,\varepsilon\, \text{con}_R : (\exists \gamma) \,.\, \gamma \,\varepsilon\, \text{pnt}_R \,.\, \gamma \subset \alpha \cap \beta\} \quad \text{Df}$$

$$(\kappa \mid \lambda, \mu, \nu)_R \,.\!=:\, \kappa, \lambda, \mu, \nu \,\varepsilon\, \text{pnt}_{\bar{R}} \,.\, \lambda \cap \mu \cap \nu \subset \kappa$$

$$\text{cln}_R = \hat{\varpi}\{(\exists \kappa, \lambda, \mu) : \varpi = \iota`\kappa \cup \iota`\lambda \cup \iota`\mu : (\kappa \mid \lambda, \kappa, \mu)_R \,.\vee.\, (\lambda \mid \kappa, \mu, \mu)_R \,.\vee.\, (\mu \mid \kappa, \kappa, \mu)_R\} \quad \text{Df}$$

$$\text{cpl}_R = \hat{\varpi}\hat{\rho}\{(\exists \lambda, \mu, \nu, \sigma, \tau, \upsilon) : \varpi = \iota`\lambda \cup \iota`\mu \cup \iota`\nu \,.\, \rho = \iota`\sigma \cup \iota`\tau \cup \iota`\upsilon : (\lambda \mid \sigma, \tau, \upsilon)_R \,.\, (\mu \mid \sigma, \tau, \upsilon)_R \,.$$
$$(\nu \mid \sigma, \tau, \upsilon)_R \,.\vee.\, (\sigma \mid \lambda, \tau, \upsilon)_R \,.\, (\mu \mid \sigma, \tau, \upsilon)_R \,.\, (\nu \mid \sigma, \tau, \upsilon)_R \,.\vee \ldots\ldots \vee.\, (\upsilon \mid \sigma, \tau, \lambda)_R \,.\, (\upsilon \mid \sigma, \tau, \mu)_R \,.$$
$$(\upsilon \mid \sigma, \tau, \nu)_R \,.\vee.\, (\sigma \mid \mu, \nu)_R \,.\, (\tau \mid \lambda, \mu, \nu)_R \,.\, (\upsilon \mid \lambda, \mu, \nu)_R \vee \ldots \vee.\, (\nu \mid \lambda, \mu, \sigma)_R \,.\, (\nu \mid \lambda, \mu, \tau)_R \,.$$
$$(\nu \mid \lambda, \mu, \upsilon)_R\} \quad \text{Df}$$

$$\text{pln}_R(\lambda, \mu, \nu) = s`\overrightarrow{\{\text{cpl}_R \triangleright \text{cln}_R\}_*}`(\iota`\lambda \cup \iota`\mu \cup \iota`\nu) \quad \text{Df}$$

$$\text{pln}_R = D`[s`\overrightarrow{\{\text{cpl}_R \triangleright \text{cln}_R\}_*}] - \iota`\Lambda \quad \text{Df}$$

Let R be the relation of intersection between closed convex surfaces.

$$\mathrm{pnt}_R = \hat{\alpha}\{\alpha = p`\overrightarrow{R}``\alpha\} \quad \mathrm{Df}$$

$$\mathrm{cont}_R = \dot{-}(R \mid \dot{-}R) \quad \mathrm{Df}$$

$$\mathrm{tng}_R = \hat{x}\hat{y}\{\mathrm{pnt}_R \cap \overleftarrow{\epsilon}`x \cap \overleftarrow{\epsilon}`y \, \epsilon \, 1\} \quad \mathrm{Df}$$

$$\mathrm{nucl}_R = \mathrm{cont}_R \dot{-} R \mid \mathrm{tng}_R \quad \mathrm{Df}$$

$$\mathrm{bet}_R(x,y,z) . = . (\exists u) . uRx . uRz . \overrightarrow{\mathrm{cont}_R}`u \cap \overrightarrow{R}`x \cap \overrightarrow{R}`z \subset \overrightarrow{R}`y \quad \mathrm{Df}$$

$$\mathrm{con}_{R,x} = \hat{\alpha}\hat{\beta}\{\alpha = \hat{y}\{z \epsilon \beta . \supset_z : \mathrm{bet}_R(x,y,z) . v . \mathrm{bet}_R(y,z,x) . v . \mathrm{bet}_R(z,x,y)\}\} \quad \mathrm{Df}$$

$$\mathrm{env}_R = \hat{\alpha}\hat{\beta}\{(\exists \gamma, x, y) . \gamma \epsilon \mathrm{pnt}_R . y \, \mathrm{nucl}_R \, x . \alpha \, \mathrm{con}_{R,x} \, \gamma . \beta \, \mathrm{con}_{R,y} \, \gamma\} \quad \mathrm{Df}$$

$$\mathrm{cn}_R = \hat{\alpha}\{(\exists \beta, x) . \alpha \, \mathrm{con}_{R,x} \, \beta\} \quad \mathrm{Df}$$

$$\bar{R} = \hat{\alpha}\hat{\beta}\{\gamma \, \mathrm{env}_R \, \alpha . \delta \, \mathrm{env}_R \, \beta . \supset_{\gamma,\delta} . \exists ! \mathrm{pnt}_R \cap \mathrm{Cl}`(\gamma \cap \delta)\} \restriction \mathrm{cn}_R \quad \mathrm{Df}$$

($\mathrm{pnt}_{\bar{R}}$ is the class of all points, finite or at infinity)

$$\infty_R = \mathrm{pnt}_{\bar{R}} - \hat{\kappa}\{(\exists \alpha) . \hat{\beta}\{(\exists x) . \beta \, \mathrm{con}_{R,x} \, \alpha\} \subset \kappa\} \quad \mathrm{Df}$$

(∞_R is the plane at infinity)

$$\mathrm{cplr}(\kappa, \lambda, \mu, \nu) . = :. \kappa, \lambda, \mu, \nu \, \epsilon \, \mathrm{pnt}_{\bar{R}} :. \hat{\beta}\{(\exists x) . \beta \, \mathrm{con}_{R,x} \, \alpha\} \subset \kappa : \gamma \epsilon -\lambda-\mu :$$
$$(\exists y) . \gamma \, \mathrm{con}_{R,y} \, \alpha : \supset_{\alpha,\gamma} . \gamma \sim \epsilon \, \nu \quad \mathrm{Df}$$

(this is the relation of coplanarity)

$$\mathrm{pl}_R = \hat{\varpi}\{(\exists \lambda, \mu, \nu) . \lambda, \mu, \nu \, \epsilon \, \mathrm{pnt}_{\bar{R}} . \varpi = \iota`\lambda \cup \iota`\mu \cup \iota`\nu \cup \hat{\kappa}\{\mathrm{cplr}_R(\kappa, \lambda, \mu, \nu)\}\} \quad \mathrm{Df}$$

$pln_R = \iota` \infty_R \cup \overrightarrow{max_C}`pl_R$ Df (This is the class of all planes

$lin_R = s` pln_R \underset{"}{\cap}`` pln_R - pln_R$ Df (This is the class of all ~~lin~~ lines)

$l_R = (lin_R \cup \iota`\hat{\alpha}\hat{\beta}\{\alpha, \beta \varepsilon pnt_{\overline{R}} : \alpha, \beta \varepsilon \kappa . \supset_\kappa : \kappa \sim\varepsilon\, lin_R\}) - \iota`\Lambda$ Df

$irr_R = \hat{\kappa}\hat{\lambda}\{\kappa, \lambda \varepsilon l_R : \exists ! \, a \cap Cl`(\kappa \cap \lambda) . \vee .$ ~~(μ, ν) … $\mu \cap \nu = \lambda \cap \mu \cap \nu = \Lambda :$~~
$(\exists \mu, \nu) . \kappa \cap \lambda = \kappa \cap \mu \cap \nu = \lambda \cap \mu \cap \nu = \Lambda . \exists ! \kappa \cap \mu . \exists ! \kappa \cap \nu . \exists ! \lambda \cap \mu . \exists ! \lambda \cap \nu\}$ Df.

$fs_R = s_\varepsilon \| [\overrightarrow{(irr_R)_*}]_\varepsilon$ Df

$ints_R = \hat{\kappa}\hat{\lambda}\{\exists ! \, a \cap Cl`(\kappa \cap \lambda)\} \upharpoonright s`\overrightarrow{(fs_R)_*}`l_R$ Df

$\zeta_R = D`\overrightarrow{ints_R}$ Df

If R is the relation of intersection between convex surfaces, we shall have $\zeta_R = lin_R$. However, whatever R may be, ζ_R will always have certain important projective properties which lin_R may or may not have. Thus, any two members of $pnt_{\overline{R}}$ will belong to ~~one~~ a member of ζ_R and one only, and every two members of ζ_R that intersect will possess only one member of $pnt_{\overline{R}}$ in common, and the triangle transversal axiom will hold of ζ_R. To assure that ~~lin~~ ζ_R is the class of all lines in a real projective geometry, assumptions $E_0 - E_3'$, H, C, and R of Veblen and Young (Am. Journ. of Math., Vol 30) are necessary and sufficient. $E_0 -$ ~~E_3~~ become, in our symbolism and that of the *Principia*,

(Most of these are simplified and some are slightly strengthened)

E_0. $\min_\iota ` Nc``\zeta_R > 2$

E_1 & E_2 $Nc`\zeta_R > 1$

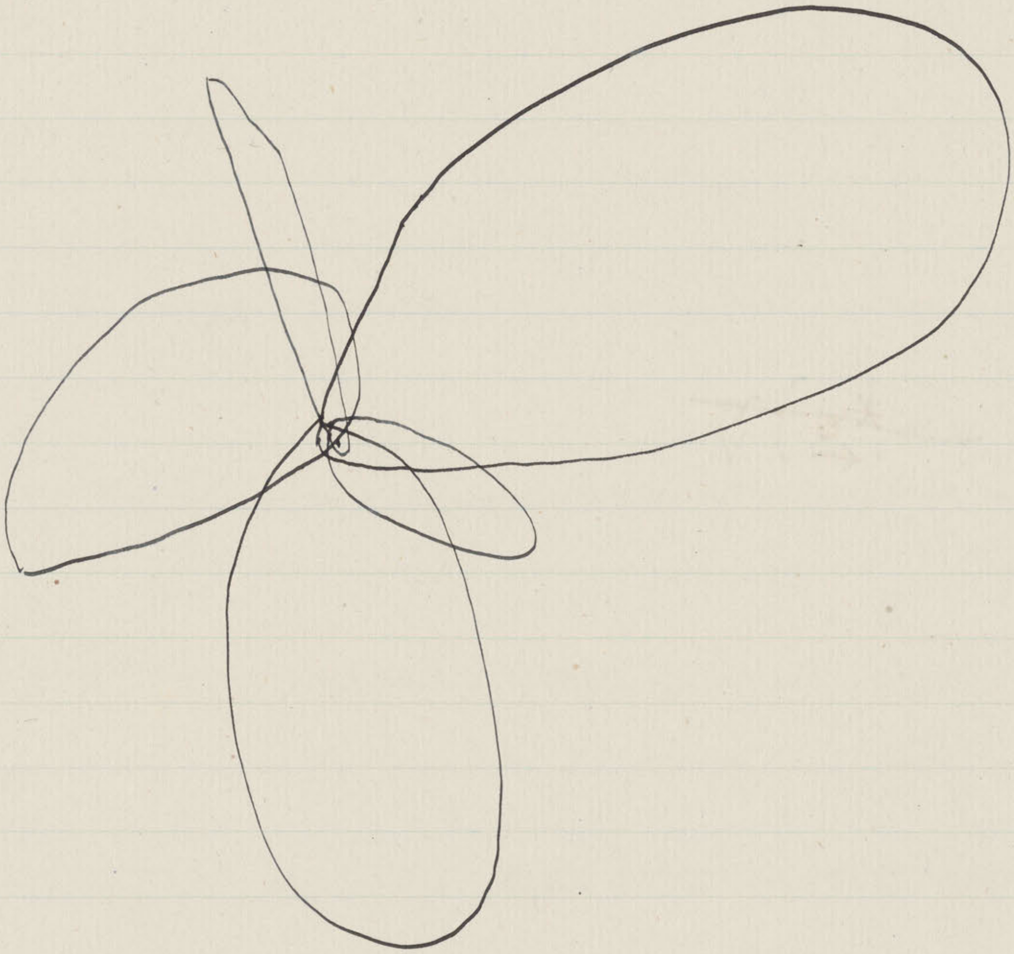
E_3 $\Lambda \,\varepsilon\, s`\zeta_R \cap$ '' ζ_R

E_3'. $\kappa, \lambda, \mu \,\varepsilon\, \zeta_R . \supset_{\kappa, \lambda, \mu} : (\exists \nu) . \nu \,\varepsilon\, \lambda_R . \exists ! \kappa \cap \nu . \exists ! \lambda \cap \nu . \exists ! \mu \cap \nu .$

It says that 'if there is any harmonic sequence, there is one such that every point of it is distinct from all the points of the sequence which precede it, C gives the property of continuity roughly speaking, to members of ζ_R, and R differentiates a real projective geometry from a complex one.

Diagram for

pnt_R



~~In such a case as this, the just-noticeable difference seems to be a more natural unit. We obtain this for a unit if we replace the k and the d, respectively, in $meas_{k,d}$ by Vct_q and~~

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~~Let κ be class of convex bodies in space at a given time,~~
~~where a body is regarded as t~~

~~R_* stands for, 'being a convex body contained in'~~
~~$\breve{R}_*|R_*$ stands for, 'overlaps'~~

~~$bet_R(x, y, z) . \equiv . (\exists u) . u \breve{R}_*|R_* x . u \breve{R}_*|R_* y . u \breve{R}_*|R_* z . \overrightarrow{R}_*{}^{\prime}u \wedge \breve{R}_*{}^{\prime\prime}\overrightarrow{R}_*{}^{\prime}x \wedge \breve{R}_*{}^{\prime\prime}\overrightarrow{R}_*{}^{\prime}y \subset \breve{R}_*{}^{\prime\prime}\overrightarrow{R}_*{}^{\prime}z$~~

~~$D^{\prime}R_{\varepsilon}|\overrightarrow{R} \cup \hat{\alpha}\{$~~

$u \cap$

$\in$

~~$\hat{u}\{\overrightarrow{R}_*{}^{\prime}u \wedge$~~

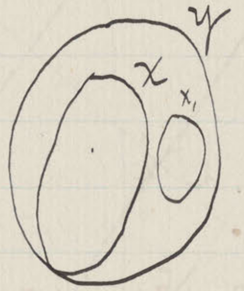
Diagrams representing

(1) $x \ \mathrm{cont}_R \ y$

(2) $x \ \mathrm{tng}_R \ y$

(3) $x \ \mathrm{nucl}_R \ y$

1



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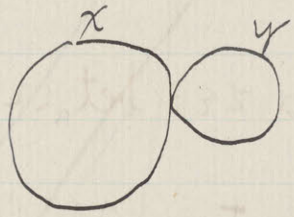

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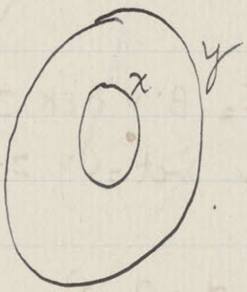


Let R be the relation 'overlaps', applying to closed convex solids. Let us make the following definitions:

$pnt_R = \hat{\alpha}\{\alpha = p^{\prime}\overrightarrow{R}^{\prime\prime}\alpha\}$ Df

~~$bet_R(x,y,z).\equiv$~~

$cont_R = \dot{-}(\breve{R}\,|\dot{-}R)$ Df

$bet_R(x,y,z).\equiv:(\exists u).uRx.uRy.uRz.\overrightarrow{cont_R}^{\prime}u \cap \overrightarrow{R}^{\prime}x \cap \overrightarrow{R}^{\prime}y \subset \overrightarrow{R}^{\prime}z$ Df

$collin_R(\alpha,\beta,x).\equiv:.(\exists y,z):y\varepsilon\alpha:z\varepsilon\beta: bet_R(x,y,z).v.bet_R(y,z,x).v.bet_R(z,x,y)$ Df

$lin_R = \hat{\alpha}\{(\exists\beta,\gamma).\beta \neq \gamma.\beta,\gamma\varepsilon pnt_R.\alpha = \hat{x}\{collin_R(\gamma,\beta,x)\}\}$ Df

$pln_R = \hat{\alpha}\{(\exists\beta,\gamma,\delta):\beta,\gamma,\delta\varepsilon pnt_R:\beta,\gamma,\delta\varepsilon\kappa.\supset_\kappa \kappa \sim\varepsilon^{\prime\prime} lin_R:\alpha = \hat{x}\{(\exists y):.collin_R(\beta,\gamma,y): z\varepsilon\delta.\supset_z: bet_R(x,y,z).v.bet_R(y,z,x).v.bet_R(z,x,y)\}\}$ Df

~~$dir_R = D^{\prime}sg^{\prime}\hat{\alpha}\hat{\beta}\{\alpha,\beta\varepsilon pln_R.\alpha\cap\beta = \Lambda\}$~~ Df

~~$Ct = \hat{\alpha}\hat{\beta}\{\exists!\alpha\cap\beta\}$ Df~~

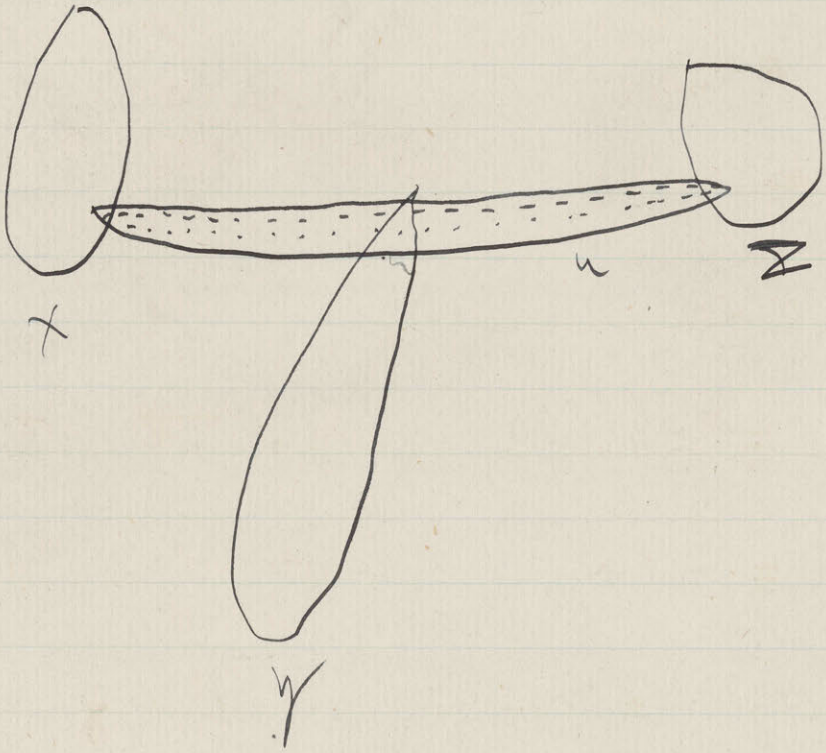
~~$drct_R = \hat{\kappa}\{\kappa = p^{\prime}\overrightarrow{Ct}^{\prime\prime}\kappa\}$ Df~~

~~$irr_R = \hat{\alpha}\hat{\beta}\{\kappa\varepsilon drct_R.\supset_\kappa:\alpha\sim\varepsilon\kappa.v.\beta\sim\varepsilon\kappa:\alpha\cap\beta\sim\varepsilon lin_R\}\upharpoonright pln_R$ Df.~~

$pl_R = \hat{\alpha}^{\prime\prime}\hat{\kappa}\{\kappa = p^{\prime}\overrightarrow{irr_R}^{\prime\prime}\kappa\}$ Df

Diagram representing

$bet_R(x, y, z)$



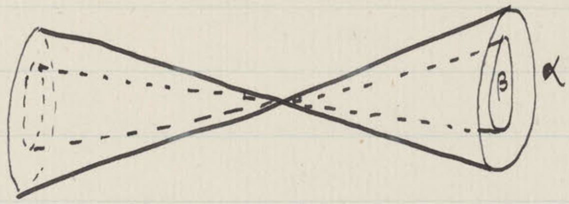
~~$v = x \cap y$~~

~~$pnt_R = \hat{\alpha}\{\alpha = p^c \overrightarrow{R}``\alpha\}$ Df~~

~~$cox_R = \hat{\alpha}\{x, y \varepsilon \alpha . x \varepsilon \beta . y \varepsilon \gamma . \beta, \gamma \varepsilon pnt_R . \supset_{x,y,\beta,\gamma} . \exists ! \beta \cap \gamma \cap \alpha\}$ Df~~

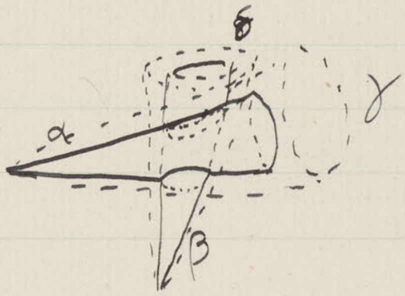
~~$\mathfrak{I}_R\, ac_R =$~~

Diagram representing $\alpha \; env_2 \; \beta$

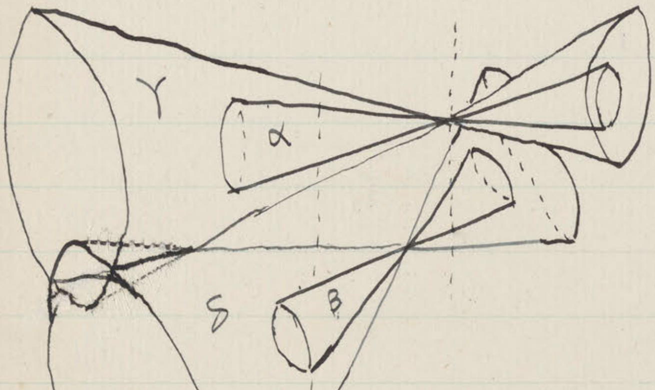


Diagrams representing two cases of $\alpha \bar{R} \beta$

(1). α intersects β in a finite part of plane

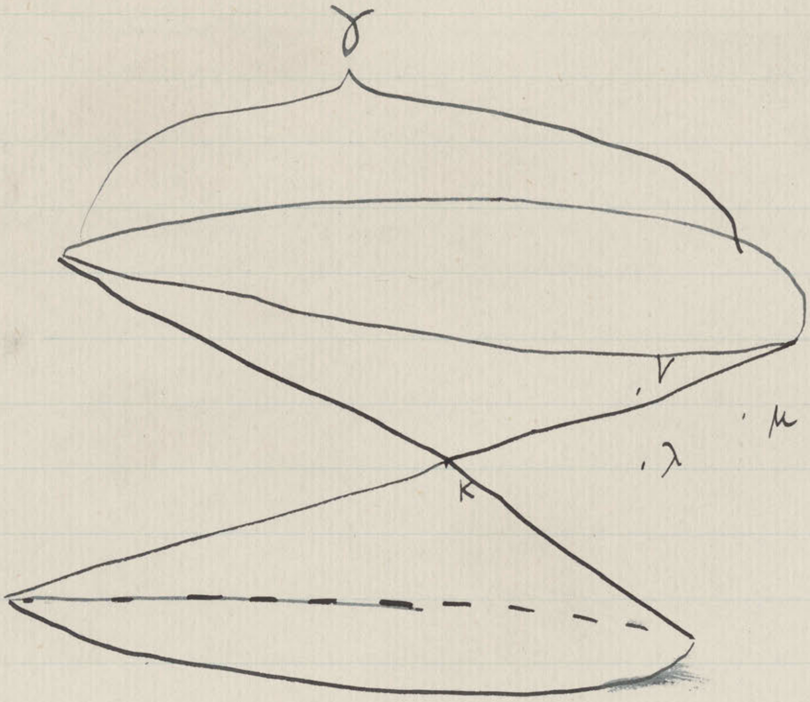


(2) α 'intersects' β at infinity only

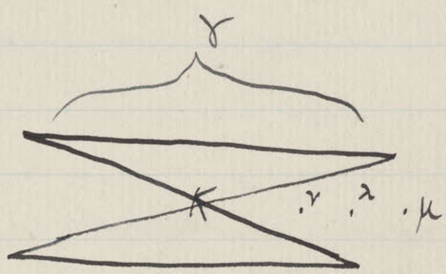


$$\Vert_{R} = \hat{\alpha}\hat{\beta}\{(\exists\gamma).\gamma\,\varepsilon\,|\ell_{R}.\alpha\subset\gamma\,|\,\beta\subset\gamma$$

Diagram illustrating cplr$(\kappa, \lambda, \mu, \nu)$



View from a point slightly above ~~with reference to axis of γ~~ plane of κ, λ, μ, and ν



View from point in plane of κ, λ, μ, and ν.

Definitions of Geometrical Measurement in Terms of the Apparent Overlapping of Convex ~~Surfaces~~ Solids

Norbert Wiener

Let xRy stand for 'x seems to overlap y' where x and y are convex solids.

$$\pi_R = \hat{\alpha}\{\alpha = p`\overrightarrow{R}``\alpha\} \qquad Df$$

π_R is the class of all points, where a point is regarded as the class of convex surfaces which seem to pass through it. Whether they actually do pass through it or not has, according to us, no meaning at this stage. In general, we shall regard a ~~surface~~ solid as containing all the points which it seems to contain. Every two members of a member of π_R seem to overlap, while any convex surface seeming to overlap all members of a member of π_R belong to that member. Two surfaces may seem to overlap even when they are not yet in contact.

$$\sigma_R = \hat{\kappa}\{(\exists\alpha,\beta).\alpha,\beta\,\epsilon\,\pi_R.\alpha\neq\beta.\kappa=\pi_R\cap\overleftarrow{C}`\alpha\cap\beta\} \qquad Df$$

σ_R is the class of all linear segments ~~belong~~ defined by R. Compare this definition with that of Huntington in his article in the <u>Mathematische Annalen</u>. A segment is the class of all points which contain all the convex solids ~~surfaces~~ common to two given points.

$$Ov_R = \hat{\kappa}\hat{\lambda}\{\kappa, \lambda \,\varepsilon\, \sigma_R . (\exists \alpha, \beta) . \alpha \neq \beta . \alpha, \beta \,\varepsilon\, \kappa \cap \lambda\}_* \quad Df$$

Ov_R is the relation between two segments when they belong to the same line. Two segments belong to the same line when they can be connected by a finite number of segments, each having two points in common with the previous one.

$$lin_R = s``D`\overrightarrow{Ov_R}$$

This is the class of lines and is made up of the logical sums of the classes of linear segments such that each class is the class of all the segments which are part of a given line.

~~$Cn_\kappa(\alpha, \beta, \gamma) . \equiv : \alpha, \beta, \gamma \,\varepsilon\, \kappa : (\exists \delta, \xi, \eta, \vartheta, \lambda, \mu) . \delta, \xi, \eta, \vartheta, \lambda, \mu \,\varepsilon\, \kappa . \delta \cap \xi \cap \eta =$~~

~~$\delta \cap \eta \cap \vartheta = \delta \cap \vartheta \cap \lambda = \delta \cap \lambda \cap \mu = \delta \cap \xi \cap \vartheta = \delta \cap \eta \cap \lambda = \delta \cap \vartheta \cap \mu =$~~

~~$\delta \cap \xi \cap \lambda = \delta \cap \eta \cap \mu = \delta \cap \xi \cap \mu = \xi \cap \eta \cap \vartheta = \xi \cap \vartheta \cap \lambda = \xi \cap \lambda \cap \mu =$~~

~~$\xi \cap \eta \cap \lambda = \xi \cap \vartheta \cap \mu = \xi \cap \eta \cap \mu = \eta \cap \vartheta \cap \lambda = \eta \cap$~~

$\delta, \ldots, \mu \, \varepsilon \, \nu$

$$Cn_\nu(\alpha, \beta, \gamma) . = : \alpha, \beta, \gamma \, \varepsilon \, \nu . (\exists \delta, \xi, \eta, \vartheta, \kappa, \lambda, \mu) : \delta \neq \vartheta .$$

$$\xi \neq \kappa . \eta \neq \lambda . \delta \neq \xi . \vartheta \neq \kappa . \exists ! \alpha \cap \delta \cap \eta . \exists ! \beta \cap \delta \cap \xi . \exists ! \gamma \cap \xi \cap \eta .$$

$$\exists ! \alpha \cap \vartheta \cap \lambda . \exists ! \beta \cap \vartheta \cap \kappa . \exists ! \gamma \cap \kappa \cap \lambda . \exists ! \delta \cap \vartheta \cap \mu .$$

$$\exists ! \eta \cap \lambda \cap \mu . \exists ! \xi \cap \kappa \cap \mu . \alpha \neq \beta . \beta \neq \gamma . \alpha \neq \gamma . \quad Df$$

Three lines are concurrent, we shall say, if they join corresponding angles of two triangles in perspective from a given line — that is, if they bear one another the relation Cn_{lin_R}

$$\Pi_R = \hat{\nu}\{\kappa, \lambda \, \varepsilon \, \nu ; \kappa \neq \lambda . \supset_{\kappa, \lambda} : \mu \, \varepsilon \, \nu . \lambda \neq \mu . \kappa \neq \mu . \equiv_\mu . Cn_{lin_R}(\kappa, \lambda, \mu)\} - 0 - 1 - 2 \quad Df$$

This is the class of all general points, whether at infinity or not. A general point is a class of lines whereof any ~~three~~ are concurrent, while any line concurrent with two lines belonging to the point belongs to the point, provided such a class contains at least three lines.

$$\zeta_R = \hat{\sigma}\{(\exists \nu, \varpi) . \sigma = \hat{\rho}\{Cn_{\Pi_R}(\nu, \varpi, \rho)\} \cup \iota`\nu \cup \iota`\varpi\} - 1 \quad Df$$

A general line (whether at infinity or not) is a class consisting of all the general points collinear with two given general points, including these points themselves, where collinearity is determined by a

Desarguesian construction.

$$y \leftarrow R \rightarrow x = \hat{\alpha}\{(\exists \beta, \gamma) . \beta, \gamma \varepsilon \pi_R . x \varepsilon \beta . y \varepsilon \gamma . \beta \cap \gamma \subset \alpha . \alpha \varepsilon \pi_R\} \quad Df$$

This is the class of points lying between y and x, where points are taken in their simpler sense, and x and y are solids. If x and y are spheres of the same size, $y \leftarrow R \rightarrow x$ defines a region made up of a right cylinder with two hemispherical caps.

$$cyl_{R,x,y} = y \leftarrow R \rightarrow x \cap \Omega ' \{ lin_R \cap \hat{\kappa} \{ \kappa \cap \sout{bet_{R,x,y}}\ y \leftarrow R \rightarrow x \ \varepsilon 1 \}\} \quad Df$$

This is the surface of $y \leftarrow R \rightarrow x$ — i.e. that part of it which can be reached by lines cutting it in only one point.

$$penc_{R,x,y} = lin_R \cap \hat{\kappa}\{\exists ! Cl ' \kappa \cap \sigma_R \cap Cl ' cyl_{R,x,y}\} \quad Df$$

This is the class of external common tangents of x and y, if x and y are spheres, and is made up of those lines which have segments in common with $cyl_{R,x,y}$. If x and y are equal spheres, $penc_{R,x,y}$ is the set of elements of a right cylinder.

$$\Pi_{R,x,y} = \Pi_R \cap \hat{\nu}\{Nc ' (\nu \cap penc_{R,x,y}) > 1\} \quad Df$$

This is the class of intersections of external common tangents of x and y, if x and y are spheres. If x and y are equal spheres, the only member of $\Pi_{R,x,y}$ is the point at infinity on the line joining the centers of x and of y.

$$\infty_{R,x,y,u,v} = \Pi_R \cap \hat{v}\{(\exists \varpi, \rho, \sigma). \varpi \varepsilon \Pi_{R,x,y} . \rho \varepsilon \Pi_{R,x,u} .$$

$$\sigma \varepsilon \Pi_{R,x,v} . \exists ! [s'(\daleth_R \cap \overleftarrow{\varepsilon}'v \cap \overleftarrow{\varepsilon}'\varpi) \cap s'(\daleth_R \cap \overleftarrow{\varepsilon}'\rho \cap \overleftarrow{\varepsilon}'\sigma)]\} \quad Df$$

That is, a point at infinity with reference to the four unit spheres, x, y, u, and v, is a point which is coplanar with a set of points at infinity which they determine, where coplanarity is defined by the triangle-transversal method. $\infty_{R,x,y,u,v}$ is the region at infinity so defined

$$\|_{R,x,y,u,v} = \hat{\sigma}\hat{\tau}\{\sigma, \tau \varepsilon \daleth_R . \exists ! \sigma \cap \tau \cap \infty_{R,x,y,u,v}\} \quad Df$$

Two lines are parallel, that is to say, if they intersect at infinity.

$$=_{R,x,y,u,v} = \hat{R}\hat{S}\{(\exists \mu, \nu, \varpi, \rho, \sigma, \tau, \upsilon, \omega). R = \mu \downarrow \nu . S = \varpi \downarrow \rho . \mu, \nu \varepsilon \sigma .$$

$$\nu, \varpi \varepsilon \tau . \varpi, \rho \varepsilon \upsilon . \rho, \mu \varepsilon \omega . \sigma \|_{R,x,y,u,v} \upsilon . \tau \|_{R,x,y,u,v} \omega\} \quad Df$$

Two intervals on parallel lines are, that is, equal when they are intercepted by parallel lines

$\mathrm{Vec}_{R,x,y,u,v} = D`(\overrightarrow{=_{R,x,y,u,v}})_*$ Df

A distance-vector is the relation between two points when the distance between them is of a certain size and direction. The class of these vectors is $\mathrm{Vec}_{R,x,y,u,v}$, when the plane at infinity is determined by x, y, u, and v.

$\mathrm{insd}_R = \hat{\nu}\hat{x}\{(\exists\alpha).\ \alpha\varepsilon\pi_R.\ x\varepsilon\alpha.\ \mathrm{lin}_R \cap \overline{\varepsilon}`\alpha \subset \nu.\ \nu\varepsilon\Pi_R\}$ Df

$\overrightarrow{\mathrm{insd}_R}`x$ is the class of members of Π_R lying in x.

$\mathrm{Meas}_{R,\underline{x},y,u,v} = \hat{\mu}\hat{\nu}\{\mu = H``(1/2) +_a \overset{\text{limin}}{\mathrm{Min}}_{\Theta}`\mathrm{Inx}_{\mathrm{Vec}_{R,x,y,u,v},\overrightarrow{\mathrm{insd}_R}`x}``$

$\mathrm{Vec}_{R,x,y,u,v} \cap \breve{C}``(\downarrow\nu``\overrightarrow{\mathrm{insd}_R}`x) -_a \overset{\text{limin}}{\mathrm{Min}}_{\Theta}`\mathrm{Inx}_{\mathrm{Vec}_{R,x,y,u,v},\overrightarrow{\mathrm{insd}_R}`x}``$

$\mathrm{Vec}_{R,x,y,u,v} \cap \breve{C}``(\downarrow\nu``\Pi_R - \overrightarrow{\mathrm{insd}_R}`x)\}$ Df

This represents the distance of a point from the center of the unit sphere x. $\mathrm{Inx}_{\kappa,\alpha}$ is defined in my paper on measurement. $\mathrm{Meas}_{R,\underline{x},y,u,v}`\nu$ is made up of three terms: (1) the real number, $\frac{1}{2}$; (2), the least distance of ν from points of x; (3) the negative of the least distance of ν from points not belonging to x. In the case where we actually deal with the intersection of convex surfaces, either 2 or 3 is always zero. Note the relevance of the definition of $\operatorname{Inx}_{\kappa,\alpha}$ to this context.

$$(x \leftrightarrow y)_{u,v} = \lim \mathrm{Max}_{\Theta}\text{`` }\operatorname{Inx}_{\mathrm{Vec}_{R,x,y,u,v},\ \overrightarrow{\mathrm{insd}}_R\text{' }x}\text{`` }\mathrm{Vec}_{R,x,y,u,v} \frown$$

$$\breve{C}\text{``}(s\text{' }\overrightarrow{\mathrm{insd}}_R\text{' }x \downarrow_{\text{''}} \overrightarrow{\mathrm{insd}}_R\text{' }y) -_n H\text{''}(I/1)\, D_f$$

This gives us the distance between the center of x and the center of y, if x, y, u, and v are unit spheres. With these data we are enabled to set up a system of Cartesian coördinates, in the following manner. Let us call $(x \leftrightarrow y)_{u,v}$ λ_1, and $(x \leftrightarrow u)_{y,v}$ λ_2, and $(x \leftrightarrow v)_{y,u}$ λ_3, and $(y \leftrightarrow u)_{x,v}$ λ_4, and $(y \leftrightarrow v)_{x,u}$ λ_5, and $(u \leftrightarrow v)_{x,y}$ λ_6. Let us [redacted] ~~as its coordinate. Let us assign to y the number triad $(\lambda_1, 0, 0)$, to u the number triad~~ Let us define h as $\dfrac{\lambda_2^2 - \lambda_4^2 - \lambda_1^2}{2\lambda_1}$, and k as $\sqrt{\lambda_2^2 - h^2}$, and ~~to v the~~ [redacted] l as $\dfrac{\lambda_3^2 - \lambda_5^2 - \lambda_1^2}{2\lambda_1}$, n as $\dfrac{\lambda_3^2 - \lambda_6^2 + h^2 + k^2 - 2lh}{2k}$, and m as $\sqrt{\lambda_3^2 - l^2 - n^2}$. Then,

if $\mathrm{Meas}_{R,\underline{x},y,u,v}$'$r$ be a, if $\mathrm{Meas}_{R,\underline{y},u,v,x}$'$r$ be b, if $\mathrm{Meas}_{R,\underline{u},v,x,y}$'$r$ be c, and $\mathrm{Meas}_{R,\underline{v},x,y,u}$ be d, we shall say the coordinates of r are the triad (p, q, r), where $p = \dfrac{a^2 - b^2 - \lambda_1^2}{2\lambda_1}$, $r = a^2 - c^2 + h^2 + k^2 - 2xh$, and $q = \sqrt{a^2 - p^2 - r^2}$ if d have its minimum value for a given determination of a, b, and c, while

$q = -\sqrt{a^2 - p^2 - r^2}$ if the contrary is the case.

What we have practically done is to translate the tripunctual coordinates of v in terms of its distances from the centers of x, y, and u, where the side of the plane determined by these points on which v lies is obtained by considering its distance from the center of v, into the system of cartesian coordinates the X-axis of which is the line joining the centers of x and of y, the origin of which is x, and the ZX-plane of which is the plane containing the centers of x, y, and u, where the Y-coordinate of v is positive.

It may be possible for two members of Π_R to have the same coordinates. ~~In such~~ [redacted] If we wish our points to be uniquely determined by their coordinates, we shall call a coordinate point a class of members of Π_R having a given coordinate. It may happen that there are sets of coordinates without points to fill them. In such a case, if the coordinates in question be μ, ν, and ϖ, respectively, we shall call the class [redacted] $(\mu \downarrow \nu \downarrow \varpi)$[1] the ideal point whose coordinates are (μ, ν, ϖ) and we shall bring our coordinate-points

[1] See my reduction of relations to classes in my article, <u>A Simplification of the Logic of Relations</u>, Proc. Camb. Phil. Soc.,

up to the same type as $(\mu \downarrow \nu \downarrow \varpi)$ by the prefixion of a sufficient number of ι's.

It will be observed that we are able to define a categorical system — that of ~~ordina~~ ordinary geometry — in terms of a relation about which we have assumed nothing, and an arbitrary selection of four solids as unit-spheres. It should be noticed, of course, that if the resulting system of measurement is to be at all natural — if, for example, we are to have distinct coördinate-points on every line — we must submit our selection of the unit-spheres to certain conditions — they must not coincide, their centers must not be coplanar, etc. All this, however, has nothing to do with the formal geometrical character of the resulting system.

Another point to call attention to is our definition of the planes at infinity. Our natural expectation would be that we should simply say that two non-intersecting coplanar lines are parallel. Now our experience gives us only a bounded portion of space. If we say that two lines are parallel simply when they intersect outside this region, then our parallelogram criterion of equality breaks down entirely, and any two segments are connected by a chain of segments, each equal to the next, as may readily be proved. If, on the other hand, we look for some other criterion of parallelism, we find that the collineation-properties of a bounded region of space are the same, whatever line not meeting this region may be ~~the~~ a line at infinity. We therefore must use some more or less arbitrary criterion of the plane at infinity. Now, the part of our theory which deals with measurement demands the use of a sphere as a means of comparing distances in different directions, and it becomes very convenient there to use not one but several equal spheres for this purpose. We therefore have a criterion of parallelism ready-made, for two equal spheres determine a cylinder which determines a point at infinity.