Characterization of Real Numbers

Symbols	Undefined Words
A	Which satisfies the following assumption(s)
C	(Assumptions are inclosed in brackets or parentheses)
C	A certain, meaning the same in every case
E	Number or element in the real number system
I	Is in
N	Not or no
O	Can be put into one to one corespondence with
R	Or
S	Subclass of real number system
W	and also
big X	Exists
Y	Any, meaning that the thing it modefies can be any one which exists
+	Add
small ×	Multiply
<	Is less than
=	Equals
Any small letter	refers to the identical thing that E, YE, CE S, YS or CS, with the same letter after it and in same postulate, did. In other words they are just giving names to the element or subclass which first stand with E, S, YE, CE, YS, CS and then alone

Refers to

Postulates

1. If $a \neq b$ then $a < b$ or $b < a$ — $YEa < YEbA(a = Nb) \quad R \quad b < a$

2. If $a < b$ then $a \neq b$ — $YEa = N\ YEbA(a < b)$

3. If $a < b$ and $b < c$ then $a < c$ — $YEa < YEcA(a < Eb\ W\ b < c)$

4. Dedekind's Postulate — $EaA(YEbA[b < a] | YSmA[YEcA(c|) < YEdA(d|Nm)]\ W\ YEeA[a < e] | SnA[d|n])X$

5. Postulate of Linearity — $CSrA(EaA[a < YEb\ W\ YEcA(c < b) < a] | r\ W\ rOCStA[EdA$
$(YEeA[e|t\ W\ e < d] | YSmA[YEfA(f|m)|t\ W\ f < YEgA$
$(g|t\ W\ g|Nm]\ W\ YEhA[h|t\ W\ d < h] | SnAg|n])|t\ W$
$EiA(YEjA[j < i\ W\ YEkA(k|t) < j] | Nt)|t\ W\ CElA(l < YEq$
$A[\overset{Wq\ |t}{q = Nl}])|t\ W\ EoA(YEpA[o < p\ W\ p < YEsA(s|t\ W\ s = Nl)] | Nt)|t])X$

6. no first element — $YEaA(YEb(b = Na) < a)XN$

7. No last element — $YEaA(a < YEbA(b = Na))XN$

8. addition unique — $(YE + YE)X$

9. multiplication unique — $(YE \times YE)X$

10. addition associative — $YEa + (YEb + YEc) = (a + b) + c$

11. multiplication associative — $YEa \times (YEb \times YEc) = (a \times b) \times c$

12. 0 defined — $CEaA(a + YEb = b)X$

13. 1 defined — $CEaA(a \times YEb = b)X$

14. Inverse exists — $EaA(YEb + a = CEcA(c + b = b))X$

15. Inverse exists — $EaA[YEbA(b = NCEcA(YEd + c = d)) \times a = CEeA.(e \times d = d)]X$

16. Distributive law — $YEa \times (YEb + YEc) = (a \times b) + (a \times c)$

17. addition commutative — $YEa + YEb = b + a$

18. multiplication commutative — $YEa \times YEb = b \times a$

19. If $a < b$ then $a + c < b + c$ — $YEa + YEb < YEcA(a < c) + b$

20. If $a > 0$ and $b > 0$ then $ab > 0$ — $CEaA(a + Yb = b) < YEcA(a < c) \times YEdA(a < d)$

x lies between x_{n1} and x_{n2}, an <u>interval</u>. We shall admit the possibility of intervals with infinite bounds. If we call the above interval i, then, in accordance with (1), we shall call $m(i)$ or the <u>measure</u> of i the quantity

$$\frac{1}{\sqrt{\pi^n t_1(t_2-t_1)(t_3-t_2)\cdots(t_n-t_{n-1})}}\int_{x_{11}}^{x_{12}}\int_{x_{21}}^{x_{22}}\cdots\int_{x_{n1}}^{x_{n2}} e^{-\left(\frac{\xi_1^2}{t_1}+\frac{(\xi_2-\xi_1)^2}{t_2-t_1}+\cdots+\frac{(\xi_n-\xi_{n-1})^2}{t_n-t_{n-1}}\right)}d\xi_1\,d\xi_2\cdots d\xi_n.$$

Clearly, by (1), if $i_1, i_2, \ldots, i_k$ are a set of mutually exclusive intervals constituting i, $m(i) = \sum_1^k (i_n)$.

Let us build up a set I_n of intervals in the following way: for $t = h/2^n$ $(h \leq 2^n)$ we consider intervals of the form $\tan\frac{k\pi}{2^n} \leq x \leq \tan\frac{(k+1)\pi}{2^n}$.[1] We associate these intervals in all possible manners, taking one interval for each integral value of h up to and including 2^n. Clearly, there are a finite number of intervals belonging to each I_n, while every interval of I_n is made up of a finite number of intervals of I_{n+1}. This fact, in connection with the ~~additive~~ distributive property of m, assures that the system $\{I_n\}$ satisfies the first two conditions for a 'division', as laid down in my previous paper.

The third condition ~~reads~~ demands that if S_n is the set of all functions in a number of intervals from I_n, and if S_{n+1} is always contained in S_n, either there is an element common to every S_n, or the total measure of S_n approaches 0 as n grows without limit. This condition is also satisfied. To demonstrate this, consider the function $2\sum_k^\infty h\,2^{-\frac{h}{4}} = \psi_h(u)$, where k is the largest integer such that ~~x~~ $1/2^k > u$. In the first place, $\psi_h(u)$ is well-defined, for the series $2\sum_k^\infty h\,2^{-\frac{h}{4}}$ converges, as its test-ratio is $2^{-1/4}$. In the second place, for this same reason, $\lim_{u \doteq 0} \psi_h(u) = 0$. Now, consider all those functions f such that $f(0) = 0$ and $|f(t_1) - f(t_2)| \leq \psi_h|t_2 - t_1|$. These functions are manifestly what Ascoli calls equally continuous[1] and

[1] k is an integer

[1]

are bounded as a set, and so form what Fréchet calls a compact set.[1] There is no difficulty in proving

that those functions satisfying this condition which lie in a given closed interval form a closed compact set. It results that if every S_n contains intervals ~~such~~ which contain elements satisfying the given condition of equal ~~convergence~~ continuity, there is a term common to every S_n[2]. On the other hand,

suppose that every element in some S_k does not satisfy the given condition. Then there are two elements, t_1 and t_2, such that for every f in S_k,

$$|f(t_1)-f(t_2)| > \psi_n |t_2 - t_1|$$

Represent t_1 and t_2 as binary fractions. If we make use of the two ways of representing a terminating binary fraction, ~~there~~ and define k as the largest integer such that $1/2^k > |t_2 - t_1|$, there is a terminating binary fraction agreeing with t_1 and also with t_2 up to the kth place after the point. By comparing the sequence of ~~sums~~ terminating binary fractions by which t_1 and t_2 can be defined with the series for $\psi_n(u)$ it results that ~~at~~ it is possible to find ~~an~~ integers l and m such that $m+1 \leq 2^l$ and

$$(2) \qquad \left|f\left(\frac{m}{2^l}\right) - f\left(\frac{m+1}{2^l}\right)\right| > \frac{h}{2^{\frac{l}{4}}}$$

We shall take $h>1$. For a given l, the total weight of the functions satisfying (2) may readily be shown not to be greater than

$$\frac{2^{l+1}}{\sqrt{\frac{\pi}{2^l}}} \int_{\frac{h}{2^{\frac{l}{4}}}}^{\infty} e^{-2^l x^2} dx = \frac{2^{l+1}}{\sqrt{\pi}} \int_{h \cdot 2^{\frac{l}{4}}}^{\infty} e^{-x^2} dx.$$

This is not greater than

$$\frac{2^{l+1}}{\sqrt{\pi}} \int_{h \cdot 2^{\frac{l}{4}}}^{\infty} e^{-x} dx = \frac{2^{l+1}}{\sqrt{\pi}} e^{-h\, 2^{\frac{l}{4}}}$$

For all l's, therefore, the total weight of all functions satisfying (2) is not greater than

$$\omega_n = \sum_{l=1}^{\infty} \frac{2^{l+1}}{\sqrt{\pi}\, e^{n \cdot 2^{\frac{l}{4}}}} .$$

This series may be shown to converge by the ~~test~~-ratio test. Furthermore, $\lim_{n=\infty} \omega_n = 0$. Hence, given any positive quantity ε, there is some S_k with total weight less than ε.

We are thus in a position to apply the work of Daniell on summable functions, as I showed in my previous article. We ~~have those thus defined~~ may therefore define the class T_0, consisting of all functionals which are constant over ~~all the~~ each interval of some I_n. If F is such a functional, we shall define ~~Λ~~ $\iota(F)$ as

$$\frac{\sum m(i_k)\, F(f)}{~~\sum~~},$$

when f lies in i_k, and i_k takes all possible values in I_n. Now, by the methods of Daniell, ~~we~~ we may extend ι to the limit of any bounded sequence of functions to which ι is applicable. In the.

$$= \frac{\sum u_m v_m}{\sum u_m^2} = \cos \angle \xi\eta.$$

It results from this that $\angle \xi(\xi \oplus \eta)$ is the half of $\angle \xi\eta$ in the first or fourth quadrant. Now, let ~~ξ~~ ξ and η be any two vectors of equal magnitude, with the only restriction that $\xi \oplus \eta \neq 00\xi$. Form the vector ζ, which shall be a positive multiple of $\xi \oplus \eta$ with the same magnitude as ξ. Interpolate a term ϑ in the same way between ξ and ζ, and κ between η and ζ. ~~Call the~~ Consider the sequence $\xi, \vartheta, \zeta, \kappa, \eta$. There is no difficulty in proving that every such angle as $\angle \xi\vartheta$ has a positive cosine which is not ~~greater~~ less than $\frac{\sqrt{2}}{2}$, for it is not ~~greater~~ less than $\sqrt{\frac{1+\sqrt{\frac{1}{2}}}{2}}$, as results from the repeated use of the formula $\cos \phi/2 = \sqrt{\frac{1+\cos\phi}{2}}$.

ABC
BCA
CAB

ACD
A=C
A=D
~~DCA~~
CDA
CAD

BCA

~~ACD, BDA~~
~~ACD. DAB~~
~~ACD, DAB~~
~~A=D, ABC~~
~~A=D B≠C, A≠B~~
~~A=D, BCA~~
~~CDA, C=A~~
~~CDA, C=B~~
CDA, ABC
~~CDA, BCA~~, ADB
DCB
~~CAD, C=A~~
~~CAD, C=B~~
CAD, ABC
CAD, BCA
CBD

ACD. ABD
~~ACD. A=D~~
ACD B≠D
ACD BDA
ACD. DAB

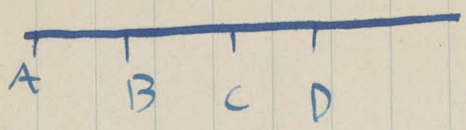
A≠D, ABC
A=D, B=C, A≠B
A=D, BCA
~~A=D, CAB~~

~~CDA, CAB~~
CDA, C=A
CDA, C=B
CDA, ABC
CDA, BCA

A=C, ABD
A=C, B≠D
A=C, BDA
A=C, DAB

CAD, CAB
CAD

CDA
ABC
C1
~~CD~~
~~CBD~~
~~BCD~~
BDC



ABC . ∨ . ACB . ∨ . BAC .

ABC . $\supset$. ACB ✓

ABC . $\supset$. CBA ✓

~~AB~~ C

ABC . BCD . $\supset$. ACD

~~ABC . BCD . $\supset$ ABD~~

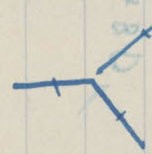
ABC . ABD . $\supset$. ~~BCD . ∨ . BDC~~ ∼ CBD ✓

ABC . ACD . $\supset$ ~~ABD~~ BCD ✓

~~ABC . ABD . $\supset$. ACD . ∨ . ADC . ∨ CAD~~

~~A~~

~~ACD . ABD . $\supset$. ABC . ∨ ACB ∨ BAC~~



CAB . DCA

~~DC~~

Given system P_K. Let $L_1, L_2, \ldots$ be sets of points such that $K - L_n$ can be enclosed in a set of intervals of weight $\leq \varepsilon_n$, where $\lim \varepsilon_n = 0$.

Total weight $< \dfrac{2^{n+1}}{\sqrt{\pi}} e^{-\frac{1}{2} 2^{\frac{n}{4}}}$

Definitions

(T_1), (Ve), (R) as before

(Li) = system satisfying I–IX

(Sp) = system satisfying I, II, III, IV, ~~V~~ VI, VIII, IX, and

~~X~~ ~~VI~~' There is at least one connected set with interior and exterior elements ~~and at least two boundary elements~~.

(H) = system satisfying ~~Hahns~~ Hausdorff's 'Umgebungsaxiome'.

(Ve_1) = set that is (1) Ve, ~~(2) Separable~~, (2) such that if α and β are vectors and $\|\alpha\| = \|\beta\|$, there is a finite set of vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\|\alpha_k\| = \alpha$ for all k, and $\|\alpha - \alpha_1\| < \|\alpha\|$, $\|\alpha_1 - \alpha_2\| < \|\alpha\|$, $\ldots$, $\|\alpha_k - \alpha_{k+1}\| < \|\alpha\|$, $\|\alpha_n - \beta\| < \|\alpha\|$

Theorems.

Every (Sp) is an (H), a (T_1), an (R).

Every (Ve_1) is an (Sp)

Every one of the following spaces is a (Ve_1)

(1) E_n

(2) The Hilbert space

(3) The space of continuous functions with uniform distance.

(4) The space of ~~bounded~~ continuous functions with distance defined in accordance with the formula

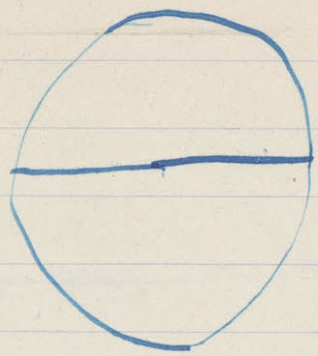
$$\sqrt{\int_0^1 [x(t) - y(t)]^2 \, dt},$$

~~only those functions for w~~

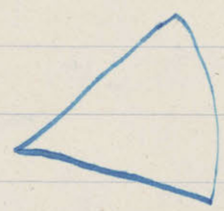
(Li) is an (Sp).

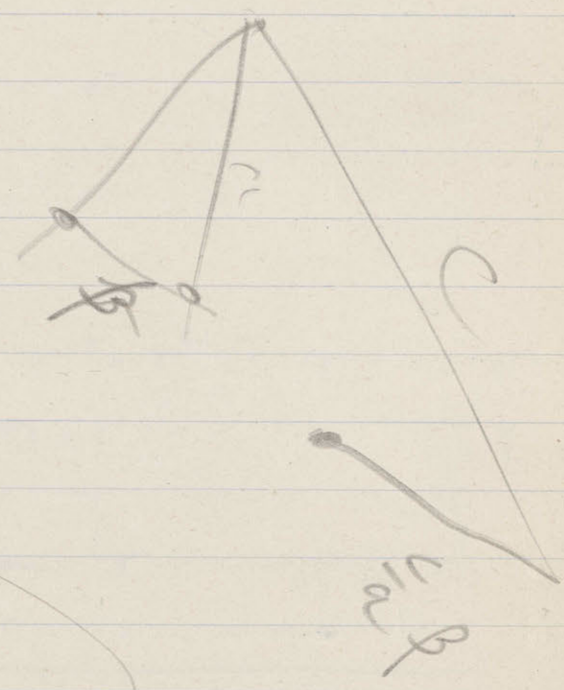
(5) The space of $\aleph_0$ dimensions with uniform distance.

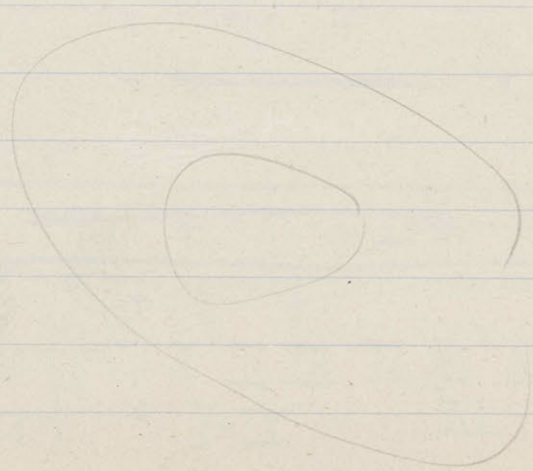
$$\cos(\xi\eta) = u_1 v_1 + u_2 v_2 + \cdots$$



$a\xi + b\eta$







~~D~~ A   1 1 → 2

~~ABC~~ ADC
~~ACB~~ ~~ACD~~
~~BCA~~ CD.

~~1~~ ABC   ADC   ABD   BCD
BCA   DCA   BDA   CDB-
CAB   CAD   DAB   DBC

1 2 x x
1 1 x x
3 x 3 x
2 x 3 x

x 3 x 3
x 2 x 3
x x 2 2
x x 2 1

~~x x~~ 2 1

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1121
1122
1211
~~1212~~
~~1221~~
~~1222~~
2111
2112
2121
2122
2211
2212
2221
2222

~~1111~~
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~~1121~~
~~1122~~
~~1123~~
~~1131~~
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-2211
3212
~~3322~~

DAC, DBC
DBC. DCA
BDC. BDA
BCD BDA

BCA	DCA	ABD	BCD

CAB   DCA   ABD
CBD

CAB   CAD   BDA   CDB

~~1,2,3~~),

~~4 Two pairs equivalent~~

5 ~~Exists connected set with 2 boundary elements~~

6. ~~Connected set det by 2 boundary elements if at least 2~~

~~7, 8~~ ~~7 (R)~~

~~8 Separable~~

9 ~~10~~. Every neighborhood contains segment

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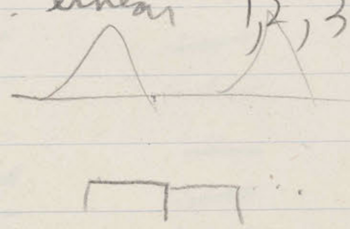
a tr R conn. if ~~leaves inv~~ changes conn. region.
If R = S/T, at least one element changed by both S & T.

~~10 becomes Every point~~

9 ~~10~~ becomes — if there is a tr. ch. A & leaving E inv, then there is conn tr. ch. A & leaving E inv.

In decent form — 1, 2, 3, 4, 7, 8, 9.

Not peculiar. linear 1, 2, 3, 4, 7, 8, 9.



$$|f(x_2) - f(x_1)| \leq K|x_2 - x_1|$$

for every pair of numbers x_1 and x_2 in the interval from $-\pi$ to π inclusive. Such a function is obviously of limited total variation in the interval. It may be seen on inspection that T_0 satisfies conditions (1)–(4).

Besides a class T_0, Daniell's theory involves the existence of an operation I satisfying the following conditions:

(C) $I(cF) = cI(F)$, if c is any constant;

(A) $I(F_1 + F_2) = I(F_1) + I(F_2)$;

(P) $I(F) \geq 0$ if $F(p) \geq 0$ for all p.

(L) If $F_1 \geq F_2 \geq \cdots \geq 0 = \lim F_n$ for all p, then

$$\lim I(F_n) = 0.$$

In defining our operation I, we shall make use of a functional G having the following properties:

(α) If f and g are any two continuous functions defined over a range from $-\pi$ to π inclusive, and g satisfies a Lipschitz condition, $\lim_{n \doteq 0} G\{f + ng\} = G\{f\}$;

(β) If $\{f_n\}$ is a sequence of continuous functions defined over a range from $-\pi$ to π inclusive, ~~with~~ converging uniformly to the continuous limit f, then $\lim G\{f_n\} \leq G\{f\}$;

(γ) If f is any continuous function defined from $-\pi$ to π inclusive, $|G\{f\}| \leq 1$;

(δ) If $|G\{f\}| \geq \varepsilon > 0$ and $-\pi \leq x_1 \leq \pi$, $-\pi \leq x_2 \leq \pi$, then there is a function φ independent of x_1 and x_2 and of f such that $|f(x_2) - f(x_1)| \leq \varphi(\varepsilon)|x_2 - x_1|$;

(ε) If $|G\{f\}| \geq \varepsilon > 0$, then there is a function ψ independent of f, such that for any x between $-\pi$ and π, inclusive, $|f(x)| \leq \psi(\varepsilon)$.

An example of such a functional G is $\dfrac{1}{1 + \max\left|\frac{\Delta f(x)}{\Delta x}\right| + \max|f(x)|}$,

where $\max\left|\frac{\Delta f(x)}{\Delta x}\right|$ means the largest value of $\left|\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right|$, for $x_2 \neq x_1$, $-\pi \leq x_1 \leq \pi$, $-\pi \leq x_2 \leq \pi$, and where $\max|f(x)|$ is taken for $-\pi \leq x \leq \pi$. Conditions (γ), (δ), and (ε) need no comment, while (α) needs only the remark that if $|g(x_2) - g(x_1)| \leq K|x_2 - x_1|$, then

$$nK + \left|\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right| \geq \left|\frac{f(x_2) + ng(x_2) - f(x_1) - ng(x_1)}{x_2 - x_1}\right| \geq \left|\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right| - nK.$$

As to (β), it is clear that, for any positive ε there is some number s such that if $n \geq s$, $|f(x) - f_n(x)| < \frac{\varepsilon}{2}$. Then $\lim \max\left|\frac{\Delta f_n(x)}{\Delta x}\right| \geq \max\left|\frac{\Delta f(x)}{\Delta x}\right| - \varepsilon$. As ε is arbitrarily small, $\lim \max\left|\frac{\Delta f_n(x)}{\Delta x}\right| \geq \max\left|\frac{\Delta f(x)}{\Delta x}\right|$. Since $\lim \max|f_n(x)| = \max$

$$|f(x_2) - f(x_1)| = K|x_2 - x_1|$$

for every pair of numbers x_1 and x_2 in the interval from $-\pi$ to π inclusive. Such a function is obviously of limited total variation in the interval. It may be seen on inspection that T_0 satisfies conditions (1)-(4).

Besides a class T_0, Daniell's theory involves the existence of an operation I satisfying the following conditions:

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(A) $I(F_1 + F_2) = I(F_1) + I(F_2)$;

(P) $I(F) \geq 0$ if $F(p) \geq 0$ for all p;

(L) If $F_1 \geq F_2 \geq \dots \geq 0 = \lim F_n$ for all p, then $\lim I(F_n) = 0$.

In defining our operation I, we shall make use of a functional G, defined for all functions continuous from $-\pi$ to π, inclusive, and enjoying the following properties:

(α) If g satisfies a Lipschitz condition,

$$\lim_{n=0} G\{f + ng\} = G\{f\};$$

(β) If $\{f_n\}$ is a sequence of continuous functions converging uniformly to f,

$$\lim G\{f_n\} \leq G\{f\};$$

(γ) $$0 \leq G\{f\} \leq 1;$$

(δ) There is a φ such that if $G\{f\} \geq \varepsilon > 0$ and $-\pi \leq x_1 \leq \pi$, $-\pi \leq x_2 \leq \pi$,

$$|f(x_2) - f(x_1)| \leq \varphi(\varepsilon)|x_2 - x_1|.$$

(ε) There is a ψ such that if $|G\{f\}| \geq \varepsilon > 0$ and $-\pi \leq x \leq \pi$,

$$|f(x)| \leq \psi(\varepsilon).$$

An example of such a functional G is $1/\left\{1 + \max\left|\frac{\Delta f(x)}{\Delta x}\right| + \max|f(x)|\right\}$, where $\max\left|\frac{\Delta f(x)}{\Delta x}\right|$ means the ~~largest value~~ upper bound of $\left|\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right|$ for $x_2 \neq x_1$, $-\pi \leq x_1 \leq \pi$, $-\pi \leq x_2 \leq \pi$, and where $\max|f(x)|$ is taken for $-\pi \leq x \leq \pi$. Conditions (γ), (δ), and (ε) need no comment, while (α) needs only the remark that if $|g(x_2) - g(x_1)| \leq K|x_2 - x_1|$, then

$$\left|\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right| + nK \geq \left|\frac{f(x_2) + ng(x_2) - f(x_1) - ng(x_1)}{x_2 - x_1}\right| \geq \left|\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right| - nK.$$

As to (β), it is clear that for any positive ε there is some number s such that if $n \geq s$, $|f(x) - f_n(x)| < \varepsilon/2$. Then

$$\lim \max\left|\frac{\Delta f_n(x)}{\Delta x}\right| \geq \max\left|\frac{\Delta f(x)}{\Delta x}\right| - \varepsilon.$$

As ε is arbitrarily small,

$$\lim \max\left|\frac{\Delta f_n(x)}{\Delta x}\right| \geq \max\left|\frac{\Delta f(x)}{\Delta x}\right|.$$

Since $\lim \max|f_n(x)| = \max|f(x)|$, (β) follows at once.

We shall define I(F) by the formula

(J). It will also be observed that in a (J) there is always a group of bicontinuous, biunivocal operations generated by the members of Σ. It does not follow that these cases are identical as to their limit-properties, for though a bicontinuous, biunivocal transformation which keeps invariant every element of E will also transform every limit-element of E into a limit-element of E, it does not follow that it will leave every limit-element of E invariant.

The case when Σ consists precisely of the group of all bicontinuous, biunivocal transformations is especially ~~inter~~ interesting; we shall denominate it (J_1). If we call a set E closed if it contains all its limit-elements — if, that is, it contains all those elements that are left invariant by every transformation which belongs to Σ and leaves every element of E invariant, then it is easy to see that a bicontinuous transformation is precisely one which leaves all closed sets closed. We thus get as the necessary and sufficient condition that Σ should contain all bicontinuous, biunivocal transformations,

A. If R is a biunivocal transformation of C such that ~~leaves~~ R and $\breve{R}$ both leave all closed sets closed, it belongs to Σ.

If Σ consists of all bicontinuous, biunivocal transformations, it must also satisfy the group conditions

B. If R belongs to Σ, so does $\breve{R}$, and

C. If R and S belong to Σ, so does R|S.

A, B, and C are moreover sufficient to secure that Σ consists in all bicontinuous, biunivocal transformations, for it results from B and C that if S belongs to Σ, then if T belongs to Σ, so do S|T|$\breve{S}$ and $\breve{S}$|T|S, so that S and ~~not~~ $\breve{S}$ simply permute the operations of Σ, and change no limit-properties. Needless to say, every (J_1) is a (J).

4. The next problem is to determine under what circumstances a system in which limit-element is defined belongs to one of the classes (J) or (J_1). To begin with, we shall search for the necessary condition that ~~a (S)~~ a (V), or system in which ~~non-sequential limit~~ neighborhood is defined, be a (J). We may make use of the fact that our notion of limit necessarily satisfies the two conditions:

1. Every limit-element of a set E is also limit-element of every set containing E;

2. The fact that an element A is or is not a limit-element of a set E is not affected by adjoining A to E.

Fréchet* has shown that under these conditions, if we define a neighborhood of A as a set V_A of elements such that the set of all elements not in V_A does not have A as a limit-element, then the necessary and sufficient condition for a set E to have A as a limit-element is for E to have elements in every V_A. In terms of Σ

*V., p. 4.

~~*V., p. 4.~~

$$I(F) = \int_0^1 \cdots \int_0^1 F(a_0, a_1, \ldots, a_n, b_1, \ldots, b_n)\, G_n(a_0, a_1, \ldots, a_n, b_1, \ldots, b_n)\, \cancel{G\{f\}}\; da_0\, da_1 \cdots da_n\, db_1 \cdots db_n,$$

where

$$f(x) = a_0/2 + \sum_1^\infty (a_n \cos nx + b_n \sin nx),$$

and $F\{f\} = F(a_0, a_1, \ldots, a_n, b_1, \ldots, b_n)$ is a functional belonging to T_0, and $G_n(a_0, a_1, \ldots, a_n, b_1, \ldots, b_n)$ is $G\{f_n\}$, where $f_n(x) = a_0/2 + \sum_1^n (a_n \cos nx + b_n \sin nx)$. The integrand is, by ($\alpha$), a continuous function of the a's and b's, so that $I(F)$ will always exist. That I satisfies conditions (C), (A), and (P) may be seen on inspection. The only one of the Daniell conditions involving any discussion is (L).

Now,

$$|I(F)| \leq \max_f \{|F\{f\}|\, G_n\{f\}\}.$$

Therefore

$$\lim_{n \doteq \infty} I(F_n) \leq \lim_{n \doteq \infty} \max_f \{F_n\{f\}\, G_n\{f\}\}.$$

Consequently (L) will be satisfied if $\max_f \{F_n\{f\}\, G_n\{f\}\}$ approaches 0 with $1/n$. Assume that it does not. Then we can find a $k > 0$ such that

$$\max_f \{F_n\{f\}\, G_n\{f\}\} \geq k$$

for all n. However, ~~max~~ $F_n\{f\} G_n\{f\}$ is continuous in the a's and b's, and therfore attains its maximum

~~those in I_1, and so~~ on Call the resulting set J_n. Clearly, J_n is contained in J_{n+1}. ~~Now, form K_n, consisting of those intervals in I_n, but not excluding all elements in J_n, together with the set of all functions in J_n. Weight~~ Let i_n be an interval from I_n. Divide it into parts $i_{n,0}$, $i_{n,1}$, $i_{n,2}$, $\dots$, where $i_{n,k}$ consists of all the elements in i_n and J_k but not J_{k+1}, and $i_{n,0}$ consists of all the elements in i_n and in no J_κ. Let j_k consist of all elements in J_k. Weight j_k ~~as sum of weights of intervals, and~~ and $i_{n,k}$ after fashion of Borel measure. Form

$$K_p = i_{p,1}, \dots, i_{p,p}, j_p, \text{ for every } i_p$$

$$\|P'Q'\| \geq \left| \|PQ\| - \frac{\|BC\|}{\|AB\|} \left| \|AQ\| - \|AP\| \right| \right|$$

$$\geq \|PQ\| (1 - \|BC\| / \|AB\|).$$

It follows from these inequalities that, to put it ~~roughly~~, $P'Q'$ is small when and only when PQ is small, and that a set of points approaching indefinitely close to a given point is transformed into a set approaching indefinitely close to the transform of the given point, and vice versa. In other words, our transformation leaves limit properties invariant in both directions, and so belongs to Σ. Moreover, our transformation changes B into the point D such that

$$AD = AB \oplus \left\{ \frac{\|AB\|}{\|AB\|} \odot BC \right\} = AB \oplus BC = AC,$$

or in other words, into C. We thus have completed our proof for the equivalence of point-pairs by a consideration of "rotations".

<u>Proof of</u> (4).

From I, $\underline{\text{VII}}$, and the fact that no single ~~point~~ element has a limit, it follows that there is at least one segment E with an interior element. Let A and B be two boundary elements of this segment. Then by $\underline{\text{IX}}$, the segment contains every connected set with A and B as boundary-elements. Now, let C be any boundary-element of the segment other than A and B; I say that there is no connected set in $\bar{E}$ having C for a limit-element. $\bar{E}$ may be divided into components, in accordance with a theorem given by Hausdorff"; let F be such a component. It cannot have A and C simultaneously for limit-elements, for this would contradict $\underline{\text{IX}}$. Neither can it have B and C simultaneously for limit-elements. If it has C alone for a limit-element, and not A and B, then, by assumption 2 of the Riesz-Fréchet set, $E+F$ will still have A and B for boundary-elements, so that by $\underline{\text{IX}}$, F will have to vanish.

On the other hand, at least one boundary-element of E is a limit-element of a connected set from $\bar{E}$. Let P be an element in E, Q an element not

$$\frac{\xi \oplus \eta}{2} = \cancel{u_1 + v_1 \ldots u_2 + v_2}$$

$$(u_1 + v_1,\ u_2 + v_2,\ \ldots\ \ldots)$$

$$(u_k + v_k)\sqrt{\frac{\sum u_n^2}{\sum (u_n + v_n)^2}}$$

$\cancel{\Sigma}\ v$

$\cancel{\theta =}$

$$\angle \xi\eta = \cos^{-1}\frac{u_1 v_1 + u_2 v_2 + \cdots + u_k v_k + \cdots}{u_1^2 + u_2^2 + \cdots}$$

$\cancel{\angle \xi\eta + \angle \eta\theta = \cos^{-1}\left(u_1 v_1^2 v_1 + \cdots\right.}$

$\|\xi \ominus \eta\|$

$= \sqrt{2\|\xi\|^2[1 - \cos\angle\xi\eta]}$

A C B

$\angle \alpha\beta = 2 \angle \xi\eta$

$\angle \alpha\beta < 90^\circ$

~~To prove~~

$$\frac{(u_1 v_1 + u_2 v_2 + \cdots)^2}{(u_1^2 + u_2^2 + \cdots + u_n^2 + \cdots)(v_1^2 + v_2^2 + \cdots)} \leq 1$$

$2\sum(u_n^2 + u_n v_n)\ [\sum u_n^2 + \sum u_n v_n]$

$\cos^2$

$\sum u_m^2 \sum u_n^2 + 2\sum u_n^2 \sum u_m v_m + \sum u_n v_n \sum u_m v_m$

in E, and F a connected set containing P and Q, which must exist on the basis of $\underline{\mathrm{I}}$, $\underline{\mathrm{VI}}$, and $\underline{\mathrm{VII}}$. Let G consist of those elements in F and at the same time in $\overline{E}$, and let H be that component of F containing Q.

The Theory of Sets of Points in Terms of Continuous Transformation.

$$\xi = (u_1, u_2, \dots, u_k, \dots)$$

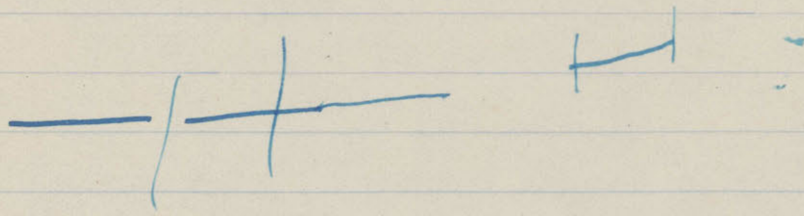
$$\eta = (v_1, v_2, \dots, v_k, \dots)$$

$$a\xi + b\eta = (au_1 + bv_1 \quad , \dots$$

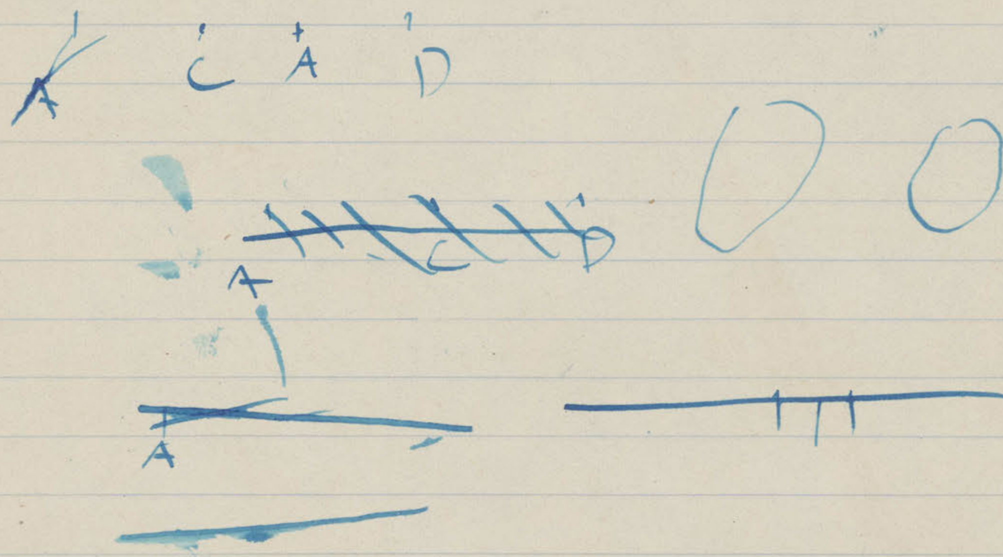
$$\|(a \odot \xi) \oplus (b \odot \eta)\| = \sqrt{a^2\|\xi\|^2 + b^2\|\eta\|^2 + 2ab}$$

Theorem VII. If ABC and BCD, then ACD.

Proof. By theorem VI, we have DAC, ADC, or ACD. If A is ~~in~~ interior to (D, C) and B is in (A, C), then, by theorem IV, B is interior to ~~in~~ (D, C), which, by theorem III, contradicts BCD. If ADC, then, by theorem IV



QRP



APB

XY YZ XZ

~~XY~~

XY XZ YZ or ZY

Theorem XIII. If M and N are two classes of elements exhausting K, and such that ~~if A belon~~ there are two fixed elements C and D such that if A belongs to M and B belongs to N, $AB|CD$, then there is an element P such that if Q belongs to M and R belongs to N and $Q \neq P \neq R$, QPR.

Proof

It follows from theorem I that K is connected. Hence there is at least one element P that ~~either is in M and~~ is a limit-element of both M and N, ~~or is in N and is a limit-element of M. It follows from postulate VII~~ that If Q belongs to M and R to N, ~~then by theorem~~ and $Q \neq P \neq R$, then by theorem VI, either PQR or QPR or QRP. If PQR, then, since the segment QR is a connected set, there is at least one elements in QR which

<u>Proof</u>. Let K′ be the set to which reference is made in postulate ~~theorem~~ VIII. Then every element is a limit-element of K′. It follows from ~~VII and~~ the fact that a segment is closed

I K consists of only one element; Σ identity transf.

II K line, Σ continuous transf preserving direct.

III Adjoin to Σ transf consisting of adding or subt. 2 to rat'l & 3 to irrat.

IV

V ~~#~~ Bicontin, biuniv Transf that leave whole segment fixed.

VI Two lines

VII

VIII "

IX Plane

I. An event is simultaneous with another if neither precedes the other

II An instant is a class of events whereof none precedes any other, while any event simult with all members of an instant belongs to the instant.

III One instant precedes another if one member of the first precedes one member of the second

IV. One instant is an ancestor of another if it precedes it, or precedes som instant preceding it, or so on indefinitely

V A maximum chain is a class of ~~tempal~~ instants whereof each is a temporal ancestor of each other, such that no ~~class of~~ instants not bel to class is a temporal ancestor of some member of class & also temporal descendant of some member of the class.

VI A generalized instant is the logical sum of ~~the members of~~ a maximum chain

VII A gen inst prec another if ~~the max ch. cont~~ an inst. forming part of it is a temporal ancestor of an inst forming part of the other.

I was born in New England, I have been lived here, I will probably grow old and die in these parts; and yet, in spite of intimate associations, I am not a New Englander — no, thank Goodness, I am not a New Englander. These rugged hills, where I have lived and grown have impressed themselves deep on my nature, in these wild forest lie my happiest memmories, its sub-classes will be supposed to be given.

A <u>weighted division</u> of K will be defined as a division of K to each term of which a number not always 0 is assigned — its <u>weight</u>. A weighted division corresponding to α_K will be represented by such a symbol as α_K^U, α_K^V, α_K^W, etc. The weight of $t_j(\alpha_K)$ will be written $U_\alpha\{t_j(\alpha_K)\}$, $V_\alpha\{t_j(\alpha_K)\}$, or $W_\alpha\{t_j(\alpha_K)\}$, respectively.

A <u>division-sequence</u> of K is a sequence of weighted divisions $\alpha_K^{U_1}(1)$, $\alpha_K^{U_2}(2)$, ..., $\alpha_K^{U_n}(n)$, ..., such that

(1) Every member of $\alpha_K(n+1)$ is wholly contained in one member of $\alpha_K(n)$ and one only,

(2) $U_n\{t_j(\alpha_K(n))\}$ is the sum of all the numbers $U_{n+1}\{t_e(\alpha_K(n+1))\}$ such that $t_e(\alpha_K(n+1))$ is contained in $t_j(\alpha_K(n))$.

A <u>chain</u> with respect to this division-sequence is a sequence consisting of one term $t_{j_n}(\alpha_K(n))$ from every $\alpha_K(n)$ belonging to P_K, such that $t_{j_n}(\alpha_K(n))$ contains $t_{j_{n+1}}(\alpha_K(n+1))$.

A <u>partition</u> P_K is a division-sequence such that if $\alpha_K^{U_n}(n)$ is the nth member of it, $\Sigma U_n\{t_j(\alpha_K(n))\}$ for all the t_j's which do not form members of a chain all the members of which contain at least one K-element in common, is a function of n that approaches 0 as n increases without limit. Every $\alpha_K(n)$, $\alpha_K^{U_n}(n)$, and $t_j(\alpha_K(n))$ is said to <u>belong to</u> P_K.

A function defined for all elements of K is said to be a P_K <u>step-function</u> if there is an $\alpha_K(n)$ belonging to P_K such that the function is constant over every

To be called
for by

Dr N. Wiener

(See separate sheet
written for questions
set)

Questions set on Wednesday, 15. XII. 1915.

(1) State carefully the different ways in which Fullerton (ch. V) distinguishes between 'appearance' and 'reality'.

What is the main point of his criticism of the distinction?

2. How much of the Self as analysed by James can be conceived as pre-existing before birth or as surviving death?

Note: I explained to the class that question (2) called for, first, a brief statement of the results of James's analysis; second, an answer to the question (designed to elicit their 'reaction') how much of what James sets down as belonging to the Self is bound up with this life 'on earth', and how much could be carried on into the general conception of pre-existence or survival.

I received this information
~~known age~~
I will not hear of this
Who will prevent me
I'll not give unless this

If you should happen to find
it would you kindly return
it to me.
I can't do without it.
Oh nonsense you can get along
very well without it.
I heard your brother was ill.
He was but he is getting on
very well now.