Studies in Synthetic Logic.

By Norbert Wiener, Ph. D

§1. In a recent article of mine in the Proceedings of the Cambridge Philosophical Society*, I showed how we could regard the series of the instants of time as a construction from the non-serial relation of complete temporal succession between events, ~~and~~ only making a few simple presuppositions concerning the formal properties of this relation of complete succession, and, in a foot-note, I showed further how, without making any assumptions concerning the formal properties of a given relation, R, we can construct another relation from R in a perfectly determinate manner, so that this latter relation will always be a series.

In this article I wish to extend this method of series-building in two directions. I first ~~wish~~ mean to bring the definitions of order through triadic and tetradic relations under a single very general heading, and to show that Frege's theory of hereditary relations and the theory of series-synthesis developed in my ~~last~~ former article can be generalized so as to apply to these. Then I shall give an alternative method of constructing series from non-serial relations, which bears much the same relation to the various series of sensation intensities that the method of my previous article bears to the series of temporal (and spatial) sensation-extensities.

~~In order to treat of polyadic relations, we must~~

In general, our symbolism will be that of the

* A Contribution to the Theory of Relative Position, vol. XVII, Part 5.

Principia Mathematica of Whitehead and Russell, and we shall take the theorems established in that book for granted. But as we shall have much to do with polyadic relations, and as those parts of the *Principia* which shall treat of general polyadic relations are not yet in print, it will be necessary for us to develop a symbolism of our own here, and to take for granted such properties of polyadic relations as have precise analogues in the theories of classes and of binary relations. Moreover, as we shall want to speak of properties of relations between *any* number of terms, and as in Mr. Russell's system,* relations between m terms belong to different types than relations between n terms, if $m \neq n$, so that no propositional functions ~~of~~ whose arguments range over m-adic and n-adic relations exist, we shall have to permit a certain logical laxity in our symbolism. Whereas our theorems really demand a separate, though precisely parallel, proof when the relations are m-adic, n-adic, p-adic, etc., we shall treat all these proofs as a single proof. Whereas every relation holds between a definite number of terms, we shall permit lines of dots to occupy the places of an indefinite number of these terms. Whereas the analogues of $\dot{\cap}$, $\dot{\cup}$, $\dot{-}$, etc. are different ~~for~~ with each different sort of relation with which they have to do, we shall represent them all by the symbols we use for binary relations. To the reader acquainted with symbolic logic, there will be no difficulty whatever in reducing ~~each~~ any particular case of the theorems I prove to strict logical form.

* See, however, my article, *A Simplification of the Logic of Relations*, *Proc. Camb. Phil. Soc.*, vol XVII, Part 5. The method of this article can be extended to n-adic relations in general.

6

STUDIES IN SYNTHETIC LOGIC

BY

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Studies in Synthetic Logic. By NORBERT WIENER, Ph.D. (Communicated by Mr G. H. Hardy.)

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§ 1. In a recent article of mine in the *Proceedings of the Cambridge Philosophical Society**, I showed how we can regard the series of the instants of time as a construction from the non-serial relation of complete temporal succession between events in time, and how only a few simple presuppositions concerning the formal character of this relation of complete temporal succession sufficed to establish the seriality of the relation of succession between instants; and, in a foot-note, I showed further how, *without making any assumptions* concerning the formal properties of a given relation, P, we can construct another relation from P in a perfectly determinate manner, so that this latter relation will always be a series.

In this article, I wish to extend this method of series-construction in two different directions. I first mean to bring the definitions of order through triadic and tetradic relations under a single very general heading, and to show that Frege's theory of hereditary relations and the theory of series-synthesis developed in my former article can be generalized so as to apply to these. Then I shall give an alternative method of constructing series from non-serial relations which bears much the same relation to the various series of sensation-*intensities* that the method of my previous article bears to the series of instants that constitutes one sort of *extension*, time.

In general, our symbolism will be that of the *Principia Mathematica* of Whitehead and Russell, and we shall take the theorems established in that book for granted. But as we shall have much to do with polyadic relations, and as the parts of the *Principia* which will treat of general polyadic relations are not yet in print, it will be necessary for us to develop a symbolism of our own here. Such properties of polyadic relations as have precise analogues in the theory of classes we shall take for granted. Moreover, as we shall want to speak of properties of relations among *any* number of terms, and as in Mr Russell's system†, relations among m terms belong to different types than relations among n terms, if $m \neq n$, so that no propositional functions whose arguments range over

* "A Contribution to the Theory of Relative Position," vol. XVII, Part 5, pp. 441—9.

† See, however, my article, "A Simplification in the Logic of Relations," *Proc. Camb. Phil. Soc.*, vol. XVII, Part 5, pp. 387—90. The method of this article can be extended to n-adic relations in general.

m-adic and n-adic relations exist, we shall have to permit a certain logical laxity in our symbolism. Though our theorems really demand a separate, though precisely parallel, proof when the relations dealt with are m-adic and when they are n-adic, we shall have to treat these proofs as one. Though every relation holds among a definite set of terms, we shall permit dots to fill the places of an indefinite number of these. Though the analogues of $\dot{\cap}$, $\dot{\cup}$, $\dot{s}$, etc. are different with each different sort of relation with which they have to do, we shall represent them all by the symbols we use in the case of binary relations. To the reader acquainted with symbolic logic, there will be no difficulty in reducing any particular case of the theorems I prove to a strictly rigorous form.

§ 2. Let us write the proposition, '$a_1, a_2, \dots, a_n$ are in the n-adic relation R,' as $R\{a_1, a_2, \dots, a_n\}$. I shall call a property of an n-adic relation, R, an *n-transitivity* of R when it can be written in the form

$$(1)\quad (\exists b_1, b_2, \dots, b_k) \,.\, T_R\{a_1, a_2, a_3, \dots, a_n, b_1, b_2, \dots, b_k\} \,.\, \supset_{a_1, a_2, \dots, a_n} . \, R\{a_1, a_2, \dots, a_n\},$$

where T_R is the logical disjunction of a number of expressions in the form

$$R\{c_1, c_2, \dots, c_n\} \,.\, R\{c_1', c_2', \dots, c_n'\} \,.\, R\{c_1'', c_2'', \dots, c_n''\} \dots R\{c_1^{(l)}, c_2^{(l)}, \dots, c_n^{(l)}\},$$

where l is not necessarily the same in each of these expressions, and $c_1, c_2, \dots, c_n, c_1', c_2', \dots, c_n', \dots, c_1^{(l)}, c_2^{(l)}, \dots, c_n^{(l)}$, which are not all distinct from one another, are to be found among $a_1, a_2, \dots, a_n$, $b_1, b_2, \dots, b_k$. Ordinary binary transitivity is an example of a 2-transitivity; the property of 'betweenness,' which may be written

$$(\exists d) : abd \,.\, bdc \,.\, \vee \,.\, abd \,.\, bcd \,.\, \vee \,.\, adc \,.\, dbc \,.\, \vee \,.\, abd \,.\, acd \,.\, bac \,.\, \vee \,.\, dab \,.\, dac \,.\, bac \,.\, \vee \,.\, bca : \supset_{a,b,c} . \, abc,$$

is a 3-transitivity; the property of Vailati's separation-relation, which may be written

$$(\exists e) : ab \| dc \,.\, \vee \,.\, cd \| ab \,.\, \vee \,.\, ab \| ec \,.\, ae \| cd : \supset_{a,b,c,d} . \, ab \| cd,$$

is a 4-transitivity. From these examples it is obvious that the transitivity-properties of relations are of very great logical interest, and that a method which shall point out significant analogies between the various sorts of transitivity is not without importance.

One property which all sorts of n-transitivity have in common is this: if R is any n-adic relation whatever, then it is always possible, given any particular form of n-transitivity, to construct in a perfectly determinate manner a relation, R', including R, forming a well-defined function of R, having the desired sort of transitivity.

This is proved as follows: let the n-transitivity in question be the one given in (1). Decompose $T_R\{a_1, \ldots, a_n, b_1, \ldots, b_k\}$, as indicated, into a sum of expressions of the form

$$R\{c_1, c_2, \ldots, c_n\} \,.\, R\{c_1', c_2', \ldots, c_n'\} \ldots R\{c_1^{(l)}, c_2^{(l)}, \ldots, c_n^{(l)}\}.$$

Let there be, say, f such expressions, the pth one always with l_p R's. Replace each of these R's by one and one only of the variable relations $X_1, X_2, \ldots, X_m$, with the same arguments as the R it replaces, and let $m = \sum_{p=1}^{p=f} l_p$. We shall thus transform T_R into a relation which is a function of the m variable relations $X_1, X_2, \ldots, X_m$. Let us call this relation $\dfrac{T}{X_1 X_2 \ldots X_m}$. Now, let us define the relation $\underline{X_1 X_2 \ldots X_m}_T$ as follows:

$$(2) \quad \underline{X_1 X_2 \ldots X_m}_T\{a_1, a_2, \ldots, a_n\} \,.=.\, (\exists b_1, b_2, \ldots, b_k) \,.$$
$$\frac{T}{X_1 X_2 \ldots X_m}\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_k\} \quad \text{Df.}$$

Like $\dfrac{T}{X_1 X_2 \ldots X_m}$, $\underline{X_1 X_2 \ldots X_m}_T$ is a function of $X_1, X_2, \ldots, X_m$, where the latter may assume any values which are n-adic relations. Now, I define the class of T-powers of R, or, as I write it, $\overrightarrow{T_{\mathrm{pr}}}\text{‘}R$, as follows:

$$(3) \quad T_{\mathrm{pr}} = \hat{S}\hat{R}\{X_1, X_2, \ldots, X_m \in \mu \,.\supset_{X_1, X_2, \ldots, X_m} .$$
$$\underline{X_1 X_2 \ldots X_m}_T \in \mu : R \in \mu :\supset_\mu . S \in \mu\} \quad \text{Df.}$$

I make the further definition,

$$(4) \quad R_T = \dot{s}\text{‘}\overrightarrow{T_{\mathrm{pr}}}\text{‘}R \quad \text{Df.}$$

Now, R_T includes R and is a function of it, and has the desired sort of n-transitivity.

First, R_T includes R. For, since, as may be seen on inspection, $R T_{\mathrm{pr}} R$, $R \in \overrightarrow{T_{\mathrm{pr}}}\text{‘}R$. Since every member of a class is included in the sum of the class, $R \mathrel{\mathsf{G}} \dot{s}\text{‘}\overrightarrow{T_{\mathrm{pr}}}\text{‘}R \mathrel{\mathsf{G}} R_T$. Secondly, as R_T is derived from R by a process which is really perfectly definite (though I admit that some of the stages of the process by which I have derived R_T from R are not uniquely determined, a little reflection will convince one that all the possible determinations of $\dfrac{T}{X_1 X_2 \ldots X_m}$ yield the same value of R_T), it is a function of R, and

of R alone, once T is determined. Thirdly, R_T has the desired sort of n-transitivity. For we can write

$$T_{R_T}\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_k\}$$

as a sum of products of the form

$$R_T\{c_1, c_2, \ldots, c_n\} \,.\, R_T\{c_1', c_2', \ldots, c_n'\} \ldots R_T\{c_1^{(l)}, c_2^{(l)}, \ldots, c_n^{(l)}\}.$$

Now to say $R_T\{d_1, d_2, \ldots, d_n\}$ is, by the definition of R_T, the same as to say that there is some S such that $S T_{\mathrm{pr}} R$, and $S\{d_1, d_2, \ldots, d_n\}$. Therefore

$$T_{R_T}\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_k\}$$

is equivalent to

$$(\exists S_1, S_2, \ldots, S_m) \,.\, \frac{T}{S_1 S_2 \ldots S_m}\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_k\} \,.$$
$$S_1 T_{\mathrm{pr}} R \,.\, S_2 T_{\mathrm{pr}} R \,.\, S_3 T_{\mathrm{pr}} R \ldots S_m T_{\mathrm{pr}} R.$$

Therefore

$$(5) \quad \vdash :::. (\exists b_1, b_2, \ldots, b_k) \,.\, T_{R_T}\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_k\} ::$$
$$\equiv :: (\exists S_1, S_2, \ldots, S_m) \,.\, \underline{S_1 S_2 \ldots S_m}_T\{a_1, a_2, \ldots, a_n\} \,.$$
$$S_1 T_{\mathrm{pr}} R \,.\, S_2 T_{\mathrm{pr}} R \,.\, S_3 T_{\mathrm{pr}} R \ldots S_m T_{\mathrm{pr}} R ::$$
$$\equiv :: (\exists S_1, S_2, \ldots, S_m) :. \underline{S_1 S_2 \ldots S_m}_T\{a_1, a_2, \ldots, a_n\} :.$$
$$X_1, X_2, \ldots, X_m \in \mu \,.\supset_{X_1, X_2, \ldots, X_m} . \underline{X_1 X_2 \ldots X_m}_T \in \mu :$$
$$R \in \mu :\supset_\mu . S_1, S_2, \ldots, S_m \in \mu ::$$
$$\supset :: (\exists S_1, S_2, \ldots, S_m) :. \underline{S_1 S_2 \ldots S_m}_T\{a_1, a_2, \ldots, a_n\} :.$$
$$X_1, X_2, \ldots, X_m \in \mu \,.\supset_{X_1, X_2, \ldots, X_m} . \underline{X_1 X_2 \ldots X_m}_T \in \mu :$$
$$R \in \mu :\supset_\mu . \underline{S_1 S_2 \ldots S_m}_T \in \mu ::$$
$$\supset :: (\exists S_1, S_2, \ldots, S_m) \,.\, \underline{S_1 S_2 \ldots S_m}_T\{a_1, a_2, \ldots, a_n\} \,.$$
$$\underline{S_1 S_2 \ldots S_m}_T T_{\mathrm{pr}} R ::$$
$$\supset :: (\exists U) \,.\, U\{a_1, a_2, \ldots, a_n\} \,.\, U \in \overrightarrow{T_{\mathrm{pr}}}\text{‘}R ::$$
$$\supset :: (\dot{s}\text{‘}\overrightarrow{T_{\mathrm{pr}}}\text{‘}R)\{a_1, a_2, \ldots, a_n\} ::$$
$$\supset :: R_T\{a_1, a_2, \ldots, a_n\}.$$

This is what we wished to prove, for, if we compare this with (1), it shows that R_T has the desired sort of transitivity.

When the transitivity in question is ordinary binary transitivity R_T becomes R_{po}. In general, the appropriate form of R_T performs the function of R_{po} in systems whose order is given by a triadic or tetradic or other polyadic relation.

§ 3. There is another important sort of property which the ordinary serial relation, the 'between' relation on a given line, and the separation-relation have in common. For the binary serial relation, it is ordinary connexity; for the 'between' relation on a given line it may be expressed in symbols as

$$(\exists m, n) : amn \,.\vee.\, man \,.\vee.\, mna : bmn \,.\vee.\, mbn \,.\vee.\, mnb :$$
$$cmn \,.\vee.\, mcn \,.\vee.\, mnc : \supset_{a,b,c} :$$
$$a = b \,.\vee.\, b = c \,.\vee.\, c = a \,.\vee.\, abc \,.\vee.\, bca \,.\vee.\, cab ;$$

for the separation-relation it is

$$(\exists m, n, o) : am \,\|\, no \,.\vee.\, ma \,\|\, no \,.\vee.\, mn \,\|\, ao \,.\vee.\, mn \,\|\, oa :$$
$$bm \,\|\, no \,.\vee.\, mb \,\|\, no \,.\vee.\, mn \,\|\, bo \,.\vee.\, mn \,\|\, ob :$$
$$cm \,\|\, no \,.\vee.\, mc \,\|\, no \,.\vee.\, mn \,\|\, co \,.\vee.\, mn \,\|\, oc :$$
$$dm \,\|\, no \,.\vee.\, md \,\|\, no \,.\vee.\, mn \,\|\, do \,.\vee.\, mn \,\|\, od :$$
$$\supset_{a,b,c,d} : a = b \,.\vee.\, b = c \,.\vee.\, c = d \,.\vee.\, d = a \,.\vee.\, a = c \,.\vee.\, b = d \,.\vee.$$
$$ab \,\|\, cd \,.\vee.\, ac \,\|\, bd \,.\vee.\, ad \,\|\, bc.$$

For the sake of brevity, let us generalize the notion of 'field' in the following manner:

$$(6) \quad C = \hat{\alpha}\hat{R}\,\{\alpha = \hat{x}\,\{(\exists a_1, a_2, \ldots, a_{n-1})\} :$$
$$R\,\{x, a_1, a_2, \ldots, a_{n-1}\} \,.\vee.\, R\,\{a_1, x, a_2, \ldots, a_{n-1}\} \,.\vee \ldots$$
$$\vee.\, R\,\{a_1, a_2, \ldots, a_{n-1}, x\}\} \quad \text{Df.}$$

Now, I shall define a property of an n-adic relation, R, as an *n-connexity* of that relation if it can be written in the form

$$(7) \quad a_1, a_2, \ldots, a_n \,\epsilon\, C\text{‘}R : l \neq m \,.\supset_{l,m} : \sim(a_l \neq a_m) : \supset_{a_1, a_2, \ldots, a_n} :$$
$$R\,\{a_1, a_2, \ldots, a_n\} \,.\vee.\, R\,\{a_1', a_2', \ldots, a_n'\} \,.\vee.$$
$$R\,\{a_1'', a_2'', \ldots, a_n''\} \,.\vee \ldots \vee.\, R\,\{a_1^{(p)}, a_2^{(p)}, \ldots, a_n^{(p)}\},$$

where $a_1' \ldots a_n', a_1'' \ldots a_n'', \ldots, a_1^{(p)} \ldots a_n^{(p)}$ are each definite permutations of $a_1 \ldots a_n$. It is obvious that ordinary binary connexity is, by this definition, a 2-connexity, and that the properties of 'between' and separation which we have just mentioned are, respectively, 3- and 4-connexities.

Now, I wish to raise with regard to n-connexities the precise analogue of the question which we raised with regard to n-transitivities in the last section: is it possible, given any n-adic relation and any n-connexity, to form by a perfectly definite method an n-adic relation genuinely dependent on this relation, having the desired sort of n-connexity?

As in the former case, I shall answer this question by actually

constructing such a relation. I shall define the relation $R_{\sigma\lambda}$ as the relation such that $R\,\{a_1, a_2, \ldots, a_n\}$ when, and only when,

$$a_1, a_2, \ldots, a_n \,\epsilon\, C\text{‘}R,$$

and the conclusion of (7) is false*.

I shall define the class, ϖ_R, as follows:

$$(8) \quad \varpi_R = \hat{\alpha}\,\{x, y \,\epsilon\, \alpha \,.\, a_1, a_2, \ldots, a_{n-2} \,\epsilon\, C\text{‘}R \,.\supset_{x, y, a_1, a_2, \ldots, a_{n-2}} .$$
$$R_{\sigma\lambda}\,\{x, y, a_1, a_2, \ldots, a_{n-2}\} \,.\, R_{\sigma\lambda}\,\{x, a_1, y, a_2, \ldots, a_{n-2}\} \,. \ldots$$
$$R_{\sigma\lambda}\,\{x, a_1, a_2, \ldots, a_{n-2}, y\} \,.\, R_{\sigma\lambda}\,\{y, x, a_1, a_2, \ldots, a_{n-2}\} \,.$$
$$R_{\sigma\lambda}\,\{a_1, x, y, a_2, \ldots, a_{n-2}\} \,. \ldots R_{\sigma\lambda}\,\{a_1, x, a_2, \ldots, a_{n-2}, y\} \,. \ldots$$
$$R_{\sigma\lambda}\,\{y, a_1, a_2, \ldots, a_{n-2}, x\} \,. \ldots R_{\sigma\lambda}\,\{a_1, a_2, \ldots, a_{n-2}, x, y\} ::$$
$$c \,\epsilon\, \alpha \,.\supset_c : R_{\sigma\lambda}\,\{c, b_1, b_2, \ldots, b_{n-1}\} \,.\vee.\, R_{\sigma\lambda}\,\{b_1, c, b_2, \ldots, b_{n-1}\} \,.\vee \ldots$$
$$\vee.\, R_{\sigma\lambda}\,\{b_1, b_2, \ldots, b_{n-1}, c\} :. \supset_{b_1, b_2, \ldots, b_{n-1}} .\, b_1, b_2, \ldots, b_{n-1} \,\epsilon\, \alpha\} \quad \text{Df.}$$

Next, I define ins as follows:

$$(9) \quad \text{ins} = \hat{P}\hat{Q}\,\{P\,\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \,.\equiv_{\alpha_1, \alpha_2, \ldots, \alpha_n} : \alpha_1, \alpha_2, \ldots, \alpha_n \,\epsilon\, \varpi_Q :$$
$$(\exists a_1, a_2, \ldots, a_n) \,.\, a_1 \,\epsilon\, \alpha_1 \,.\, a_2 \,\epsilon\, \alpha_2 \ldots a_n \,\epsilon\, \alpha_n \,.\, Q\,\{a_1, a_2, \ldots, a_n\}\} \quad \text{Df.}$$

Now, I claim, ins‘R possesses the desired sort of n-connexity, whatever R may be.

For did it not, by (7), it would be possible to find n distinct α's, say $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that none of those relations hold between them which can be made from those in the conclusion of (7) by substituting ins‘R for R, and each α for the a with the same number; while, as we learn from (9), each α is a member of ϖ_R. That is to say, if we pick out one member from α_1, say x_1, one from α_2, say x_2, and so on till we come to α_n, from which we pick out x_n, then $x_1, x_2, \ldots, x_n$ will stand to one another in none of the relations mentioned in the conclusion of (7), and hence will stand to one another in the relation $R_{\sigma\lambda}$. This will be true whatever the values that x_1 takes in α_1, x_2 in α_2, etc. It is easy to see that from this and the second half of the proposition in the brackets in (8), we can conclude that $\alpha_1 = \alpha_2 = \ldots = \alpha_n$, which contradicts our hypothesis. Hence, ins‘R always possesses the n-connexity expressed in (7).

Another and equally important property possessed by ins‘R is that, if (ins‘R) $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are all distinct.* For suppose that (ins‘R) $\{\alpha_1, \alpha_2, \ldots, \alpha_1, \ldots, \alpha_n\}$. Then we shall have to have, by the definition of ins, $R\,\{a_1, a_2, \ldots, b, \ldots, a_n\}$, where a_1 belongs to α_1, a_2 to α_2, etc., b to α_1, and so on till we get to a_n, which belongs to α_n; $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all, by the definition of ins, members of ϖ_R. Therefore, by the definition of ϖ_R, we shall

* It will be seen, of course, that $R_{\sigma\lambda}$, ϖ_R, and ins are essentially functions of the particular sort of n-connexity asserted in (7).

have $R_{\sigma\lambda}\{a_1, a_2, \ldots, b, \ldots, a_n\}$. We are thus led into a contradiction. It will be noted that this property too is characteristic of ordinary binary serial relations, of ternary relations such as the 'between' relation, and although in this case not clearly stated, of Vailati's separation-relations.

§ 4. Now two interesting questions arise: first, what hypothesis is necessary concerning the n-adic relation R if ins‘R is to have a given sort of n-transitivity? and secondly, is it possible to build a function of R which has any given sort of n-transitivity, any given sort of n-connexity, and is such that if this function holds between $\kappa_1, \kappa_2, \ldots, \kappa_n$, the κ's are all distinct? The first question is exceedingly easy to answer. Let the transitivity in question be that of (1), and the connexity that of (7). Modify (1) in the following manner: if in any of the products that, added, make up T_R, a term, say x, occurs as argument to several R's, replace it in all but one of its occurrences by some term, so that in no two occurrences is it replaced by the same term; multiply the product in which it occurs by all the expressions which can be formed by taking $R_{\sigma\lambda}$ [derived from the connexity expressed in (7)], and giving it as arguments any n (not all necessarily distinct) of the terms which replace x, including x itself; and introduce the terms, other than x itself, which replace x, as apparent variables, in such a manner that their range is the whole left side of (1), and that they are preceded by an $\exists$. If we transform (1) in this way, it is easy to see, though tedious to prove, that we obtain a sufficient condition for ins‘R's possessing the sort of n-transitivity indicated in (1) and the sort of n-connectedness indicated in (7).

As to the second question, it is almost self-evident that ins‘[(ins‘$R)_T$] possesses the sort of n-transitivity indicated in (1), the sort of n-connexity indicated in (7), and that if

$$\{\text{ins‘}[(\text{ins‘}R)_T]\}\,\{\kappa_1, \kappa_2, \ldots, \kappa_n\},$$

and $\kappa_i, \kappa_j, i = j$. The two latter properties follow simply from the fact that this relation is an ins of something; the fact that it has the former quality follows obviously from the following considerations. If Q has any sort of n-connexity, and $Q \subseteq P$, then P, a fortiori, has the same sort of n-connexity, if its field is that of Q; for the hypothesis of (7) (with R changed throughout to Q), remains unchanged, while, if

$$Q\,\{a_1^{(s)}, a_2^{(s)}, \ldots, a_n^{(s)}\}, \text{ then } P\,\{a_1^{(s)}, a_2^{(s)}, \ldots, a_n^{(s)}\},$$

so that the conclusion of (7) is true of P if it is true of Q. Therefore, (ins‘$R)_T$ has the desired sort of n-connexity and n-transitivity, though it may be possible for us to have $i \neq j$, $\kappa_i = \kappa_j$, and (ins‘$R)_T\,\{\kappa_1, \kappa_2, \ldots, \kappa_n\}$. Since (ins‘$R)_T$ is connected in the way determined by (7), $[(\text{ins‘}R)_T]_{\sigma\lambda}$ can only hold between $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$

when $\alpha_1 = \alpha_2 = \ldots = \alpha_\kappa$. Therefore, $\varpi_{(\text{ins‘}R)_T}$ is made up exclusively of unit-classes. Now, we can write the condition for the n-transitivity of ins‘[(ins‘$R)_T$] as follows:

$$(10)\quad (\exists\lambda_1, \lambda_2, \ldots, \lambda_k)\, T_{\text{ins‘}[(\text{ins‘}R)_T]}\,\{\kappa_1, \kappa_2, \ldots, \kappa_n, \lambda_1, \lambda_2, \ldots, \lambda_k\}\,.\, \supset_{\kappa_1, \kappa_2, \ldots, \kappa_n}.\, \text{ins‘}[(\text{ins‘}R)_T]\,\{\kappa_1, \kappa_2, \ldots, \kappa_n\}.$$

The expression in the form $T_{\text{ins‘}[(\text{ins‘}R)]}$ is here the sum of products of terms of the form ins‘[(ins‘$R)_T$] $\{\mu_1, \mu_2, \ldots, \mu_n\}$, where the μ's are to be found among the κ's and λ's, and all the λ's appear somewhere as arguments to ins‘[(ins‘$R)_T$]. Therefore, since

$$C‘\text{ins‘}[(\text{ins‘}R)_T] \subset \varpi_{(\text{ins‘}R)_T},$$

all the κ's and λ's are unit classes. Therefore, since

$$\{\text{ins‘}[(\text{ins‘}R)_T]\}\,\{\nu_1, \nu_2, \ldots, \nu_n\}$$

holds when and only when $\nu_1 \ldots \nu_n$ are members of $\varpi_{(\text{ins‘}R)_T}$, and there is an α_1 belonging to ν_1, an α_2 belonging to ν_2, …, an α_n belonging to ν_n, we may write (10) as follows:

$$(11)\quad (\exists\beta_1, \beta_2, \ldots, \beta_k)\,.\,\beta_1, \beta_2, \beta_k, \alpha_1, \alpha_2, \ldots, \alpha_n \in \breve{\iota}‘‘\varpi_{(\text{ins‘}R)_T}\,.\, T_{(\text{ins‘}R)_T}\{\alpha_1, \alpha_2, \ldots, \alpha_n; \beta_1, \beta_2, \ldots, \beta_k\}\,.\supset_{\alpha_1, \alpha_2, \ldots, \alpha_n}.\,(\text{ins‘}R)_T\,\{\alpha_1\ \alpha_2, \ldots, \alpha_n\}.$$

From (5) it follows that (11) is identically satisfied, and hence that ins‘[(ins‘$R)_T$] has the desired sorts of n-connexity and n-transitivity, and never, to put it roughly, relates a member of its field to itself, *whatever R may be.* Hence, if we have a system whose postulates can be put in the form of three propositions, one asserting a certain n-transitivity, another a certain n-connexity of a given n-adic relation, P, and the third asserting that P never relates a member of its field to itself, then, given any n-adic relation, R, we can construct a function of R having the desired properties of P. Moreover, it is easy to see that if R itself has the desired properties, the constructed relation will be, so we may put it, of the same formal properties as R, but two types higher.

§ 5. Now, there are very important sorts of relations whose definitions may be put in the above form. The general 'between' relation between members of a series is, it is easy to see, *completely* determined as to its formal properties by the three propositions

$$(\exists d) : abd\,.\,bdc\,.\vee.\,abd\,.\,bcd\,.\vee.\,adc\,.\,dbc\,.\vee.\,abd\,.\,acd\,.\,bac\,.\vee.\,dab\,.\,dac\,.\,bac\,.\vee.\,bca : \supset_{a,b,c}.\,abc,$$

$$(\exists m, n) : amn\,.\vee.\,man\,.\vee.\,mna : bmn\,.\vee.\,mbn\,.\vee.\,mnb : cmn\,.\vee.\,mcn\,.\vee.\,mnc : \supset_{a,b,c} : a = b\,.\vee.\,b = c\,.\vee.\,c = a\,.\vee.\,abc\,.\vee.\,bca\,.\vee.\,cab,$$

$$abc\,.\supset_{a,b,c}.\,a \neq b\,.\,b \neq c\,.\,a \neq c.$$

Similarly, if we understand the separation-relation to hold only between four distinct terms, the general separation-relation is *completely* determined by the three following propositions:

$$(\exists e) : ab \,\|\, dc \,.\lor.\, cd \,\|\, ab \,.\lor.\, ab \,\|\, ec \,.\, ae \,\|\, cd : \supset_{a,b,c,d} .\, ab \,\|\, cd,$$

$$\begin{aligned}(\exists m, n, o) : {} & am \,\|\, no \,.\lor.\, ma \,\|\, no \,.\lor.\, mn \,\|\, ao \,.\lor.\, mn \,\|\, oa : \\ & bm \,\|\, no \,.\lor.\, mb \,\|\, no \,.\lor.\, mn \,\|\, bo \,.\lor.\, mn \,\|\, ob : \\ & cm \,\|\, no \,.\lor.\, mc \,\|\, no \,.\lor.\, mn \,\|\, co \,.\lor.\, mn \,\|\, oc : \\ & dm \,\|\, no \,.\lor.\, md \,\|\, no \,.\lor.\, mn \,\|\, do \,.\lor.\, mn \,\|\, od : \\ \supset_{a,b,c,d} : {} & a=b \,.\lor.\, b=c \,.\lor.\, c=d \,.\lor.\, d=a \,.\lor.\, a=c \,.\lor.\, b=d \,.\lor. \\ & ab \,\|\, cd \,.\lor.\, ac \,\|\, bd \,.\lor.\, ad \,\|\, bc,\end{aligned}$$

$$ab \,\|\, cd \,.\supset_{a,b,c,d} .\, a \neq b \,.\, a \neq c \,.\, a \neq d \,.\, b \neq c \,.\, c \neq d \,.\, b \neq d.$$

That is, *from any triadic or tetradic relation, we are able to construct a between-relation or a separation-relation, respectively.* This fact should play much the same part in explaining how the regular relations of space may be derived from the irregular relations to be found in our experience that the analogous fact concerning dyadic relations plays in showing how the serial relation of the instants of time may be derived from the non-serial relation of complete succession between events*. Logically too this fact has a considerable interest, for it gives a hint of another method of defining mathematical systems than by the use of postulates; given our fundamental logical postulates to start with, we may be able to select the fundamental 'indefinables' of a mathematical system in such a manner that whatever values they may assume within their range of significance, the fundamental formal properties of the system will remain invariant.

§ 6. Of course, *all* the formal properties of a triadic or tetradic relation are not determined when the relation is completely determined as a between or separation relation. Hence there remain interesting and important questions yet as to whether simple properties of R may be given which will give $\text{ins}\text{‘}R$ or R_T or $\text{ins}\text{‘}[(\text{ins}\text{‘}R)_T]$ properties analogous to density or 'Dedekindianness,' etc. If density with respect to a given transitivity, say that of (1), be the property of a relation R which holds when the implication in (1) is converted, then it requires little proof to see that if the converse of (1), modified in the manner that (1) is modified in the first paragraph of § 4, is true of R, and if $C\text{‘}R \subset s\text{‘}\varpi_R$, then $\text{ins}\text{‘}R$ will have the required sort of density. I know of no simpler property of R, however, by which we can replace $C\text{‘}R \subset s\text{‘}\varpi_R$, and, at any rate, if R is a between or separation relation, this sort of density will not be the property which we would naturally call by that name. If $R\{a, b, c\}$ means 'b is between a and c,' then what we would naturally call density would be the property of R which can be written

$$(a, c) :: a, c \in C\text{‘}R \,.\supset :.\, (\exists b) \,.\, R\{a, b, c\} : \lor \,.\, a = c.$$

Provided that $C\text{‘}P \subset s\text{‘}\varpi_P$, then if P is any triadic relation having this property, then $\text{ins}\text{‘}P$, and hence, as may be seen easily, $(\text{ins}\text{‘}P)_T$ and $\text{ins}\text{‘}[(\text{ins}\text{‘}P)_T]$, will have this property.

§ 7. Let us now turn to the second topic to be treated in this paper, the problem of the synthesis of the series of sensation-intensities from the relations between sensations given in experience. This problem, in itself, is not one of pure logic or of pure mathematics, but its solution depends upon the solution of a purely logical and mathematical problem. In my previous article*, as I said at the beginning of this paper, I showed how from the relation of complete succession between the events in time, we can construct the series of the instants in time. The method was the following: we make the definitions:

$$(12) \qquad P_{\text{se}} = (\dot{-} P \,\dot{-}\, \breve{P}) \upharpoonright C\text{‘}P \qquad \text{Df.}$$

$$(13) \qquad \tau_P = \hat{\alpha}\{\alpha = p\text{‘}\overrightarrow{P_{\text{se}}}\text{“}\alpha\} \qquad \text{Df.}$$

$$(14) \qquad \text{inst} = \hat{Q}\hat{P}\{Q = (\breve{\epsilon} \,;\, P) \upharpoonright \tau_P\} \qquad \text{Df.}$$

If P is the relation between two events, x and y, when x is over before y begins, then P_{se} is the relation between two events which occur together at some moment; τ_P is the class of all instants of time—that is, the class of all those classes, α, such that α is made up of events in such a manner that every two events in α occur together at some moment, and if an event occurs at the same moment with every member of α, then it belongs to α; and $\text{inst}\text{‘}P$ is the relation between two members of τ_P—that is, instants—when some event at the first instant is over before some event at the second instant begins: that is, it is the relation between an instant and a succeeding instant. If $P \,|\, P_{\text{se}} \,|\, P \in P$, whether P is a temporal relation or not, $\text{inst}\text{‘}P$ will be a series. Now, let P stand for the relation, say, between any coloured object and a noticeably brighter one. Then P_{se} will be the relation between two coloured objects when the first is apparently of the same brightness as the second, for it is the relation between two members of the field of P—that is, coloured objects—when neither is in the relation P to the other. Now, it is obvious that when $xP \,|\, P_{\text{se}}y$, x must be, noticeably or unnoticeably, more bright than y, for this proposition says that x is noticeably brighter than some object which, at the brightest, is indistinguishable from y. Therefore, it is obvious that if $xP \,|\, P_{\text{se}} \,|\, Py$, x is brighter than something noticeably brighter than y, and hence is noticeably brighter than

* See *Proc. Camb. Phil. Soc.*, vol. XVII, Part 5, pp. 441—9.

* See *Proc. Camb. Phil. Soc.*, vol. XVII, Part 5, pp. 441—9.

y, and $P \mid P_{se} \mid P \Subset P$. inst‘$P$ is therefore here also a series, and nothing would seem more natural than for us to call it the series of sensation intensities.

But there are serious objections against this method of procedure, and here a genuine logical problem arises. For, although it is natural to regard a sensation-intensity as a class of sense-objects—the class of sensations ‘of a certain intensity’—we naturally consider the intensity of a given sensation as uniquely determined, and the relations between two sensations, x and y, when x is of the same intensity as y, as a transitive, symmetrical, reflexive relation. Now, in general, τ_P is not a class of mutually exclusive classes, and the relation between two terms which belong to the same member of τ_P is not transitive. The fact that a certain river was flowing during the Siege of Troy, and is flowing while I am writing this article, does not mean that I was writing this article during the Siege of Troy, yet if we take P as the relation between one event and another which completely follows it, my writing this article and the flowing of the river will both belong to some member of τ_P; the Siege of Troy and the flowing of the river will both belong to some other member of τ_P. So we have the definite mathematical problem before us: given a relation, P, fulfilling certain conditions, not sufficient to make it a series, we wish to construct from it a serial relation in such a manner that the terms of this series shall form a class of mutually exclusive classes.

I shall first give the method by which this series may be derived from the relation between x and y when x is of noticeably greater intensity than y; then I shall state a set of conditions sufficient to secure the serial character of the derived relation, and finally I shall interpret conditions and results. Perhaps the best method logically would be first to formulate all the conditions to which the original relation must be subject, and then to treat the problem as a purely formal one, but the logical gain would hardly compensate us for the loss in clarity. So I first make the following definitions:

(15) $P_s = (\breve{\overrightarrow{P_{se}}} \mid \overrightarrow{P_{se}}) \upharpoonright C\text{‘}P$ Df.

(16) $\lambda_P = \mathrm{D}\text{‘}\overrightarrow{P_s}$ Df.

(17) $\mathrm{int} = \hat{Q}\hat{P}\{Q = [\breve{\epsilon}\,;(P_{se} \mid P)] \upharpoonright \lambda_P\}$ Df.

If P is the relation between x and y when x is, say, noticeably brighter than y, then P_{se} is the relation between two things which are not distinguishable as concerns their brightness, and P_s is the relation between two things possessing brightness when each of the things which is indistinguishable from the one in brightness is also indistinguishable from the other, and *vice versa*.

It follows at once from the definition of P_s that it is transitive, symmetrical, and reflexive, whatever P may be, and hence in this respect it satisfies the requirements we have set up for the relation between two members of a sensation-intensity.

λ_P is the class of brightness-intensities, where P is the relation ‘noticeably brighter than.’ Since λ is defined as $\mathrm{D}\text{‘}\overrightarrow{P_s}$, it follows that it must always be a class of mutually exclusive classes; for suppose that two members of λ_P, say $\overrightarrow{P_s}\text{‘}x$ and $\overrightarrow{P_s}\text{‘}y$, had the term z in common. Then we would have zP_sx and zP_sy. From the definition of P_s it is symmetrical, so we get xP_sz and zP_sy, which, on account of the transitivity of P_s, gives us xP_sy, and, hence, $\overrightarrow{P_s}\text{‘}x \subset \overrightarrow{P_s}\text{‘}y$. In just the same way, we get $\overrightarrow{P_s}\text{‘}y \subset \overrightarrow{P_s}\text{‘}x$, or, finally, $\overrightarrow{P_s}\text{‘}y = \overrightarrow{P_s}\text{‘}x$. int‘$P$ is the relation between two members of λ_P when a member of one is in the relation $P_{se} \mid P$ with a member of the other. Whatever P is, int‘$P \Subset J$. For suppose that $\alpha\,(\mathrm{int}\text{‘}P)\,\alpha$. Then, since α must belong to λ_P, every term of α stands in the relation P_s to every term of α. However, from the definition of int‘P, there must be two terms of α, x and y, such that $xP_{se} \mid Py$. This may be written as

$$(\exists z) \,.\, xP_{se}z \,.\, zPy.$$

From this and the definition of P_{se}, we get $(\exists z) \,.\, xP_{se}z \,.\, z \sim P_{se}y$, or $\overrightarrow{P_{se}}\text{‘}x \neq \overrightarrow{P_{se}}\text{‘}y$, which may be written $x \mathbin{\dot{-}} P_sy$. Thus, the assumption that $\sim(\mathrm{int}\text{‘}P \Subset J)$ is self-contradictory.

A condition which will ensure the transitivity of int‘P is $P_{se} \mid P \,\epsilon\, \mathrm{trans}$. For it follows from the definitions of P_s, λ_P, and int that if $\alpha\,(\mathrm{int}\text{‘}P)^2\,\beta$, $\alpha[\{\breve{\epsilon}\,;(P_{se} \mid P \mid \breve{\overrightarrow{P_{se}}} \mid \overrightarrow{P_{se}} \mid P_{se} \mid P)\} \upharpoonright \lambda_P]\beta$.

Now,

$$(18) \quad \vdash \,.\, \breve{\overrightarrow{P_{se}}} \mid \overrightarrow{P_{se}} \mid P_{se} = \hat{x}\hat{z}\,\{(\exists \alpha, y) \,.\, \alpha = \overrightarrow{P_{se}}\text{‘}y \,.\, \alpha = \overrightarrow{P_{se}}\text{‘}x \,.\, yP_{se}z\}$$

$$= \hat{x}\hat{z}\,\{(\exists y) \,.\, \overrightarrow{P_{se}}\text{‘}y = \overrightarrow{P_{se}}\text{‘}x \,.\, z \,\epsilon\, \overrightarrow{P_{se}}\text{‘}y\} = P_{se}$$

Therefore, $P_{se} \mid P \mid \breve{\overrightarrow{P_{se}}} \mid \overrightarrow{P_{se}} \mid P_{se} \mid P$ is simply $P_{se} \mid P \mid P_{se} \mid P$. If $P_{se} \mid P$ is transitive, then we find that $\alpha\,(\mathrm{int}\text{‘}P)^2\,\beta$ implies that $\alpha[\{\breve{\epsilon}\,;(P_{se} \mid P)\} \upharpoonright \lambda_P]\beta$, which is simply $\alpha\,(\mathrm{int}\text{‘}P)\beta$. A hypothesis which will make $P_{se} \mid P$ transitive is $P \mid P_{se} \mid P \Subset P$. This is the same condition which we found to suffice for the transitivity of inst‘P.

When will int‘P be connected? Under what conditions, that is, will it be true that

$$\alpha, \beta \,\epsilon\, C\text{‘}\mathrm{int}\text{‘}P \,.\, \alpha \neq \beta \,.\, \supset_{\alpha,\beta} : \alpha\,(\mathrm{int}\text{‘}P)\,\beta \,.\, \mathrm{v} \,.\, \beta\,(\mathrm{int}\text{‘}P)\,\alpha\,?$$

This condition is manifestly implied by

$$\alpha, \beta \epsilon \lambda_P \,.\, \alpha \neq \beta \,.\, \supset_{\alpha,\beta} : \alpha (\text{int‘}P) \beta \,.\, \vee \,.\, \beta (\text{int‘}P) \alpha.$$

Since $\alpha (\text{int‘}P) \beta$ merely demands that α and β should be members of λ_P, and that *some* member of α should bear the relation $P_{se} | P$ to *some* member of β, and since if x and y are both members of α, and $\alpha \epsilon \lambda_P$, $xP_s y$, int‘P will be connected if

$$x \mathrel{\dot{-}} P_s y \,.\, \supset_{x,y} : xP_{se} | Py \,.\, \vee \,.\, yP_{se} | Px.$$

Now,

$$(19) \quad \vdash :: x \mathrel{\dot{-}} P_s y :. \supset_{x,y} :. \overrightarrow{P_{se}}‘x \neq \overrightarrow{P_{se}}‘y :.$$

$$\supset_{x,y} :. (\exists z) : zP_{se} x \,.\, z \mathrel{\dot{-}} P_{se} y \,.\, \vee \,.\, zP_{se} y \,.\, z \mathrel{\dot{-}} P_{se} x :.$$

$$\supset_{x,y} :. (\exists z) : zP_{se} x . zPy . \vee . zP_{se} x . yPz . \vee . zP_{se} y . zPx . \vee . zP_{se} y . xPz :.$$

$$\supset_{x,y} :. xP_{se} | Py \,.\, \vee \,.\, yP | P_{se} x \,.\, \vee \,.\, yP_{se} | Px \,.\, \vee \,.\, xP | P_{se} y.$$

If $P | P_{se} \subseteq P_{se} | P$, this reduces at once to the condition that we have just shown to be sufficient for the connectedness of int‘P.

§ 8. We have seen, then, that if

$$P_{se} | P \,\epsilon\, \text{trans} \quad \text{and} \quad P | P_{se} \subseteq P_{se} | P, \quad \text{int‘}P \,\epsilon\, \text{ser}.$$

Now the questions arise, what do these conditions mean when P is, for example, the relation 'noticeably brighter than'? and, are they true of such relations? The meaning of $P_{se} | P \,\epsilon\, \text{trans}$ in such a case is clear, as is also its truth; $P_{se} | P$ is the relation between two objects, x and y, when x is not merely apparently, but actually brighter than y, for $xP_{se} | Py$ says that x is only subliminally different, if at all different, in brightness from something that is supraliminally brighter than y. Now, the transitivity of the relation, 'brighter than,' is obvious: at least as obvious, at any rate, as the existence of a series of brightnesses.

The meaning of $P | P_{se} \subseteq P_{se} | P$, however, is not quite so obvious. This condition demands that if x be noticeably brighter than something indistinguishable from y, it shall be indistinguishable from something noticeably brighter than y. We may interpret this demand as saying: if x is noticeably brighter than everything noticeably less bright than y, then y is noticeably less bright than everything noticeably brighter than x. A little reflection will convince us that this proposition is probably true: moreover, it is easy to see that its truth, and the truth of analogous propositions concerning all sorts of sensory intensity, form necessary conditions for the truth of the Weber-Fechner law. For suppose that this proposition were false: we might then have, to put it crudely, x and y both just noticeably brighter than x, and u just noticeably brighter than x, but subliminally different from y. Let a be the objective strength of the stimulus produced by z; then, by Weber's law, the strength of the stimulus produced by x or y will be $a(1+c)$, where c is a constant

independent of the value of a. Since u is just noticeably brighter than x, the strength of stimulus produced by u will be

$$a(1+c)(1+c) = a(1+2c+c^2).$$

But since u is only subliminally different from y in brightness, the strength of the stimulus produced by u is less than $a(1+2c+c^2)$.

Hence, we are landed in the contradiction,

$$a(1+2c+c^2) < a(1+2c+c^2).$$

A little reflection will convince the reader that any other way of violating the condition, $P | P_{se} \subseteq P_{se} | P$, would likewise be incompatible with Weber's law.

This seems the proper place to call attention to the fact that if P be the relation of complete precedence between the events in time, $P | P_{se} \subseteq P_{se} | P$ is *false*. For suppose that at this present moment two events begin, one of which lasts five minutes and the other ten. It is clear that neither event can be simultaneous with an event which wholly precedes the other: that is, neither bears to the other the relation $P_{se} | P$. Now suppose that one minute after the shorter event is ended, some event begins. This bears the relation P_{se} to the longer event, and the shorter event bears to it the relation P. Therefore, the shorter event bears to the longer event the relation $P | P_{se} \mathrel{\dot{-}} P_{se} | P$. So we have proved nothing in this article which entitles us to say that if P is the relation of complete precedence among the instants of time, int‘P is a series. And, as a matter of fact, it is not a series. If, however, we limit the field of P to events, say, that last exactly five minutes, then $P | P_{se} \subseteq P_{se} | P$, and int‘$P$ is a series.

In case P is the relation, 'noticeably brighter than,' one can readily see that int‘P is not only a series, but the series we mean when we speak of the series of brightnesses. For, if Weber's law is true, or even if some quantitatively different law of the same general form is true, P_s is exactly the relation which holds between two things of the same brightness, for $xP_s y$ says, practically, the limina of distinguishability from x are the limina of distinguishability from y, and it can be deduced from this and Weber's law that this is true when and only when x and y produce stimuli of the same intensity, and hence it follows further from Weber's law, x and y must be of the same sensation-intensity. λ_P is therefore the class of all classes containing all the things of the same brightness as a given thing, and hence can be fittingly called the class of all brightnesses; and what could be more natural than to say that a given brightness is greater than another when and only when a thing of the first brightness is brighter than a thing of the second?

If we want to secure the compactness of int'P, it is sufficient to assume the compactness of $P_{se} \mid P$, though not, as far as I know, necessary. Similarly, $P_{se} \mid P \in \text{Ded}$ is a condition sufficient to assure the Dedekindian character of int'P.

The interest and importance of this work on sensation-intensities lies in the fact that it is often naively assumed by psychologists that the series of sensation-intensities is in some wise a datum of experience, and not a construction. As a result, they are led into the most grotesque interpretations of such numerical formulae as Weber's law. A series of sensation-intensities is often treated as if it were, in some sense or other, a series of sensation-*quantities,* without any analysis whatsoever of the basis on which this series is put into one-one correspondence with the series of 0 and the positive real numbers, in order of magnitude. It is at any rate a necessary preliminary to this exceedingly complex problem to know what the series of sensation-intensities really are, and what their relation to our experience is: without this analysis, no scientific psychophysics is possible.