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Baker

A Simplification of the Logic of Relations.

By N. Wiener, Ph.D.

Two axioms, known as the axioms of reducibility, are stated on page 174 of the first volume of the *Principia Mathematica* of Whitehead and Russell. One of these, ✱12.1, is essential to the treatment of identity, descriptions, classes, and relations: the other, ✱12.11, is involved only in the theory of relations. ✱12.11 is applied directly only in ✱20.701.702.703 and ✱21.12.13.151.3.701.702.703. It states that, given any propositional function φ of two variable individuals, there is another propositional function of two variable individuals, involving no apparent variables, and having the same truth-value as φ for the same arguments, or in symbols:

$$\vdash :(\exists f):\varphi(x,y). \equiv . f!(x,y).$$

In ✱20 and ✱21.701.702.703 all that is done with ✱12.11 is to extend it to cases where the arguments of φ and f are classes and relations: ✱12.11 is essential to the development of the calculus of relations only owing to its applications in ✱21.12.13.151.3. Here it is needed to make the transition between the definition of a binary relation and its uses. This is due to the fact that a binary relation itself is not defined, but only propositions about it, and ✱12.11 is needed to assure us that these propositions about it behave as if there were a real object with which they concern themselves. The authors of the *Principia* wish to treat a binary relation as the extension of a propositional function of two variables: that is, when they speak about the relation between x and y when $\varphi(x,y)$, they mean to speak of any propositional function which holds of those values of x and y, and only those values, of which φ holds. Now, as it leads one into vicious-circle paradoxes to speak directly of 'any propositional function which holds of those values of x and y, and those only, of which φ holds', they first define a proposition concerning the relation between x and y when $\varphi(x, y)$ as a proposition concerning a propositional function involving no apparent variables which holds of x and y when and only when $\varphi(x, y)$. Then they need to use $*12.11$ to assure us that whatever φ may be, there always is some such propositional function. Now, if we can discover a propositional function ψ of one variable so correlated with φ that its extension is determined uniquely by that of φ, and vice versa — if, to put it in symbols, when ψ' bears to φ' the same relation that ψ bears to φ, $\vdash :: \varphi'(x, y) . \equiv_{x,y} : \varphi(x, y) : \equiv : \psi'\alpha . \equiv_{\alpha} \psi\alpha$ —, we can entirely avoid the use of $*12.11$, and interpret any proposition concerning the extension of φ as ~~as concern~~ if it concerned the extension of ψ; for the existence of the extension of a propositional function of one variable is assured to us by $*12.1$, quite as that of one of two variables is by $*12.11$. Now, such a ψ is the propositional function

$$(\exists x, y) . \varphi(x, y) . \alpha = \iota'(\iota'\iota'x \cup \iota'\Lambda) \cup \iota'\iota'\iota'y .$$

For it is clear that for each ordered pair of values of x and y there is one and only one value of α, and vice versa. On the one hand, as $\iota'(\iota'\iota'x \cup \iota'\Lambda)$ is determined uniquely by x, and $\iota'\iota'\iota'y$ is determined uniquely by y, ~~so that~~ $\iota'(\iota'\iota'x \cup \iota'\Lambda) \cup \iota'\iota'\iota'y$ is determined uniquely by x and y. On the other hand, if $\iota'(\iota'\iota'x \cup \iota'\Lambda) \cup \iota'\iota'\iota'y = \iota'(\iota'\iota'z \cup \iota'\Lambda) \cup \iota'\iota'\iota'w$, either $\cancel{\iota'\iota'\iota'y.}$ $\iota'\iota'y = \iota'\iota'z \cup \iota'\Lambda$ or $\iota'\iota'y = \iota'\iota'w$. The former supposition is clearly impossible, for, as $\iota'z \neq \Lambda$, $\iota'\iota'z \cup \iota'\Lambda$ is not a unit class. From the latter alternative we conclude immediately that $y = w$. Similarly, $x = z$.

Therefore, when x and y are of the same type, we can make the following definition:

$$\hat{x}\hat{y}\varphi(x,y) = \hat{\alpha}\{(\exists x,y).\varphi(x,y).\alpha = \iota'(\iota'\iota'x \cup \iota'\Lambda) \cup \iota'\iota'\iota'y\} \quad Df.$$[1]

[1] This may seem circular, as ι is a relation, defined in the <u>Principia</u> as $\vec{I}$, but it really is not circular, for $\iota'x$ may be defined directly as the class, $\hat{y}(y = x)$.

It will be seen that in this definition of $\hat{x}\hat{y}\varphi(x,y)$ it is essential that the x and the y should be of the same type, for if they are not $\iota'(\iota'\iota'x \cup \iota'\Lambda)$ and $\iota'\iota'\iota'y$ will not be, and $\iota'(\iota'\iota'x \cup \iota'\Lambda) \cup \iota'\iota'\iota'y$ will be meaningless. To overcome this limitation, and secure typical ambiguity for domain and converse domain of $\hat{x}\hat{y}\varphi(x,y)$ separately, we make the following definitions.

$$\hat{\alpha}\hat{y}\varphi(\alpha,y) = \hat{\kappa}\{(\exists\alpha,y).\varphi(\alpha,y).\kappa = \iota'(\iota'\iota'\alpha \cup \iota'\Lambda) \cup \iota'\iota'(\iota'\iota'y \cup \iota'\Lambda)\} \quad Df.$$

$$\hat{\kappa}\hat{y}\varphi(\kappa,y) = \hat{\mu}\{(\exists\kappa,y).\varphi(\kappa,y).\mu = \iota'(\iota'\iota'\kappa \cup \iota'\Lambda) \cup \iota'\iota'[\iota'(\iota'\iota'y \cup \iota'\Lambda) \cup \iota'\Lambda]\} \quad Df.$$

etc.

$$\hat{x}\hat{\beta}\varphi(x,\beta) = \hat{\kappa}\{(\exists x,\beta).\varphi(x,\beta).\kappa = \iota'[\iota'(\iota'\iota'x \cup \iota'\Lambda) \cup \iota'\Lambda] \cup \iota'\iota'\iota'\beta\} \quad Df.$$

$$\hat{x}\hat{\lambda}\varphi(x,\lambda) = \hat{\mu}\{(\exists x,\lambda).\varphi(x,\lambda).\mu = \iota'\{\iota'[\iota'(\iota'\iota'x \cup \iota'\Lambda) \cup \iota'\Lambda] \cup \iota'\Lambda\} \cup \iota'\iota'\iota'\lambda\} \quad Df.$$

etc.

Though these definitions may seem to conflict with oneanother, they really do not conflict, for where one of them is applicable, the others are meaningless, ~~for~~ since they define relations between objects of different types. Moreover, it is easy to see that our definitions are so chosen that $\vdash : \hat{\mu}\hat{\nu}\varphi(\mu,\nu) = \hat{\varpi}\hat{\rho}\psi(\varpi,\rho) . \supset . t'D'\hat{\mu}\hat{\nu}\varphi(\mu,\nu) = t'D'\hat{\varpi}\hat{\rho}\psi(\varpi,\rho) . t'\text{D}'\hat{\mu}\hat{\nu}\varphi(\mu,\nu) = t'\text{D}'\hat{\varpi}\hat{\rho}\psi(\varpi,\rho)$. This is important, as we might easily have given a definition of a relation which would permit it to have several domains or converse domains of different types. This is why we did not define $\hat{\alpha}\hat{y}\varphi(\alpha,y)$ simply as

$$\hat{\kappa}\{(\exists\alpha,y).\varphi(\alpha,y).\kappa = \iota'(\iota'\iota'\alpha \cup \iota'\Lambda) \cup \iota'\iota'\iota'y\},$$

for this would also represent $\hat{\alpha}\hat{\beta}\{(\exists y).\varphi(\alpha,y).\beta = \iota'y\}$

It will be seen that what we have done is practically ~~is~~

to revert to Schröder's treatment of a relation as a class of ordered couples. The complicated apparatus of ι's and Λ's of which we have made use is simply and solely devised for the purpose of constructing a class which shall depend ~~solely on~~ only on an ordered pair of values of x and y, and which shall ~~depend on~~ correspond to only one such pair. The particular method selected of doing this is largely a matter of choice: for example, I might have substituted V, or any other constant ~~typically ambiguous class~~ class not a unit class, and existing in every type of classes, in every place I have written Λ.

Our changed definition of $\hat{x}\hat{y}\,\varphi(x,y)$ renders it ~~necess~~ necessary to give new definitions of several other symbols fundamental to the theory of relations. I give the following table of such definitions.

$$\mathrm{Rel} = \hat{\kappa}\{\kappa \subset \hat{x}\hat{y}(x=x\,.\,y=y)\} \quad \mathrm{Df}.$$

$$xRy\,.=.\,\hat{z}\hat{w}\{z=x\,.\,w=y\} \subset R\,.\,R\,\varepsilon\,\mathrm{Rel}\quad \mathrm{Df}.$$

$$\varphi R\,.=.\,(\exists\alpha)\,.\,\alpha=R\,.\,\alpha\,\varepsilon\,\mathrm{Rel}\,.\,\varphi\alpha \quad \mathrm{Df}.$$ "

"We shall understand in this way any propositional functions containing capital letters in the positions proper to their arguments. Thus $\sim\varphi R$ shall be understood as

$$(\exists\alpha)\,.\,\alpha=R\,.\,\alpha\,\varepsilon\,\mathrm{Rel}\,.\sim\varphi\alpha,$$

and not as

$$(\alpha) :.\ \alpha=R\,.\supset:\alpha \sim\varepsilon\,\mathrm{Rel}\,.\vee.\sim\varphi\alpha.$$

We make this definition as well as the two following ones because ~~the significance of~~ a propositional function of ~~x~~ a class of the sort we have defined as a relation may significantly take as arguments classes of the same type which are not relations, and we wish to ~~prevent~~ define propositional functions of relations in such a manner as to ~~exclude~~

~~the~~ require that their arguments be relations

$$(R) . \varphi R :=: \dot{\alpha} \varepsilon \mathrm{Rel} . \supset_\alpha . \varphi\alpha \qquad Df.$$

$$(\exists R) . \varphi R : = . (\exists \alpha) . \alpha \varepsilon \mathrm{Rel} . \varphi \alpha \qquad Df.$$

The first two and the last two of these definitions replace *21.01.02 and *21.07.071 respectively. From these definitions and the laws of the calculus of classes it is an exceedingly simple matter to deduce any of the propositions of ~~*12.1~~ *21 which are not explicitly used for the purpose of deriving the properties of relations from the particular definition of relations given ~~the~~ there, and from this it is easy to prove that the formal properties of the ~~objec~~ objects I call relations are ~~ess~~ essentially the same as those of the relations of the Principia.

But it is obvious that since they are also classes, our relations will possess some formal properties not possessed by those of the Principia. I give in conclusion a table of some of the more interesting of these.

$$\vdash . R \cup S = R \mathbin{\dot{\cup}} S$$

$$\vdash . R \cap S = R \mathbin{\dot{\cap}} S$$

$$\vdash : R \subset S . \equiv . R \mathbin{\dot{\subset}} S$$

$$\vdash . R - S . \equiv . R \mathbin{\dot{-}} S$$

$$\vdash . \dot{V} \subset V$$

$$\vdash . \Lambda = \dot{\Lambda}$$

$$\vdash . \mathrm{Rel} \subset \mathrm{Cls}$$

$$\vdash : R\, p\kappa . \equiv . R\, \dot{p}\kappa$$

$$\vdash : R\, s\kappa . \equiv . R\, \dot{s}\kappa$$

$$\vdash . \alpha + \beta \ \mathrm{sm}\ s'\alpha \uparrow \beta$$

$$\vdash . \alpha \times \beta \ \mathrm{sm}\ \alpha \uparrow \beta$$

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A Simplification of the Logic of Relations.

By N. WIENER, Ph.D. (Communicated by Mr G. H. Hardy.)

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Two axioms, known as the axioms of reducibility, are stated on page 174 of the first volume of the *Principia Mathematica* of Whitehead and Russell. One of these, ∗12·1, is essential to the treatment of identity, descriptions, classes, and relations: the other, ∗12·11, is involved only in the theory of relations. ∗12·11 is applied directly only in

∗20·701·702·703 and ∗21·12·13·151·3·701·702·703.

It states that, given any propositional function ϕ of two variable individuals, there is another propositional function of two variable individuals, involving no apparent variables, and having the same truth-value as ϕ for the same arguments, or in symbols:

$$\vdash : (\exists f) : \phi(x, y) \,.\equiv.\, f!(x, y).$$

In ∗20 and ∗21·701·702·703 all that is done with ∗12·11 is to extend it to cases where the arguments of ϕ and f are classes and relations: ∗12·11 is essential to the development of the calculus of relations only owing to its application in ∗21·12·13·151·3. Here it is needed to make the transition between the definition of a binary relation and its uses. This is due to the fact that a binary relation itself is not defined, but only propositions about it, and ∗12·11 is needed to assure us that these propositions about it behave as if there were a real object with which they concern themselves. The authors of the *Principia* wish to treat a binary relation as the extension of a propositional function of two variables: that is, when they speak about the relation between x and y when $\phi(x, y)$, they mean to speak of any propositional function which holds of those values of x and y, and only those values, of which ϕ holds. Now, as it leads one into vicious-circle paradoxes to speak directly of "any propositional function which holds of those values of x and y, and those only, of which ϕ holds," they first define a proposition concerning the relation between x and y when $\phi(x, y)$ as a proposition concerning *a propositional function involving no apparent variables* which holds of x and y when and only when $\phi(x, y)$. Then they need to use ∗12·11 to assure us that, whatever ϕ may be, there always is some such propositional function. Now, if we can discover a propositional function ψ of one variable so correlated with ϕ that its extension is determined uniquely by that of ϕ, and vice versa—if, to put it in symbols, when ψ' bears to ϕ' the same relation that ψ bears to ϕ, $\vdash :. \phi'(x, y) . \equiv_{x,y} . \phi(x, y) : \equiv : \psi'\alpha . \equiv_\alpha . \psi\alpha$—, we can entirely avoid the use of $*12{\cdot}11$, and interpret any proposition concerning the extension of ϕ as if it concerned the extension of ψ; for the existence of the extension of a propositional function of one variable is assured to us by $*12{\cdot}1$, quite as that of one of two variables is by $*12{\cdot}11$. Now, is such a ψ the propositional function

$$(\exists x, y) . \phi(x, y) . \alpha = \iota\text{‘}(\iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\Lambda) \cup \iota\text{‘}\iota\text{‘}\iota\text{‘}y.$$

For it is clear that for each ordered pair of values of x and y there is one and only one value of α, and vice versa. On the one hand, as $\iota\text{‘}(\iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\Lambda)$ is determined uniquely by x, and $\iota\text{‘}\iota\text{‘}\iota\text{‘}y$ is determined uniquely by y, $\iota\text{‘}(\iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\Lambda) \cup \iota\text{‘}\iota\text{‘}\iota\text{‘}y$ is determined uniquely by x and y. On the other hand, if

$$\iota\text{‘}(\iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\Lambda) \cup \iota\text{‘}\iota\text{‘}\iota\text{‘}y = \iota\text{‘}(\iota\text{‘}\iota\text{‘}z \cup \iota\text{‘}\Lambda) \cup \iota\text{‘}\iota\text{‘}\iota\text{‘}w,$$

either $\iota\text{‘}\iota\text{‘}y = \iota\text{‘}\iota\text{‘}z \cup \iota\text{‘}\Lambda$ or $\iota\text{‘}\iota\text{‘}y = \iota\text{‘}\iota\text{‘}w$. The former supposition is clearly impossible, for, as $\iota\text{‘}z \neq \Lambda$, $\iota\text{‘}\iota\text{‘}z \cup \iota\text{‘}\Lambda$ is not a unit class. From the latter alternative we conclude immediately that $y = w$. Similarly, $x = z$.

Therefore, when x and y are of the same type, we can make the following definition:

$$\hat{x}\hat{y}\phi(x, y) = \hat{\alpha}\{(\exists x, y) . \phi(x, y) . \alpha = \iota\text{‘}(\iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\Lambda) \cup \iota\text{‘}\iota\text{‘}\iota\text{‘}y\} \quad \text{Df.*}$$

It will be seen that in this definition of $\hat{x}\hat{y}\phi(x, y)$ it is essential that the x and the y should be of the same type, for if they are not $\iota\text{‘}(\iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\Lambda)$ and $\iota\text{‘}\iota\text{‘}\iota\text{‘}y$ will not be, and $\iota\text{‘}(\iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\Lambda) \cup \iota\text{‘}\iota\text{‘}\iota\text{‘}y$ will be meaningless. To overcome this limitation, and secure typical ambiguity for domain and converse domain of $\hat{x}\hat{y}\phi(x, y)$ separately, we make the following definitions:

$$\hat{\alpha}\hat{y}\phi(\alpha, y) = \hat{\kappa}\{(\exists \alpha, y) . \phi(\alpha, y) . \kappa = \iota\text{‘}(\iota\text{‘}\iota\text{‘}\alpha \cup \iota\text{‘}\Lambda) \cup \iota\text{‘}\iota\text{‘}(\iota\text{‘}\iota\text{‘}y \cup \iota\text{‘}\Lambda)\} \quad \text{Df.}$$

$$\hat{\kappa}\hat{y}\phi(\kappa, y) = \hat{\mu}\{(\exists \kappa, y) . \phi(\kappa, y) . \mu = \iota\text{‘}(\iota\text{‘}\iota\text{‘}\kappa \cup \iota\text{‘}\Lambda) \cup \iota\text{‘}\iota\text{‘}[\iota\text{‘}(\iota\text{‘}\iota\text{‘}y \cup \iota\text{‘}\Lambda) \cup \iota\text{‘}\Lambda]\} \quad \text{Df.}$$

etc.

$$\hat{x}\hat{\beta}\phi(x, \beta) = \hat{\kappa}\{(\exists x, \beta) . \phi(x, \beta) . \kappa = \iota\text{‘}[\iota\text{‘}(\iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\Lambda) \cup \iota\text{‘}\Lambda] \cup \iota\text{‘}\iota\text{‘}\iota\text{‘}\beta\} \quad \text{Df.}$$

$$\hat{x}\hat{\lambda}\phi(x, \lambda) = \hat{\mu}\{(\exists x, \lambda) . \mu = \iota\text{‘}\{\iota\text{‘}[\iota\text{‘}(\iota\text{‘}\iota\text{‘}x \cup \iota\text{‘}\Lambda) \cup \iota\text{‘}\Lambda] \cup \iota\text{‘}\Lambda\} \cup \iota\text{‘}\iota\text{‘}\iota\text{‘}\lambda\} \quad \text{Df.}$$

etc.

* This may seem circular as ι is a relation, defined in the *Principia* as $\overrightarrow{I}$, but it really is not circular, for $\iota\text{‘}x$ may be defined directly as the class, $\hat{y}(y = x)$.

Though these definitions may seem to conflict with one another, they really do not conflict, for where one of them is applicable, the others are meaningless, since they define relations between objects of different types. Moreover, it is easy to see that our definitions are so chosen that

$$\vdash : \hat{\mu}\hat{\nu}\phi(\mu, \nu) = \hat{\varpi}\hat{\rho}\psi(\varpi, \rho) \,.\supset .\, t`\mathrm{D}`\hat{\mu}\hat{\nu}\phi(\mu, \nu)$$
$$= t`\mathrm{D}`\hat{\varpi}\hat{\rho}\psi(\varpi, \rho) \,.\, t`\mathrm{Cl}`\hat{\mu}\hat{\nu}\phi(\mu, \nu) = t`\mathrm{Cl}`\hat{\varpi}\hat{\rho}\psi(\varpi, \rho).$$

This is important, as we might easily have defined relations so that they might have several domains or converse domains of different types. This is why we did not define $\hat{\alpha}\hat{y}\phi(\alpha, y)$ simply as

$$\hat{\kappa}\{(\exists \alpha, y) \,.\, \phi(\alpha, y) \,.\, \kappa = \iota`(\iota`\iota`\alpha \cup \iota`\Lambda) \cup \iota`\iota`\iota`\iota`y\},$$

for this would also represent

$$\hat{\alpha}\hat{\beta}\{(\exists y) \,.\, \phi(\alpha, y) \,.\, \beta = \iota`y\}.$$

It will be seen that what we have done is practically to revert to Schröder's treatment of a relation as a class of ordered couples. The complicated apparatus of ι's and Λ's of which we have made use is simply and solely devised for the purpose of constructing a class which shall depend only on an ordered pair of values of x and y, and which shall correspond to only one such pair. The particular method selected of doing this is largely a matter of choice: for example, I might have substituted V, or any other constant class not a unit class, and existing in every type of classes, in every place I have written Λ.

Our changed definition of $\hat{x}\hat{y}\phi(x, y)$ renders it necessary to give new definitions of several other symbols fundamental to the theory of relations. I give the following table of such definitions:

$\mathrm{Rel} = \hat{\kappa}\{\kappa \subset \hat{x}\hat{y}(x = x \,.\, y = y)\}$	Df.
$xRy \,.=.\, \hat{z}\hat{w}\{z = x \,.\, w = y\} \subset R \,.\, R \in \mathrm{Rel}$	Df.
$\phi R \,.=.\, (\exists \alpha) \,.\, \alpha = R \,.\, \alpha \in \mathrm{Rel} \,.\, \phi\alpha$	Df.*
$(R) \,.\, \phi R :=: \alpha \in \mathrm{Rel} \,.\supset_\alpha .\, \phi\alpha$	Df.
$(\exists R) \,.\, \phi R :=.\, (\exists \alpha) \,.\, \alpha \in \mathrm{Rel} \,.\, \phi\alpha$	Df.

The first two and the last two of these definitions replace ∗21·03·02 and ∗21·07·071 respectively. From these definitions and the laws

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$$(\exists \alpha) \,.\, \alpha = R \,.\, \alpha \in \mathrm{Rel} \,.\, \sim\phi\alpha,$$

and not as

$$\alpha = R \,.\, \alpha \in \mathrm{Rel} \,.\supset_\alpha .\, \sim\phi\alpha.$$

We make this definition as well as the two following ones because a propositional function of a class of the sort we have defined as a relation may significantly take as arguments classes of the same type which are not relations, and we wish to define propositional functions of relations in such a manner as to require that their arguments be relations.

of the calculus of classes it is an exceedingly simple matter to deduce any of the propositions of $*21$ which are not explicitly used for the purpose of deriving the properties of relations from the particular definition of relations given there, and from this it is easy to prove that the formal properties of the objects I call relations are essentially the same as those of the relations of the *Principia*.

But it is obvious that since they are also classes, our relations will possess some formal properties not possessed by those of the *Principia*. I give in conclusion a table of some of the more interesting of these:

$$\vdash . R \cup S = R \mathbin{\dot\cup} S$$
$$\vdash . R \cap S = R \mathbin{\dot\cap} S$$
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$$\vdash . R - S . \equiv . R \mathbin{\dot-} S$$
$$\vdash . \dot{\mathrm{V}} \subset \mathrm{V}$$
$$\vdash . \Lambda = \dot\Lambda$$
$$\vdash . \mathrm{Rel} \subset \mathrm{Cls}$$
$$\vdash : Rp\kappa . \equiv . R\dot{p}\kappa$$
$$\vdash : Rs\kappa . \equiv . R\dot{s}\kappa$$
$$\vdash . \alpha + \beta \ \mathrm{sm} \ s`\alpha \uparrow \beta$$
$$\vdash . \alpha \times \beta \ \mathrm{sm} \ \alpha \uparrow \beta$$