A Contribution to the Theory of Relative Position.[1]

By Norbert Wiener, Ph.D

One of the most interesting departments of the new Mathematical Logic is the theory of [redacted] relations. Perhaps the most thoroughly investigated of all relations are the so-called serial relations: the relations, that is, which are contained in diversity, transitive, and connected. Cantor, [redacted] Dedekind, Frege, Schröder, Burali-Forti, Whitehead and Russell, and Huntington, among others, have investigated the properties of series almost exhaustively, from many different standpoints. On the other hand, there is a class of relations intimately related to series which has been very little studied. An example of this class of relations is the relation between one event and another when the first wholly precedes the second. This relation [redacted] differs from a serial relation proper in that it is not connected: that is [redacted] it is not true that of two distinct events, one must precede [redacted] every part of the other. Two distinct events may overlap or completely coinside in the time of their occurrence. In general, the relation between one stretch [redacted] of terms in a series and another which follows but does not overlap it will be of this sort

Since a relation of this sort is a relation between the stretches of terms in a series, it might seem that we should deduce the properties [redacted] of these relations from those of series proper. However, we shall find that it is possible

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[1] The subject of this paper was suggested to me by Mr. Bertrand Russell, and the paper itself is the result of an attempt to simplify and generalize certain notions used by him in his treatment of the relation between the 'series' of events in time and the series of instants.

to reverse this procedure, and develop the properties of a series from those of the relation between [redacted] a stretch of the series and another which follows, but does not overlap it. Since the latter sort of relation is more general than serial relation, which it includes as a special case, and since its properties have not received from most mathematical logicians the attention which they deserve, I shall derive the properties of series from those of relations between stretches, and not vice versa.

Those, such as Schröder[1], who have treated of the relations [illegible] those between events in time have generally thought it necessary to make use of two fundamental relations — partial or complete simultaneity, and complete precedence [redacted]. Now, in order to secure the proper formal properties for the series of events, [redacted] the following propositions or their equivalent must be assumed to be true concerning simultaneity and precedence [redacted] (which we shall symbolize by S and P, respectively).

$$\vdash . S \dot{\cap} P = \Lambda$$

$$\vdash . S \dot{\cup} P \dot{\cup} \breve{P} = C`S \uparrow C`S$$

$$\vdash . S \subseteq \breve{S}$$

$$\vdash . C`S = C`P$$

From these it can be deduced without the slightest difficulty that

$$\vdash . S = (\dot{-}P \dot{-} \breve{P}) \upharpoonright C`P$$

Since this is the case, we can dispense entirely with S as a fundamental relation, and *define* it as $(\dot{-}P \dot{-} \breve{P}) \upharpoonright C`P$. That is, one [redacted] entity is defined [redacted] to be simultaneous with another when [redacted] each entity precedes or follows something, and neither precedes the other. Let us [redacted] make

¶ the following definitions:

$*0.01$[2] $\quad P_{cm} = (\dot{-}P \dot{-} \breve{P}) \upharpoonright C`P \quad$ Df

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1)

2) I [redacted] follow the method of the *Principia Mathematica* in numbering my definitions and propositions

define $\alpha \downarrow y$ as $\iota`(\alpha \cup \iota`y)$ and $\kappa \downarrow y$ as $\iota`(\kappa \cup \iota`\iota`y)$. It will be noted, however, that if we took these latter definitions of $\alpha \downarrow y$ and $\kappa \downarrow y$, $\kappa \downarrow y$ and $\kappa \downarrow \iota`y$ would be ~~indistinguishable, which in~~ identical, and this might be an inconvenience. In our definitions of $\downarrow$, it will readily be seen that no such confusion can happen: our $\downarrow$ has just the same degree of typical ambiguity as that of the Principia, and no more.

It will be noted that we give a definition, not of the uses of the symbol $\hat{x}\hat{y}\{\phi(x,y)\}$, but of the symbol itself, as far as this can be done in terms of classes, ~~so~~ that we avoid the use of *12.11. It ~~not~~ might seem that the use of the relation ι, which is defined as $\overrightarrow{I}$, ~~not~~ makes our definition circular, but this is not so, for we can define $\iota`x$ as the class $\hat{y}(y = x)$. If, then, it can be shown that the objects we have just defined possess all the formal properties of their namesakes in the Principia, we ~~will~~ shall have shown how to avoid the use of *12.11.

All the propositions in the Principia which concern themselves with ~~relatives~~ relations are to be deduced from those in *21. Of these, *21.15.18.2. 21.22.23.24 ~~.3.31.32.33.4.41.42.43~~ and 3—64 inclusive embrace all the information of which use is later made. It is an easy matter to see that ~~the~~ objects which we treat as ~~relatives~~ relations satisfy all these propositions as they stand, save that in some, as the ~~relatives~~ relations of a given type are only some among the classes of that type, ~~so~~ that the conditions of significance of a proposition ~~about relatives do not~~ ~~in themselves~~ ~~demand that~~ do not in themselves determine whether it shall deal with relations, ~~the~~ such a hypothesis as $R \varepsilon \mathrm{Rel}$ or $S \varepsilon \mathrm{Rel}$ has to be added. To show how ~~some of~~ these propositions are verified under our definition of relations, I ~~will~~ shall prove *21.3, which asserts that $\vdash : x\{\hat{x}\hat{y}\psi(x,y)\}y \,.\equiv .\, \psi(x,y)$

By our definition of xRy, $\vdash : x\{\hat{x}\hat{y}\psi(x,y)\}y \equiv .\ \hat{x}\hat{y}\psi(x,y) \varepsilon \mathrm{Rel} \,.\, x\downarrow y \subset \hat{x}\hat{y}\psi(x,y)$.

This becomes:

A Contribution to the Theory of Relative Position[1]

By Norbert Wiener, Ph.D.

One of the most interesting departments of the new mathematical logic is the theory of relations. [redacted] The so-called serial relations have been ~~investigated~~ studied more thoroughly than those of any other form: Cantor, Dedekind, Frege, Schröder, Burali-Forti, Huntington, Whitehead, and Russell have investigated the more fundamental properties of series almost exhaustively. On the other hand, a class of relations closely allied to series — the class of [redacted] such relations as that which one event bears to another when it completely precedes it, or that which one segment of a line bears to [redacted] a non-overlapping segment of the same line to the left of it — [redacted] has received very scant attention from the mathematical logicians.[2] Relations of this sort differ in general from serial relations in not being connected: that is, of two distinct events, it is not necessary that one should wholly precede the other, and of two distinct segments of a line, it is not necessary that one should not overlap the other and should be to the left of it. Nevertheless, each such relation has a series in some manner correlated to it: the relation of succession between events is evidently in close connection with the series of the instants of time, and the [redacted] the relation of being-to-the-~~right~~-left-of-and-not-overlapping between the segments of a line is intimately related to

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1) The subject of this paper was suggested to me by Mr. Bertrand Russell, and the paper itself is the result of an attempt to simplify and generalize certain notions used by him in his treatment of the relation between the 'series' of events in time and the series of instants.

2)

the serial relation of to-the-left-of which holds between the points of the line.

In studying the connection between these two sorts of relations, two courses are open to us: we may construct the relation between events or [redacted] segments as a [redacted] function of that between instants or points, respectively, or vice versa[redacted]. It is largely a matter of indifference which method we choose, but since the properties of series have been studied very thoroughly, whereas relations of the form of those between events and segments have not, and since the latter have very interesting properties apart from their connection with series, we shall [redacted] consider the serial relations as derived and the relations between events and segments as primary.[1] The question then arises, given events or segments and relations between them, how shall we define an instant [redacted] or a point? It is obvious that we must define it by the class of events or segments which occur at it or include it, respectively: that is, we must define it as that class of events or instants. But not every class of events or segments is a point or instant. What characteristic is possessed, then, by all those classes of events or instants which define moments or points, and by no others? It is clear (1) that every event or segment at a given moment or point overlaps every other event or segment at that moment or point, and (2) that every event or segment which overlaps every event or segment at that moment or point is a member of that moment or point. Writing α for a moment or point, and S for the relation of

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[1] Another motive for doing so is that, according to Mr. Russell, the temporal relations we actually experience subsist between events and not instants, and the instants are mere constructions.

overlapping, we may write (1) and (2) together as

$$\alpha = p`\vec{s}``\alpha,$$ [1]

and we may define the class, γ of all instants or points by

$$\gamma = \hat{\alpha}\{\alpha = p`\vec{s}``\alpha\} \quad Df.$$

If we do this, however, we shall need to take two different relations — simultaneity and precedence, or overlapping and being-to-the-left-of — as [redacted] fundamental in the theory of relations between events or [redacted] segments, [redacted] for it is obvious that if we know what events [redacted] are simultaneous with what events, or what segments of a line overlap what segments, we may nevertheless be ignorant of the order of precedence between the different sets of simultaneous events or overlapping segments. This is, as a matter of fact, what most authors on the subject have actually done. But if we start with complete precedence or being-entirely-to-the-left-of as our fundamental notion, we can easily define simultaneity and overlapping in terms of it. For, writing S for simultaneity or overlapping and P for precedence or being-entirely-to-the-left-of, respectively, the following propositions are obviously true:

$$\vdash . \; S \cap P = \Lambda$$

$$\vdash . \; S \cup P \cup \breve{P} = C`S \uparrow C`S$$

$$\vdash . \; S \subset \breve{S}$$

$$\vdash . \; C`S = C`P$$

From these it is an easy matter to deduce that

$$\vdash . \; S = (-P \dot{-} \breve{P}) \upharpoonright C`P.$$

This obviates the necessity to regard simultaneity or overlapping

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1) This definition is due to Mr. Russell

as a fundamental notion: an event [redacted] is simultaneous with another when [redacted] each entirely precedes or follows some other event, and when neither entirely precedes the other. We may then write the definition of [redacted] the class of all instants or points as:

$$\tau = \hat{\alpha}\{\alpha = p`\overrightarrow{\{(\dot{-}P\dot{-}\breve{P})\upharpoonright C`P\}}``\alpha\} \quad Df.$$

What, then, is the relation of precedence of instants or points in terms of that of events or segments? An instant, it is easy to see[1], comes before another when and only when some event belonging to it comes entirely before some event belonging to the other. Similarly, a [redacted] point on a line comes before another only when some segment on which it lies comes entirely before some [redacted] segment on which the other lies. In symbols, α precedes β when and only when:

$$\alpha, \beta \varepsilon \tau . (\exists x, y) . x \varepsilon \alpha . y \varepsilon \beta . x P y.$$

Writing A for the relation between one moment and a following one, we have

$$\vdash A = (\breve{\varepsilon} | P) \upharpoonright \tau \quad Df.$$

The question now arises, [redacted] to what conditions [redacted] must we subject P if A is to be a series? To simplify our calculations, and to express S, τ, and A as functions of P, let us make the following definitions:

$*.01$[2] $\vdash . \; P_{ent} = (\dot{-}P\dot{-}\breve{P}) \upharpoonright C`P \qquad Df.$

$*.02 \vdash . \; \tau_P = \hat{\alpha}\{\alpha = p`\overrightarrow{P_{ent}}``\alpha\} \qquad Df.$

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[1] We suppose that at no two instants [redacted] is the totality of events in the universe the same. This is a necessary hypothesis if we are to regard the relation of succession between instants as a function of that between events.

[2] I follow the method of the *Principia Mathematica* in numbering my definitions and propositions.

$\vdash . [*35.21.48.481, *34.21, *36.01] . \supset \vdash . R|(\dot{-}R \dot{-} \breve{R})|R = R|[(\dot{-}R \dot{-} \breve{R}) \upharpoonright C'R]|R$

$[*21.42, *36.13, *23.33.35.05] \quad = R|\hat{x}\hat{y}(\sim xRy . \sim yRx . x, y \in C'R)|R . \quad (1)$

$\vdash . [(1), *202.103, \text{Transp}] . \supset \vdash : R \in \text{connex} . \supset . R|(\dot{-}R \dot{-} \breve{R})|R = R|\hat{x}\hat{y}(x = y)|R$

$[*50.1] \quad = R|I|R$

$[*50.4] \quad = R|R \quad (2)$

$\vdash . [(2), *204.1] . \supset \vdash : R \in \text{ser} . \supset . R|(\dot{-}R \dot{-} \breve{R})R \subset R . R \in Rl'J .$

$[*0.01] \quad \supset . R \in \text{cs} \quad (3)$

$\vdash . (3) . \supset \vdash . \text{ser} \subset \text{cs}$

Moreover, in the specific instance of time it has been shown [redacted] by Mr. Russell that it is advantageous for methodological purposes to regard the instants of time [redacted] as constructions from its events. This is an additional reason for starting from the members of cs and [redacted] forming certain members of ser as functions of them. Let us, then, agree that an instant, for example, is to be regarded as a class of events, and a point on a line, as a [redacted] class of segments of the line, etc. in the rest of this paper. The question then arises, when is a class of events an instant, and when is a class of segments a point, etc.? It is obvious on inspection that not every class of events is an instant: all the [redacted] events which make up a given instant must be simultaneous with another, and all the events which are simultaneous with every member of [redacted] an instant must belong to that instant. Moreover, Λ must not be an instant. It also can readily be seen that any class satisfying these conditions will be an instant. That is, if P is the relation of an event to an event which completely follows it, it is a simple matter to show that the class of all instants is[2]

$$\hat{\alpha}\{\alpha = p'\overrightarrow{P_{sl}}``\alpha\}.$$

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[1] The references are to theorems in the _Principia Mathematica_

[2] This definition is due to Mr. Russell.

[*61.12] $\supset\vdash.\ \mathrm{inst}\text{‘}P\,\epsilon\,\mathrm{Rl}\text{‘}J.$

[*190.15, *71.163.28] $\supset\vdash.\ \mathrm{inst}\text{“}\hat{R}\{R\,|\,R_{se}\,|\,R\subset R\}\subset\mathrm{Rl}\text{‘}J$ (3)

$\vdash.$ (1) *11.03.54 $\supset\vdash:\alpha\,\mathrm{inst}\text{‘}P\beta.\beta\,\mathrm{inst}\text{‘}P\gamma.\supset.\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha.\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta.\gamma=p\text{‘}\overrightarrow{P_{se}}\text{“}\gamma.(\exists x,y,u,v).x\,\epsilon\,\alpha.y,u\,\epsilon\,\beta.v\,\epsilon\,\gamma.xPy.uPv.$

[*11.45] $\supset.\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha.\gamma=p\text{‘}\overrightarrow{P_{se}}\text{“}\gamma.(\exists x,y,u,v).x\,\epsilon\,\alpha.y,u\,\epsilon\,\beta.\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta.xPy.uPv.v\,\epsilon\,\gamma.$

[*13.101, *3.31] $\supset.\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha.\gamma=p\text{‘}\overrightarrow{P_{se}}\text{“}\gamma.(\exists x,y,u,v).x\,\epsilon\,\alpha.y\,\epsilon\,p\text{‘}\overrightarrow{P_{se}}\text{“}\beta.u\,\epsilon\,\beta.xPy.uPv.v\,\epsilon\,\gamma.$

[*40.14, *37.62, *32.12] $\supset.\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha.\gamma=p\text{‘}\overrightarrow{P_{se}}\text{“}\gamma.(\exists x,y,u,v).x\,\epsilon\,\alpha.y\,\epsilon\,\overrightarrow{P_{se}}\text{‘}u.xPy.uPv.v\,\epsilon\,\gamma$

[*32.18, *34.1, *11.23.54] $\supset.\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha.\gamma=p\text{‘}\overrightarrow{P_{se}}\text{“}\gamma.(\exists x,v).x\,\epsilon\,\alpha.v\,\epsilon\,\gamma.x\,P\,|\,P_{se}\,|\,P\,v$ (4)

$\vdash.$(4). *10.26.28. *23.1. $\supset\vdash:P\,|\,P_{se}\,|\,P\subset P.\mathrm{Hyp}4.\supset:.\alpha\,\mathrm{inst}\text{‘}P\beta:$

[*3.3] $\supset\vdash:.P\,|\,P_{se}\,|\,P\subset P:\supset:\mathrm{Hyp}4.\supset.\alpha\,\mathrm{inst}\text{‘}P\beta:$

[*201.1, *34.5, *23.01, *10.23] $\supset:\mathrm{inst}\text{‘}P\,\epsilon\,\mathrm{trans}:.$

[*37.17, *30.31] $\supset\vdash.\ \mathrm{inst}\text{“}\hat{R}\{R\,|\,R_{se}\,|\,R\subset R\}\subset\mathrm{trans}$ (5)

[*35.9, *36.29, 33.252] $\vdash::\alpha,\beta\,\epsilon\,C\text{‘}\mathrm{inst}\text{‘}P:\supset:.\alpha,\beta\,\epsilon\,\tau_P$ [redacted] $:.$

[redacted]

[*0.03] $\supset::\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha.\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta:.$

[*40.14, *32.12, *37.62] $\supset::\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha:\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta:x\,\epsilon\,\alpha.\supset_x\alpha\subset\overrightarrow{P_{se}}\text{‘}x:y\,\epsilon\,\beta.\supset_y.\beta\subset\overrightarrow{P_{se}}\text{‘}y:.$

[*33.152, *36.29, *35.9, *33.252] $\supset::\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha:\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta:x\,\epsilon\,\alpha.\supset_x\alpha\subset C\text{‘}P:y\,\epsilon\,\beta.\supset_y.\beta\subset C\text{‘}P:.$

[*10.23.252, *24.12.15] $\supset::\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha.\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta.\alpha\subset C\text{‘}P.\beta\subset C\text{‘}P:.$

[*0.02, *23.05.3335, *36.13] $\supset::\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha:\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta::x\,\epsilon\,\alpha.y\,\epsilon\,\beta:\supset_{x,y}xPy.\vee.yPx.\vee.xP_{se}y:.$

[*331.14, *1.01] $\supset::\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha:\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta:.x\,\epsilon\,\alpha:y\,\epsilon\,\beta:\sim xPy:\sim yPx:\supset_{x,y}$ [redacted] $xP_{se}y:.$

[*10.26.11, *11.06.01, *4.71.01, *3.26] $\supset::\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha::\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta:.x\,\epsilon\,\alpha.y\,\epsilon\,\beta.\supset_{x,y}\sim xPy.\sim yPx:\supset:x\,\epsilon\,\alpha.y\,\epsilon\,\beta.\supset_{x,y}xP_{se}y:.$

[*40.51, *11.05] $\supset:.\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha::\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta:.x\,\epsilon\,\alpha.y\,\epsilon\,\beta.\supset_{x,y}\sim xPy.\sim yPx:\supset:x\,\epsilon\,\alpha.\supset_x x\,\epsilon\,p\text{‘}\overrightarrow{P_{se}}\text{“}\beta:.$

[*22.1] $\supset::\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha:\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta:.x\,\epsilon\,\alpha.y\,\epsilon\,\beta.\supset_{x,y}\sim xPy.\sim yPx:\supset:\alpha\subset\beta$ (6)

Similarly, $\vdash::\alpha,\beta\,\epsilon\,C\text{‘}\mathrm{inst}\text{‘}P:.\supset:.\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha::\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta:.x\,\epsilon\,\alpha.y\,\epsilon\,\beta.\supset_{x,y}\sim xPy.\sim yPx:\supset:\beta\subset\alpha$ (7)

$\vdash$(6)(7) $\supset\vdash::\alpha,\beta\,\epsilon\,C\text{‘}\mathrm{inst}\text{‘}P:.\supset:.\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha::\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta:.x\,\epsilon\,\alpha.y\,\epsilon\,\beta.\supset_{x,y}\sim xPy.\sim yPx:\supset:\alpha=\beta:.$

[*4.4, *5.6, *4.13, *11.01.03, [redacted] *10.23.42] $\supset:.\alpha=p\text{‘}\overrightarrow{P_{se}}\text{“}\alpha::\beta=p\text{‘}\overrightarrow{P_{se}}\text{“}\beta:.(\exists x,y).x\,\epsilon\,\alpha.y\,\epsilon\,\beta.xPy:\vee:(\exists x,y).x\,\epsilon\,\alpha.y\,\epsilon\,\beta.yPx:\vee:\alpha=\beta:.$

[(1)] $\supset:.\alpha\,\mathrm{inst}\text{‘}P\beta.\vee.\beta\,\mathrm{inst}\text{‘}P\alpha.\vee.\alpha=\beta$ (8)

$\vdash . (8) . {*}202.103 . \supset \vdash . \mathrm{inst}`P \in \mathrm{connex}.$

$[{*}37.17, {*}30.31] \supset \vdash . \mathrm{inst}``\hat{R}\{R|R_{se}|R \subset R\} \subset \mathrm{connex}$ (9)

$\vdash . (3) . (5) . (9) . \supset \vdash . \mathrm{inst}``\hat{R}\{R|R_{se}|R \subset R\} \subset \mathrm{Rl}`J \cap \mathrm{trans} \cap \mathrm{connex}.$

$[{*}204.01] \supset \vdash . \mathrm{inst}``\hat{R}\{R|R_{se}|R \subset R\} \subset \mathrm{ser} \supset \vdash \mathrm{Prop}.$

${*}0.2$ $\vdash . \mathrm{inst}``\mathrm{cs} \subset \mathrm{ser}$

<u>*Dem*</u>

$\vdash . {*}0.01 . \supset \vdash : R \in \mathrm{cs} . \supset_R . R|(\dot{-}R\dot{-}\breve{R})|R \subset R.$

$[{*}35.48.481.21, {*}36.01] \supset_R . R|[(\dot{-}R\dot{-}\breve{R}) \upharpoonright C`R]|R \subset R$ $\cancel{(1)}$

$[{*}0.02]$ [redacted] $\supset_R . R|R_{se}|R \subset R$ (1)

$\vdash . (1) . \supset \vdash . \mathrm{cs} \subset \hat{R}\{R|R_{se}|R \subset R\}.$

$[{*}37.2] \supset \vdash . \mathrm{inst}``\mathrm{cs} \subset \mathrm{inst}``\hat{R}\{R|R_{se}|R \subset R\}.$

$[{*}23.49] \supset \vdash . \mathrm{Prop}.$

It will be noticed that two of the three serial properties of inst'P — its being contained in diversity and its connexity — are independent of the properties of P itself. It is especially noticeable that no use is made of [redacted] $P \subset J$ ~~is made~~ in proving $\mathrm{inst}`P \subset J$, nor, indeed, in deducing any of the serial properties of inst'P. [redacted] inst is a valuable tool for what Mr. Russell calls 'fattening out' a relation: i.e., deriving from a non-serial relation a relation with many of the properties of a series.

It is interesting to consider under what conditions inst'P will be compact. If we define csd as follows:

${*}0.21$ $\mathrm{csd} = \mathrm{cs} \cap \hat{R}\{R \subset R|R_{se}|R\} \cap \hat{R}\{\breve{R}|R_{se} \subset \breve{R}\,|\,\mathrm{min}_R|\overrightarrow{R}_{se}\}$ Df

we shall find that $R \in \mathrm{csd}$ is a sufficient condition for the density of R. This condition says (1) R is a relation of complete succession, (2) if x precedes y by the relation R, there are a u and a v, which are simultaneous with reference to R, such that xRu and vRy, (3) if x follows by R some R-contemporary of y, it follows some <u>initial</u> R-contemporary of y. This latter condition, ~~which~~

$\vdash$. $\quad \supset\vdash : x\, \breve{P}|P_{se}\, x \,.\supset.\, (\exists y) .\, y P x \,.\, y P_{se} x.$

$\qquad .\supset.\, (\exists y) .\, y P x \,.\, y \dot{-} P x \qquad (4)$

$\vdash.(4). \quad \supset\vdash .\, x \dot{-} \breve{P}|P_{se}\, x \qquad (5)$

$\vdash.(3).(5). \quad \supset\vdash :: P \varepsilon \mathrm{csd} \,.\, x \varepsilon C\text{‘}P :.\supset:: z P_{se} x \,.\, z \dot{-} P|P_{se}\, x \,.\supset_z .\, y P_{se} z :\supset.\, y P_{se} x \qquad (6)$

$\vdash.(1).(2).(6) \quad \supset\vdash.$ Prop

It will be observed that the only portions of the hypothesis of ~~which we a~~ $P \varepsilon \mathrm{csd}$ of which we actually make use in this theorem are $P \subset J$ and $\breve{P}|P_{se} \subset \breve{P}|min_P|\vec{P}_{se}$. The theorem ensures us that $C\text{‘}P \subset s\text{‘}\tau_P$: that is, ~~that every~~ in the case of time, that each event shall be at at least one instant — the instant at which it begins. For, since $P \subset J$, $I \restriction C\text{‘}P \subset P_{se}$. This ensures that $x \varepsilon \vec{P}_{se}\text{‘}x$ if $x \varepsilon C\text{‘}P$ [redacted] Moreover, as we saw in the proof of the last theorem, $x \dot{-} \breve{P}|P_{se}\, x$, or $x \varepsilon -\breve{P}\text{“}\vec{P}\text{‘}x$. Therefore, if $x \varepsilon C\text{‘}P$, $x \varepsilon \vec{P}_{se}\text{‘}x \cap C\text{‘}P - \breve{P}\text{“}\vec{P}_{se}\text{‘}x$, which is $\overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}x$. As we have proved in $*0.22$ that $\overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}x \varepsilon \tau_P$, $C\text{‘}P \subset s\text{‘}\tau_P$.

I now wish to prove that [redacted] $\mathrm{inst}\text{“}\mathrm{csd} \subset \mathrm{comp}$.

$*0.23 \qquad \vdash .\, \mathrm{inst}\text{“}\mathrm{csd} \subset \mathrm{comp}$

Dem

$\vdash *0.1,(1) \quad \supset\vdash :: P \varepsilon \mathrm{csd} :.\supset:.\, \alpha\, \mathrm{inst}\text{‘}P\beta :\supset: \alpha = p\text{‘}\vec{P}_{se}\text{“}\alpha \,.\, \beta = p\text{‘}\vec{P}_{se}\text{“}\beta \,.\, (\exists x, y) .\, x \varepsilon \alpha \,.\, y \varepsilon \beta \,.\, x P y :$

$\supset: \alpha = p\text{‘}\vec{P}_{se}\text{“}\alpha \,.\, \beta = p\text{‘}\vec{P}_{se}\text{“}\beta \,.\, (\exists x, y) .\, x \varepsilon \alpha \,.\, y \varepsilon \beta \,.\, x\, P|P_{se}|P\, y :$

$\supset: \alpha = p\text{‘}\vec{P}_{se}\text{“}\alpha \,.\, \beta = p\text{‘}\vec{P}_{se}\text{“}\beta \,.\, (\exists x, y) .\, x \varepsilon \alpha \,.\, y \varepsilon \beta \,.\, x\, P|[min_P|\vec{P}_{se}]|P\, y :$

$\supset: \alpha = p\text{‘}\vec{P}_{se}\text{“}\alpha : \beta = p\text{‘}\vec{P}_{se}\text{“}\beta : (\exists x, y) : x \varepsilon \alpha : y \varepsilon \beta : (\exists u, v) .\, x P u \,.\, v \varepsilon \overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u \,.\, v P y :$

$\supset: (\exists u, v) : \alpha = p\text{‘}\vec{P}_{se}\text{“}\alpha : \beta = p\text{‘}\vec{P}_{se}\text{“}\beta : \overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u \varepsilon p\text{‘}\vec{P}_{se}\text{“}\overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u : (\exists x, y) .\, x \varepsilon \alpha \,.\, y \varepsilon \beta.$

$v \varepsilon \overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u \,.\, u P_{se} u \,.\, u \dot{-} \breve{P}|P_{se}\, u \,.\, x P u \,.\, v P y :$

$\supset: (\exists u, v) : \alpha = p\text{‘}\vec{P}_{se}\text{“}\alpha : \beta = p\text{‘}\vec{P}_{se}\text{“}\beta : \overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u \varepsilon p\text{‘}\vec{P}_{se}\text{“}\overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u : (\exists x, y) .\, x \varepsilon \alpha \,.\, y \varepsilon \beta.$

$u, v \varepsilon \overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u \,.\, x P u \,.\, v P y :$

$\supset: (\exists u, v) : \alpha = p\text{‘}\vec{P}_{se}\text{“}\alpha : \beta = p\text{‘}\vec{P}_{se}\text{“}\beta : \overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u \varepsilon p\text{‘}\vec{P}_{se}\text{“}\overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u : (\exists x, m) .\, x \varepsilon \alpha \,.$

$m \varepsilon \overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u \,.\, x P m : (\exists n, y) .\, n \varepsilon \overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u \,.\, y \varepsilon \beta \,.\, n P y :$

$\supset: (\exists u) .\, \alpha\, \mathrm{inst}\text{‘}P\, (\overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u) \,.\, (\overrightarrow{min_P}\text{‘}\vec{P}_{se}\text{‘}u)\, \mathrm{inst}\text{‘}P\beta :$

$\supset: (\exists \gamma) .\, \alpha\, \mathrm{inst}\text{‘}P\gamma \,.\, \gamma\, \mathrm{inst}\text{‘}P\beta :$

$\supset: \alpha\, (\mathrm{inst}\text{‘}P)^2\, \beta \qquad (1)$

$\vdash$ (1) $\quad \supset\vdash : P \varepsilon \mathrm{csd} . \supset . \mathrm{inst}`P \varepsilon \mathrm{comp}:$

$\supset\vdash$. Prop.

It will be noticed that it is not true that csd $\subset$ comp. ~~Applied~~ For example, if P is the relation of complete succession between one-inch stretches on a line, then if x begins a quarter of an inch ~~from~~ before the end of y, and y begins a quarter of an inch before the end of z, then z will completely ~~succeed~~ precede x, but y will be a complete antecedent or successor neither of x nor of z, and, indeed, there will be no one inch stretch completely preceding x and completely following z. Nevertheless, P will be a member of csd. ~~$P|P_{sd}|P = P$ is a weaker hypothe~~ $P \subset P|P_{sd}|P$ is a weaker hypothesis than $P \subset P^2$, which implies it if $P \subset J$

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2 proofs to W. L. Matthews

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Slip 51

"A Contribution to the Theory of Relative Position"

By Norbert Wiener, Ph.D. (Communicated by Mr. G. H. Hardy)

[Received 14 March 1914].

The theory of relations is one of the most interesting departments of the new mathematical logic. The relations which have been most thoroughly studied are the series: that is, relations which are contained in diversity, transitive, and connected, or in Mr. Russell's symbolism, those relations R of which the following proposition is true.

[redacted] $R \subset J \,.\, R^2 \subset R \,.\, R \cup \breve{R} \cup I \restriction C'R = C'R \uparrow C'R$

Cantor, Dedekind, Frege, Schröder, Burali-Forti, Huntington, Whitehead, and Russell, are among those who have helped to give us an almost exhaustive account of the more fundamental properties of series. There is a class of relations closely allied to series, however, which has received very scant attention [redacted] from the mathematical logicians[1]. Examples of the sort of relation to which I am referring are the relation between two events in time when one completely precedes the other, or the relation between two intervals on a line when one lies to the left of the other, and does not overlap it, or, in general, the relation between two stretches [redacted] α and β, of [redacted] terms of a series R, when any term lying in α bears the relation R to any term lying in β. Relations of this sort, which I shall call relations of complete sequence, differ in general from series in not being connected:

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1) The subject of this paper was suggested to me by Mr. Bertrand Russell, and the paper itself is the result of an attempt to simplify and generalize certain notions used by him in his treatment of the relation between the series of events and the series of instants.

that is, for example, it is not necessary that of two distinct events, each of which wholly precedes or follows some other event, one should <u>wholly</u> precede the other, for [redacted] the times of their occurrence may overlap [redacted]. But in all the instances we have given, the relation of complete sequence is closely bound up with some serial relation: the relation of succession between the events of time is intimately related to the series of its instants, the relation between two intervals on a line, one of which lies completely to the other's left, is intimately related to the series of its points, and so on. These considerations lead us to the general questions, (1) what are the formal properties which characterise relations of the sort we have called relations of complete sequence? and (2) what is the nature of the connection between relations of complete sequence and series?

One very general property which belongs to relations of the sort we have called relations of complete sequence is that they never hold between a given term and itself. This property — that of being contained [redacted] in diversity — they share with series proper. Writing cs for the class of relations of complete sequence, we can [redacted] represent in the symbolism of the Principia Mathematica of Whitehead and Russell this fact by the formula

$$[redacted]\ cs \subset Rl`J.$$

Another property they share with series is that of transitivity. If, for example, the event x wholly precedes the event y, while the event y wholly precedes the event z, the event x wholly precedes the event z. But they possess another [redacted] property more powerful logically, which may be called a generalized form of transitivity. If, for example, the event x wholly precedes the event y, if the event y neither wholly precedes nor wholly follows the event z, while the event z wholly precedes the event w, then the event x will wholly precede the event w. All the other relations which we have mentioned as examples of relations of complete precedence will be found to possess the same property, which, moreover, will be satisfied by all those relations which we would naturally call relations of complete precedence. We may, then, [redacted] so define 'relation of complete precedence' as to regard it as a [redacted] property common to all [redacted] such relations. In symbols, we will then have

$$\vdash . \mathrm{cs} \subset \hat{R}\{R|(\dot{-}R\dot{-}\breve{R})|R \mathrel{\dot{\subset}} R\}.$$

The relation $(\dot{-}R\dot{-}\breve{R})$, with its field limited to that of R, is what we ordinarily know as simultaneity. In most theories of time and of relations of complete precedence, it has been thought necessary to treat [redacted] precedence and simultaneity as coördinate primitive ideas. Nevertheless, those who hold such theories have to assume such propositions as the following, in order to make simultaneity and [redacted] precedence possess the appropriate formal properties[1]:

$$\vdash . S \mathbin{\dot{\cap}} P = \dot{\Lambda}$$

$$\vdash . S \mathbin{\dot{\cup}} P \mathbin{\dot{\cup}} \breve{P} = C\text{'}S \uparrow C\text{'}S$$

$$\vdash . S \mathrel{\dot{\subset}} \breve{S}$$

$$\vdash . C\text{'}S = C\text{'}P.$$

From these it is an easy matter to deduce that

$$\vdash . S = (\dot{-}P\dot{-}\breve{P}) \upharpoonright C\text{'}P$$

while on the hypothesis that $P \mathrel{\dot{\subset}} J$, the [redacted] converse deduction can readily be made. Therefore, we may define simultaneity as that relation which holds between x and y when both either follow or precede something and neither precedes the other. [redacted] The second property of relations of complete sequence may, then, be interpreted to state that if R is such a relation, then if xRy, y is-simultaneous-with-respect-to-R[2] to z, and zRw, then xRw.

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[1] In the following list of propositions, S stands for 'is simultaneous with', and P for 'precedes'.

[2] In this paper, 'x is-simultaneous-with-respect-to-R to y' will be interpreted as meaning $x[(\dot{-}R\dot{-}\breve{R}) \upharpoonright C\text{'}R]y$.

We shall find that most of the properties of relations of the sort of complete temporal succession between events follow from the two conditions which we have mentioned above — indeed, many of the most important ones follow from the second alone —, so that we shall <u>define</u> a relation of complete succession as one which satisfies those two conditions: in other words, we shall make the following definition:

*0.01[1] $cs = Rl\text{‘}J \cap \hat{R}\{R|(\dot{-}R\dot{-}\breve{R})|R \mathrel{\dot{\subset}} R\}$ Df

Moreover, as we shall have frequent cause to refer to the relation $(\dot{-}P\dot{-}\breve{P})\restriction C\text{‘}P$, and as this expression is rather unwieldy, we shall abbreviate it as follows:

*0.02 $P_{sl} = (\dot{-}P\dot{-}\breve{P})\restriction C\text{‘}P$ Df

Now the question arises, how are the members of cs related to series? How, for example, is the relation between an event and another that completely succeeds it related to the relation between an <u>instant</u> and another that follows it? Two methods of procedure are open to us: we may define an event as a class of instants, and derive succession between events from that between instants, or we may define [redacted] an instant as the class of all the events that occur at it. Both these methods seem to have certain inherent disadvantages: if we choose the first method, then we cannot consider the possibility of several events occurring with [redacted] the same times of beginning and ending, whereas if we choose the second alternative, we cannot consider the possibility of [redacted] all the events of one moment happening also at another and vice versa. However, we shall choose the latter method of procedure, since cs is a more general notion than ser. This can be proved as follows: [redacted]

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[1] I follow the method of the <u>Principia</u> <u>Mathematica</u> of Whitehead and Russell.

$\vdash . \; R|(\dot{-}R \dot{-} \breve{R})|R = R|[(\dot{-}R \dot{-} \breve{R}) \upharpoonright C`R]|R$

$= R|\hat{x}\hat{y}(x \dot{-} R y \,.\, y \dot{-} R x \,.\, x, y \,\varepsilon\, C`R) R \qquad (1)$

$\vdash : R \,\varepsilon\, \text{connex} \,.\, \supset .\, R|(\dot{-}R \dot{-} \breve{R})|R = R|\hat{x}\hat{y}(x = y)|R$

$= R|I|R$

$= R|R \qquad (2)$

$\vdash : R \,\varepsilon\, \text{ser} \,.\, \supset .\, R|(\dot{-}R \dot{-} \breve{R})|R \subset R \,.\, R \,\varepsilon\, Rl`J \,.$

$\supset .\, R \,\varepsilon\, \text{cs}$

$\vdash . (3) . \supset \vdash .\, \text{ser} \subset \text{cs}.$

Moreover, it has been shown by Mr. Russell that it is advantageous for purposes of methodological simplicity to regard the instants of time as constructions from its events. This is an additional reason for starting from the members of cs and forming certain members of ser as functions of them. Let us, then, agree that an instant, for example, is to be regarded as a class of events, and a point on a line as a class of the segments of the line, for the purposes of this paper. The question then arises, when is a class of events an instant, and when is a class of segments a point? It is obvious on inspection that not every class of events is an instant: all the events which make up a given instant must be simultaneous with one another, and all the events which are simultaneous with every member of the instant must belong to that instant. Moreover, Λ must not be an instant. It can also be seen readily that any class satisfying these conditions will be an instant. That is, if P is the relation of an event to an event which completely follows it, it is a simple matter to show that the class of all instants is

$\hat{\alpha}\{\alpha = p`\overrightarrow{P_{se}}``\alpha\}$[1]

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[1] This definition is due to Mr. Russell.

One instant precedes another when and only when some event belonging to the one entirely precedes some event belonging to the other. That is, calling the relation of precedence between instants inst'P, we can easily show that we have:

$$\vdash .\ \mathrm{inst}'P = (\breve{\epsilon};P) \upharpoonright \hat{\alpha}\{\alpha = p'\overrightarrow{P_{sl}}``\alpha\}.$$

Let me now make the following definitions for any value of P:

$*0.03$ $\qquad \gamma_P = \hat{\alpha}\{\alpha = p'\overrightarrow{P_{sl}}``\alpha\} \qquad Df$

$*0.04$ $\qquad \mathrm{inst} = \hat{Q}\hat{P}\{Q = (\breve{\epsilon};P) \upharpoonright \gamma_P\} \qquad Df.$

I wish to show that

$$\vdash .\ \mathrm{inst}``\hat{R}\{R|R_{sl}|R \subset R\} \subset \mathrm{ser}$$

and hence that

$$\vdash .\ \mathrm{inst}``\mathrm{cs} \subset \mathrm{ser}.$$

This shows us how we can construct a serial relation from any relation of the same sort as complete succession; or, indeed, from any relation agreeing with it in only one respect.

[redacted]

$*0.1.$ $\vdash . \text{inst}`` \hat{R}\{R | R_{se} | R \subset R\} \subset \text{ser}$

<u>Proof</u>

It is easy to show that

$$\vdash : \alpha \,\text{inst}`P\beta . \equiv . \alpha = p`\overrightarrow{P_{se}}``\alpha . \beta = p`\overrightarrow{P_{se}}``\beta . (\exists x, y) . x \,\varepsilon\, \alpha . y \,\varepsilon\, \beta . xPy, \qquad (1)$$

from the definitions of inst and γ_P. From this we can deduce

$$\vdash : \alpha \,\text{inst}`P\beta . \supset . \beta = p`\overrightarrow{P_{se}}``\beta . (\exists x, y) . x \,\varepsilon\, \alpha . y \,\varepsilon\, \beta . \sim x P_{se} y,$$

since, by the definition of P_{se}, xPy and $xP_{se}y$ are incompatible. This reduces to

$$\vdash : \alpha \,\text{inst}`P\beta . \supset . \beta = p`\overrightarrow{P_{se}}``\beta . (\exists x) . x \,\varepsilon\, \alpha . \sim(x \,\varepsilon\, p`\overrightarrow{P_{se}}``\beta),$$

from which we can deduce

$$\vdash : \alpha \,\text{inst}`P\beta . \supset_{\alpha, \beta} \alpha J \beta,$$

or $\quad \vdash . \text{inst}`P \,\varepsilon\, Rl`J. \qquad (2)$

Also, we find [redacted] from (1) that

$$\vdash : \alpha \,\text{inst}`P\beta . \beta \,\text{inst}`P\gamma . \supset . \alpha = p`\overrightarrow{P_{se}}``\alpha . \beta = p`\overrightarrow{P_{se}}``\beta . \gamma = p`\overrightarrow{P_{se}}``\gamma . (\exists x, y, u, v) . x \,\varepsilon\, \alpha . y, u \,\varepsilon\, \beta . v \,\varepsilon\, \gamma . xPy . uPv.$$

This implies

$$\vdash : \alpha \,\text{inst}`P\beta . \beta \,\text{inst}`P\gamma . \supset . \alpha = p`\overrightarrow{P_{se}}``\alpha . \gamma = p`\overrightarrow{P_{se}}``\gamma . (\exists x, v) . x \,\varepsilon\, \alpha . v \,\varepsilon\, \gamma . x P | P_{se} | P v.$$

This, together with (1), gives us

$$\vdash . \text{inst}`` \hat{R}\{R | R_{se} | R \subset R\} \subset \text{trans}. \qquad (3)$$

By the definitions of inst and γ_P, we find that

$$\vdash : \alpha, \beta \,\varepsilon\, C`\text{inst}`P . \supset . \alpha = p`\overrightarrow{P_{se}}``\alpha . \beta = p`\overrightarrow{P_{se}}``\beta .$$

By an easy deduction, we can arrive, from this proposition and the definition of P_{se}, to the proposition

$$\vdash :: \alpha, \beta \,\varepsilon\, C`\text{inst}`P . \supset :. \alpha = p`\overrightarrow{P_{se}}``\alpha . \beta = p`\overrightarrow{P_{se}}``\beta :. x \,\varepsilon\, \alpha . y \,\varepsilon\, \beta : \supset_{x, y} : xPy . \vee . yPx . \vee . x P_{se} y,$$

whence we get

$$\vdash :: \alpha, \beta \,\varepsilon\, C`\text{inst}`P . \supset :. \alpha = p`\overrightarrow{P_{se}}``\alpha . \beta = p`\overrightarrow{P_{se}}``\beta :. x \,\varepsilon\, \alpha . y \,\varepsilon\, \beta . \supset_{x, y} \sim xPy . \sim yPx : \supset : u \,\varepsilon\, \alpha . \supset_u u \,\varepsilon\, p`\overrightarrow{P_{se}}``\beta,$$

or $\quad \vdash :: \alpha, \beta \,\varepsilon\, C`\text{inst}`P . \supset :. \alpha = p`\overrightarrow{P_{se}}``\alpha . \beta = p`\overrightarrow{P_{se}}``\beta :. x \,\varepsilon\, \alpha . y \,\varepsilon\, \beta . \supset_{x, y} \sim xPy . \sim yPx : \supset . \alpha \subset \beta .$

By an exactly similar argument,

$$\vdash :: \alpha, \beta \,\varepsilon\, C`\text{inst}`P . \supset :: \alpha = p`\overrightarrow{P_{se}}``\alpha . \beta = p`\overrightarrow{P_{se}}``\beta :. x \,\varepsilon\, \alpha . y \,\varepsilon\, \beta . \supset_{x, y} \sim xPy . \sim yPx : \supset . \beta \subset \alpha .$$

Combining these, we get

$\vdash :: \alpha, \beta \epsilon C`\mathrm{inst}`P . \supset :. \alpha = p`\overrightarrow{P_{se}}``\alpha . \beta = p`\overrightarrow{P_{se}}``\beta :. x \epsilon \alpha . y \epsilon \beta . \supset_{x,y} \sim x P y . \sim y P x : \supset . \alpha = \beta .$

This we may write as

$\vdash :: \alpha, \beta \epsilon C`\mathrm{inst}`P . \supset :. \alpha = p`\overrightarrow{P_{se}}``\alpha . \beta = p`\overrightarrow{P_{se}}``\beta :. (\exists x, y) . x \epsilon \alpha . y \epsilon \beta . x P y : v : (\exists x, y) . x \epsilon \alpha . y \epsilon \beta . y P x : v : \alpha = \beta .$

By (1), this becomes

$\vdash :. \alpha, \beta \epsilon C`\mathrm{inst}`P . \supset : \alpha \; \mathrm{inst}`P \beta . v . \beta \; \mathrm{inst}`P \alpha . v . \alpha = \beta$

or $\vdash . \mathrm{inst}`P \epsilon \mathrm{connex}$. (4)

¶ Combining (2), (3), and (4) we get the desired conclusion: namely,

$\vdash . \mathrm{inst}``\hat{R}\{R | R_{se} | R \subset R\} \subset \mathrm{ser}.$

From this we can easily conclude that

$\vdash . \mathrm{inst}``\mathrm{cs} \subset \mathrm{ser}.$

It will be noticed that two of the three serial properties of inst'P — its being contained in diversity and its connexity — are independent of the properties of P itself. It is especially noticeable that no use is made of $P \subset J$ in proving $\mathrm{inst}`P \subset J$, nor, indeed, in deducing any of the serial properties of inst'P. inst is a valuable tool for what Mr. Russell calls 'fattening out' a relation: i.e. [redacted] deriving from a non-serial relation a relation with many of the properties of series.

It is interesting to consider under what conditions inst'P will be compact. If we define csd as follows:

$*0.2 \qquad \mathrm{csd} = \mathrm{cs} \cap \hat{R}\{R \subset R | R_{se} | R . \breve{R} | R_{se} \subset \breve{R} | \mathrm{min}_R | \overrightarrow{R_{se}}\} \quad \mathrm{Df},$

we shall find that $R \epsilon \mathrm{csd}$ is a sufficient condition for the density of inst'R. This condition says that (1) R is a relation of complete sequence, (2) if x precedes y by the relation R, there are two members of the field of R neither of which bears the relation R to the other, while x precedes the one by R, while the other precedes y by R, (3) if x follows by R some R-contemporary of y, it follows some <u>initial</u> R-contemporary of y. This latter condition, which was first formulated by Mr. Russell, ensures that if $x \epsilon C\text{‘}R$ and $R \epsilon \mathrm{csd}$, $\overrightarrow{min}_R\text{‘}\overrightarrow{R}_{se}\text{‘}x \epsilon \gamma_R$. This I now wish to prove.

*0.21. $\vdash : P \epsilon \mathrm{csd} . x \epsilon C\text{‘}P . \supset . \overrightarrow{min}_P\text{‘}\overrightarrow{P}_{se}\text{‘}x \epsilon \gamma_P$

<u>Proof</u>

It follows from the definition of $p\text{‘}\kappa$ and $\overrightarrow{min}_P\text{‘}\alpha$ that

$$\vdash . p\text{‘}\overrightarrow{P}_{se}\text{“}\overrightarrow{min}_P\text{‘}\overrightarrow{P}_{se}\text{‘}x = \hat{y}\{\alpha \epsilon \overrightarrow{P}_{se}\text{“}[\overrightarrow{P}_{se}\text{‘}x \cap C\text{‘}P - \breve{P}\text{“}\overrightarrow{P}_{se}\text{‘}x] . \supset_\alpha . y \epsilon \alpha\}.$$

Since it follows from the definition of P_{se} that $\vdash . C\text{‘}P_{se} \subset C\text{‘}P$, this reduces to

$$\vdash . p\text{‘}\overrightarrow{P}_{se}\text{“}\overrightarrow{min}_P\text{‘}\overrightarrow{P}_{se}\text{‘}x = \hat{y}\{\alpha \epsilon \overrightarrow{P}_{se}\text{“}[\overrightarrow{P}_{se}\text{‘}x - \breve{P}\text{“}\overrightarrow{P}_{se}\text{‘}x] . \supset_\alpha . y \epsilon \alpha\}.$$

This becomes by a little manipulation

$$\vdash . p\text{‘}\overrightarrow{P}_{se}\text{“}\overrightarrow{min}_P\text{‘}\overrightarrow{P}_{se}\text{‘}x = \hat{y}\{z P_{se} x . z \dot{-} \breve{P}|P_{se} x . \supset_z . y P_{se} x\}. \qquad (1)$$

On the other hand, it follows from the definition of $\overrightarrow{min}_P\text{‘}\alpha$ that

$$\vdash . \overrightarrow{min}_P\text{‘}\overrightarrow{P}_{se}\text{‘}x = \hat{y}\{y P_{se} x . y \dot{-} \breve{P}|P_{se} x\}.$$

Since by definition any R which belongs to csd satisfies the condition, $\breve{R}|R_{se} \subset \breve{R}|min_R|R_{se}$, we get

$$\vdash : P \epsilon \mathrm{csd} . \supset . \overrightarrow{min}_P\text{‘}\overrightarrow{P}_{se}\text{‘}x = \hat{y}\{y P_{se} x . y \dot{-} \breve{P}|min_P|\overrightarrow{P}_{se} x\}$$

From this we may deduce

$$\vdash : P \epsilon \mathrm{csd} . \supset . \overrightarrow{min}_P\text{‘}\overrightarrow{P}_{se}\text{‘}x = \hat{y}\{y P_{se} x :: z P_{se} x . z \dot{-} \breve{P}|P_{se} x : \supset_z : y P z . v . y \dot{-} P z . z \dot{-} P y . y, z \epsilon C\text{‘}P\}.$$

But if we had in the conclusion of the ~~bracketed~~ second proposition in the brackets $y P z$, this would, together with $y P_{se} x$, give us $z \breve{P}|P_{se} x$, which contradicts the hypothesis. Hence, by the definition of P_{se}, we have

$$\vdash : P \epsilon \mathrm{csd} . \supset . \overrightarrow{min}_P\text{‘}\overrightarrow{P}_{se}\text{‘}x = \hat{y}\{y P_{se} x : z P_{se} x . z \dot{-} \breve{P}|P_{se} x . \supset_z . y P_{se} z\} \qquad (2)$$

Now, it is part of the hypothesis $P \epsilon \mathrm{csd}$ that $P \subset J$. From this it is easy to deduce that $I \restriction C\text{‘}P \subset P_{se}$, or that $x \epsilon C\text{‘}P . \supset_x . x P_{se} x$. Moreover, it follows from the definition of P_{se} that $y P x$ and $y P_{se} x$ are incompatible hypotheses, and hence that $x \dot{-} \breve{P}|P_{se} x$. This fact, combined with (1), gives us

$$\vdash : P \epsilon \mathrm{csd} . x \epsilon C\text{‘}P . \supset . p\text{‘}\overrightarrow{P}_{se}\text{“}\overrightarrow{min}_P\text{‘}\overrightarrow{P}_{se}\text{‘}x = \hat{y}\{y P_{se} x : z P_{se} x . z \dot{-} \breve{P}|P_{se} x . \supset . y P_{se} z\} \qquad (3)$$

From (2), (3), and the definition of γ_P, we have:

$$\vdash : P \epsilon \mathrm{csd} . x \epsilon C\text{‘}P . \supset . \overrightarrow{min}_P\text{‘}\overrightarrow{P}_{se}\text{‘}x = p\text{‘}\overrightarrow{P}_{se}\text{“}\overrightarrow{min}_P\text{‘}\overrightarrow{P}_{se}\text{‘}x . \supset . \overrightarrow{min}_P\text{‘}\overrightarrow{P}_{se}\text{‘}x \epsilon \gamma_P.$$

This is the desired proposition.

It will be observed that the only portions of the hypothesis of [redacted] $P \varepsilon csd$ of which we actually make use in this theorem are $P \subset J$ and $\breve{P}|P_{se} \subset \breve{P}|\min_P|\vec{P}_{se}$. The theorem ensures us that $\vdash . C`P \subset s`\tau_P$: that is in the case of time, that each event shall be [redacted] at some instant [redacted] the instant at which it begins. For, since $P \subset J$, $I \restriction C`P \subset P_{se}$. This ensures that $x \varepsilon \vec{P}_{se}`x$. Moreover, as we have just seen, $x \,\dot{-}\, \breve{P}|P_{se} x$, or $x \varepsilon - \breve{P}``\vec{P}`x$. Therefore, if $x \varepsilon C`P$, $x \varepsilon \vec{P}_{se}`x \cap C`P - \breve{P}``\vec{P}_{se}`x$, or $x \varepsilon \overrightarrow{\min_P}`\vec{P}_{se}`x$. As we have proved in *0.21 that $\overrightarrow{\min_P}`\vec{P}_{se}`x \varepsilon \tau_P$, we get the formula, $\vdash . C`P \subset s`\tau_P$.

I now wish to prove that inst"csd $\subset$ comp.

*0.22 $\quad \vdash .$ inst"csd $\subset$ comp.

<u>Proof</u>

As we saw in *0.1, (1),

$$\vdash : \alpha \,\text{inst}`P\beta . \equiv . \alpha = p`\vec{P}_{se}``\alpha . \beta = p`\vec{P}_{se}``\beta . (\exists x, y) . x \varepsilon \alpha . y \varepsilon \beta . x P y .$$

Since $R \varepsilon csd$, by definition, implies $R \subset R|R_{se}|R$, this gives us

$$\vdash :. P \varepsilon csd . \supset : \alpha \,\text{inst}`P\beta . \supset . \alpha = p`\vec{P}_{se}``\alpha . \beta = p`\vec{P}_{se}``\beta . (\exists x, y) . x \varepsilon \alpha . y \varepsilon \beta . x P|P_{se}|P y .$$

Since $R \varepsilon csd$ also implies $\breve{R}|R_{se} \subset \breve{R}|\min_R|\vec{R}_{se}$, this becomes

$$\vdash :. P \varepsilon csd . \supset : \alpha \,\text{inst}`P\beta . \supset . \alpha = p`\vec{P}_{se}``\alpha . \beta = p`\vec{P}_{se}``\beta . (\exists x, y) . x \varepsilon \alpha . y \varepsilon \beta . x P|\overbrace{[\min_P|\vec{P}_{se}]}|P y$$

$x P|\overbrace{[\min_P|\vec{P}_{se}]}|P y$ says that there are a u and a v such that $x P u$, $v P y$, and [redacted] $v \min_P|\vec{P}_{se} u$. This latter proposition is equivalent to $v \varepsilon \overrightarrow{\min_P}`\vec{P}_{se}`u$. We have just seen, moreover, that $u \varepsilon \overrightarrow{\min_P}`\vec{P}_{se} u$, and that $\overrightarrow{\min_P}`\vec{P}_{se} u \varepsilon \tau_P$. This gives us

$$\vdash :: P \varepsilon csd . \supset :. \alpha \,\text{inst}`P\beta : \supset : (\exists u, v) : \alpha = p`\vec{P}_{se}``\alpha . \beta = p`\vec{P}_{se}``\beta : \overrightarrow{\min_P}`\vec{P}_{se}`u \varepsilon p`\vec{P}_{se}``\overrightarrow{\min_P}`\vec{P}_{se}`u : (\exists x, y) . x \varepsilon \alpha . y \varepsilon \beta .$$
$$u, v \varepsilon \overrightarrow{\min_P}`\vec{P}_{se}`u . x P u . y P v .$$

From this and *0.1, (1) it is an easy matter to deduce that

$$\vdash :. P \varepsilon csd : \supset : \alpha \,\text{inst}`P\beta . \supset . (\exists \gamma) . \alpha \,\text{inst}`P\gamma . \gamma \,\text{inst}`P\beta .$$

This gives us immediately

$\vdash .$ inst"csd $\subset$ comp.

It will be noticed that it is not true that csd $\subset$ comp. For example, if P is the relation of complete succession between one-inch stretches on a line, P will be a member of csd, and an inch stretch beginning half an inch after the end of another will bear the relation P to it, yet there will be no inch stretch to which the first bears the relation P and which bears the relation P to the second. $P \subset P|P_{se}|P$ is a weaker hypothesis than $P \subset P^2$, which implies it if $P \subset J$.