An Alternative to the Method of Postulates

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§1 The process of mathematical reasoning which was initiated by Euclid, and which has been brought to a high degree of perfection and subtlety by the modern logistical mathematicians, together with such geometers as Hilbert and Veblen, has been described and analysed by Dr. A. N. Whitehead[1] in the following

[1] Mathematical concepts of the Material World, Phil. Trans. of the Royal Soc. of London, vol. 205 A, p. 469-470

terms: "In the study of every concept [i. e. of every 'complete set of axioms, together with the appropriate definitions and the resulting propositions] there are four logical stages of progress. The first stage consists of the definition of those entities which are capable of definition in terms of the fundamental relations. These definitions are logically independent of any axioms concerning the fundamental relations, though their convenience and importance are chiefly dependent upon such axioms. The second stage consists of the deduction of those properties of the defined entities which do not depend upon the axioms. The third stage is the selection of the group of axioms which determines that concept.... The fourth stage is the deduction of propositions which involve among their hypotheses some or all of the axioms of the third stage". The notable point concerning the derivation of a system from axioms is that, as Dr. Whitehead points out, there are two distinct stages in the derivation of a logical system from a set of axioms or definitions where deductive reasoning is involved: we may draw certain conclusions about the properties of the objects and relations etc. we define in terms of our fundamental relation from their definitions alone, without the aid of any additional hypothesis which still further limits their properties, or we may deduce the consequences of such additional hypotheses. It is manifest, moreover, that the first of these two kinds of deductive reasoning is prior to and independent of the second, whereas the second is bound up with and dependent on the first. The question, then, naturally arises, in the exposition of such a branch of mathematics as geometry, for example, as a deductive system, to what extent can that portion of the deduction which rests merely on the definitions of geometrical concepts be made to assume the functions which, in most systems of geometry, are attributed [redacted] to the part of the geometrical reasoning which rests on the axioms, and to what extent can the importance of this latter side of geometry be minimised by an adroit choice of definitions?

This question may seem to the casual

observer trifling, formal, and technical, but it is really of very considerable logical importance. A problem which has perplexed great philosophers and great physicists alike is, how are we to know whether space is amenable to a [redacted] geometrical treatment? When geometry is considered as the set of deductions from a given group of postulates concerning a given relation or class of classes, this question reduces itself to the two questions, how is this relation, or class, to be analysed in terms of the space-relations and space-qualities which are most directly given to us? and, how are we to know that our relation, or class, when ~~so~~ analysed, will fulfil the requirements laid down for it in ~~the~~ our set of postulates for geometry? Now, it is obvious that the less work we make the geometrical postulates assume, and the more that we put to the account of our definitions, the easier will the second of the two above problems be. If, finally, we are able, to all intents and purposes, to eliminate <u>all</u> geometrical postulates from our system of geometry, we shall have obtained a complete solution of the second problem.

To the reader who is acquainted with the modern theory of postulates, certain apparent difficulties in carrying out such a program will suggest themselves. A completely adequate [redacted] set of postulates for geometry must be, among other things, what is called a <u>categorical</u> set: that is, all the ordinal properties of the defining relation or class of classes" are determined by the postulates.

"An ordinal property of a relation is one which is possessed by all relations similar to one which possesses it. An ordinal property of a class of classes is one possessed by all classes doubly similar to any class which possesses it. A relation is similar to another if the fields of the two can be put in a one-one correspondence of such a nature that if a certain set of terms are in one of the relations, the correspon-ding terms are in the other relation. Two classes of classes are doubly similar (*) if the logical sum of one may be put into one-one correspondence with the logical sum of the other in such a manner that the members of a class belonging to the first always correspond to the members of a class belonging to the second.

Now, it is manifestly impossible for a set of postulates containing no postulates to be categorical. However, a system defined by a "set of definitions" only may have a character analogous to categoricalness. If one consider the matter carefully, it will be seen that the categoricalness of a set of postulates is primarily, not an attribute of the set of postulates but of the fundamental relation or class of classes chosen for geometry. Now, although a fundamental relation or class about which nothing is specified cannot, of course, be categorically determined, some function of it, involving in its definition logical constants only, may be categorically determined. Thus, to give a crude example, whatever class of classes κ may be, if I remove from κ all the members of κ, I shall get a class whose formal properties are categorically determined, and are independent of κ — the null-class. It is thus possible that though a system without postulates cannot be, in its totality, categorically determined, some definite portion of it may be, and the same holds true of a system with postulates, but with a non-categorical set.

In determining a function of the fundamental relation or class of classes of geometry which shall be categorically determined, perhaps the most natural method to follow is to choose some function of this defining relation or class which will be identical with this ~~fu~~ relation or class when the relation or class satisfies a complete set of geometrical postulates, but which will always satisfy these postulates, whether the original class or relation does so or not".

"Cf. E. Schroeder, *Algebra der Logik*, B. III, 1, §12, s 169–70. Here an analogous requirement is formulated which a solution of an equation in the algebra of binary relatives must satisfy. It is really a particular case of our method that Schroeder uses here.

Instead, then, of postulating the desired propositions concerning the fundamental class or relation, we form this function of it, and proceed from it as we should have proceeded from the original relation if we had assumed the necessary postulates concerning it

§2. Perhaps the best way of illustrating the significance of the method described above is to give a concrete instance of it, and show in detail how this function which ~~is to~~ takes the place of a set of postulates is to be built up. The case which we shall discuss is that of projective geometry and the set of postulates which we shall endeavour to replace as far as possible will be that given by Dr. A. N. Whitehead in his _The Axioms of Projective Geometry_, which follow more or less the lines laid down by Pieri. We shall, however, alter the enunciation of these postulates slightly, so as to render them more suitable for the direct application of our method. The fundamental class of classes of this system, which we shall term κ, is the class of all lines. The postulates then read as follows:

I. κ is not the null-class ~~whose only member is the null class~~.

II. The class of terms belonging to members of κ is not a unit-class.

III If x and y are members of classes belonging to κ, there is a class belonging to κ which contains them both.

IV If α and β are distinct members of κ, α and β have not more than one member in common.

V The cardinal numbers of members of κ are greater than two.

VI κ is not a unit-class.

VII If α and β have x in common, ~~and~~ β and γ have y in common, and α and γ have z in common, while δ has a term in common with α other than x or z, and a term in common with β other than x or y, but no term in common with δ, then if α, β, γ, and δ are members of κ, α is identical with β.

VIII If α and β belong to κ, and have a member in common, and x is a member of α other than that member, then there is a member of κ containing x, but no member of β

IX If $\alpha, \beta, \gamma, \delta, \xi, \eta$ are six distinct members of κ, such that α; δ; and ξ; α, β, and η; β, γ, and ξ; and γ, δ, and η, have each a common member, then if x belongs both to α and γ, y both to ξ and η, and z both to β and δ, there is no member of κ containing x, y, and z together.

X There are a pair of members of κ, α and β, say,

$$\varpi (\mathrm{Am}_\varphi{}^{\backprime}\lambda) \beta \;:\equiv:\; \varpi, \beta \,\varepsilon\, \Finv r_{\varphi *}{}^{\backprime\backprime}\lambda \,.\, \exists !\, \Finv r_{\varphi *}{}^{\backprime\backprime}\lambda \cap \mathrm{Cl}^{\backprime}\varpi \cap \mathrm{Cl}^{\backprime}\beta \,:$$

$$\equiv:\; (\exists \mu, \nu) \,.\, \mu, \nu \,\varepsilon$$

possessing a member in common, such that there is a member, say $\underline{x}$, of some member of κ, such that if γ is a member of κ containing $\underline{x}$, there is a member of κ, containing a member of α, and a member of β distinct from this, and a member of γ