Operations in Complex Algebra Isomorphic with Addition and Multiplication

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In his paper in Monographs on Topics of Modern Mathematics[1], entitled The Fundamental Propositions of Algebra, Professor E. V. Huntington develops a set of postulates for ordinary complex algebra in terms of the operations of addition and multiplication and the relation '~~greater~~ less than' among real numbers. In the course of this paper, three examples are given of mathematical systems isomorphic with ordinary algebra. In each of these, the field of the system is the same as that of ordinary complex algebra, with the exception that the infinity of our original algebra may be represented by some term not playing the part of infinity in our new system, and vice versa. If we represent the analogue of ~~the~~ + operation by $\oplus$, that of $\times$ by $\odot$, and that of < by $\otimes$, the defining formulae for these operations are in the first example

$$a \oplus b = a + b,$$

$$a \odot b = 5(a \times b),$$

and $\otimes = <$;

in the second,

$$a \oplus b = \frac{a \times b}{a + b},$$

with proper allowances when a or b or a+b = 0,

$$a \odot b = a \times b,$$

while $\otimes$ is the relation between x and y when x is negative and y is positive, or when both are of one sign and $y < x$; and finally in the third example,

$$a \oplus b = a + b + 1,$$

$$a \odot b = a \times b + a + b,$$

and $\otimes = <$.

As Professor Huntington remarks in a note[2] "Each of these systems is obtained from the ordinary complex plane by a projective transformation":– the first by the transformation

$G(@)$, for if T be a one-one transformation which turns $@$ to $\#$, it will ~~turn~~ 1, 0, and -1 into the entities which satisfy the corresponding definitions with $@$ altered to $\#$, ~~it~~ and will turn any iteration of $@$ on a given set of terms into the corresponding iteration of $\#$ on the analogues of these terms. Therefore, any ~~one-one~~ isomorphic transformation of $@$ into another operation generates an isomorphic transformation of $+$ and $\times$. ~~This transformation will render the analogues of + and × algebraic operations, provided that the analogue of @ is algebraic, for it may be readily shown that the relations x + y = z and x y = z will then result from the solution of a set of simultaneous algebraic equations, which problem is always reducible to the solution of a single algebraic equation.~~ The converse of this statement follows in a precisely similar manner from the fact that $x @ y = \dfrac{y}{1+x-y}$, ~~while this operation~~

which clearly involves no operations not defineable in terms of $+$ and $\times$. It follows

OPERATIONS IN COMPLEX ALGEBRA ISOMORPHIC WITH ADDITION AND MULTIPLICATION.

Norbert Wiener.

In a paper entitled The Fundamental Propositions of Algebra,1) Professor E. V. HUNTINGTON develops a set of postulates for ordinary complex algebra in terms of the operations of addition and multiplication and the relation 'less than' among real numbers. In the course of this paper, three examples are given of mathematical systems isomorphic with complex algebra. If we adjoin to the ordinary complex number-system the number infinity, whose formal properties we determine in an appropriate manner, then the field of each of HUNTINGTON'S systems is the same as that of ordinary algebra. If we represent the analogue of addition by $\oplus$, that of multiplication by $\odot$, and that of $<$ by $\otimes$, the defining formulae for these are in the first example

$$a \oplus b = a + b,$$

$$a \odot b = 5(a \times b),$$

and $\otimes = <$;

in the second,

$$a \oplus b = \frac{a \times b}{a + b}$$

crete empirical data compiled therein. These are ~~those~~ the properties of the table which result from the methods employed in schematizing and arranging the data presented. This suggests the possibility hinted at by Poincaré, that the certainty of geometry may be due to the manner in which it orders our spatial experiences, and not to these experiences themselves. Geometry would then be, in a way, the study of 'the form of our external sense', but would owe both its certainty and its practical value to the fact of its being a form imposed on this external sense with a practical end in view, rather than a form inherent in our experience itself. Its practical value, that is, is due to the fact that it is a form chosen by us with the definite purpose of simplifying the expression of the laws of physics, which would assume an inordinately complicated form if we were to express them directly in terms of our immediate experiences. The sort of schematism employed in deriving Space from experience may be discovered if we regard the processes of measurement, etc. that are employed in physics rather as <u>definitions</u> of such entities as distances than as modes of discovering them.

It is a simple matter to prove that if $x \# k$ is of period 3, $k = 1$. 1 is thus defineable in terms of #. 0 may be defined as $1 \# 1$. xy may be defined as $[(x \# y) \# 1] \# 1$, for

$$[(x \# y) \# 1] \# 1 = 1 - \cfrac{1}{1.\left(1 - \cfrac{1}{1.\left(1 - \cfrac{1}{xy}\right)}\right)}$$

$$= 1 - \cfrac{1}{1 - \cfrac{xy}{xy - 1}}$$

$$= 1 - \frac{xy - 1}{(xy - 1) - xy}$$

$$= 1 + (xy - 1)$$

$$= xy.$$

This definition of xy will evidently give an indeterminate result when x or y or $x \# y$ or $(x \# y) \# 1$ is 0. In the first two cases, we define xy as 0, in the second, as 1, and in the third, as $x \# y = 1 - \frac{1}{xy} = 1$, $xy = -1$, which we define as that number x, other than 1, which makes $x \# x = 1$. In case x or y is infinite, while the other factor is not 0, we define xy as infinity, while we assign no meaning to the product of zero by infinity.

$x \quad y$ may be defined as $y\{(x \# 1)[(x \# 1) \# 1] \# y\}$, for

$$y\left\{(x \# 1)\left[(x \# 1) \# 1\right] \# y\right\} = y\left\{\left(\frac{x-1}{x}\right)\left(1 - \frac{1}{\frac{x-1}{x}}\right) \# y\right\}$$

$$= y\left\{\left(\frac{x-1}{x}\right)\left(1 - \frac{x}{x-1}\right) \# y\right\}$$

$$= y\left\{\frac{x-1}{x} \cdot \frac{-1}{x-1} \# y\right\}$$

$$= y\left(-\frac{1}{x} \# y\right)$$

$$= y\left\{1 - \frac{1}{-\frac{1}{x} \cdot y}\right\}$$

$$= y\left(1 + \frac{x}{y}\right)$$

$$= x + y.$$

This definition breaks down if either x, y, $x \# 1$, $(x \# 1) \# 1$, or $(x \# 1)\left[(x \# 1) \# 1\right] \# y$ is either zero or infinity, or is indeterminate. In all cases where the expression which we have defined as $x + y$ is indeterminate, if the expression which, by our definition, is $y + x$ is determinate, we shall define $x + y$ as the latter. The only cases which remain, as a little computation will show, are those where either x or y is infinity or zero, or where x and y are finite, and $(x \# 1)\left[(x \# 1) \# 1\right] \# y = 0$. We define $x + 0$ and $0 + x$ as x, $\infty + x$ and $x + \infty$ as ∞ (~~provided $x \neq \infty$~~ unless $x = \infty$, in which case we do not define $x + \infty$

nor x), while if x ~~is~~ and y are finite and (x # 1)[(x # 1) # 1] # y = o, we shall agree to call x + y 1. The propriety of this last definition is ~~[illegible]~~ evident, for under the circumstances when we have agreed to use it, 1 + $\frac{x}{y}$ will equal 0, and x will equal -y.

We have thus given a definition of multiplication involving nothing but # and logical constants, and a definition of addition ~~wh~~ which can ultimately be expressed in the same manner, since it in

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There are several different methods in accordance with which we may consider a mathematical system. A method that has recently been much used for the characterization of a mathematical system is that of postulates. A mathematical system has been regarded as determined as to all its mathematical properties when certain relationships between certain of the entities of which it treats have been specified explicitly in a given small number of propositions. This method is unquestionably one of the utmost mathematical and logical value, but it nevertheless is obviously ~~unable~~ liable to tell us both too little and too much about the mathematical system which it specifies. It is a familiar fact, first, that a multitude of sets of postulates, dealing with quite different operations and relations, may define the same system, and secondly, that there are scarcely any two mathematical systems so diverse that it is not possible to arrange some construction out of the elements of the one which will fulfill the conditions of the other. For these reasons, a more precise specification of a mathematical system may be obtained by considering, not the formal properties of any single operation or relation entering into its constitution, but a definition which will enable us to determine whether <u>any</u> given operation or relation belongs to the system; and ~~a~~ by determining ~~not the properties of any single element of~~ those properties which any element whatsoever of the system possesses by virtue of its position in the system. This may be done by a set of postulates, but not every set of postulates does this. For example, as KEMPE pointed out #), the algebra of logic may retain its formal

#)

properties unchanged under transformations that alter any element of the system into any other, yet most of the postulates for the algebra of logic single out two elements of the system under the names of 0 and 1, and correspondingly concern themselves with certain specific operations dependent on these entities. If we desire to replace such a set of postulates by one which does not thus over-specify the entities with which it deals, and which consequently gives us a truer idea of ~~t~~ the internal structure of the system it defines, one of the first preliminary steps for us to take is naturally that of making a survey of the system to determine to what extent we have been over-specifying the system in our postulates. This naturally involves the question of just precisely what entities, relations, and operations of the system defined by our ~~postulates~~

postulates possess the formal properties of the entities, relations, and operations dealt with in our postulates. Of course, the most obvious of these formal properties are precisely those formulated in the postulates themselves. We thus obtain the following fundamental question concerning any specified system of relations and operations which is regarded as generated by a set of postulates concerning certain of these: What operations and relations of the system may be substituted for those with which the postulates concern themselves without altering the truth-value of the postulates? I have already dealt with this question in the form in which it appears in Boolean algebras #). I propose to devote this paper to the discussion of the

#). In an article which will have appeared in these TRANSACTIONS.

corresponding problem in the ordinary algebra of complex quantities.

A set of postulates has been developed for this algebra by HUNTINGTON #). This set concerns itself with the operations of addition

#). See Monograph IV. in Monographs on Topics of Elementary Mathematics, edited by J. W. A. Young.

and multiplication among complex numbers and the relation of "greater than" among real numbers. Now, both the operation of addition and that of multiplication can be derived from the iteration of the operation

$$x @ y = \frac{y}{1 + x - y}$$

on the numbers to be added or multiplied and given constant numbers. This may be shown by the following formulae:

$$x @ 1 = \frac{1}{x}$$

$$\frac{1}{x @ y} = \frac{1 + x}{y} - 1$$

$$\frac{1}{0 @ y} = \frac{1}{y} - 1$$

$$\frac{1}{0 @ \frac{1}{y}} = y - 1$$

$$\frac{1}{(x - 1) @ y} = \frac{x}{y} - 1$$

$$\frac{x}{\left(\frac{1}{y}\right)} - 1 = xy - 1$$

$$-1(xy-1) - 1 = -xy$$

$$-(1)(-xy) = xy$$

$$x(y - 1) = xy - x$$

$$x.\frac{y}{x} - x = y - x$$

$$y - x(-1) = x + y$$

It is clear, then, that the operation @ and the pair of operations, addition and multiplication, determine one another reciprocally in such a manner that the question of what pairs of algebraic operations possess the same formal properties as addition and multiplication reduces itself to the question as to what algebraic operations possess the same formal properties as @, and that it is possible to construct a set of postuates for complex algebra in terms of @ and the relation "greater than" alone.

Since complex algebra is a categorically determined system #),

#). Ibid, sec. 32.

any formal properties of @ which do not involve the naming of any special algebraic entities, but simply the statement that all or no or some such entities have a certain property, follow from the postulates of the algebra itself, and must be possessed by any algebraic operation satisfying a set of postulates for @. If we wish to investigate just what operations in algebra have the same formal properties as @, however, we may restrict our investigation somewhat. It is a familiar fact that a projective transformation changes every algebraic operation into another algebraic operation, and that if we adjoin the entity ∞ to our number-system, making the appropriate alterations in our definition of an algebraic operation, etc., a projective transformation will be isomorphic, and will not alter any of the formal properties of the operations it transforms. Furthermore, a projective transformation may be found which will transform any three numbers into any other three. We may therefore restrict our search for operations of the same formal properties as @ to those operations which bear the ~~sa~~