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"Galvanophone: A Hearing apparatus . . ." etc. 1951

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[ca 1951]

GALVANOPHONE, A Hearing Apparatus for the Investigation of Very Small Electric Phenomena in Living Body.

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During my investigation on the statistical nature of brain-waves, I was frequently called attention by some medical scientists, among whom may be mentioned Prof. Motokawa of the Tohoku University and Prof. Minoshima of the Hokkaido University, on the disirability of an apparatus by which one may directly hear the wave forms of brain-waves, and obtain facilities for clinical diagnosis as well as for research works.

As the brain-waves are electrical fluctuations of extremely low frequencies the main components of which being near 10 cycles per second or much lower in some cases, the ear cannot percieve them in their natural vibration frequencies. One of the possible methods to change them into audible frequencies is to make a magnetic or a film recording and play back with a speed sufficiently many times higher than that of recording. This method, however, requires special manipulations, and so cannot give immediate informations as the things are going on. A second method is to modulate the amplitude or the frequency of an audible sound of suitable frequency by the wave form of the brain-waves. This idea was put into practice on Oct. 1947.

Brain-waves were amplified by the usual resistance-capacity coupled amplifier, the output of which was used to modulate the amplitude of a sinusoidal wave of 360 c.p.s. The sound thus produced showed all the charasteristics of brain-waves very clearly, and unexperienced hearers who attended my experiment were able to distinguish indivisual brain-waves without difficulty. The result of these experiments were reported at the meeting of the Brain-wave Researchers' Association held on Nov. 1947. The first public presentation of the "brain-wave sound" was made on May 1948 at the annual meeting of the Japanese Physiological Society held at Niigata.

Frequency modulation was also tried. The effect seemed somewhat better in various respects than that of amplitude modulation. Three identical sets of amplifiers were constructed for the purpose of amplifying simultaneously three different phenomena, and their outputs were used to modulate three different frequencies which were related to each other by some suitable chord. On listening to the "chorus" thus produced, one may with some practice grasp the general characteristics of the phenomena under investigation.

All the electrical devices used in these trial experiments were operated from D.C. batteries. In practical applications, however, decided advantages are obtained by operating them from commercial A.C. line. For this purpose a battery elliminator was constructed which gave D.C. outputs of 150 volts and 6.3 volts respectively for plate and filament supply. Difficulties arising from A.C. induction were completely elliminated by careful electric and magnetic shielding, but occasional fluctuation of line voltage gave disturbing effect which was very difficult to elliminate. This elliminator together with the amplifier and modulators _ the two modes of modulation being interchageable _ were constructed to form a portable single set, and was shown to the members of Brain-wave Researchers' Association on Oct. 1948.

At this stage of experiment, I happened to have in hand an all A.C. operating direct current amplifier called "Iron Detector", which uses an extremely stable electric interruptor in the input circuit for the purpose of transforming the input voltage into an intermittent electrical vibration of about 600 c.p.s., which is amplified by a resonance amplifier. This amplifier responds steadily to 1 μV, but as the input impedance is coparatively low and so somewhat large current must be applied, this excellent apparatus is unsuitable for such cases as brain-wave or action current of heart beating where high input impedance and extremely low current require special attention.

A project was initiated to modify the Iron Detector so as to be used as a general purpose A.C. operating amplifier of very small electric phenomena especially in living bodies. With collaborations of the members of the physical and physiological sections of the Research Institute of Applied Electicity and those of the maker of the Iron Detector, an experimental set was completed on March 1949, which seemed to fulfill nearly all the requirements for practical application, and was named "Galvanophone" The details of this apparatus are described in the following.

In Fig.1 is shown a block diagramme of the new apparatus. (I) is a double T type wave filter which, consisting entirely of resistances and capacities, elliminates practically all the electrostatic pick-ups from A.C. line, so that one may dispense with those inconvenient devices for shielding the input circuit which have been for example used in the brain-wave study. Only in the worst condition one is needed to rearrange the general lay out or to apply simple devices for shielding.

The vibrating interruptor (II) is an electromagnetic vibrator, on the vibrating reed of which is attached an electric contact device made of special matal. The original form of the contact mechanism as supplied by the maker is shown in Fig.2 (A). In order to avoid jumping effect produced by the collision of the vibrating reed with the fixed electrode, the amplitude of vibration is so adjusted that the reed just touches the fixed electrode at its extreme position. The proportion of the duration of make to that of brake is very small, so that the resulting wave form is the so-called impulse waves. As only the fundamental component of this wave form is amplified in the following stages, and its amplitude is very small as compared to the actual height of the indivisual impulse, this mode of interrupting is very disadvantageous for our purpose. Fig.2 (B) shows an improved contact mechanism which gives nearly equal duration of make and brake, and the jumping effect is avoided by a simultaneous motion of both electrodes with the same phase but with slightly different amplitudes. The square shaped wave thus produced is very steady and its fundamental component is sufficient enough to give necessary amplification.

Between the interruptor and the amplifier is inserted a high pass filter (III) which passes freely the frequencies to be amplified and stops the low frequency components of the grid current of the first amplifying tube from entering into the interruptor. The filter consists simply of two ~~seice~~ stages of series capacity and shunt resistance.

The amplifier (IV) is of the usual resistance-capacity coupled type using three 6C6 type tubes, the only difference being that relatively small coupling capacities are used so that only the higher frequencies (above 500 c.p.s.) are amplified. The gain is about 100-120 DB. Noises from various sources, especially those from the first tube are amply present at this stage. A.C. ham originating from the filament of the first tube also becomes considerable in spite of the above precaution. They are, however, completely elliminated by the following heterodyne filter (V), except those which cannot be avoided in principle.

The circuit in (V) is essentially the same as that of the usual heterodyne frequency converter, using 6L7 as the mixer tube. The frequency to be mixed is identical with that of the amplified one, and its voltage is taken from the oscillator (XI) which was used to excite the interruptor. In the plate circuit of the mixer tube, a low pass filter is inserted which just allows those narrow frequency band which are contained in the input electrical variation _ for example 20 c.p.s. in the case of brain-waves _ to pass through. Thus we have an amplified waves identical in shape with those of the input except for a small amount of noises which have passed through the heterodyne filter. Although the noises may be reduced to any extent as the width of the pass band is made narrower, but this means a sacrifice of faithful reproduction.

In order to make the output of the heterodyne filter audible, this is again used to modulate an audio frequency, the same frequency being employed as the previous one. For this purpose a balanced vacuum tube modulator operating on square law characteristics is used (VI). After one stage of voltage amplification (VII), the modulated wave is rectified by a diode tube which is negatively biassed so as to suppress the low noise level described above. The last stage (IX) is a power amplifier, and the output is ready to operate ~~on~~ a meter, a speaker or an oscillograph.

In addition to those described above, a vacuum thermopile is provided which is used to produce a small D.C, voltage for the purpose of zero balancing (X). The entire circuit diagramme is given in Fig. 3.

The Galvanophone responds to small electrical fluctuations of frequencies 0-20 c.p.s. with voltages as low as 5μV. It is all A.C. operating, and without any shielding of the input leads, no troubles arouses from A.C. pick ups. Probable fields of application, with some modification when necessary, are

(i) clinical application of action currents produced by heart beating,
(ii) brain-wave study and its practical application,
(iii) various electro-physiological studies,
(iv) measurement of temperature by thermocouples,

and so on. The apparatus was exhibited at the annual meeting of Japanese Physiological Society held at Kyoto this year.

(Sept. 25, 1949.)

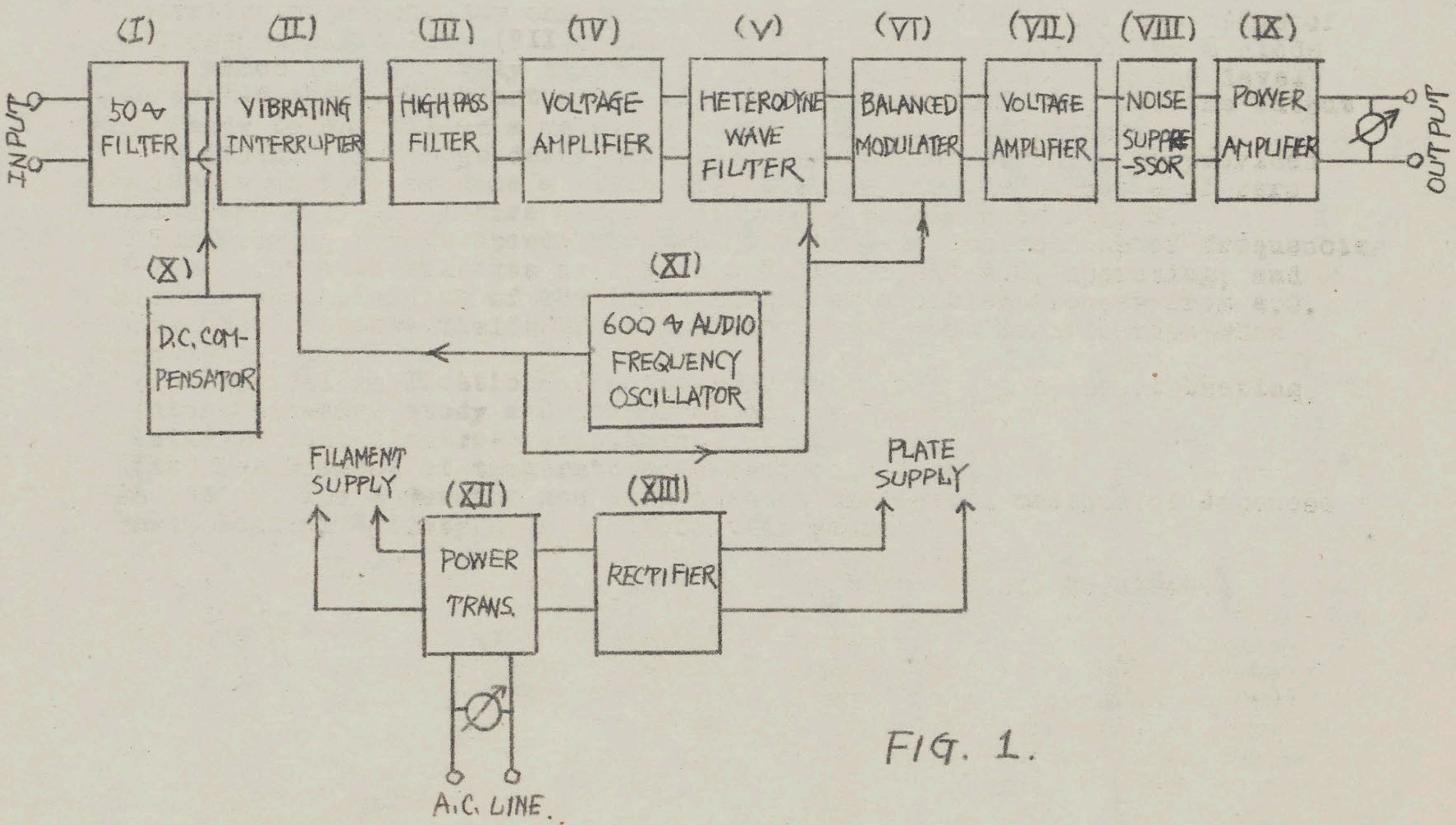

FIG. 1.

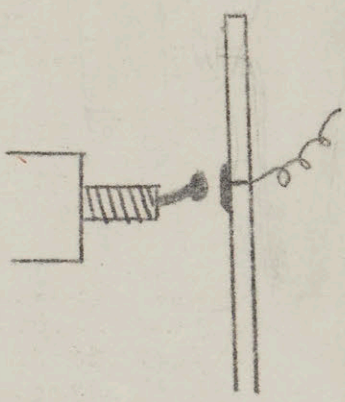

FIG 2 (A)

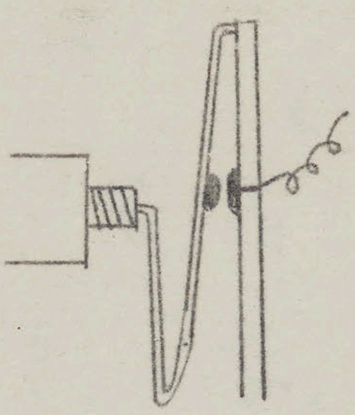

FIG 2 (B)

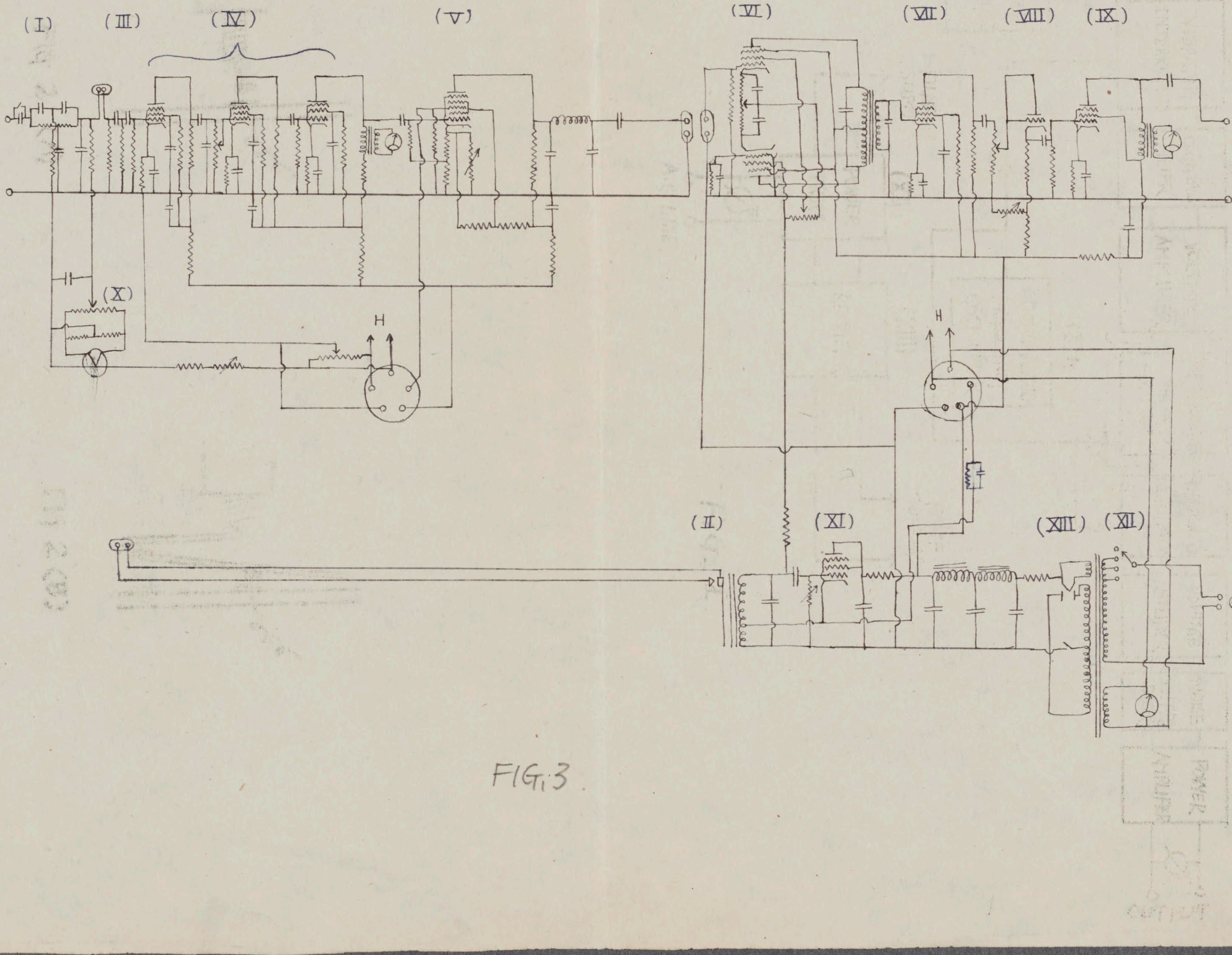

FIG. 3

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ON THE LONG PERIOD FORECASTING BY MEANS OF HARMONIC ANALYSIS

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ON THE LONG PERIOD FORECASTING BY MEANS OF HAPMONIC ANALYSIS

by
Katsumi Imahori and
Teisaku Kobayashi

1. Introduction

The most systematic formulation which has ever been made of the theory of prediction of stationary time series is, to the best of the anthors' knowledge, that of Kolmogoroff and Wiener[1]) which has recently been developed independently in U.S.S.R. and in U.S.A. Their theory is essentially a minimization problem in which a linear transformation is sought such that when applied to the past and present values of a stationary time series it guves the future values of the time series concerned with as small errors as possible. It is mathematically rigorous, and covers wide field of applications, so that it appears as if no room is left for any essentially new contributions except for possible extentions and applications following the lines established by the above mentioned authorities.

Meanwhile in the metorological practice of weather forecasting there are two leading principles which characterize the various existing methods of forecasting. The one uses statistical methods such for example as the correlation coefficient between rainfall at a particular district and temperature of sea water at another. Method of periodogramme analysis may also be classified into this category. These statistical methods have one characteristic feature in that they can do without having any regard to possible physical mechanisms or causality relations between the quantities concerned. By the other principle of forecasting on the other hand one seeks for some physical law which governs the quantities entering into the phenomena in question, and which may be effectively used for the purpose of prediction. The two principles are of course not independent. Various "theories" put forward for the purpose of weather forecasting are approximate in the sense that they can not take all the variables into account which have some interconnection with the phenomena under consideration, so that one must necessarily resort to the statistical method.

In the statistical formulation of the prediction problem, which might well be said to have been grought up to almost completeness by the hand of Kolmogoroff and Wiener, the physical bases or assumptions on which all the mathematical theories are built, and the physical meaning of various functions and formula occurring in them are apt to be left out of consideration.

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1) N. Wiener: Extrapolation, Interpolation, and Smoothing of Stationary Time Series, New York, 1949.

Statiscical treatnemts of e. g. meteorological data may sometime lead to deterministic physical laws in case where the probability becomes unity, but these are special cases of minor importance. In the theory of e.g. Brownian motion, the observed irregular motion of a particle has a certain statistical regularity which may be expressed by the well-known Langevin equation, so that in general one might expect some physical law which, although unable to give precise prediction in a deterministic sense, expresses the interrelation of the mechanism existing in the phenomena under consideration and enables one to draw conclusions as to the effect produced under given conditions.

In their study on the statistical analysis of brain-waves, one of the present authors and Dr. K. Suhara[2)] have formulated a theory in which a linear operator is sought such that when operated on the observed brain-wave this is transformed into a completely random time series. The operational equation established in this way reduces in the case of Brownian motion to the Langevin equation, and thus it is to be regarded in general as a tentative physical law in the above sense. While in the case of brain waves nothing is known at present as to the mechanism of generation, so that a purely statistical attack has been the only one available for any systematic formulation, there are many examples in which informations from different sources can be utilized in assuming a physical model which is governed by some known physical law. In view of the most effective application of the statistical theory to the meteorological forecasting, the most interesting problem is how to combine these two methods of attack into a single formulation.

Although some methodological consideration on this problem has been made by the same author in another field of study 3), the application of the same method to meteorological forecasting was not put into practice until last summer when Dr. K. Takahashi of the Meteorological Research Institute visited Sappero and held a lecture on the method of periodogramme analysis applied to his researches on seasonal forecasting. When the authors' theory has been formulated to a certain extent and some numerical results obtained as to the probable temperature of this winter at Sapporo, the authors were made aware of the above mentioned works of Kolmogoroff and Wiener. The present paper is a revised formulation of the manuscript prepared for presenting to the annual meeting of the Japanese Meteorological Society held in Nov. 1950. The authors do not pretend to have given a completed theory, but it is hoped that their contribution adds something new to the development of the prediction theory as a physical science.

2. Fundamental Assumptions

Let the quantities which are used to describe the state of the system under consideration be expressed by functions $x(t, \gamma)$'s of time t, in which a parameter γ, assuming continuous or discrete set of values, is used to distinguish different quantities. In case the variable x depends in a completely deinite way on the independent variable t, $x(t)$ is said to be a causal process, and the procedure by which this isdetermined from a set of given conditions may be formulated as follows.

2) K. Imahori and K. Suhara: Folia Psych. et Neul. Jap. Vol. 3, No. 2. 137. 1949.
3) K. Imahori: Bulletin of the Res. Inst. Appl. Elect., Vol. I, No. 1, 1949.

A system of finite or denumerably infinte number of functions $q_1(t), q_2(t), \cdots q_n(t)$ is introduced which are derived from $x(t)$ by a set of transformations

$$q_i(t) = K_i\{x(t)\}, \quad i = 1, 2, \cdots n \qquad (1)$$

where $K_1, K_2, \ldots$ are operators which transform the function $x(t)$ into $q_1(t), q_2(t), \cdots$ respectively. The n-dimensional space defined by the variables $q_1, q_2, \ldots, q_n$ may be used to represent possible states of the system, and is called the phase space of the system. Starting from a point $q_{10}, \cdots q_{n0}$ on which the system finds itself at a particular time t = 0, one may successively follow the path of the representatice point as time proceeds, provided that the limit of the rate of change in coordinates in a small time interval Δt exists for $\Delta t \to 0$ and is defined as a single valued function of coordinates 4), i.e.,

$$\frac{dq_i}{dt} - F_i(q) = 0, \quad i = 1, 2, \cdots n \qquad (2)$$

The problem thus reduces to the solution of these simultaneous differential equations under given initial conditions. In dynamical systems they correspond to the equation of motion expressed in Hamilton's canonical form, and the functional forms of $F_1, F_2, \cdots F_n$ are determined by the dymamical structure of the system. In the present case the equations (2) are also called equations of motion of the system, and the functions F_i's are regarded to be characteristic of the system considered. The number of dimensions n should also be characteristic of the system in order that the above requirement of unique determination of the process is to be fulfilled, while it is to a certain extent a matter of convenience, what kind of transformations which were introduced in (1) is to be adopted. Linear trnasformations are generally used, such for example as

$$q_i(t) = \frac{d^{i-1}x}{dt^{i-1}}, \quad i = 1, 2, \cdots n \qquad (3)$$

$$q_i(t) = x(t + \overline{i-1}\,\tau), \quad i = 1, 2, \cdots n \qquad (4)$$

So much for the causal process. A random process is one which is not a causal process, so that the variables are not determined uniquely as functions of the time, the only available information being their probability distributions when the measurement is repeated a sufficient number of times. Using the same transformation (1), the increment Δq_i of each variable in a short time Δt are distributed according to some probability law. This may be expressed by a conditional probability

4) The more general case where the functions F_i's contain time explicitly is not considered here although the generalization might not be very difficult. In equation (2) q stands for $q_1, q_2, \cdots q_n$, the same convention will be frequently used throughout this paper.

function depending upon the coordinates q_i, $q_i' = q_i + \Delta q_i$ andthe time interval Δt 5), such that when the coordinates are known to be $q_1, q_2, \cdots q_n$ at time t , the probability that they ~~ø~~lie between $q_1', q_2', \cdots q_n'$ and $q_1'+dq_1', \cdots$ at time $t+\Delta t$ is given by

$$P(q_1, q_2, \cdots q_n / q_1', q_2', \cdots q_n' ; \Delta t)\, dq_1' \cdots dq_n' , \qquad (5)$$

Here is involved the assumption that the process is a Markoff process in which the dpendence of the distribution function on coordinates is restricted only to the initial coordinates whatever may be the history previous to it. The plausibility of this assumption might be seen in the similar situation as stated in the case of causal processes.

Using (5) the first and second moments of the changes in the coordinates in a small time interval Δt are given by

$$\left.\begin{aligned} a_i(q, \Delta t) &= \int \cdots \int (q_i' - q_i)\, P(q, q', \Delta t)\, dq_1' \cdots dq_n' , \\ b_{ij}(q, \Delta t) &= \int \cdots \int (q_i' - q_i)(q_j' - q_j)\, P(q, q', \Delta t)\, dq_1' \cdots dq_n' \\ i, j &= 1, 2, \cdots n \end{aligned}\right\} \qquad (6)$$

It is assumed that in the limit $\Delta t \to 0$, all the a_i's and b_{ij}'s become proportional to Δt , so that

$$\left.\begin{aligned} A_i(q) &= \lim_{\Delta t \to 0} \frac{a_i(q, \Delta t)}{\Delta t} , \\ B_{ij}(q) &= \lim_{\Delta t \to 0} \frac{b_{ij}(q, \Delta t)}{\Delta t} , \end{aligned}\right\} \quad i, j = 1, 2, \cdots n \qquad (7)$$

exist. Then it can be shown that the generalized Fokker-plank equation

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left[A_i(q) \cdot P\right] + \frac{1}{2} \sum_{kl} \frac{\partial^2}{\partial q_k \partial q_l}\left[B_{kl}(q) \cdot P\right] , \qquad (8)$$

holds, where P is regarded as a function of $q_1, q_2, \cdots q_n$ and t , and the initial values of coordinates are contained as parameters. Thus if the functional forms of A_i's and B_{ij}'s are assumed to be known, the problem reduces to the solution of the diffusion equation (8) under the initial condition:

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5) Dependence upon the absolute position in time is also left out of constideration. C.f . foot-note on page 3 .

$$P(q_1, q_2, \cdots, q_n, 0) = \delta(q_1 - q_{10}, \cdots, q_n - q_{n0}), \quad (9)$$

where $\delta(q)$ is the so-called ~~Dd~~ Dirac's δ-function.

The direct physical meaning of the functions A_i's and B_{ij}'s is obvious from their definitions, but another interesting interpretation may be obtined in connection with a possible physical law by which the changes in time of the coordinates may be described. While the mean rate of change of the coordinate q_i is given by $A_i(q)$, the actual rate of change dq_i/dt will differ from it by a quantity which is totally unperdictable, so that one may write

$$\frac{dq_i}{dt} - A_i(q) = p_i(t), \quad i = 1, 2, \cdots n. \quad (10)$$

in which p_i's as functiins of t have the following properties:

$$\left.\begin{aligned} &\overline{p_i(t)} = 0, && i = 1, 2, \cdots n \\ &\overline{p_i(t')\, p_j(t'')} = B_{ij}\, \delta(t' - t''), && i, j = 1, 2, \cdots n \end{aligned}\right\} \quad (11)$$

The use in the second equation of the same notation B_{ij}'s as in (7) is justified by calculating the second moments. Thus from (10) one gets

$$\Delta q_i = A_i(q)\, \Delta t + \int_0^{\Delta t} p_i(t)\, dt,$$

$$\overline{\Delta q_i\, \Delta q_j} = A_i(q)\, A_j(q)\, (\Delta t)^2 + \iint_0^{\Delta t} \overline{p_i(t')\, p_j(t'')}\, dt'\, dt'',$$

$$\lim_{\Delta t \to 0} \frac{\overline{\Delta q_i\, \Delta q_j}}{\Delta t} = B_{ij},$$

which was to be shown.

It is interesting to note that the equation (10) may be regarded as a generalization of the equations (2), the functions $A_i(q)$ of the former corresponding to the functions $F_i(q)$ of the latter, and the functions $p_i(t)$ resembling the external random "forces" in the case of random processes. They play the same role as the so-called Langevin equations in the theory of Brownian motion, and thus can be regarded as representing a possible physical model of the system.

3. Theory of Linear Prediction

The differential equation (10) of the preceding article can not always be considered as linear, because one has no a priori knowledge as to the reason why the functional form of $A_i(q)$ should assume some particular structure except when this is given or at least assumed from the known structure of the system in question. It is easy to give simple examples in which the the phenomena are governed by nonlinear laws, and so their complete formulations have not yet been obatined. There seems however to exist one way to get rid of this difficulty. The key point is that a statistical ensemble of any physical systems, linear or non-linear, might be considered as equaivalent to a another one of appropriately chosen linear systems. Some consideration along this line are now being made, but the details will not be described here, and the present report will deal only with the case where a linear law can be assumed to exist.

Assuming that $A_i(q)$ is linear in coordinates, one puts

$$A_i(q) = \sum_{j=1}^{n} a_{ij} q_j \quad , \quad i=1,2,\cdots n$$

so that the equation (10) becomes

$$\frac{dq_i}{dt} - \sum_j a_{ij} q_j = p_i(t) , \quad i=1,2,\cdots n \tag{12}$$

where the coefficients $a_{ij}'s$ are now considered as constants characteristic of the system. The corresponding Fokker-Planck equation may be written

$$\frac{\partial P}{\partial t} = -\sum_{ij} a_{ij} \frac{\partial}{\partial q_i}[q_j P] + \frac{1}{2}\sum_{ij} B_{ij} \frac{\partial^2 P}{\partial q_i \partial q_j} . \tag{13}$$

The solution of (12) or (13) can be obtained in various forms. It is convenient to begin with an orthogonal transformation defined by

$$z_i = \sum_{j=1}^{n} c_{ij} q_j \quad , \quad i=1,2,\cdots n \tag{14}$$

such that

$$\sum_j c_{ij} a_{jk} = \lambda_i c_{ik} \quad , \quad i,k=1,2,\cdots n \tag{15}$$

where $\lambda_i's$ are solutions of a determinantal equations

$$\mathrm{Det.}\,(a_{ij} - \lambda\,\delta_{ij}) = 0 . \tag{16}$$

Then the differential equation will be reduced to

$$\frac{dz_i}{dt} - \lambda_i z_i = \pi_i(t) \quad , \quad i=1,2,\cdots n \tag{17}$$

where

$$\pi_i(t) = \sum_j c_{ij}\, p_j(t) , \tag{18}$$

so that

$$\overline{\pi_i(t')\pi_j(t'')} = \sigma_{ij}\,\delta(t'-t'') = \sum_{kl} c_{ik} c_{jl} B_{kl} \cdot \delta(t'-t''), \tag{19}$$

and the Fokker-planck equation becomes

$$\frac{\partial P}{\partial t} = -\sum_i \lambda_i \frac{\partial}{\partial z_i}[z_i P] + \frac{1}{2}\sum_{ij} \sigma_{ij} \frac{\partial^2 P}{\partial z_i \partial z_j} . \tag{20}$$

The solution of the last equation was given by Ming Chen Wang and G. E. Uhlenbeck [6], thus

$$F(\xi,t) = \operatorname{Exp}\left[-i\sum_j \xi_j z_{j0} e^{\lambda_j t} + \frac{1}{2}\sum_{jk}\sigma_{jk}\frac{\xi_j\xi_k}{\lambda_j+\lambda_k}\left\{1-e^{(\lambda_j+\lambda_k)t}\right\}\right], \tag{21}$$

where $F(\xi,t)$ is the Fourier transform of $P(z,t)$, and z_{j0} the initial value of z_j. The probability function $P(z,t)$ is thus an n-dimensional Gaussian distribution with the average value

$$\bar{z}_i = z_{i0}\, e^{\lambda_i t}, \quad i = 1, 2, \ldots, n \tag{22}$$

and the variances

$$\overline{(z_i-\bar{z}_i)(z_j-\bar{z}_j)} = -\frac{\sigma_{ij}}{\lambda_i+\lambda_j}\left[1-e^{(\lambda_i+\lambda_j)t}\right]. \tag{23}$$

The solution of the Langevin equation (17) can also be obtained easily:

$$z_i(t) = \int_0^\infty \pi_i(t-t')\, e^{\lambda_i t'} dt' = \int_{-\infty}^{t} \pi_i(t')\, e^{\lambda_i(t-t')} dt', \tag{24}$$

$$i = 1, 2, \ldots n$$

It is interesting to note that the auto-and cross- correlation functions for z_i's are intimately related to the above expressions (22) and (23), which characterize the probability function. One obtains from (24)

$$Z_{ij}(\tau) = \left[z_i(t+\tau)\, z_j(t)\right]_t = \iint_0^\infty \left[\pi_i(t+\tau-t')\,\pi_j(t-t'')\right]_t e^{\lambda_i t'+\lambda_j t''} dt'\,dt''$$

$$= \begin{cases} -\dfrac{\sigma_{ij}}{\lambda_i+\lambda_j}\, e^{-\lambda_j \tau}, & \text{for } \tau<0 \\[2ex] -\dfrac{\sigma_{ij}}{\lambda_i+\lambda_j}\, e^{\lambda_i \tau}, & \text{for } \tau>0. \end{cases} \tag{25}$$

in which it is assumed that the real parts of λ_i's are negative, and $[\;]_t$ means average with respect to t. Thus the function $Z_{ij}(t)$ satisfies the differential equation

$$\frac{dZ_{ij}}{dt} = \lambda_i Z_{ij}, \quad \text{for } t>0. \tag{26}$$

which is obtained by equating the righthand side of (12) equal to zero. This property can be extended to more general cases in which only the linearity is assumed for the Langevin equation. Note that the average "motion" given by the equation (22) has also the same property.

From the above description it is seen that the problem of prediction has been essentially solved. Given the initial coordinates z_0's the average value of z_i's and the variances at time t later can be calculated from (22) and (23), provided that λ_i's are known. Transforming back to the original coordinates one obtains the following expressions:

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6) Ming Chen Wang and G. E. Uhlenbeck: Rev. Mod. Phys. 17, 323. 1935.

$$\left.\begin{aligned}
&P = \frac{1}{(2\pi)^{\frac{n}{2}}\Delta^{\frac{1}{2}}}\,\mathrm{Exp.}\left[-\frac{1}{2}\sum_{ij}\tilde{D}_{ij}(q_i-\bar{q}_i)(q_j-\bar{q}_j)\right],\\
&\Delta = \mathrm{Det.}(D_{ij}), \quad (\tilde{D}_{ij}) = (D_{ij})^{-1},\\
&\bar{q}_i = \sum_{jk}\tilde{c}_{ij}c_{jk}q_{k0}e^{\lambda_j t}, \quad (\tilde{c}_{ij}) = (c_{ij})^{-1}.\\
&\overline{(q_i-\bar{q}_i)(q_j-\bar{q}_j)} = D_{ij} = -\sum_{kl}\tilde{c}_{ik}\tilde{c}_{jl}\frac{\sigma_{kl}}{\lambda_k+\lambda_l}\left[1-e^{(\lambda_k+\lambda_l)t}\right].
\end{aligned}\right\} \quad (27)$$

$$q_i(t) = \sum_j \tilde{c}_{ij}\int_0^{\infty}\pi_j(t-t')e^{\lambda_j t'}dt' = \sum_j \tilde{c}_{ij}\int_{-\infty}^{t}\pi_j(t')e^{\lambda_j(t-t')}dt', \quad (28)$$

$$\left.\begin{aligned}
Q_{ij}(\tau) &= \left[q_i(t+\tau)q_j(t)\right]_t\\
&= \begin{cases}\displaystyle\sum_{kl}\frac{\tilde{c}_{ik}\tilde{c}_{jl}\sigma_{kl}}{\lambda_k+\lambda_l}e^{-\lambda_l\tau}, & \text{for } \tau<0\\[2ex] \displaystyle\sum_{kl}\frac{\tilde{c}_{ik}\tilde{c}_{jl}\sigma_{kl}}{\lambda_k+\lambda_l}e^{\lambda_k\tau}, & \text{for } \tau>0\end{cases}
\end{aligned}\right\} \quad (29)$$

$$\frac{dQ_{ij}}{dt} - \sum_k a_{ik}Q_{kj} = 0, \quad \text{for } t>0. \quad (30)$$

The last equations may be utilized to determine n^2 constants a_{ij}'s from the experimentally obtainable functions Q_{kl}'s, so that by solving (15) and (16) one obtains the coefficients c_{ij}'s of diagonal transformation and the principal values λ_i's. The "diffusion" constants σ_{ij}'s are determined from (29).

The formal procedure of prediction sketched above will be seen to be in agreement with that used by Dr. Ogawara in some of the applications of this theory of stochastic extrapolation. 7)

In practical application of the above theory, howecer, one is sure to be perplexed with the calculations involving high order determinants, for in most cases the order n of the determinants must be taken so large that neither the evaluation of these determinants nor the solution of the determinantal equation (16) can be carried on.
While Dr. Ogawara assumed a very small number of dimensions (n = from 5 to 10), in Wiener's theory all the present and past values are needed for the prediction of the future. The solution of these difficulties may ~~will~~ be found in one or more of the following projects:

(1). Device of an automatic calculator.
(2). Incorporation of various existing theories on meteorological phenomena for the deterination of system constants.
(3). Formulation of some approximate procedure.
In the remaining part of this report will be described some considerations on the 3rd problem listed above.

4. Decomposition of Time Series and the Application of Harmonic Analysis 8)

Let the set of n time series $q_i(t)$'s be so chosen that one is possible to devide them into sets of functions $q_1(t), q_2(t), \cdots q_m(t)$ and $q_{m+1}(t), \cdots q_n(t)$, where the functions of each set belong to different frequency regions, so that using the formal Fourier transforms of each time series

$$q_k(t) = \int_{-\infty}^{\infty} f_k(\nu)\, e^{2\pi i \nu t}\, d\nu, \quad k=1,2,\cdots n \qquad (31)$$

one has

$$f_i(\nu)\cdot f_j(\nu) = 0, \quad i=1,2,\cdots m;\ j=m+1,\cdots,n. \qquad (32)$$

In this case the time series belonging to different sets are statistically independent. In particular one has

$$[q_i(t+\tau), q_j(t)]_t = 0, \quad i=1,2,\cdots m;\ j=m+1,\cdots n. \qquad (33)$$

It is further assumed, as is often practically the case, that the characteristic solutions

$$\left.\begin{aligned} z_i &= z_{i0}\, e^{\lambda_i t}, && \text{for } t>0 \\ &= 0, && \text{for } t<0 \end{aligned}\right\} \quad i=1,2,\cdots n \qquad (34)$$

have their amplitude spectra falling entirely into one or other of the two frequency regions. Let $\lambda_1, \lambda_2, \cdots \lambda_m$ belong to the first set and the remaining ones to the second. Then from (28) one has

$$q_i(t) = \sum_{j=1}^{m} \tilde{c}_{ij} \int_0^\infty \pi_j(t-t')\, e^{\lambda_j t'}\, dt', \quad i=1,2,\cdots,m$$

$$= \sum_{j=m+1}^{n} \tilde{c}_{ij} \int_0^\infty \pi_j(t-t')\, e^{\lambda_j t'}\, dt', \quad i=m+1,\cdots,n.$$

so that the Langevin equation will also be separated,

$$\frac{dq_i}{dt} - \sum_{j=1}^{m} a'_{ij}\, q_j = p_i(t), \quad i=1,2,\cdots m,$$

$$\frac{dq_i}{dt} - \sum_{j=m+1}^{n} a'_{ij}\, q_j = p_i(t), \quad i=m+1,\cdots,n.$$

Thus the problem is reduced to ones of lower order. This reduction may be carried on until the overlapping of the amplitude spectra of different characteristic solutions (34) violates the assumption used inthe above argument. Even in the latter case one may proceed further under tolerable approximations. In favorable cases one would be possible to attain complete separation of the different λ_i's by the above procedure without appreciable errors, so that it is finally reducedto a number of simplest types of Brownian motion.

At this point it would easily be recognized that the method of harmonic analysis plays an important role in the practical applications of the present theory. In the following will be

7)M. Ogawara: Reports from the Central Met. Obs. No. 24. 1949.
M. Ogawara and T. Fujita: Forecasting of Wolf's sunspot Numbers by Stochastic Extrapolation (unpubl.)

8) The simplest case of the prediction of a single time series is considered here. Extension to mutiple time series will be trivial.

First start with the observed time series $x(t)$ whose values are supposed to be known from sufficently large negative value of time $-T$ to $t=0$. When the values of $x(t)$ are given for discrete set of times, one may either regard them to be substituted by a continuous curve which is obtained by the usual method of curve fitting, or replace various integrals in the formulation for continuous case by proper summation in the discrete case. As the nature of errors introduced by these modifications may be computed by the well-established method of Fourier integrals, it will be left out of considertion here.

Given the function $x(t)$, one may easily calculate its Fourier transform

$$A(\nu) = \int_{-T}^{0} x(t)\, e^{-2\pi i \nu t}\, dt, \tag{35}$$

and its absolute magnitude $|A(\nu)|$. It must be noted here that the amplitude spectrum thus obtained contains a certain amount of indefiniteness in the sense that its fine structures in the frequency bands within a definite frequency difference $\Delta\nu = 1/T$ are physically meaningless. The amplitude spectrum given by $|A(\nu)|$ will usually consist of a number of maxima and minima showing more or less conspicuous predominancies in certain frequency bands, each one of which corresponding to one or more characteristic values discussed above. The frequency difference of adjacent maxima must of course be greater than $1/T$ owing to the above mentioned indaterminacy, however, only those maxima should be considered as significant ones whose adjacent frequency differences are appreciably greater than $1/T$, so that if there exists a definite upper bound in frequency — this is the case for example in discrete time series — the number of characteristic values should be taken at most equal to $2|\nu_{max}|T$.

Now if some of the minima in the spectral curve are found to be negligibly small and almost touch the zero axis, one may safely take them as deviding points by which the characteristic solutions (34) are completely separated into a number of groups, As the other extreme case one may consider the one in which two or more characteristic "resonance" frequencies overlap each other so that they fuse into a single maximum.
The real situation is, howecer, that one does not know a priori how many resonance frequencies there are in a given frequency band, but it is rather the spectral curve itself that gives any information about them. One "assumes" that a single maximum in the spectral curve corresponds to a single resonance frequency. The amount ~~if-erresponds-to-a-single-resonance-~~of error due to this assumption may be calculated in some special cases where some known functional forms are substitured in place of the experimental curves.

The intermediate case in which the adjacent maxima in the spectral curve are partly resolved but not completely, are the ones where most of the ambiguities occur. As an approximate procedure for these cases, one may take either of the two ways: one assumes a single characteristic value λ for a group of subsidiary maxima which as a shole constitute a single broader maximum; or one ascribes different values of characteristic solutions to each of the partly resolved maxima. It would be convenient to have some numerical criteria as to which one of the two alternativew should ve taken, but the details will be treated in a subsequent paper .

It thus came to the conclusion that in any case one can with different degrees of approximation reduce the problem into a set of harmonically bounded Brownian motions including as a special case that of free particle. Using real variables, the Langevin equations for them can be written as :

$$\frac{d^2q}{dt^2} + \beta\frac{dq}{dt} + \omega_0^2 q = p(t), \tag{36}$$

or

$$\frac{dq}{dt} + \beta q = p(t) \tag{37}$$

with

$$\overline{p(t')\,p(t'')} = 2D\,\delta(t'-t'')$$

The solutions are well known [6], i.e. the conditional probability functions are Gaussion functions with the averages and variances:

$$\left.\begin{aligned}\bar{q} &= \frac{\dot{q}_0}{\omega_1}e^{-\frac{1}{2}\beta t}\sin\omega_1 t + \frac{q_0}{\omega_1}e^{-\frac{1}{2}\beta t}\left(\omega_1\cos\omega_1 t + \frac{\beta}{2}\sin\omega_1 t\right),\\ \overline{(q-\bar{q})^2} &= \frac{D}{\beta}\left[1-\frac{1}{\omega_1^2}e^{-\beta t}\left(\omega_1^2+\frac{1}{2}\beta^2\sin^2\omega_1 t-\beta\omega_1\sin\omega_1 t\cos\omega_1 t\right)\right]\\ \omega_1^2 &= \omega_0^2-\beta^2/4\end{aligned}\right\} \tag{38}$$

and

$$\left.\begin{aligned}\bar{q} &= q_0 e^{-\beta t}\\ \overline{(q-\bar{q})^2} &= \frac{D}{\beta}\left[1-e^{-2\beta t}\right]\end{aligned}\right\} \tag{39}$$

respectively for (36) and (37). Those for the original time series can be calculated by linear superposition of $\bar{q}$'s and summing up the $\overline{(q-\bar{q})^2}$'s thus obtained for different components.

In Figs. 1—3 are shown a few exemples of practical applications, which are intended not to be used as any routine works, but only to show how effective and promising the present method is. Fig. 1a is a plot of monthly mean temperatures at Sapporo from Sep. 1944 to Feb. 1951, expressed in deviations from the mean annual change. The amplitude spectrum are shown in Fig. 1b, obtained by the usual method of harmonic analysis using 72 ordinates. The predicted errors shown by the shaded area. The circles are the observed values.

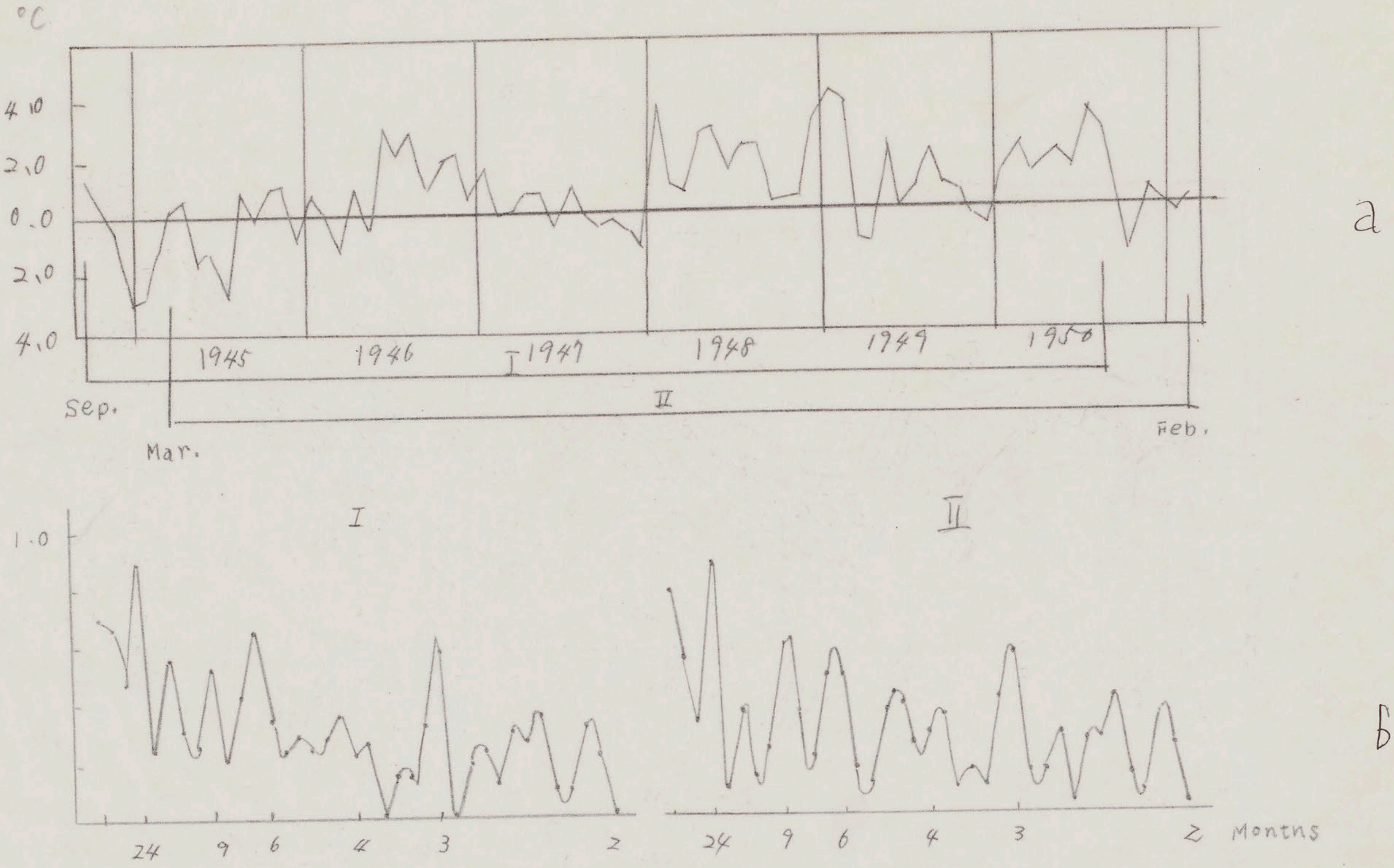
In Fig. 2 is shown another example of prediction of the monthly mean temperature obtained by a different procedure. In this case, the 72 values of mean temperature of Jan. shown in Fig.2a were subjected to harmonic analysis and used to calculate the probable temperature of Jan. in the future. The same procedures were repeated for Feb., etc. The agreement with observation-dotted circles—is fairly good.

It would be interesting to note the approximate coincidence of the two predictions based upon different periods.

values are given in Fig. 1c together with their probable-

The last example is shown in Figs . 3 a, b, c for the case of prediction of the five days mean temperature obtained by similar calculations. While all these examples are based on the method of harmonic analysis using 72 ordinates, it would naturally be expected that better prediction should result by increasing the number of ordiantes in the analysis.

In conclusion the authous wish to express their hearty thanks to Prof. T. Hori, the director of the Institue to which they belong, for the interest he have had in this work, and also to Dr. K. Takahashi and Dr. M. Ogawara, both of the Meteorological Research Institute, for the valuable discussions held about the present method. The data used for calculation in this paper were supplied by Mr. Y. Morita, of the Meteorological Observatory at Sapporo, to whom also the authors' appreciation should be expressed.


°C
4.0
2.0
0.0
2.0
4.0
1945
1946
I 1947
1948
1949
1950
II
Sep.
Mar.
Feb.
a
I
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1.0
24
9
6
4
3
2
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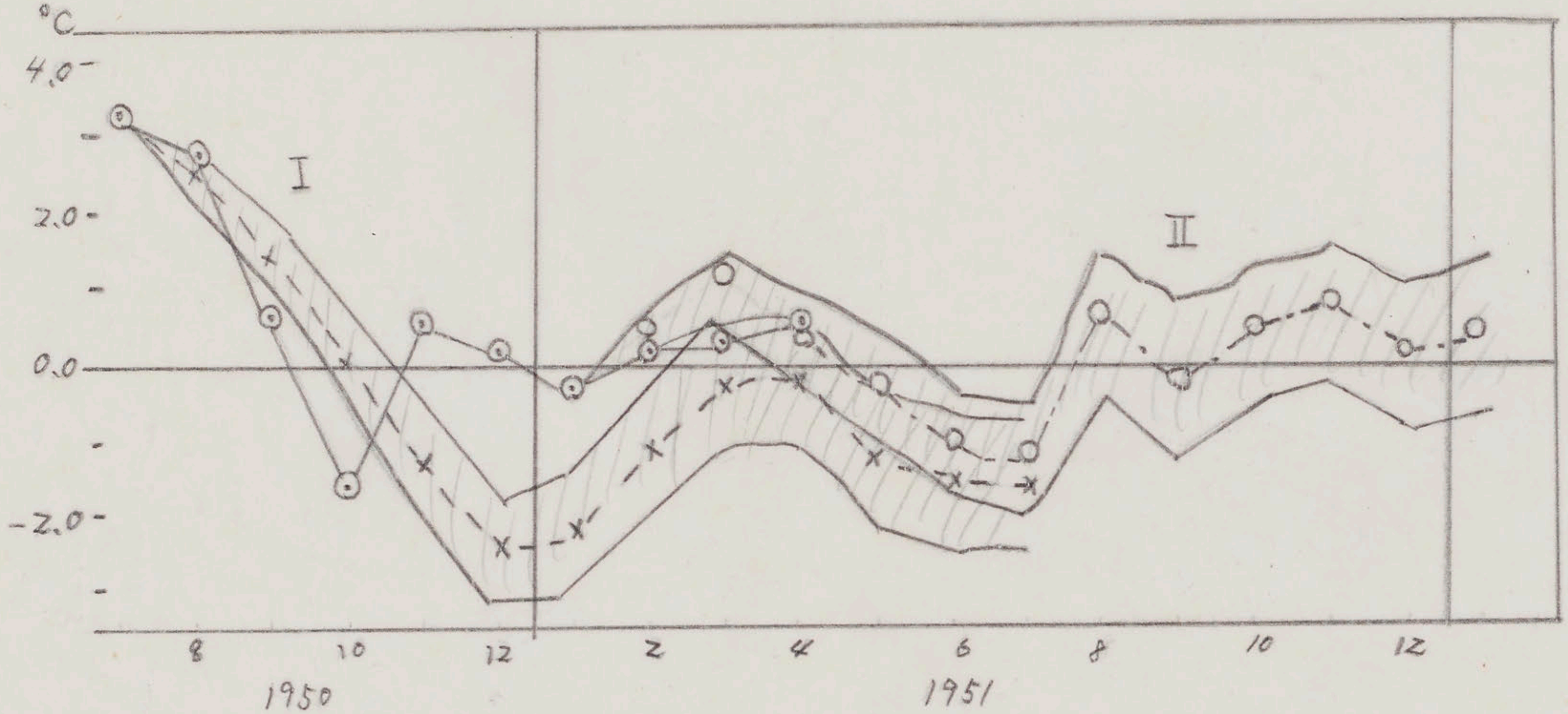
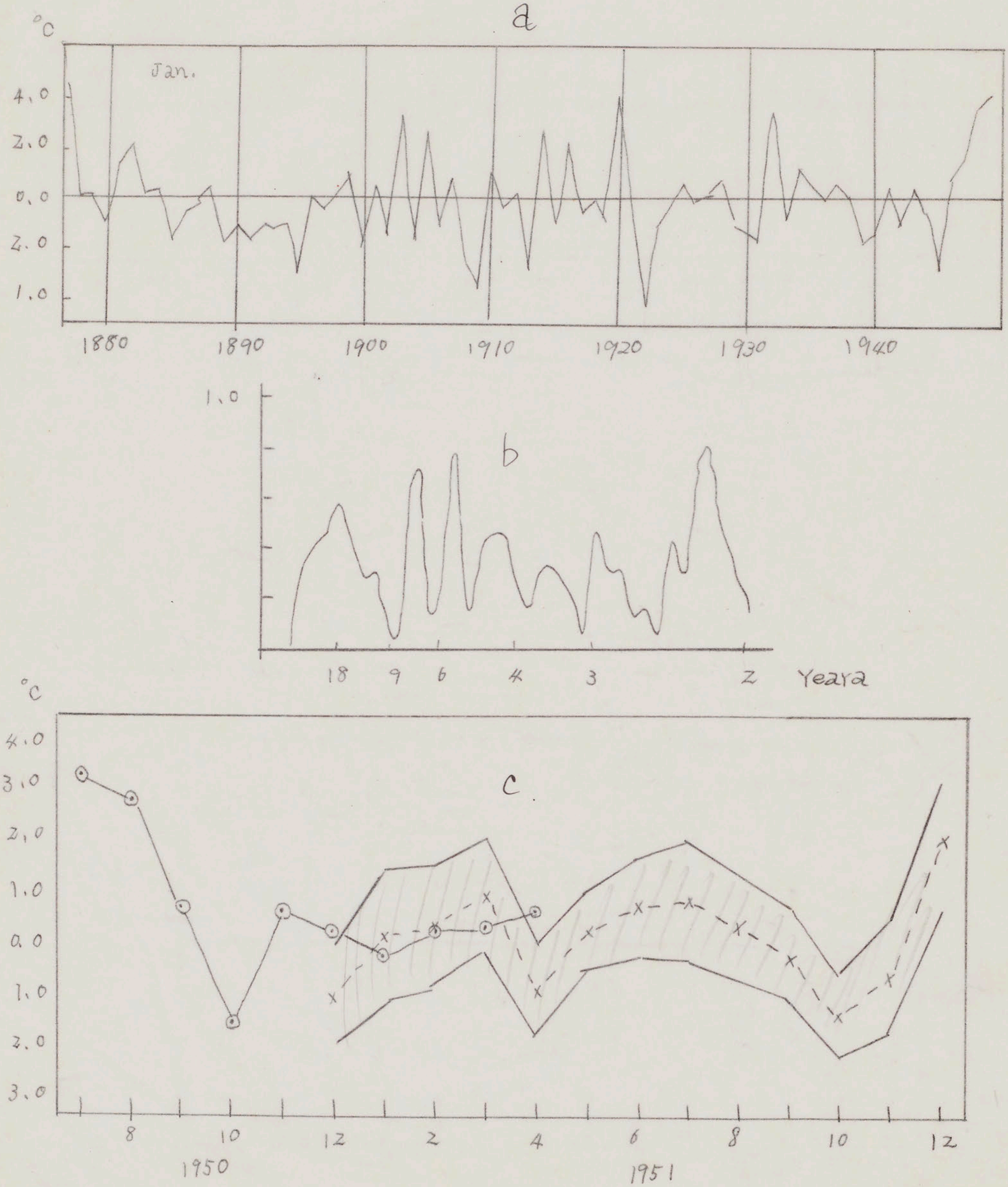
Months
б

Fig. 1.



C.

Fig. 1.


a
°C
Jan.
4.0
2.0
0.0
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1.0
1880
1890
1900
1910
1920
1930
1940
b
1.0
18
9
6
4
3
2
Yeara
c.
°C
4.0
3.0
2.0
1.0
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8
10
12
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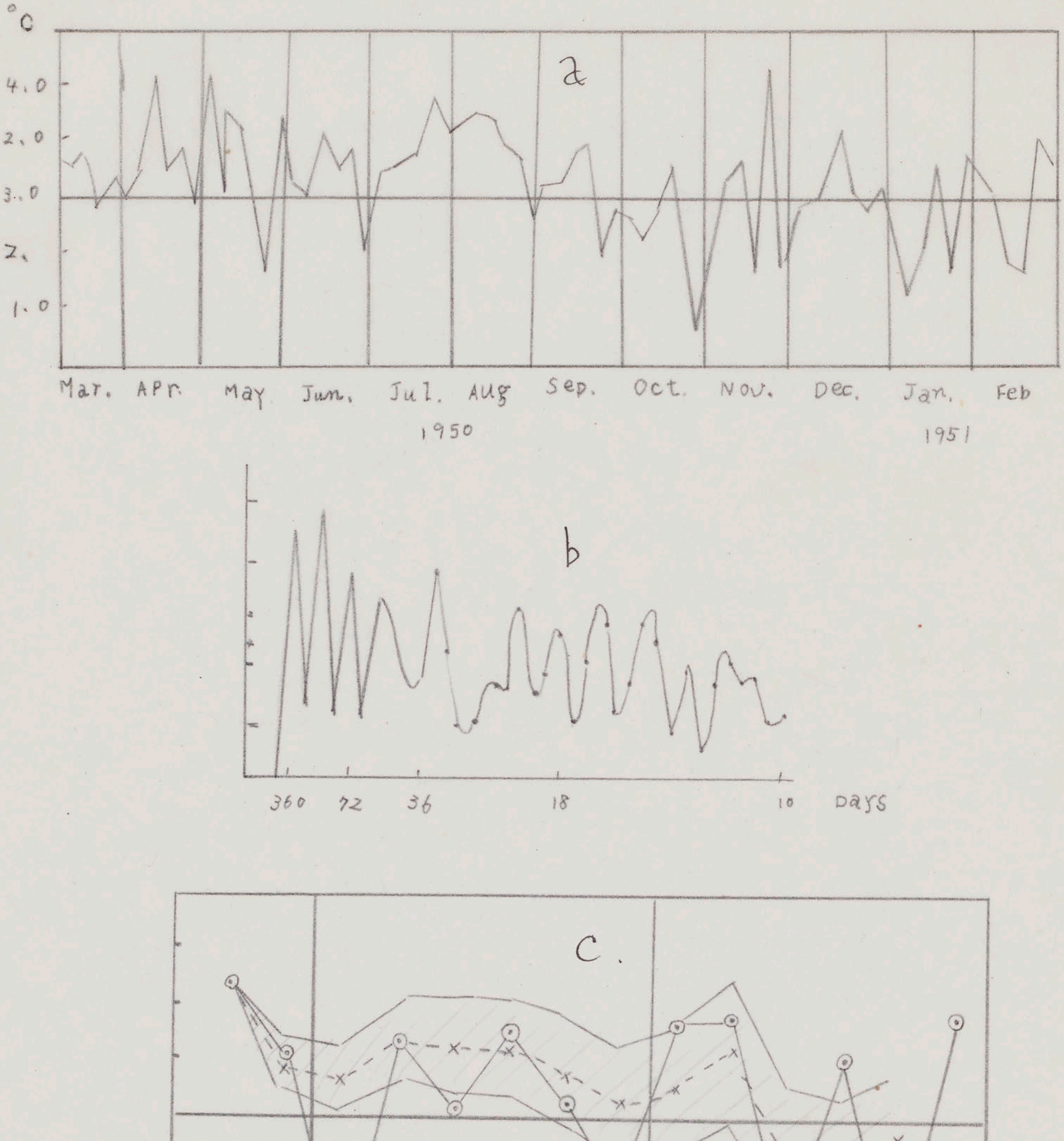
10
12
1950
1951

Fig. 2.


°C
4.0
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3.0
2.
1.0
a
Mar. Apr. May Jun. Jul. Aug Sep. Oct. Nov. Dec. Jan. Feb
1950
1951
b
360 72 36 18 10 Days
c.
Feb. Mar. Apr.

Fig. 3.

ON THE DIFFUSION BY TURBULENT MOTION

By

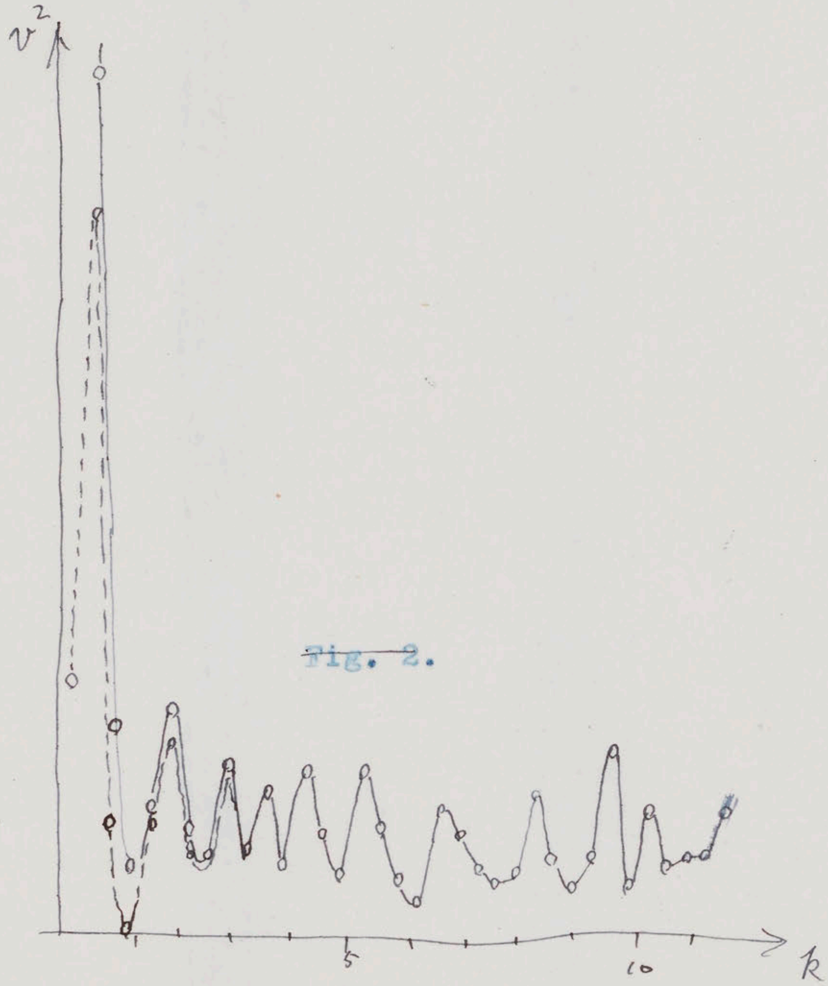
Katsumi Imahori and Jun-ichi Hori

I. Introduction.

It is a well-known fact that in treating the phenomena of diffusion in the atmosphere caused by turbulent motion, we have to set limits to the scale of turbulence according to the nature of the problem concerned. For example, the effect of turbulence produced by a woods on the diffusion of fog or heat in the atmosphere should be described in terms of those turbulent motion whose scales are comparatively smaller than the dimensions of the space in which the diffusion takes place; i.e. the average interval between trees or branches or leaves in the case of diffusion in the woods, the average height of trees in the case of diffusion behind the woods, the average extension of the woods in the case of diffusion in the horizontal plane, or still larger scales in the case of diffusion in the upper atmosphere. Those turbulence whose scales are larger than the respective dimensions have only to be taken into account as changes in the mean flow.

In the case of homogeneous turbulence extending infinitely in space, the local difference in the distribution of scales of turbulence does not come into play, as in the analogous case of white spectrum in optical phenomena, and thereby the theoretical treatment becomes considerably simple. The existing theories of turbulence has been almost confined to such cases. When, however, we consider the turbulent phenomena occurring near or in the woods, the turbulence having some particular scale plays an important role, as in the case of selective absorption in optics. A marked example of such phenomena was provided by our observation at Ochiishi which was carried out in July, 1950. Fig. 1 is the map of the region near Ochiishi, in which the shaded portion represents the woods we chose for observation. Arrows indicate the direction of prevailing wind. Fig. 2A and B show the energy spectra of turbulent flow at A and B(i.e. in front of and behind the woods) respectively. These were obtained from the observations each lasting 10 minutes and simultaneously carried out at A and B , by harmonic analysis for 72 terms and averaging over three data corresponding to observations at three slightly different points. Eliminating the spectra which correspond to the homogeneous isotropic turbulence in free space ($k^{-5/3}$ -law), we obtain the ones shown by broken lines. It is a very remarkable fact that Fig. 2A shows regular array of highly distinct frequency bands, while such regular structure wholly disappears in Fig. 2B. Fig. 3 and 4 show the energy spectra which were obtained from the observations within the woods near A and at a vacancy in the woods also near A, respectively. Both of these spectra have highly regular structure similar to that in Fig. 2A. Such a structure in the spectrum is presumably due to the peculiar conformation of the land along the coast. (For the verificatiin of this presumption, however, further investigations are necessary.) Closer examination of Fig. 3 and 4 reveals several interesting features, some common to both spectra and some characteristic to each spectrum. The meaning of these features which may probably be looked for in connection with the characteristics of woods, will be investigated in future.

Fig. 1.



~~Fig. 2.~~

Fig. 2A

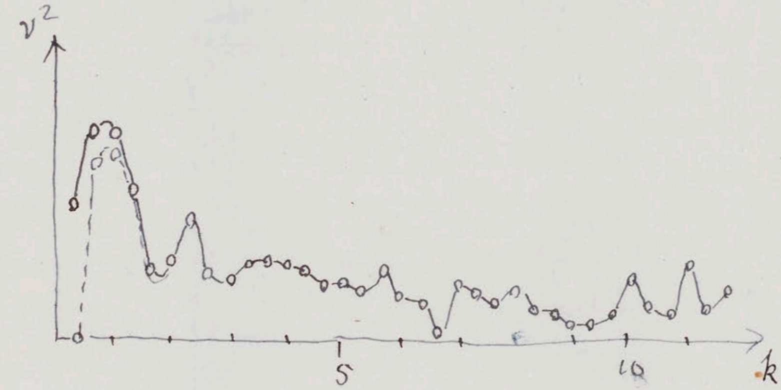


Fig. 2B.

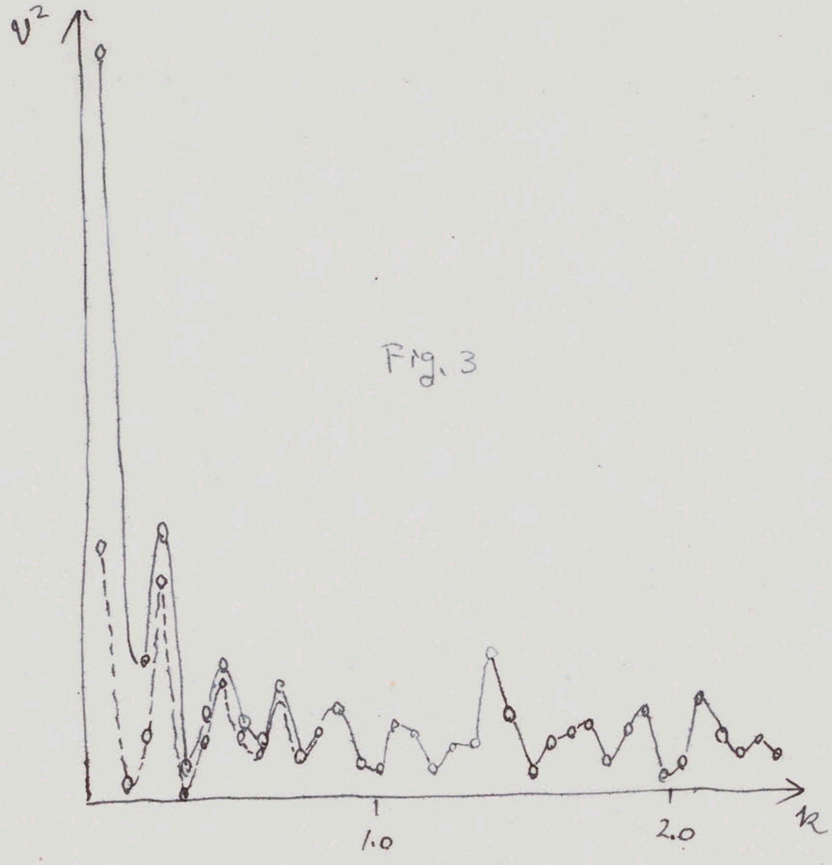


Fig. 3

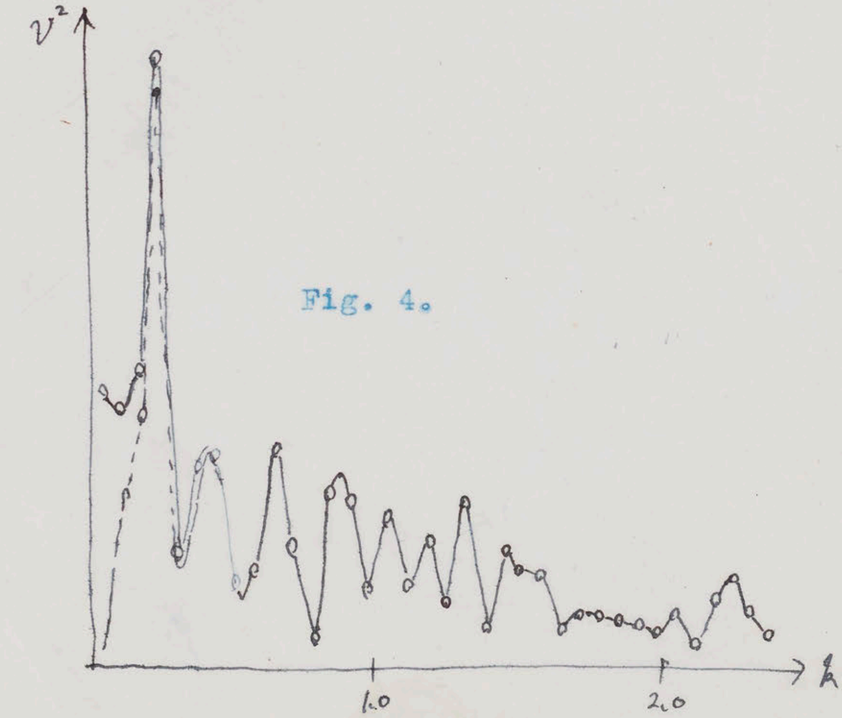


Fig. 4.

Thus it becomes very important to consider the diffusion due to some particular part of the turbulent mass which has a definite scale and is picked up from the mixture of turbulent masses having various magnitudes. The method of treating such a problem have hitherto involved so many ambiguities, that in some cases even an erroneous method has been accepted. The purpose of the present paper is to clarify these points as much as possible, and further to develop a general method for treating the problem of diffusion by turbulent motion, although it has not so far been able to reach the final step.

2. The Method of Correlation Function.

We have merely to consider the motion of the wind. For simplicity we treat here only the one-dimensional case. Extension to the 3-dimensional case will involve no serious difficulty.

Let u(t) be the component ~~ø~~ in a suitably chosen direction of wind-celocity at a fixed position, the mean value of which being assumed to be zero. As the statistical quantity which is deduced from u(t) and plays the fundamental role in the problem of diffusion, G. I. Taylor considered the function:

$$U(\tau)=\overline{u(t)u(t+\tau)}. \tag{1}$$

$U(\tau)/U(0)$ is the so-called " auto-correlation function" of u. Now put

$$\frac{dx}{dt}=u,$$

and form

$$X(\tau)=\overline{x(t)x(t+\tau)}, \tag{2}$$

determining the integration constant so that the average value of x becomes zero, then it may easily be proved that

$$\frac{d^2X}{d\tau^2}=-U(\tau). \tag{3}$$

Thus if we know $U(\tau)$, we can obtain $X(\tau)$ by solving (3). The function $x(t)$ as given above is in general different from the position of a particle which moves with the air. It may be regarded, however, as representing the motion of the particle, in as much as we ~~ø~~ do not ask for the motion itself but only the average value in some sense.

Integrating (3) we obtain:

$$X(\tau)=X(0)-\int_0^{\tau}d\tau'\int_0^{\tau'}U(\tau'')\,d\tau''. \tag{4}$$

Let l be the distance which was swept by the particle in τ seconds from the initial time t, then for its mean-square we have, since

$$\overline{\{x(t+\tau)-x(t)\}^2}=2\{X(0)-X(\tau)\},$$

$$\overline{l^2}=2\int_0^{\tau}d\tau'\int_0^{\tau'}U(\tau'')\,d\tau''. \tag{5}$$

The correlation function $U(\tau)$ is an even function which in general has maximum at $\tau = 0$ and tends to zero when τ becomes large. Fig. 5A and B shows two examples. For sufficiently small τ, (4) and (5) become

$$\left.\begin{aligned} X(\tau) &= X(0) - U(0)\frac{\tau^2}{2}, \\ \overline{l^2} &= U(0)\tau^2, \end{aligned}\right\} \qquad (6)$$

respectively, as naturally expected. On the other hand, an ambiguity occurs when τ increases indefinitely. There may be considered two distinct cases:

$$\text{(A)} \qquad \int_0^\infty U(\tau)\,d\tau = \text{const.} = L,$$

$$\text{(B)} \qquad \int_0^\infty U(\tau)\,d\tau = 0.$$

(A) (B)

Fig. 5

The case in which this integral becomes infinite does not come into question. Integrating (A) and (B) we get

$$\text{(A)} \qquad \overline{l^2} \to 2L\tau,$$

$$\text{(B)} \qquad \overline{l^2} \to \text{const.} \quad (\tau \to \infty),$$

respectively. In the case (A), which was treated by Taylor, in (4) loses its physical meaning, since in that case the ensemble of particles spreads indefinitely and the process cannot be treated by the method of correlation as a stationary one with respect to the coordinate x. Further criterion is therefore required in order to know whether (5) holds or not even in such a case. Similar argument applies in the case (B) when $\overline{l^2} \to \infty$. Thus it remains only the case (B) in which $\overline{l^2} \to$ const, where the process can be regarded as stationary also with respect to the coordinate, and since the value of the constant ray be taken as equal to $X(0)$, formula (4) has reasonable meaning also when $\tau \to \infty$.

From the above argument, it will be seen that the method of the correlation function involves a difficulty for infinite times. In next section we propose another method which is appropriate for treating the heterogeneous turbulence as explained in the introductory section.

3. The Method of Fokker-Planck.

Such a statistical phenomenon, ~~as~~ as the wind velocity $u(t)$ varying complicatedly from time to time, can no more be treated from the causal point of view, and we are naturally compelled to ~~take~~ have recourse to the probability theory. For this purpose we define a probability function $P(u_0/u, t)$, which will describe the characteristic statistical feature of the given phenomenon. $P(u\ \ /u, t)du$ gives the probability that the wind velocity at t t will have the value between u and $u + du$, when the initial velocity u_0 (at $t = 0$) is given. In the case of Markoff process, this function fulfilles the so-called Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial u}\left[A(u)P\right] + \frac{1}{2}\frac{\partial^2}{\partial u^2}\left[B(u)P\right], \tag{7}$$

where

$$A(u) = \lim_{\Delta t \to 0} \frac{\overline{\Delta u}}{\Delta t}, \qquad B(u) = \lim_{\Delta t \to 0} \frac{\overline{(\Delta u)^2}}{\Delta t}. \tag{8}$$

The function P, and hence the velocity distribution function

$$W(u_0, u; t) = W(u_0)P(u_0/u; t) \tag{9}$$

can be obtained as the solution of (7) under the given initial condition.

Now the probability function which is necessary for us is the one with respect to the coordinate x. $x(t)$ may not, however, be regarded as a simple Markoff process, as will be seen from physical considerations. (In the causal process, the future motion cannot be determined uniquely by giving only the value ~~f~~ of x at $t = 0$.) In order to be able to treat it still as a Markoff process, we consider x and u simultaneously, regarding them as components of two-dimensional (x-u)-Markoff process. Thus the probability function is given by $P(u_0, x_0/u, x; t)$, and the Fokker-Planck equation becomes:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(A_1P) - \frac{\partial}{\partial u}(A_2P) + \frac{1}{2}\left\{\frac{\partial^2}{\partial x^2}(B_{11}P) + 2\frac{\partial^2}{\partial x \partial u}(B_{12}P) + \frac{\partial^2}{\partial u^2}(B_{22}P)\right\};$$

$$A_1(x,u) = \lim \frac{\overline{\Delta x}}{\Delta t}, \qquad A_2(x,u) = \lim \frac{\overline{\Delta u}}{\Delta t}, \tag{10}$$

$$B_{11}(x,u) = \lim \frac{\overline{(\Delta x)^2}}{\Delta t}, \quad B_{12}(x,u) = \lim \frac{\overline{\Delta x \Delta u}}{\Delta t}, \quad B_{22}(x,u) = \lim \frac{\overline{(\Delta u)^2}}{\Delta t}.$$

As an example, consider the Brownian motion of a free particle. The equations of motion are then given by

$$\left.\begin{aligned} &\frac{dx}{dt} - u = 0, \\ &\frac{du}{dt} + \beta u = p(t), \end{aligned}\right\} \tag{11}$$

where $p(t)$ represents the completely random external force, and has the property such that

$$\left.\begin{aligned} \overline{p(t)p(t+\tau)} &= 0, \quad \tau \neq 0, \\ &= 2D\delta(\tau), \quad \tau = 0. \end{aligned}\right\} \qquad (12)$$

Integrating (11) with respect to t from t to t + Δ t, we obtain

$$\Delta x = u\Delta t,$$

$$\Delta u = -\beta u \Delta t + \int_t^{t+\Delta t} p(t')dt',$$

and hence

$$A_1 = u, \quad A_2 = -\beta u,$$

$$B_{11} = 0, \quad B_{12} = 0, \quad B_{22} = 2D.$$

Consequently the Fokker-Planck equation becomes

$$\frac{\partial P}{\partial t} = -u\frac{\partial P}{\partial x} + \beta\frac{\partial uP}{\partial u} + D\frac{\partial^2 P}{\partial u^2}. \qquad (13)$$

Solving this equation, we obtain the following average values and variances:

$$\overline{x} = x_0 + \frac{u_0}{\beta}(1-e^{-\beta t}), \quad \overline{u} = u_0 e^{-\beta t},$$

$$\overline{(x-\overline{x})^2} = \frac{D}{\beta^2}\left\{2t - \frac{4}{\beta}(1-e^{-\beta t}) + \frac{1}{\beta}(1-e^{-2\beta t}\right),$$

$$\overline{(u-\overline{u})^2} = \frac{D}{\beta}(1-e^{-2\beta t}), \qquad (14)$$

$$\overline{(x-\overline{x})(u-\overline{u})} = \frac{D}{\beta^2}(1-e^{-\beta t})^2.$$

Taking the limit t $\to$ 0 or t $\to \infty$, the variance of x becomes

$$\begin{aligned} \overline{(x-\overline{x})^2} &\to \frac{2D}{3}t^3, \quad t \to 0, \\ &\to \frac{2D}{\beta^2}t, \quad t \to \infty \end{aligned} \qquad (15)$$

respectively.

4. The Decomposition of Turbulence.

Let u(t) be the observed wind velocity, and $u_0(t)$ its average value taken over the time interval (t — T/2, t + T/2). If we decompose u(t) into

$$u(t)=u_0(t)+u'(t)$$

$u'(t)$ represents the turbulent part of the total flow. According to the length of the time interval T, the scale of the largest turbulent mass included in $u'(t)$ varies, so that we must choose the appropriate length of T, according to the nature of the problem, as suggested in the introduction. Extending this method of treatment and in order to investigate the effect on the diffusion of turbulent mass having a particular scale, we decompose u(t) into many parts:

$$u(t)=u_0(t)+u_1(t)+\cdots+u_n(t), \qquad (16)$$

where $u_0(t)$, $u_1(t)$,..... include the components whose frequencies lie between 0 and ν_1, ν_1 and ν_2, ..., ν_{n-1} and ν_n, respectively. These components can be considered as independent of each other, so that if we put

$$U_i(\tau)=\overline{u_i(t)u_i(t+\tau)}, \quad i=0,1,2,\cdots,n, \qquad (17)$$

it is seen that the additivity of correlation functions holds:

$$U(\tau)=\sum_{m=0}^{n} U_m(\tau). \qquad (18)$$

It will be possible to make this decomposition appropriately in such a manner, that the frequency spectrum of each of the $u_i(t)$'s can approximately be treated as that of a suitably chosen damped harmonic oscillator, whose Langevin equation is

$$\frac{d^2x_i}{dt^2}+\beta_i\frac{dx_i}{dt}+\omega_i^2x_i=p_i(t), \qquad (19)$$

where x_i is defined by $u_i = dx/dt$, such that its mean value becomes zero. Rewriting this into

$$\left.\begin{aligned} &\frac{dx}{dt}-u=0,\\ &\frac{du}{dt}+\beta u+\omega_0^2x=p(t), \end{aligned}\right\} \qquad (20)$$

we get the corresponding Fokker-Planck equation

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}(uP)+\frac{\partial}{\partial u}\left[(\beta u+\omega_0^2x)P\right]+D\frac{\partial^2P}{\partial u^2}, \qquad (21)$$

where for simplicity we omitted the indices. By solving this equation the following values for averages and variances are obtained P being two dimensional Gaussian distribution with respect to x and u.

$$\left.\begin{aligned}
&\overline{x}=\frac{u_0}{\omega_1}e^{-\frac{1}{2}\beta t}\sin\omega_1t+\frac{x_0}{\omega_1}e^{-\frac{1}{2}\beta t}(\omega_1\cos\omega_1t+\frac{\beta}{2}\sin\omega_1t),\\
&\overline{u}=\frac{u_0}{\omega_1}e^{-\frac{1}{2}\beta t}(\omega_1\cos\omega_1t-\frac{\beta}{2}\sin\omega_1t)-\frac{\omega_0^2}{\omega_1}x_0e^{-\frac{1}{2}\beta t}\sin\omega_1t,\\
&\overline{\omega_0^2(x-\overline{x})^2}=\frac{D}{\beta}\left[1-\frac{1}{\omega_1^2}e^{-\beta t}(\omega_1^2+\frac{1}{2}\beta^2\sin^2\omega_1t+\beta\omega_1\sin\omega_1t\cos\omega_1t)\right],\\
&\overline{(u-\overline{u})^2}=\frac{D}{\beta}\left[1-\frac{1}{\omega_1^2}e^{-\beta t}(\omega_1^2+\frac{1}{2}\beta^2\sin^2\omega_1t-\beta\omega_1\sin\omega_1t\cos\omega_1t)\right],\\
&\overline{\omega_0(x-\overline{x})(u-\overline{u})}=\frac{D\omega_0}{\omega_1^2}e^{-\beta t}\sin^2\omega_1t,
\end{aligned}\right\} \qquad (22)$$

where

$$\omega_1^2 = \omega_0^2 - \beta^2/4$$

When $t \to 0$ and $t \to \infty$ the variance of x becomes

$$\left.\begin{aligned} \overline{(x-\bar{x})^2} &\to \frac{2D}{3}t^3, \quad t \to 0, \\ &\to \frac{D}{\beta\omega_0^2}, \quad t \to \infty, \end{aligned}\right\} \tag{23}$$

respectively. Comparing this with (15), we see that for $t \to 0$ exactly the same result is obtained, while for $t \to \infty$, the variance now takes the constant value. Thus in this case we come to the conclusion that in stationary state there occurs no diffusion, which would at first glance highly curious. This does not, however, involve any contradiction, as may be seen, for example, from the fact that the air surrounding the earth forms a stable layer. This result amounts to saying that, to the diffusion phenomenon in the stationary state, only that component $u_0(t)$ of the temporal variation of wind velocity $u(t)$, which contain the zero frequency, makes a contribution.

5. The Coefficient of Mixing* due to Turbulence.

Decompose the wind velocity $u(t)$ into mean and turbulent velocities:

$$u(t) = \bar{u}(t) + u'(t), \tag{24}$$

and let

$$R(\tau) = \overline{u'(t)u'(t+\tau)} \tag{25}$$

be the correlation coefficient of $u'(t)$. If we define the coefficient of mixing by

$$A = \int_0^t R(\tau)\,d\tau, \tag{26}$$

according to Taylor, the diffusion equation may be written

$$\frac{\partial \bar{s}}{\partial t} + \bar{u}\frac{\partial \bar{s}}{\partial x} = \frac{\partial}{\partial x}\left(A\frac{\partial \bar{s}}{\partial x}\right). \tag{27}$$

In the stationary case the upper limit of the integral (~~30~~ 26) should be taken as infinite. Using the results of the preceding section, we obtain,

$$R(\tau) = \frac{D}{\beta}e^{-\frac{\beta}{2}t}\left(\cos\omega_1\tau - \frac{\beta}{2\omega_1}\sin\omega_1\tau\right), \tag{28}$$

so that

$$A = \frac{D}{\beta\omega_1}e^{-\frac{\beta}{2}t}\sin\omega_1 t, \tag{29}$$

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* Austauschkoeffizient.

from which it is seen that the coefficient of mixing A takes both signs alternatively as time elapses and finally becomes zero. This is of course the result corresponding to the turbulence which has a particular frequency, but even if we take into account the turbulence components having other frequencies, exactly the same result should be obtained. Fig. 6 illustrates this fact on the observed wind velocities. Two figures correspond to two different

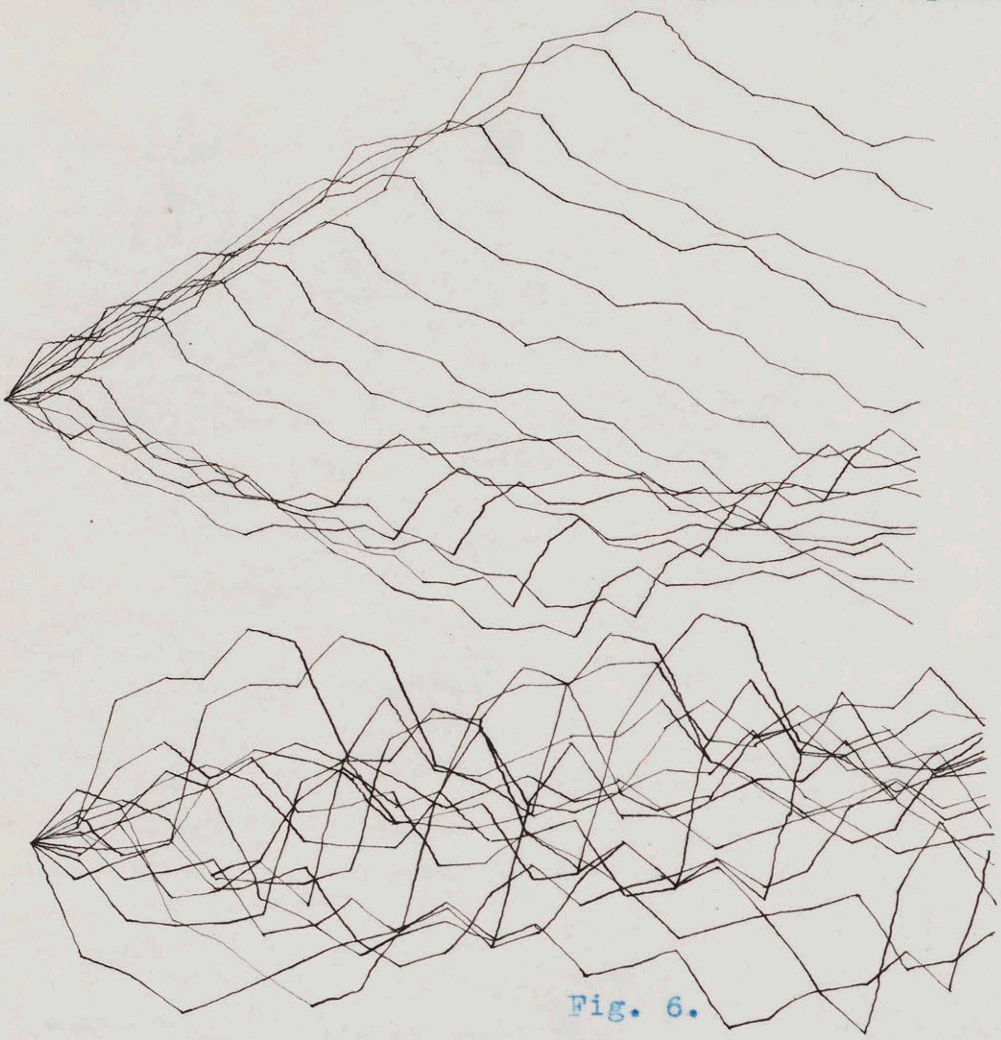

Fig. 6.

lengths of time interval used in averaging. We thus arrive at the conclusion that in the stationary field of turbulence obtained by omitting the mean flow, diffusion does not occur and the coefficient of mixing becomes zero. It is easy ~~to~~ indeed to pick up familiar examples in which diffusion actually takes place contrary to this conclusion. Such cases must however be regarded essentially as the non-stationary ones, in spite of their staionary appearance. Hence it seems natural to adopt, as the coefficient which describes the diffusion that actually occur, the value of A in (29) when it first takes the maximum rather than the value at $t \to \infty$. This is approximately equal to the value $D/\beta\omega_1$. If we accept this, it becomes possible to discuss conveniently also the diffusion due to the turbulent mass having ~~the nature~~ a particular scale. The value $D/\beta\omega_1$ can be calculated directly from the frequency spectrum of $u(t)$.

Hence we can conclude that the most reasonable method of studying the local characteristics of, and the diffusion phenomena occurring in, a turbulent field such as created by the woods, the peculiar lay of the land and so on, is to first calculate the frequency spectra from the observed temporal variation of wind velocity, at various points in such a field.

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ON THE THEORY OF ~~ANOMALOUS~~ DISPERSION OF DIELECTRICS

By

KATSUMI IMAHORI

ON THE ~~ANOMALOUS~~ THEORY OF DISPERSION OF DIELECTRICS

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1. The Dielecric Constant Considered as an Operator

The dielectric constant ε of a dielectric is usually defined as a constant which is introduced by an empirical relation

$$D = \varepsilon E, \tag{1}$$

which interconnects the displacement vector D and the field intensity E appearing in the Maxwell's electromagnetic equations. The dispersive property of a dielectric medium is described by the dependency of this ε on the frequency ω, e.g. by

$$\varepsilon(\omega) = \frac{D}{E}, \tag{2}$$

in which D and E are complex amplitudes in the sense that a stationary electric field E of frequency ω produces a displacement D of the same frequency. In the general cases of dispersive media and arbitrary time variation of the electric field, especially for example in the case of transient phenomena, one can not however apply the relation (1), but must resort to entirely different method. The same is true in the treatment of Maxwells equations.

Rewriting (2) as

$$D(\omega) = \varepsilon(\omega)\, E(\omega), \tag{2'}$$

multiplying both sides of this equation by $\frac{1}{2\pi}\cdot e^{i\omega t}$ and integrating, one has

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} D(\omega)\, e^{i\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon(\omega)\, E(\omega)\, e^{i\omega t}\, d\omega .$$

So that by putting

$$\left.\begin{aligned} D(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} D(\omega)\, e^{i\omega t}\, d\omega, \\ E(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E(\omega)\, e^{i\omega t}\, d\omega, \\ \varepsilon(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon(\omega)\, e^{i\omega t}\, d\omega, \end{aligned}\right\} \tag{3}$$

the following reation

$$D(t) = \int_{-\infty}^{\infty} \varepsilon(t')\, E(t-t')\, dt' \tag{4}$$

is obtained. This replaces (1) in the case of dispersive media. (4) may be simply expressed by an operational equation

$$D = (\varepsilon) E \tag{5}$$

where (ε) is a linear operator which operates on a physical quantity E to produce another one D. Let it be called the (ε)-operator.

Conversely one may derive (4) or (2') as an explicit representations of the (ε)-operator from the assumed general relation (5) between D and E. Let [1]

$$E = \delta(t)$$

be substituted in (5) and put

$$\varepsilon(t) = (\varepsilon)\, \delta(t), \tag{6}$$

then from the well known formula

$$E(t) = \int_{-\infty}^{\infty} E(t')\, \delta(t-t')\, dt',$$

which means that any function $E(t)$ can be expressed as a linear superposition of δ-functions, one obtains

$$D(t) = \int_{-\infty}^{\infty} \varepsilon(t-t')\, E(t')\, dt'$$

$$= \int_{-\infty}^{\infty} \varepsilon(t')\, E(t-t')\, dt'.$$

Thus $\varepsilon(t)$ may be considered as the "t-representaion" of the operator (ε). Similarly if one puts in (5)

$$E = e^{i\omega t},$$

and

$$\varepsilon(\omega)\, e^{i\omega t} = (\varepsilon)\, e^{i\omega t}, \tag{7}$$

then for any $E = E(\omega) e^{i\omega t}$ one has

$$D = (\varepsilon)\, E(\omega)\, e^{i\omega t} = E(\omega) \cdot (\varepsilon) e^{i\omega t}$$

so that

$$= E(\omega)\, \varepsilon(\omega)\, e^{i\omega t},$$

$$D(\omega) = E(\omega)\, \varepsilon(\omega).$$

The $\varepsilon(\omega)$ may be called the "ω-representation" of the (ε)-operator.

Consider for example the transient response of the circuit

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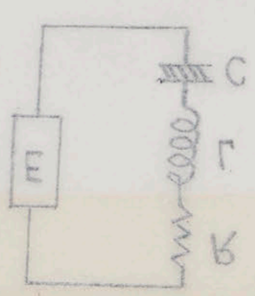
1) Here $\delta(t)$ represents the Dirac's δ-function.
See, for example, K. Imahori: Sound Analysis (in Japanese), 1949.



shown in Fig. 1. The condenser with oblique lines means that the medium is dispersive one. In place of the usual electric capacity C a linear operator (C) is used to represent the relation between the electric quantity q and the potential difference V:

$$q = (C)V. \tag{8}$$

The cicuit equation becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + (C^{-1})\,q = E(t), \tag{9}$$

where (C^{-1}) is the inverse operator of (C) and is defined by

$$V = (C^{-1})\,q. \tag{8'}$$

Solving the equation (9) by the ω-representation, one has

$$-L\omega^2 q(\omega) + Ri\omega\, q(\omega) + \frac{1}{C(\omega)}\,q(\omega) = E(\omega),$$

$$q(\omega) = \frac{E(\omega)}{1/C(\omega) - L\omega^2 + Ri\omega} = \frac{E(\omega)}{Z(\omega)},$$

so that by transforming into t-representation,

$$q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} q(\omega)\, e^{i\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E(\omega)}{Z(\omega)}\, e^{i\omega t}\, d\omega$$

is obtained. In order to solve directly by t-representation on the other hand, one has to get the inverse operator (C^{-1}) expressed in t-representation

$$C^{-1}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{C(\omega)}\, e^{i\omega t}\, d\omega,$$

and substitute in (9), obtaining an integro-differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \int_{-\infty}^{\infty} C^{-1}(t')\, q(t-t')\, dt' = E(t), \tag{10}$$

which is to be solved. Thus one sees how it is erroneous to "solve a differential equation" merely by putting $1/C(\omega)$ in place of the (C^{-1}) in (9).

2. Physical Meaning of the (ε)-Operator

In the phenomenological treatment of electromagnetic phenomena, all that is needed for its formulation concerning the physical properties of a given medium is to have its (ε)-operator expressed by an experimental t- or ω-representation. But when the equation (5) is regarded as representing a physical law which enables one to know the effect D produced by the cause E, one might ascribe a characteristic physical model to

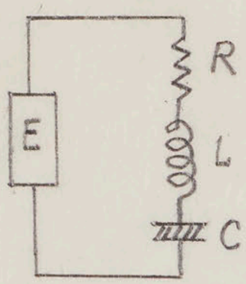


Fig. 1

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Debye's theory of dielectric dispersion is well-known, but in view of the fact that this theory agrees with experiments only in a few ideal cases and that various refined theories which has been proposed to cover more general ones are as yet in no decisive stage, it seems worth while to investigate on a possible physical model which can be derived inductively from experimental data, as contrasted to the usual deductive method in which a tentative model is first assumed and theoretical calculations are made to deduce a formula to compare with experiments. The present report deals with some considerations made on the former standpoint.

The ideal case is first considered where Debye's theory is valid. According to this theory the complex dielectric constant $\varepsilon(\omega)$ is given by

$$\varepsilon(\omega) = \varepsilon_\infty + \frac{\varepsilon_0 - \varepsilon_\infty}{1 + i\omega\tau} . \qquad (11)$$

Transforming this into t-representation, one obtains

$$\begin{aligned} \varepsilon(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\varepsilon_\infty + \frac{\varepsilon_0 - \varepsilon_\infty}{1 + i\omega\tau}\right) e^{i\omega t}\, d\omega \\ &= \varepsilon_\infty \delta(t) + (\varepsilon_0 - \varepsilon_\infty)\frac{1}{\tau} e^{-\frac{t}{\tau}}, \quad t > 0. \end{aligned} \qquad (12)$$

Now the (ε)-operator is devided up into two components, such that

$$\left.\begin{aligned} &(\varepsilon) = (\varepsilon_1) + (\varepsilon_2), \\ &\varepsilon_1(\omega) = \varepsilon_\infty, \quad \varepsilon_2(\omega) = \frac{\varepsilon_0 - \varepsilon_\infty}{1 + i\omega\tau}, \\ &D_1 = (\varepsilon_1) E, \quad D_2 = (\varepsilon_2) E, \end{aligned}\right\} \qquad (13)$$

then the inverse transformation of the last two:

$$(\varepsilon_1^{-1}) D_1 = E, \qquad (\varepsilon_2^{-1}) D_2 = E, \qquad (14)$$

may be transformed into t-representation in the following way. First starting from the ω-representation of the equations (14):

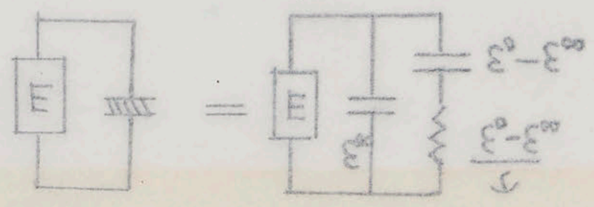
$$\frac{1}{\varepsilon_\infty} D_1(\omega) = E(\omega), \qquad \frac{1 + i\omega\tau}{\varepsilon_0 - \varepsilon_\infty} D_2(\omega) = E(\omega),$$

one can write them in t-representation as

$$\left.\begin{aligned} &\frac{1}{\varepsilon_\infty} D_1(t) = E(t), \\ &\frac{1}{\varepsilon_0 - \varepsilon_\infty}\left\{\tau \frac{d D_2(t)}{dt} + D_2(t)\right\} = E(t), \\ &D(t) = D_1(t) + D_2(t). \end{aligned}\right\} \qquad (15)$$

The physical meaning of these equations can be read from the circuit diagramme shown in Fig. 2. Thus one may picture a ~~posible~~ physical model which corresponds to the equations (15),

Fig. 2



this being regarded as a physical law which governs the displacement current D caused by the applied field E. The real existence of such model can not however be concluded from the above argument alone, it being necessary to be supplemented by evidences from other sources. Also the uniqueness of the decomposition such as given by (15) is not assured as is known in the theory of electrical networks, some consideration upon which will be described in the following articles.

3. Brune's Theory of Network Synthesis, and the Distribution Function of Fuoss and Kirkwood

The physical interpretation given in the preceding article of the (ε)-operator stands on a basis which is essentially the same as that of the method of synthesis of two terminal networks with given impedance characteristics. The latter problem in the theory of electrical networks is regarded as has been solved by O. Brune.[2] In his theory a "positive real function" $Z(\lambda)$ is defined in which the complex variable $\lambda = \gamma + i\omega$ is used instead of $i\omega$ in the impedance function given as a rational function $Z(i\omega)$ of $i\omega$. By separating successively the zeros and poles of the function $Z(\lambda)$ the process of synthesizing the network is determined. It is to be noted here that the order of separation in this procedure is quite arbitrary, so that different networks having the same characteristics might be realized. The interesting problem of finding a general relation between possible networks in this sense might be solved to some extent by the method ~~suggested by~~ of affine transformation studied by Howitt and Cauer.

Another point which must be taken into account in applying Brune's theory to the present case is that the impedance function as given by experiments is not necessarily a rational function of $i\omega$, so that the representation of it by a rational function should be regarded as an approximate one. Then it will become a matter of further consideration how should the poles and zeros of the function $Z(\lambda)$ behave when ~~this is derived~~ the procedure of approximation is varied in deriving this from the experimentally determined function $Z(i\omega)$ by replacing the variable $i\omega$ by λ. Thus the poles and zeros of $Z(\lambda)$ seem to have no physical meaning characteristic of the given $Z(i\omega)$, but in some cases it is even possible to impose some particular properties to the zeros and poles and construct a network with the desired $Z(i\omega)$. Foster's method of network synthesis by using only pure reactances as is shown by his reactance theorem is a good example of this.

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2) O. Brune: J. Math. and Phys. 10, No.3, 191, 1931.

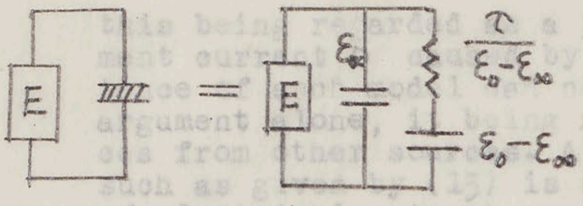

Fig. 2

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Turning now back to the problem of dielectric dispersion, from the obvious relation

$$\varepsilon(\omega) = \int_{-\infty}^{\infty} \varepsilon(\omega')\, \delta(\omega-\omega')\, d\omega' , \tag{16}$$

in which $\varepsilon(\omega)$ is the ω-representation, expressed as a complex ~~[illegible]~~ function of a real variable ω, of the (ε)-operator one may put

$$\left.\begin{aligned} D_{\omega'}(\omega) &= \varepsilon(\omega')\, \delta(\omega-\omega')\, E(\omega), \\ D(\omega) &= \int_{-\infty}^{\infty} D_{\omega'}(\omega)\, d\omega' , \end{aligned}\right\} \tag{17}$$

and obtain

$$\frac{1}{\varepsilon(\omega')} \cdot \frac{D_{\omega'}(\omega)}{\delta(\omega-\omega')} = E(\omega).$$

In order to change this into t-representation, use is made of a possible representaion

$$\varepsilon(\omega')\, \delta(\omega-\omega') = \frac{1}{(\omega'^2-\omega^2+i\kappa\omega)\, L_{\omega'}}$$

for the δ-function, then

$$L_{\omega'} \left\{ \frac{d^2 D_{\omega'}(t)}{dt^2} + \omega'^2 D_{\omega'}(t) \right\} = E(t). \tag{18}$$

This differential equation corresponds to the case in which a distribution of poles on the imaginary axis is assumed in Brune's theory.

As an example of the dispersion theory in which a distribution of poles on the real axis is considered, one may mention that of R. M. Fuoss and J. G. Kirkwood.[3] In this theory the "reduced polarisation" :

$$Q(\omega) = \frac{\varepsilon - \varepsilon_\infty}{\varepsilon_0 - \varepsilon_\infty} \tag{19}$$

is separated into real and imaginary parts

$$\left.\begin{aligned} Q(\omega) &= J(x) - iH(x), \\ x &= \log \frac{\omega_m}{\omega} , \end{aligned}\right\} \tag{20}$$

where ω_m is the ω at which H takes its maximum. Then one is possible to obtain a distribution function such that

$$\left.\begin{aligned} Q(\omega) &= \int_0^\infty \frac{G(\tau)}{1+i\omega\tau}\, d\tau , \\ \int_0^\infty G(\tau)\, d\tau &= 1 , \end{aligned}\right\} \tag{21}$$

is satisfied. Comparing (21) with (11), it can be seen that $G(\tau)$ represents a continuous distribution of relaxation time τ. ~~Thus~~

3) R.M.Fuoss and J.G.Kirkwood: J.A.C.S., 63, 385. 1941.

Thus one may set up a set of differential equations in analogy to (18) :

$$\left.\begin{aligned} \frac{1}{\varepsilon_\infty} D_1(t) &= E(t), \\ \frac{1}{\varepsilon_0-\varepsilon_\infty}\left\{\tau \frac{dD_\tau(t)}{dt} + D_\tau(t)\right\} &= E(t), \end{aligned}\right\} \qquad (22)$$

with a continuous parameter τ .

4. Expansion into Orthogonal Polynomials

In the above discussion it was assumed that the function $\varepsilon(\omega)$ which is to be derived from experiments is known for all values of its variable ω . The real situation is however that the ω at which experiments are made takes ~~p~~ in most cases only an isolated set of values, and even if a continuous experiment be assumed to be possible, its range of values is necessarily finite. This is the point where the conditions are essentially different from those of network synthesis ~~and~~ so that full care must be paid for.

Let $\varepsilon(\omega)$ be measured for values of ω lying on a region Ω — which can be several isolated points —, and consider the following problem which is set up upon this. To approximate as near as possible the experimental values of $\varepsilon(\omega)$ at all points

$$\lambda = \pm i\omega \quad (\omega \subset \Omega)$$

belonging to Ω by the expression

$$\frac{1}{\varepsilon(\lambda)} = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n, \qquad (23)$$

with properly chosen n in the sense of the method of least squares. For this purpose the method of expansion in orthogonal polynomials may be recommended.[4] Thus one may choose suitable one from

(i) Legendre's polynomials,
(ii) Hermite's polynomials,
(iii) Laguerre's polynomials,
(iv) Tschebyscheff's q-functions,

as the case may be.

Separating (23) into real and imaginary parts and expanding

$\Re\left\{\frac{1}{\varepsilon(\omega)}\right\}$ by polynomials of even order, and

$\Im\left\{\frac{1}{\varepsilon(\omega)}\right\}$ by polynomials of odd order,

the right hand side of (23) will be obtained by arranging in powers of λ . The form of (23) being thus determined the t-representation for (5) becomes in this case

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4) G. Szegö: Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ. Vol. XXIII. 1939.
Courant und Hilbert: Methoden der Mathematischen Physik, Bd. I.

$$a_0 \frac{d^n D}{dt^n} + a_1 \frac{d^{n-1} D}{dt^{n-1}} + \cdots + a_n D = E. \qquad (24)$$

The assumption of a simple polynomial in (23) is not altogether in contradiction with the positive real function in Brune's theory. For where as the $Z(\lambda)$ in Brune's theory was defined for all values of λ on the ~~re~~ imaginary axis, the $\varepsilon(\lambda)$ in the present case is defined only for a limited range of values, so that there remains some indeterminacy in constructing the function $1/\varepsilon(\lambda)$ which might result in different forms as the procedure of approximation is different. It is however desirable to generalize the present method in the direction of Brune's theory. As a concluding remark it should be mentioned that it is easy to divide up the differential equation (24) into a number of circuit equations such for examples as (15) or (22).

5. Sammary

It was shown that there are many possibilities in the physical interpretation of an experimentally given $\varepsilon(\omega)$ by the aid of physical models which can not be covered by the existing theory of electrical networks. The remaining problems are then:

(i) To formulate a unifying theory which covers all these possibities and includes in particular a general scheme of mutual transformations between any two of them. This is one of the most interesting problems which awaits further elucidation.

(ii) The degrees of freedom in the possibilities mentioned above might be reduced by formulating the result of multiple measrments concerning a number of physical quantities other than the dielectric constant for the same medium, The theory of four terminal network synthesis given by Gewertz[5] is very instructive in this sense.

(iii) To take other physical and chemical informations into account effectively in the formulation of the theory.[6] As a simple example it is remarked here that one may directly apply the present method to the analysis of dielectric constant measured on a mixture of two polar liquids each obeying the Debye's law.

~~In conclusion the author~~ ~~nor~~ developed ~~and~~ further and be

As a concluding remark the author hopes that the method describbed in this paper would be applied to wider field of natural phenomena other than dielectric dispersion, such for example as the analysis of brain-waves which is now being carried on in the author's laboratory.

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5) Charles M. Son Gewertz ; Network Synthesis, Baltimore. 1933.

6) See for example S. Takeda: On a New Interpretation of the Distribution of Relaxation Time, ~~the same~~ this volume, p.84.