MC 450 Alice Gould, class notebook

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Least Squares.

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1st Semester. (D. P. Bartlett)

A B Gould

Dependent Events

Prob. of concurrence dependent

a' ways in which 2d will follow
after 1st has happened
b' in which will not follow

all being equally likely to occur.

Ex. Urn 5 black balls 3 red 4 white
Find prob. drawing 4 black balls in succession.

$\frac{5}{12}$ ~~that~~ 1st black

$\frac{4}{11}$ 2d

$\frac{3}{10}$ 3d

$\frac{2}{9}$ 4th

$$\text{Prob.} = \frac{5 \cdot 4 \cdot 3 \cdot 2}{12 \cdot 11 \cdot 10 \cdot 9}$$

But if each had been replaced after drawing
prob would have been $\left(\frac{5}{12}\right)^4$

Ex

Four cards from a pack, find prob. that 4 aces.

If drawn clubs spades hearts diamonds

Ace clubs $\frac{1}{52}$

spades $\frac{1}{51}$

hearts $\frac{1}{50}$

diamonds $\frac{1}{49}$

Prob. $\frac{1}{52 \cdot 51 \cdot 50 \cdot 49}$ in this order

But $\underline{|4}$ orders of drawing the aces

$\therefore$ total probability $= \dfrac{\underline{|4}}{52 \cdot 51 \cdot 50 \cdot 49}$

Simpler solution

Number ways 4 cards can be drawn = no. comb 52 in sets 4 $= \dfrac{52 \cdots 49}{\underline{|4}}$

Number ways drawing 4 aces = 1

Prob. $= \dfrac{\underline{|4}}{52 \cdots 49}$

[Ex. 13. m republ. n democr, committee p+q. Find chance p rep. & q. dem.

$$\frac{\underline{|m+n}}{\underline{|p+q}\ \underline{|m+n-p-q}}$$ Combinations possible for Committee

$$\frac{\underline{|m}}{\underline{|p}\ \underline{|m-p}}$$ comb. of p republ.

$$\frac{\underline{|n}}{\underline{|q}\ \underline{|n-q}}$$ comb. of q. rep.

$$\text{Chance} = \frac{\dfrac{\underline{|m}\ \underline{|n}}{\underline{|p}\ \underline{|q}\ \underline{|m-p}\ \underline{|n-q}}}{\dfrac{\underline{|m+n}}{\underline{|p+q}\ \underline{|m+n-p-q}}} =$$

$$= \frac{\underline{|m}\ \underline{|n}\ \underline{|p+q}\ \underline{|m+n-p-q}}{\underline{|p}\ \underline{|q}\ \underline{|m-p}\ \underline{|n-q}\ \underline{|m+n}}$$

Prob throwing an ace in course of how many $= \frac{1}{2}$

Ex. In how many trials prob. ace with single die $= \frac{1}{2}$

$x =$ no trials

Prob. failing x times $= \left(\frac{5}{6}\right)^x$

not $\quad 1 - \left(\frac{5}{6}\right)^x = \frac{1}{2}$

$x\,\overline{\log 5 - \log 6} = -\log 2$

$x = 3.8$

Ex. Prob. throwing exactly 3 aces in 5 trials with single die, and prob. of at least 3 aces.

If aces first, then two others

Prob. in this order $\left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2$

$\frac{5\cdot4\cdot3}{1\cdot2\cdot3}$ ways of throwing

$\frac{5\cdot4\cdot3}{1\cdot2\cdot3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 =$ prob. exactly 3 aces.

Prob. 5 aces $= \left(\frac{1}{6}\right)^5$

4 $\quad \left(\frac{1}{6}\right)^4 \times \frac{5}{6} \times 5$

∴ Prob at least 3 aces = sum other probs

$= \frac{5\cdot4\cdot3}{1\cdot2\cdot3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 + \left(\frac{1}{6}\right)^4 \frac{5}{6} \times 5 + \left(\frac{1}{6}\right)^5$

Oct 12th Lect III

14. Find values x & y which make
$u = x^3y^2(6-x-y)$ a maximum
$y = 2 \quad x = 3$

Ex. Lottery large no tickets, prizes are to blanks = 1 : 6. In 5 drawings find prob. at least 2 prizes.

Prob. failing 5 times $= \left(\frac{6}{7}\right)^5$

4 times $\left(\frac{6}{7}\right)^4 \times 5 \times \frac{1}{7}$

Prob. failing at least 4 times = Sum

" not " 1 − sum

Extension of Taylor's

Given $f(x\ y\ z \ldots)$ expand $f(x+h\ \ y+k \ldots)$ in ascending powers hkl etc.

Consider 2 variables.

By Taylor's

$$f(x+h) = fx + h\frac{d}{dx}fx + \frac{h^2}{\lfloor 2}\frac{d^2}{dx^2}fx + \cdots\cdots$$

Consider y constant

(1) $$f(x+h, y) = f(x,y) + h\frac{d}{dx}f(x,y) + \cdots\cdots\cdots$$

Consider x const.

(2) $$f(x+h, y+k) = \underbrace{f(x+h, y)}_{(A)} + \underbrace{k\frac{d}{dy}f(x+h, y)}_{(B)}\cdots\cdots$$

A already found in (1), B by differentiating (1)

$$\frac{d}{dy}f(x+h, y) = \frac{d}{dy}f(x,y) + h\frac{d^2}{dx\,dy}f(x,y) + \cdots\cdots$$

$$\frac{d^2}{dy^2}f(x+h, y) = \frac{d^2}{dx\,dy}f(x,y) + h\frac{d^3}{dx\,dy^2}(f x, y) + \cdots\cdots$$

Substituting in (2)

$$f(x+h, y+k) = f(x,y) + h\frac{d}{dx}f(x,y) + \frac{h^2}{\lfloor 3}\frac{d^2}{dx^2}f(x,y) \cdots$$
$$+ k\frac{d}{dy}f(x,y) + kh\frac{d^2(f x,y)}{dx\,dy} \cdots\cdots$$
$$+ \frac{k}{\lfloor 2}\frac{d^2}{dy^2}f(x,y) \cdots\cdots$$

$$\therefore \cancel{u}(x+h, y+k) = u + h\frac{du}{dx} + k\frac{du}{dy} + \frac{1}{\lfloor 2}\left(h^2\frac{d^2u}{dx^2} + 2hk\frac{d^2u}{dx\,dy} + k^2\frac{d^2u}{dy^2}\right)$$
$$+ \text{etc}$$

Conditions max. & min. vals $f(x,y)$

$f(x+h, y+k) - f(x,y)$ neg for max
positive min.

$u' - u$

According to development above

$$u' - u = h\frac{du}{dx} + k\frac{du}{dy} + \frac{1}{2}\left(\qquad \right) \text{etc}$$

Neglecting powers above the first, observe sign of
$h\frac{du}{dx} + k\frac{du}{dy}$ must be same whatever sign
of h & k

$\therefore \frac{du}{dx}$ & $\frac{du}{dy}$ must be zero.

$$\left(h\frac{d}{dx}+k\frac{d}{dy}\right)$$
$$e \quad \varphi(x,y) = \varphi(x+h, y+k)$$

Ex. 1. 100 observ. time vibr. needle found between

sec	sec	
.5	.4	2 errors
.4	.3	2
3	2	4
2	1	14
1	0	26
0	-1	26
-1	-2	16
-2	-3	7
-3	-4	2
-4	-5	1
		100

B p 5

If another observ. made, find prob. error falling between each of the limits & construct a figure showing the law of error.

Read p 11-18. in Wright

∴ that any funct. max. or min. – diff. with respect to each variable & make each separately zero. Condit. probl. gen. show whether max. or min.

Least squares

Constant errors, due to fixed cause
Accidental, can not be allowed for beforehand, most important due to imperf. in observers.

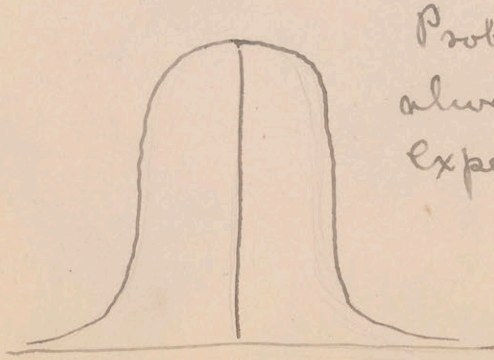
Δ = Real errors in series measurements = must be added to true value to give observed values.

v = Residual errors = must be added to ~~most to~~ prob. val found to give observed vals.

Assume Most prob. val. from direct measurements is their arithmetical mean. Sometimes attemp to prove it - quite as simple to assume.

From this follows $\Sigma v = 0$

Probability curve as always found from experiment.



Curve of error.

Ex. 1000 shots at target $5\frac{1}{2}$ ft radius

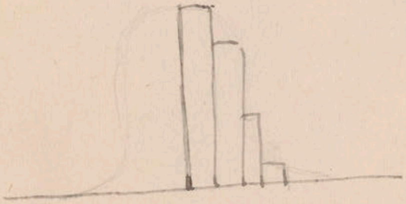
Circles 1 ft apart. There fell between $5\frac{1}{2}$ $4\frac{1}{2}$

Shots	Circles	
1	$5\frac{1}{2}$	$4\frac{1}{2}$
4	$4\frac{1}{2}$	$3\frac{1}{2}$
10	3.	2.
89	2.	1.
190	$1\frac{1}{2}$	$\frac{1}{2}$
212	$\frac{1}{2}$	$-\frac{1}{2}$ (ie below centre)
204	$-\frac{1}{2}$	$-1\frac{1}{2}$
193	" 1	" 2
79	2	3
16	3	4
2	$4\frac{1}{2}$	$5\frac{1}{2}$

(Numbers from observation)

Δ = distances fr. centre target.

Ordinates lengths prop. n° shots, absciss. dist. fr. centre.



If total area = unity, area each rect. is prob. of any special shot there falling.

1000 set

~~Prob.~~ As far as these obs. go, shot will certainly hit the target.

Since great errors do not occur – curve ought to cut $\overline{X}$ at finite distance – but impossible write eq. of such a curve – hence must make asymptotic to $\overline{X}$.

Deducing eq. of curve –

Small errors more frequent than large

$\therefore$ max. pt. curve on axis $\overline{Y}$

Positive & negative errors equally prob.

$\therefore$ curve symm. respect $\overline{Y}$

Large errors very unlikely

$\therefore$ curve asymptotic $\overline{X}$.

Frequency depends on magnitude

$\therefore$ curve possible, $y = \varphi\Delta$

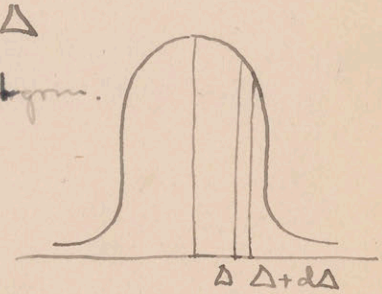
B p 11

Sat. Oct. 19th
Lect V.

To determine form of $\varphi\Delta$

Prob. between Δ & $\Delta + d\Delta$ = el. prgm.

$= y\,d\Delta = \varphi\Delta\, d\Delta$



In all discussions must suppose no. obs. very large & greater than no. unknown quantities.

*Ie Suit n obs on various combinations $z_1 z_2 z_3$ - - - . If All on one function might find most prob val $\mathfrak{u}$ but not $z_1 z_2$ - - - -.

Law holding for direct obs. on any quantity will hold for indirect series obs. on funct. of ~~diff~~ quantities since amounts to finding each unknown in terms of others.

Each method computing values will give corresp. system errors. Find prob. of system errors hence of values.

××× Ie the best values

~~Prob. of errors depends on what the errors are what they are depends on what the quantities really are.~~ Values of $z_1 z_2 z_3$ -- having been (indirectly) observed, P is a function of the real $z_1 z_2 z_3$ - - -

$z_1 z_2 z_3$ ... being undetermined are made to vary & their best value chosen

** Remember Δ = error of $\mathfrak{u}$, not of z

Let be n obs. $M_1\, M_2\, M_3$ ---- $M_1 = f_1(\text{etc})$ $M_2 = f_2(\text{etc})$
~~to~~ made upon $^* M = f(z_1\, z_2\, z_3 \cdots\cdots)$, $z_1\, z_2$ -- being the quantities whose best values really sought.*** Let errors be $\Delta_1\, \Delta_2^{**}$ ---- & their probabilities be therefore $\varphi(\Delta_1)d\Delta$ $\varphi(\Delta_2)d\Delta$ etc.

Prob. simultaneous occurrence of all these errors $= P = \varphi(\Delta_1)\varphi(\Delta_2) \cdots \varphi\Delta_n\, (d\Delta)^n$

$\log P = \log\varphi\Delta_1 + \log\varphi\Delta_2 + \cdots \log\varphi\Delta_n + n\log d\Delta$

giving probability of certain system of errors. Make this prob. max. – find corresponding ~~errors & corresp. values corrected.~~ correct values accordingly.

~~To make~~

P ~~φ~~ Δ are functions of $z_1\, z_2\, z_3 \cdots\cdots = $ best values of real z's

$\therefore$ for max val.

$$\frac{1}{\varphi\Delta_1}\frac{d\varphi\Delta_1}{dz_1} + \frac{1}{\varphi\Delta_2}\frac{d\varphi\Delta_2}{dz_1} + \frac{1}{\varphi\Delta_3}\frac{d\varphi\Delta_3}{dz_1} + \cdots = 0$$

$$\frac{1}{\varphi\Delta_1}\frac{d\varphi\Delta_1}{dz_2} + \frac{1}{\varphi\Delta_2}\frac{d\varphi\Delta_2}{dz_2} + \cdots\cdots\cdots \; 0$$

$$\frac{1}{\varphi\Delta_1}\frac{d\varphi\Delta_1}{dz_3} + \cdots\cdots\cdots\cdots \; 0$$

must have as many eq. as z.

For simplification let $\frac{1}{\varphi\Delta_1}\frac{d\varphi\Delta_1}{dz_1} = \frac{\varphi'\Delta_1}{\varphi\Delta_1}\frac{d\Delta_1}{dz_1}$
$= \psi\Delta_1 \frac{d\Delta_1}{dz_1}$

Substituting (ie writing thus)

$$\psi\Delta_1 \frac{d\Delta_1}{dz_1} + \psi\Delta_2\frac{d\Delta_2}{dz_1} + \cdots\cdots = 0$$

$$\psi\Delta_1 \frac{d\Delta_1}{dz_2} + \psi\Delta_2\frac{d\Delta_2}{dz_2} + \cdots\cdots = 0$$

Etc. $\psi\Delta_1 \frac{d\Delta_1}{dz_3} + \psi\Delta_2\frac{d\Delta_2}{dz_3} + \cdots\cdots = 0$

Since this process is perfectly general
may determine form ψ from special case.

m_1 etc observed $\therefore$ ~~const.~~ number

Σ_1 & Δ_1 dependent variables.

i.e.

In case of direct observations on a single quantity, sum of the errors is zero.

N equations each containing one of N unknown z's thus reached — solve simultaneously when ψ is known.

<u>Example</u>

n direct obs. single quantity, obs. u_1, u_2 ---

errors Δ_1, Δ_2 -----

$$z_{+} = u_1 - \Delta_1 = u_2 - \Delta_2 = \text{--------}$$

Diff.
$$1 = -\frac{d\Delta_1}{dz_1} = -\frac{d\Delta_2}{dz_2} = \text{-------}$$

Substituting in gen. eq.

$$\psi\Delta_1 + \psi\Delta_2 \text{----} = 0$$

But by Gauss' axiom $\quad z_{+} = \dfrac{u_1 + u_2 + \text{--}}{n}$

$$\therefore \Delta_1 = u_1 - \frac{u_1 + u_2 + \text{---}\, u_n}{n}$$

$$\Delta_2 = u_2 - \frac{u_1 + \text{----} - u_n}{n}$$

etc

$\therefore$ adding

$$\Sigma\Delta = \Sigma u - n\,\frac{\Sigma u}{n} = 0$$

i.e.

$$\Delta_1 + \Delta_2 + \Delta_3 + \text{------} = 0$$

But $\psi\Delta_1 + \psi\Delta_2 + \text{- - - -} = 0$

ψ being a gen. function must denote (in this example at least) multiplication by a constant. $\psi\Delta = a\Delta$

Ex. 3

[Find most prob. elevations $z_{1,2,3,4,5}$
from observations

z_1 above	0	573.08
z_2	z_1 -	. 2.60
z_2	0 -	. 575.27
z_3	z_2 -	. 167.33
z_4	z_3	3.80
z_4	z_3	170.28
z_4	z_2	425.00
z_4	z_5	319.91
z_5	0	319.75
z_5	0	

Ans. $z_1 = 572.81$
$z_2 = 575.14$
z_3 . 742.05
z_4 745.43
z_5 . 320.03]

Subst. in value $\varphi\Delta$

$$\frac{1}{\varphi\Delta}\frac{d\varphi\Delta}{dz} = \psi\Delta\frac{d\Delta}{dz} = \cancel{a}\Delta\frac{d\Delta}{dz}$$

Integrating

$$\log\varphi\Delta = \frac{1}{2}\cancel{A}\Delta^2 + C_1$$

$$\varphi\Delta = e^{\frac{1}{2}a\Delta^2 + c} = k\,e^{\frac{1}{2}a\Delta^2}$$

$$y = k\,e^{\frac{1}{2}a\Delta^2}$$

But y increases as Δ decreases

∴ exponent e negative, a negative

write $\frac{1}{2}a = -h^2$

$$y = k\,e^{-h^2\Delta^2}$$

This satisfies all conditions found necessary from inspecting curve.

VI Tues. Oct 22^d

Least Squares.

B p 15

Observations on $u = f(z, z_2 \cdot z_m)$

Must make max. in

$$P = \varphi(\Delta_1)\,\varphi\Delta_2 \cdots\cdots \varphi\Delta_n\,(d\Delta)^n$$

$$= ke^{-h^2\Delta_1^2}\,ke^{-h^2\Delta_2^2} \cdots\cdots (d\Delta)^n$$

$$= k^n e^{-h^2(\Delta_1^2+\Delta_2^2+\cdots)}(d\Delta)^n$$

ie. must make $\Delta_1^2 + \Delta_2^2 + \cdots$ minimum

($d\Delta$ is constant ∞^l)

Otherwise thus.

We have a set of equations for z & Δ

Substituting for ψ now found to be $\times$ const

we have the so-called normal equations

$$\Delta_1 \frac{d\Delta_1}{dz_1} + \Delta_2 \frac{d\Delta_2}{dz_1} + \Delta_3 \frac{d\Delta_3}{dz_1} + \cdots = 0$$

$$\Delta_1 \frac{d\Delta_1}{dz_2} + \Delta_2 \frac{d\Delta_2}{dz_2} + \Delta_3 \frac{d\Delta_3}{dz_3} + \cdots = 0$$

etc

But real errors Δ are not known - forced to use residual errors as closest approximation given by observations.

$$v_1 \frac{dv_1}{dz_1} + v_2 \frac{dv_2}{dz_1} + \cdots\cdots = 0$$

$$v_1 \frac{dv_1}{dz_2} + v_2 \frac{dv_2}{dz_2} + \cdots\cdots = 0$$

etc

By integrating, these eqs. also show that sum of squares errors (regarded as functions of unknown $z\,z_1 \ldots$) must be ~~zero~~ a minimum

Application of Method L. Sq. to adjustment observations.

1st. Obs. on linear functions of unknowns.

Example

O = the sealevel. $P_1 P_2 P_3$ points whose altitudes must be det. fr. following observations

P_1 above	O =	10 ft
P_2	P_1	7
P_2	O	18
P_2	P_3	9
P_1	P_3	2

$z_1 z_2 z_3$ = required altitudes.

Observation equations

$$z_1 - 10 = v_1$$
$$z_2 - z_1 - 7 = v_2$$
$$z_2 - 18 = v_3$$
$$z_2 - z_3 - 9 = v_4$$
$$z_1 - z_3 - 2 = v_5$$

If obs. exact, 2d members would be zero. Customary to write them zero.

Normal equations

$$(z_1 - 10)1 + (z_2 - z_1 - 7)(-1) + (z_1 - z_3 - 2)1 = 0$$
$$(z_2 - z_1 - 7)1 + (z_2 - 18)1 + (z_2 - z_3 - 9)1 = 0$$
$$(z_2 - z_3 - 9)(-1) + (z_1 - z_3 - 2)(-1) = 0$$

i.e.

$$3z_1 - z_2 - z_3 - 5 = 0$$
$$-z_1 + 3z_2 - z_3 - 34 = 0$$
$$-z_1 - z_2 + 2z_3 + 11 = 0$$

Solve

$$z_1 = 10.375$$
$$z_2 = 17.625$$
$$z_3 = 8.5$$

Solve ~~Ex~~ 2, giving to observations the following weights 25, 25, 4, 4, 4, 4, 4, 4, 1.

Ans. $z_1 = 572.98$
z_2 575.48
z_3 742.36
z_4 745.72
z_5 320.25

Rule — independent indirect linear obs.

Write obs. eqs., multiply by $\sqrt{\ }$ of weight.

Write a normal for each unknown by multiplying each obs. by coeff. of that unknown and putting sum results = zero. Solve normals for most prob. values of unknowns.

Or — Form normals & multiply each term by its weight.

Observations of unequal weight.

Difference weight may be due to any cause, but effect same as if due to repetition. Value of other observations multiplied by some factor.

Example on last page – let obs. have weights as follows

$z_1 - 10 = 0$	weight 5	= times must use in forming the normal equations. I.e. must multiply each term by its weight.
$z_2 - z_1 - 7$	3	
$z_2 - 18$	6	
$z_2 - z_3 - 9$	2	
$z_1 - z_3 - 2$	4	

$$5(z_1 - 10) - 3(z_2 - z_1 - 7) + 4(z_1 - z_3 - 2) = 0$$

$$3(z_2 - z_1 - 7) + 6(z_2 - 18) + 2(z_2 - z_3 - 9) = 0$$

$$-2(z_2 - z_3 - 9) - 4(z_1 - z_3 - 2) = 0$$

Same result if multiply each obs by square root of weight, since this multiplies the term $v\frac{dv}{dz}$ by the weight.

p being the weight, the normals are

$$p_1 v_1 \frac{dv_1}{dz_1} + p_2 \cancel{dz_2} \frac{dv_2}{dz} + \dots = 0 \text{ etc.}$$

Since so much numerical work & such large n^{os} concerned, desirable to eliminate as much as possible.

In preceding example

$Z_1 = 573.08 \quad = 573 + z_1$

$Z_2 = \quad 575 + z_2$

$Z_3 \quad 742 + z_3$

Some small correction, sign unknown; get large n^{os} fr. gen. inspection.

Observations may then be written

$$(573 + z_1) - 573.08 = 0$$
$$(575 + z_2) - (573 + z_1) - 2.6 = 0$$

.

etc

$$z_1 - 0.08 = v_1 = 0$$
$$z_2 - z_1 - 0.6 = v_2$$
$$z_2 - 0.27 = v_3$$
$$z_3 - z_2 - 0.33 = v_4$$
$$z_4 - z_2 - 1.28 = v_6$$
$$z_4 - z_3 - 0.8 = v_5$$
$$z_4 - z_5 = v_5$$
$$z_5 + 0.09 = v_7$$
$$z_5 + 0.25 = v_9$$

$z_1\ z_2 \ldots$ here stand for corrections not for values as before.

Determine in same way. Normals become

$$z_1 - 0.08 + (-1)(z_2 - z_1 - 0.6) = 0$$

etc.

Much simpler to deal with.

[Ex. 4 Angles at centre disk by 4 radial lines.

				Ans.		
A =	104°	25′	13″	104	25	2.25
B	86	33	20	86	33	9.25
C	98	13	47	98	13	36.25
D	70	48	23	70	48	12.25

Ex. 5. Angles of triagle

			wts			
36°	25′	~~44.23~~	3	36	25	44.23
90	36	47 28	6	90	36	22.46
52	57	57	4	52	57	53.31

Solve by finding corrections by gen. method]

Wright pages 11-18

71 ~~2~~ 4

Oct. 28th
Lecture VIII.

Conditioned Observations — some relation between values. Eg. 3 angles triangle, sum must be 180°.

Must satisfy obs. as nearly as possible, & satisfy exactly some other conditions.

Fixed condit. fewer than unknowns, else would be determined.

Example Angles quadrilateral

$A =$	$108° \ 13' \ 22''$	weight	3
B	$93° \ 49' \ 17$		2
C	$87 \ \ 5 \ \ 39$		2
D	$77 \ \ 52 \ \ 40$		1

$$A + B + C + D = 360$$

Find value D in terms other angles from equation of condition, subst. in observation eq. and solve for ~~others~~ most prob. vals.

If two conditions, elim. 2 quantities. Etc.

In this example use method of corrections to simplify.

$$A + B + C + D = 360° \ 0' 58'' + a + b + c + d$$

$$a + b + c + d = -0' 58''$$

modified eq. condition.

[Ex. 6.

Capacity condenser = 14 mf
Divided into 5 sections abcde
Difference b & d = 1.50 Find most
prob. capac. sections fr. following obs.

a =	2.02	wt 3		Ans.
b	4.13	2	a =	1.9651
c	2.52	5	b	4.1245
d	2.67	7	c	2.4871
e	2.84	4	d	2.6245
			e	2.7988]

Take care not combine 2 eqs of
diff wts.

Observations then become Wts

$a = 0$ 3

$b = 0$ 2

$c = 0$ 2

$[\cdot d = 0]$ wh. with condit. gives $a+b+c+58'' = 0$ 1

Normals

$3a + (a+b+c+58) = 0$ i.e. $4a + b + c + 58 = 0$

$2b + (a+b+c+58) = 0$ $\quad a + 3b + c + 58 = 0$

$2c + (a+b+c+58) = 0$ $\quad a + b + 3c + 58 = 0$

Solving

$3a - \cancel{2b} = 0$

$2a + 8b + 116 = 0$

$12a - 8b = 0$

$a = -8.29$

$a = -.8.29$
$b = -.12.43$
$c = -12.43$
$d = -24.85$

Observe that corrections are in inverse ratio of weights.

$A = 101^\circ 13' \; \cancel{12.81} \; 13.31''$

$B = 93 \quad 49 \quad 4.57$

$C = 87 \quad 5 \quad 26.57$

$D = 77 \quad 52 \quad 15.16$

Hence, for the adjustment of observations subject to single condition, divide whole discrep. among *observations in inverse ratio of weights.

Eg. Mistake in all was 58

For A $\dfrac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{2}+\frac{1}{2}+1} \times 58 = 8.29$

*Not among unknowns

[Ex. 7 Expansion alcohol with temperature.

$V = 1 + Bt + Ct^2$ $B = .00275$
$C = -.000\,000\,38$

t	V
13.947	1.0377
42.974	1.1180
67.810	1.1850
88.970	1.2411
99.210	1.2698

Wright 189–94 to end p 417.]

	x	Y	Yx	x^2	Yx^2	x^3	x^4	
Log.								
Nat	99.210	1.12698	125977	9842.62	12494.16	976490	~~976490~~	
	Σx	ΣY	ΣYx	Σx^2	ΣYx^2	Σx^3	Σx^4	

$184,127,000\ C + 2074620\ B - 5639.92 = 0$ 1st normal

$2\,074620\ C + 24397.8\ B - 66.360 = 0$

Solve for B first since C so small & so easily altered
7 place logarithms.

18

VIII Nov. 2d.
Quiz.

IX Nov. 5th.

Non-linear.

Let be series n obs. on $y = fx$
$= A + Bx + Cx^2 + \ldots$
approx. Deduce $A B \& C$

Observations

$Cx_1^2 + Bx_1 + A - Y_1 = 0$

$Cx_2^2 + Bx_2 + A - Y_2 = 0$

$\ldots\ldots\ldots\ldots$

$Cx_n^2 + Bx_n - A - Y_n = 0$

Normal eqs.

C $\quad C\Sigma x^4 + B\Sigma x^3 + A\Sigma x^2 - \Sigma Yx^2 = 0$

B $\quad C\Sigma x^3 + B\Sigma x^2 + A\Sigma x - \Sigma Y\cancel{x} = 0$

A. $\quad C\Sigma x^2 + B\Sigma x + nA - \Sigma Y = 0$

Highest power in normal eq is twice that in ~~required~~ equation.
Easiest way to tabulate. Take care retain only requisite n^o figures, in very large n^{os} may reject several before decimal pt. Care to keep same degree accuracy in all terms. May shorten very greatly. First get largest n^{os} in column & determine what may be rejected.

Nov. 9th X

Example. Most prob. eq. strt. line fr. 4 measurements of coord.

Take form eq line $y = mx + b$

$y = .5$	$x = .4$
.8	.6
1.	.8
1.2	.9

$b + .4m - .5 = 0$

$b + .6m - .8$

$b + .8m - 1 = 0$

$b + .9m - 1.2$

Normals — $4b + 2.7m - 3.5 = 0$

$2.7b + 1.97m - 2.56 = 0$

[Ex 8. Pendulum

$T = mL^n$

T tenths sec.	L centimetres
12.9	164.4
11.6	132.9
10.4	107.6
9.7	93.5
5.3	28.4
4.6	20.6]

$n = .5 \ (.499953)$

$m = 1.0044$

Eliminating $m = 1.34$ $b = -.029$ $y = 1.34x - .029$

Example Gordon's formula wrought iron columns

$C = \dfrac{S}{1+Tj^2}$ C being crushing load for unit area cross-section. j ratio length to ~~basal~~ least diam.

$S - Cj^2T - C = 0$

$C = 34650$ $j = 42$	$C = 35000$ 33	36680 $24.$	37030 19.5

Normal eqs.

$4S - \Sigma cj^2 T - \Sigma C = 0$ for S

$-\Sigma cj^2 S + \Sigma c^2 j^4 T + \Sigma c^2 j^2 = 0$ for T

ie $4S - 134{,}388{,}340T - 143{,}260 = 0$

$134{,}388{,}340\,S - 5{,}830{,}938{,}000{,}000{,}000T - 4744\,07{,}30\cancel{0{,}000}$

$\therefore\ S = 37600$ $\quad C = \dfrac{37600}{1 - .0000219 j^2}$
$T = -.0000219$

Ex. 8. $T = ml^n$, since not linear as regards m & n proceed to reduce.

$\log T = \log m + n \log l$
$\quad = k + n \log l$ $\qquad k + n\log l - \log t = 0$

Normals.

$6k + n\Sigma \log l - \Sigma \log t$ for k

$\Sigma \log l \times k + \Sigma (\log l)^2 n - \Sigma \log l \log t$ for n

Tabulate. Logs to 4 places

(See over

Ex. 8

log T log l	log T	T	l	log l	$(\log l)^2$
2.46098	1.1106	12.9	164.4	2.2159	4.91021

* Since mean of eqs. does not give mean value y, ~~must divide by coeffs of y first.~~ unless all coeffs of y are unity.

Wright § 104, 192

Do not do Ex. 8 thus.

~~Tabulate. Logs to 4 places.~~

Special methods — When coeffs of 1 unknown are all 1.

Example

$x + .35y + 1.98 = 0$

$x + .50y + 1.90$

$x + .71y + 2.00$

$x + .98y + 1.95$

$x + 1.22y + 2.00$

$x + 2.05y + 1.97 = 0$

$x + .97y + 1.97 = 0$ is mean eq. fr. adding. (if nec. mult. by wts)

If now subtr. fr. each. obs. eq., get

$-.62y + .01 = 0$

$-.47y - .07 = 0$

$-.26y + .03 = 0$

$+.01y - .02 = 0$

$.25y + .03 = 0$

$1.08y + 0 = 0$

Now form normal for y by usual process.

Must not take mean again.* if working by Least Square principle

$1.90y + 0.026 = 0$

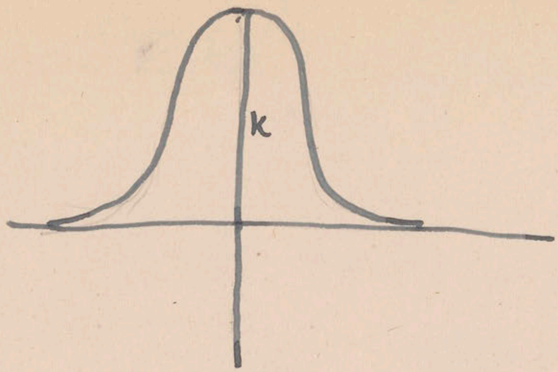
$y = -.014$

Subst. in mean eq.

$x - .97 \times 0.014 + 1.97 = 0$ $\quad x = -1.96$

If coeffs other unknown large & not very diff., this method specially useful.

Must not divide coeffs. to make all unity, because alters weights.



* which not directly integrable, must proceed as follows

‡ Since definite, product two ints same form = sq. of either

~~** v does not vary with x, hence product of integrals is double integral of products~~ I think this is wrong.

Lee Byerly p. 99

**

~~vx~~ x & v are in reality independent.

∴ product of ints. = ∬ product. ~~Right~~

v & x are used to represent h & Δ to avoid confusion.

(Over

Nov. 12th XI

Return to curve of error. $y = ke^{-h^2\Delta^2}$

To find value of k

when $\Delta = 0$, $y = k$, k is intercept on Y

thus found graphically, to find in terms h

remember total area curve $= 1$, i.e. $\int_{-\infty}^{\infty} y\,d\Delta$

$$= \int_{-\infty}^{\infty} ke^{-h^2\Delta^2} d\Delta = 1 = 2k\int_0^{\infty} e^{-h^2\Delta^2} d\Delta \; *$$

*

Let $h\Delta = t$, $d\Delta = \frac{dt}{h}$, subst. $\frac{k}{h}\int_0^{\infty} e^{-t^2} dt = \frac{1}{2}$

Let $vx^{**} = t$, ~~subst~~ then $\int_0^{\infty} e^{-v^2x^2} v\,dx = \int_0^{\infty} e^{-t^2} dt$

& mult. by ~~1st member~~ $\int_0^{\infty} e^{-v^2} dv$

$$\neq k \left[\int_0^{\infty} e^{-t^2} dt\right]^2 = {}^{*}_{*}\int_0^{\infty}\int_0^{\infty} e^{-v^2(1+x^2)} v\,dx\,dv$$

$$\equiv \int_0^{\infty}\int_0^{\infty} e^{-v^2(1+x^2)} \frac{\left(-2v\sqrt{1+x^2}\right)}{-2\sqrt{1+x^2}} dx\,dv$$

$$= \int_0^{\infty} \frac{dx}{-2(1+x^2)} \left[e^{-v^2(1+x^2)}\right]_{\infty}^{0} = \int_0^{\infty} \frac{dx}{2(1+x^2)}$$

$$= \frac{1}{2}\left[\tan^{-1} x\right]_0^{\infty} = \frac{\pi}{4}$$

$$\therefore \int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \qquad \therefore \frac{k}{h}\frac{\sqrt{\pi}}{2} = \frac{1}{2}$$

& $k = \frac{h}{\sqrt{\pi}}$ Eq. becomes $y = \frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2}$

$$\int e^{-h^2\Delta^2} 2h\, d\Delta = u$$

$\int e^{-h^2\Delta^2} d\Delta$ ~~[illegible]~~ Cannot integrate because cannot multiply by Δ under the $\int$, can however multiply by h, and since h & Δ wholly independ. may compound another integration with respect to h, hoping that this ~~transform~~ integration will make integrable as regards Δ also. But only integral in h which will not complicate second member must be of same form, ie $\int e^{-h^2} dh$ wh. will give u^2 and does give an integrable first member.

To find pts. of inflection.

Make 2d diff. zero. $\quad y = \frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2}$

$$\frac{dy}{d\Delta} = \frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2}(-2h^2\Delta) = -\frac{2h^3}{\sqrt{\pi}} e^{-h^2\Delta^2}\Delta$$

$$\frac{d^2y}{d\Delta^2} = -\frac{2h^3}{\sqrt{\pi}}\left[\Delta e^{-h^2\Delta^2}(-2h^2\Delta) + e^{-h^2\Delta^2}\right] = 0$$

$$-\frac{2h^3 e^{-h^2\Delta^2}}{\sqrt{\pi}}\left(-2h^2\Delta^2+1\right) = 0$$

either $e^{-h^2\Delta^2} = 0$ impossible by inspect. since Δ never ∞

or $+2h^2\Delta^2 = 1 \quad \therefore \Delta = \pm\frac{1}{h\sqrt{2}}$

Measure of precision

Better the obs, the grter the propot. small to large errors, i.e. the larger the intercept on Y & the steeper the curve. Intercept = K $= \frac{h}{\sqrt{\pi}}$, ∴ h increases with precision, is called the measure of.

ϑ function

Table A

t	$\vartheta t = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$	diff
0.45	0.47548	918
0.46	0.48466	909
0.47	0.49375	900
0.48	0.50275	892
0.49	0.51167	883
0.50	0.52050	874
0.51	0.52924	866
0.52	0.53790	856

A tabulates ϑ function for successive values of x (or t), B for $\frac{x}{\rho}$

* No prob. there is n multiplied by the fraction wh. denotes prob. of falling there rather than elsewhere.

$$\int u\,dv = vu - \int v\,du$$

XII Nov. 16th

Average Error

Mean of Errors, or Average Deviation

Some represent by η or a.d. ~~a.d.~~

M. of E. = mean of errors all considered +ve.

Prob. ~~in~~ single obs. that error between Δ & $\Delta + d\Delta$

$= \varphi\Delta\, d\Delta$

n observations, no prob. between Δ & $\Delta + d\Delta$

* will be $n(\varphi\Delta)d\Delta$ ~~= prob. n errors between~~

& Sum of all these errors will be $n\Delta(\varphi\Delta)d\Delta$

$\therefore$ the sum of all the errors $= \int_{-\infty}^{\infty} n\Delta\varphi\Delta\, d\Delta$

$= 2n\int_0^{\infty} \Delta\varphi\Delta\, d\Delta$; $\frac{2h}{\sqrt{\pi}}\int_0^{\infty} e^{-h^2\Delta^2}\Delta\, d\Delta = a$

i.e. $a = \frac{-1}{h\sqrt{\pi}}\left[e^{-h^2\Delta^2}\right]_0^{\infty} = \frac{1}{h\sqrt{\pi}}$

Mean Square Error, or Mean Error

denoted by ϵ or μ.

Define as $\sqrt{\ }$ mean of squares.

In n obs. no between Δ & $\Delta + d\Delta$ is $n\varphi\Delta\, d\Delta$

Sum of sqs between these limits $= n\Delta^2\varphi\Delta\, d\Delta$

of all errors $= n\int_{-\infty}^{\infty}\Delta^2\varphi\Delta\, d\Delta$

$\therefore \quad \mu^2 = 2\int_0^{\infty}\Delta^2\varphi\Delta\, d\Delta$

$= \frac{2h}{\sqrt{\pi}}\int_0^{\infty} e^{-h^2\Delta^2}\Delta^2\, d\Delta$

Integrate by parts.

$\mu^2 = \frac{2h}{\sqrt{\pi}}\left[\Delta\int e^{-h^2\Delta^2}\frac{-2h^2}{-2h^2}\Delta\, d\Delta - \frac{-1}{2h^2}\int e^{-h^2\Delta^2}d\Delta\right]_0^{\infty}$

By method evaluating indeterminates

In this case considering prob. affecting a single error, sum $\varphi\Delta d\Delta$ not $n\varphi\Delta d\Delta$ as before

24

i.e. $$\mu^2 = \frac{2h}{\sqrt{\pi}}\left[\left[\frac{\Delta e^{-h^2\Delta^2}}{-2h^2}\right]_0^\infty + \frac{1}{2h^2}\int_0^\infty e^{-h^2\Delta^2} d\Delta\right]$$

If $\Delta = \infty$ this function $= \infty \times 0$

But $\lim\limits_{\Delta=\infty}\left(\frac{\Delta}{e^{h^2\Delta^2}}\right) = \lim\limits_{\Delta=\infty}\left(\frac{1}{e^{+h^2\Delta^2} 2h^2\Delta}\right) = 0$

by differentiating ~~[illegible]~~ num & denom

$$\therefore \mu^2 = \frac{1}{h\sqrt{\pi}}\int_0^\infty e^{-h^2\Delta^2} d\Delta = \frac{1}{h^2\sqrt{\pi}}\int_0^\infty e^{-h^2\Delta^2} d(h\Delta)$$

$$= \frac{1}{h^2\sqrt{\pi}} \frac{\sqrt{\pi}}{2} \text{ (as in last lecture)}$$

$$\therefore \mu = \pm \frac{1}{h\sqrt{2}}$$

wh. already found abscissa of pt. of inflexion.

The Probable Error

Def. denoted gen. by r or by pe

Prob. exceeding which = prob. failing reach it $= \frac{1}{2}$

· Prob. of an error between Δ & $\Delta + d\Delta = \varphi\Delta\, d\Delta$

$$r \;\&\; -r = \int_{-r}^{+r} \varphi\Delta\, d\Delta$$

i.e. prob. that error is numerically less than r

$$= 2\int_0^r \varphi\Delta\, d\Delta$$

Required $= \frac{1}{2}$

$$\text{ie } \frac{2h}{\sqrt{\pi}}\int_0^r e^{-h^2\Delta^2} d\Delta = \frac{1}{2}$$

[Ex. 9

470 determinations of rt. ascension Sirius & Altair the p.e. of single obs. was found to be $r = 0''.2637$. Find no. errors wh. should have fallen between

		Ans.	Obs.
0''.0 &	0.1	95	94
.1	.2	89	88
2	3	78	78
0	1.0	465	462
over	1.0	5	8

]

whence determine r

Let $h\Delta = t$, $h\,d\Delta = dt$

$$\frac{2}{\sqrt{\pi}}\int_0^{hr} e^{-t^2}\,dt = \frac{1}{2}$$

This is a tabulated function. See Table A, interpolate if nec.

$\cancel{\ }$ $.47694 = hr = \rho$

$\cancel{hr = \rho}$. $h = \frac{\rho}{r}$,

the p.e. inversely as measure precision.

[The tabulation by expanding e^{-t^2} in series and integrating.

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{3} + \cdots\cdots$$

$$e^{-t^2} = 1 - t^2 + \frac{t^4}{\lfloor 2} - \frac{t^6}{\lfloor 3} + \cdots\cdots$$

$$\int_0^t e^{-t^2}\,dt = t - \frac{t^3}{3} + \frac{t^5}{5\lfloor 2} - \frac{t^7}{7\lfloor 3}$$]

$\underline{\text{XIII}}$ Nov. 19th

Relative values of μ; a.d.; p.e. & h (or r)

Found already

$$a.d. = \frac{1}{h\sqrt{\pi}};\quad h = \frac{\rho}{r}$$

$$= \frac{r}{\rho\sqrt{\pi}} = \frac{r}{.8453} = 1.1829\,r$$

$$\underline{r = .8453\,a}$$

$$\mu = \frac{1}{h\sqrt{2}} = \frac{r}{\rho\sqrt{2}} = \frac{r}{.6745} = 1.4826\,r$$

$$\underline{r = .6745\,\mu}$$

Table B $\quad \vartheta\left(\frac{a}{r}\right) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{r}} e^{-t^2}\,dt$

ϑ = function
not multiplier.

$\frac{a}{r}$	$\vartheta\left(\frac{a}{r}\right)$	diff
0.3	.160	53
0.4	.213	
0.6	.314	
0.7	.363	49
0.8	.411	48
1.0	.500	
1.1	.542	42
1.2	.582	40
1.3	.619	37
1.6	.719	29
1.7	.748	
2.0	.823	
2.5	.908	
3.0	.957	
3.7	.987	
3.8	.990	3.0
4.0	.993	
5.0	.999	

Wright gives on p435
Chauvenet appendix

Abscissa centre gravity of 1st half of curve

$$= \Delta_0 = \frac{\int_0^\infty y\Delta\,d\Delta}{\int_0^\infty y\,d\Delta} = \frac{\frac{2h}{\sqrt{\pi}}\int_0^\infty e^{-h^2\Delta^2}\Delta\,d\Delta}{1}$$

$$= \frac{-\frac{1}{h\sqrt{\pi}}\left[e^{-h^2\Delta^2}\right]_0^\infty}{} = \frac{1}{h\sqrt{\pi}} = a(d)$$

Convenient form

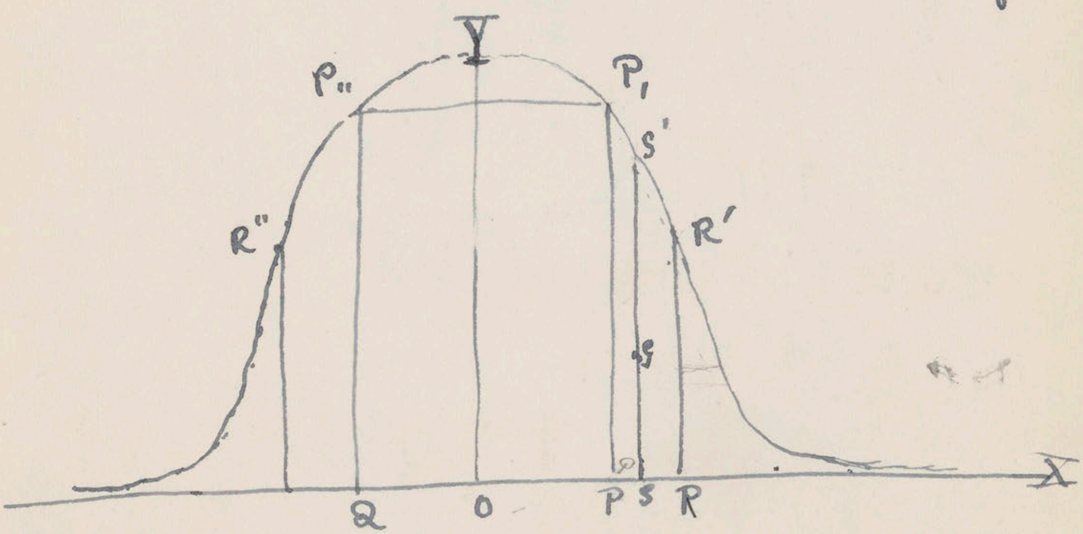
$$\mu\sqrt{2} = \frac{r}{\rho} = a\sqrt{\pi} = \frac{1}{h}$$

Tabulating

	μ	r	a.d.
μ =	1.000	1.4826	1.2533
r =	0.6745	1	0.8453
a.d =	0.7979	1.1829	1

terms in which reckoned

Representation of μ r & a.d. on curve of error



R' = pt. inflexion, P_1 & P_{11} include area $\frac{1}{2}$

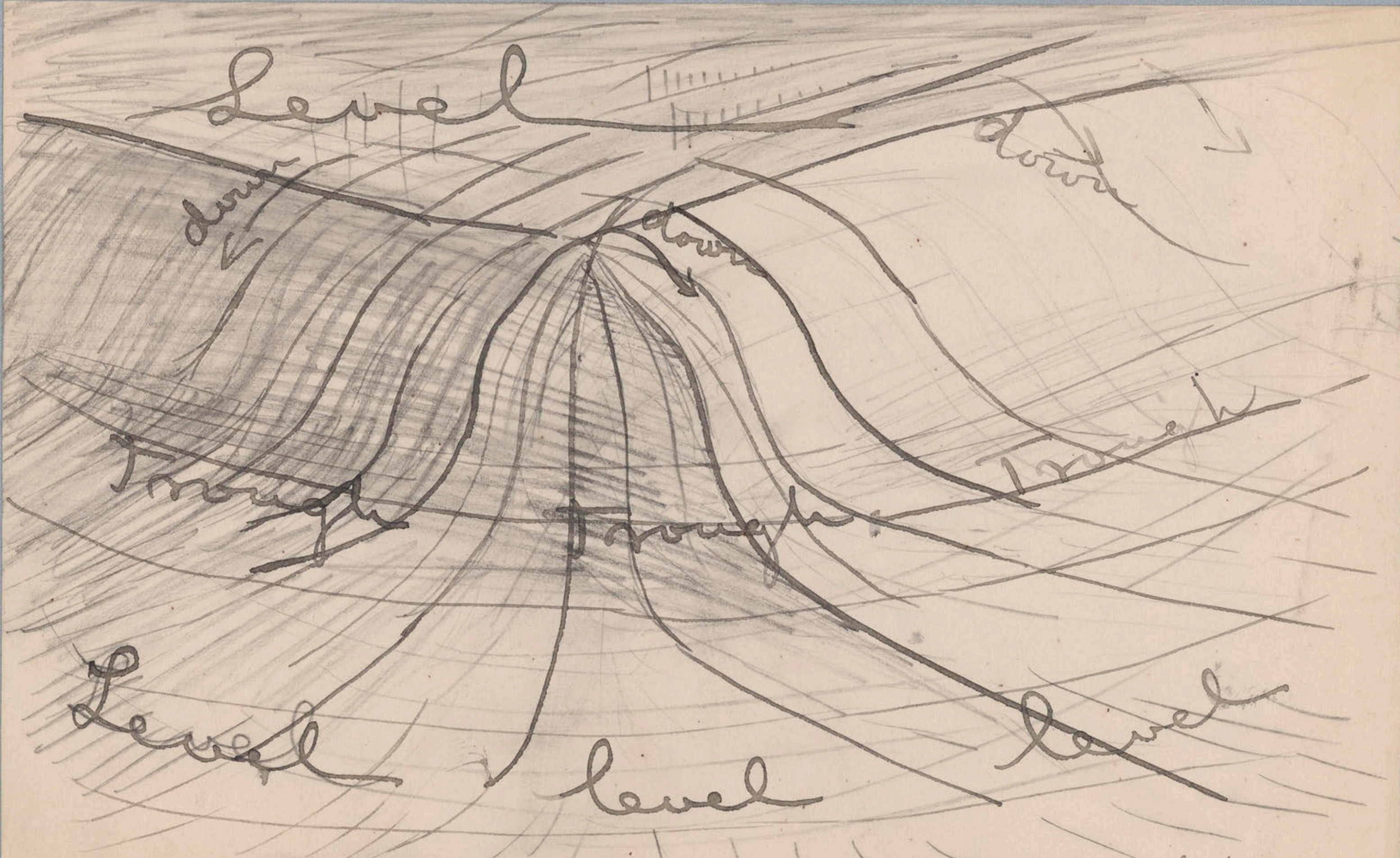
OR = $\frac{1}{h\sqrt{2}} = \mu$ OP = probable error = r

G = centre gr. a = OS

ϑ function of $x = \frac{2\cancel{x}}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$

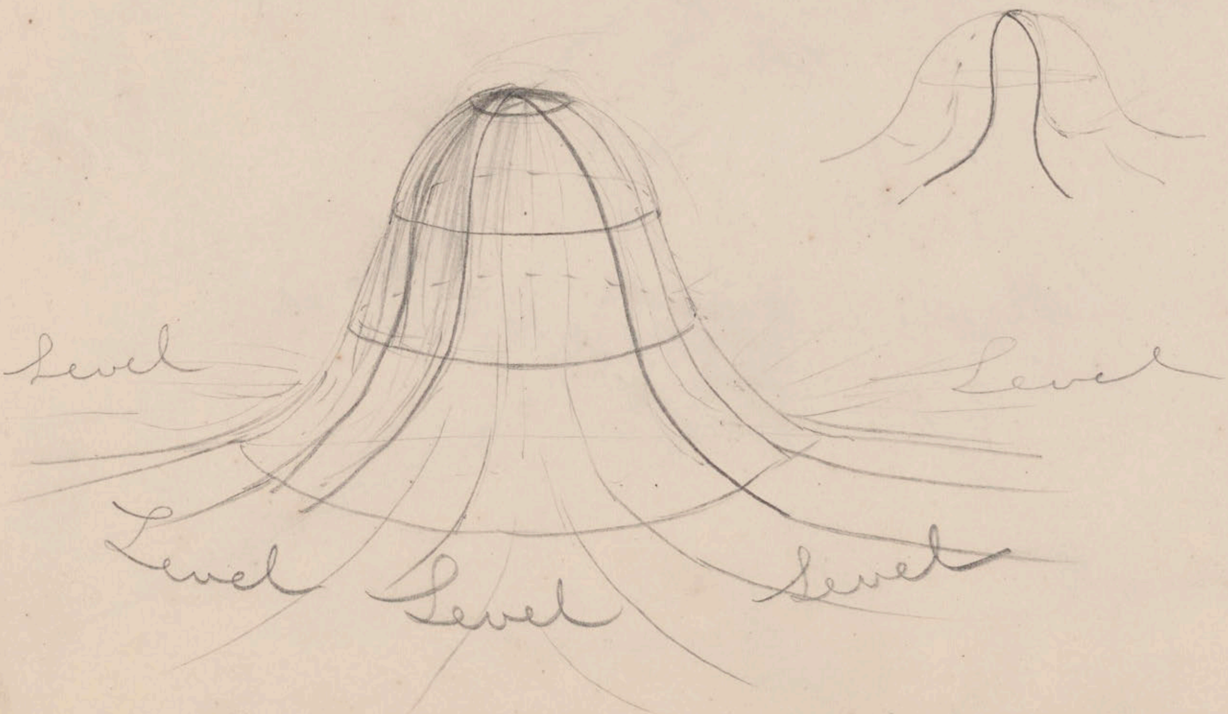
*A tabulates for values of x

B $\frac{x}{\rho}$



Surface of distribution

Surface of frequency



Comparison Theoret & Observed Results

Prob. occurrence error between a & $-a$

$$= P = \frac{2h}{\sqrt{\pi}}\int_0^a e^{-h^2\Delta^2}\,d\Delta = \frac{2}{\sqrt{\pi}}\int_0^{ha} e^{-t^2}\,dt$$

Let $h\Delta = t$

$h\,d\Delta = dt$

$h = \frac{\rho}{r}$

$$P = \frac{2}{\sqrt{\pi}}\int_0^{\rho\frac{a}{r}} e^{-t^2}\,dt = (\vartheta)\frac{a}{r}$$

ie prob. committing error less than a

Multiply by n^o obs., get prob n^o errors less than a.

Tabulated in Table B.*

1st method Given pe, to find n^o errors each part

Example

In 21 det. of correction to chronom, the pe of single obs. was found 0.08 sec. Find n^o errors between

0.0	& 0.10	(a)
1.0	0.20	(b)
over	0.20	(c)

In Table B for argument $\frac{a}{r} = \frac{.10}{.08} = 1.25$

we have

(a) $\vartheta\left(\frac{a}{r}\right)\left[= .582 + 37 \times .5\right] = .600$

21 obs. $\therefore$ n^o errors less than $0.1 = 21 \times .6 = 13$

Obs. gives 14

(b) $\frac{a}{r} = \frac{.2}{.08} = 2.5$, by table $\vartheta\frac{a}{r} = .908$

$21 \times .908 = 19.068 =$ n^o errors less than .2

$\underline{13}$

6.~~068~~ between 0.1 & .2, obs. gives 5

(c)

$21 - 19 = 2 =$ errors over .2, obs. gives 2

Wright. § 21, 22, 24-30, 37, 38, 39

~~$y = A + Bx + Cx^2$~~

[Ex. 10

A line is measured 5 times, pe = .016 ft

How many times that pe may be .004

Ex. 11. One party states length of a line = 683.4 ± 0.3

Another says = 684.8 ± 0.3

What infer fr. statements?

Ex. 12. a

Find relative wts of these measurements of area of a field

5674 ± 12

5680 ± 4

5685 ± 3

5682 ± 1

5678 ± 2]

2d method comparing, given no errors each size

Example 1 on time vibr. magnetic needle.
assume all errors between two limits fall
half away between

	errors size	.5	tenths sec.
52		1.5	
30		2.5	
11		3.5	
4		4.5	
3			

Average deviation = a d = sum ÷ no = $\dfrac{26+45+27.5+14+13.5}{100}$

= 1.26

Mean error = $\frac{1}{4}+\frac{9}{4}+\frac{25}{4}+\frac{49}{4}+\frac{81}{4}$

$\mu^2 = \dfrac{\frac{52}{4}+\frac{270}{4}+\text{etc}}{100} = \dfrac{13+67.5+\cdots}{100}$

$= 2.59$, $\mu = 1.61$

$\therefore \dfrac{\mu}{ad} = 1.28$ by observation

But from values μ & ad in terms of r, or from table $\dfrac{\mu}{ad} = 1.2533$
by theory

XIV Nov. 23d

Quiz, unexpected. very

XV Nov. 26th

Relation between weight of obs. and values of μ h r & a.d.

Problem z_1 z_2 --- unknowns. Obs. on values of n functions of give M_1, M_2 M_3 ----, Measures of precision = h_1 h_2 h_3 --- all being different. Δ_1 Δ_2 ....
To find the weights of the obs.

As found already when h constant.

Prob. occurrence Δ_1 in first obs, followed by Δ_2 in second, Δ_3 in third, etc $= P = \frac{h_1}{\sqrt{\pi}} e^{-h_1^2\Delta_1^2} d\Delta \frac{h_2}{\sqrt{\pi}} e^{-h_2^2\Delta_2^2} d\Delta \cdots$

$$= \frac{h_1 h_2 \cdots h_n}{(\sqrt{\pi})^n} e^{-[h_1^2\Delta_1^2 + h_2^2\Delta_2^2 + \cdots h_n^2\Delta_n^2]} \, d\Delta^n$$

For most prob. system, P a maximum.

$h_1^2\Delta_1^2 + \cdots h_n^2\Delta_n^2$ a minimum

$$\cancel{\Sigma} h_1^2 \Delta_1 \frac{d\Delta_1}{dz_1} + h_2^2 \Delta_2 \frac{d\Delta_2}{dz_1} \cdots h_n^2 \Delta_n \frac{d\Delta_n}{dz_1} = 0 \qquad (A)$$

$$h_1^2 \Delta_1 \frac{d\Delta_1}{dz_2} + h_2^2 \Delta_2 \frac{d\Delta_2}{dz_2} \cdots h_n^2 \Delta_n \frac{d\Delta_n}{dz_1} = 0$$

etc

Ie the series of normal equations for this case (B)

Let $h_1^2 = Cp_1$, $\quad h_n^2 = Cp_n$

Subst.

$$p_1 \Delta_1 \frac{d\Delta_1}{dz_1} + p_2 \Delta_2 \frac{d\Delta_2}{dz_1} + \cdots = 0 \qquad (C)$$

$$p_1 \Delta_1 \frac{d\Delta_1}{dz_2} + \cdots = 0$$

etcetera.

These are evidently normal eqs. from obs. of wts $p_1 p_2 p_3 \cdots p_n$

I.e. wts of obs $\propto$ sq measures of precision

Also since $r = \frac{\rho}{h}$, $\mu = \frac{1}{h\sqrt{2}}$, $ad = \frac{1}{h\sqrt{\pi}}$

wt of an obs $\propto$ inversely sq mean error, sq. p.e., sq. ad

I.e. p (wt.) $\propto \frac{1}{r^2} \propto \frac{1}{\mu^2} \propto \frac{1}{ad^2} \propto h^2$

Example.
Latitude of a station, (64 obs), $49^\circ\, 1'\, 9''.11 \pm .051''$ pe_0
What was pe. of a single obs.?

$$pe_0 = \frac{pe}{\sqrt{m}}, \quad pe = 8 \times .051 = .408$$

Example.
Two det. l, obtained $l = 427.32 \pm .04$
$l = 427.30 \pm .16$

Find relative weight observations.

$$\frac{r}{r_1} = \frac{4}{16} = \frac{1}{4}, \quad \frac{p}{p_1} = \frac{1}{16}$$

δ is not known, but approx. $= \mu_0 = \frac{\mu}{\sqrt{m}}$ mean error of arith mean.

$$\therefore \mu^2 = \frac{\Sigma v^2}{m} + \frac{\mu^2}{m}, \quad \mu^2(m-1) = \Sigma v^2, \cancel{\mu^2 =}$$

$$\mu = \sqrt{\frac{\Sigma v^2}{m-1}} \qquad \mu_0 = \sqrt{\frac{\Sigma v^2}{m(m-1)}}$$

$$r = .6745\mu$$

Relation pe single obs. & pe of arth mean

Let make m obs. of wt. unity on value of M.
then wt. of arth mean $= m$

Now by what just found

$$r^2 : r_{0\,\text{mean}}^2 = \frac{1}{1} : \frac{1}{m}, \quad r_0 = \frac{r}{\sqrt{m}}$$

Similarly

$$\mu_0 = \frac{\mu}{\sqrt{m}}$$

$$a.d._0 = \frac{a.d.}{\sqrt{m}}.$$

I.e. n^2 obs. required to give final accuracy n times as grt. as accuracy of single observation

Notice method writing pe, $\pm$ after most prob. value.

Given m obs. on M, all of equal weight, find μ ad pe of single obs. & of arith mean.

Sint obs. M_1, M_2, M_3, arith mean M_0

$\Delta_1\ \Delta_2\ \Delta_3 \cdots$

$v_1\ v_2\ v_3 \cdots$

$v_1 = M_1 - M_0$, $v_2 = M_2 - M_0$ etc, $\mu^2 = \frac{\Sigma \Delta^2}{m}$

In limiting case of accuracy, $M_0 = M$, then $v = \Delta$

$$\mu^2 = \frac{\Sigma v^2}{m}, \mu = \sqrt{\frac{\Sigma v^2}{m}}, \text{~~when~~}$$

When n^o obs. is very large, becomes practically true

When n^o small, must seek further approximation.

Let true $M = M_0 + \delta$, then $\Delta_1 = M_1 - M_0 - \delta = v_1 - \delta$

$\Delta_2 = v_2 - \delta$, $\Delta_3 = v_3 - \delta$ etc.

Squaring, adding, $\div m$

$$\frac{\Sigma \Delta^2}{m} = \mu^2 = \frac{1}{m}\left(\overline{v_1 - \delta}^2 + \overline{v_2 - \delta}^2 + \cdots\right) = \frac{\Sigma v^2 - 2\Sigma v\delta + m\delta^2}{m}$$

$$= \frac{\Sigma v^2}{m} + \delta^2 \text{ since } \Sigma v = 0$$

[Ex. 13.

Following measurements angle prism. Find most prob val angle, also mean & prob. errors and a.d, also same for a single obs.

	Ans.
34° 55′ 35″	$\mu = 20.0$
35	$r = 13.5$
20	$\mu_0 = 6.3$
05	$r_0 = 4.2$
75	
40	
10	a.d. = 14.8
30	A.D = 4.7
50	
30	$M_0 = 34° 55′ 33″$

? .8453AD

Ex. 14.

Diff longitude 2 stations. Use (36) & (39)

M	p.	Ans.
1° 4′ 30″	4	From (36)
41	1	$\mu = 17.7$ $r = 11.9$
43	1	$\mu_0 = 2.1$ r_0 1.4
37	9	From (39)
48	4	ad = 15.8 $r = 13.4$
34	16	$ad^{iv*} = 5.3$ $r^{iv} = 4.5$
25	9	AD = 1.8 $r_0 = 1.5$
46	1	
28	25	$M_0 = 1° 4′ 31.6″$
24	4	]

* of 4th observation.

Have found $\mu^2 = \frac{\Sigma v^2}{m-1}$ or $\Sigma v^2 = \overline{m-1}\,\frac{\Sigma\Delta^2}{m}$

& $v_1 = \sqrt{\frac{m-1}{m}}\,\Delta_1$ as an approximation ⟵

$v_2 = \sqrt{\frac{m-1}{m}}\,\Delta_2$ etc.

Adding, ÷ n, neglecting signs for absolute values.

$$\frac{\Sigma v}{m} = \sqrt{\frac{m-1}{m}}\,\frac{\Sigma\Delta}{m} = \sqrt{\frac{m-1}{m}}\,(ad)$$

$$\therefore ad = \frac{\Sigma v}{\sqrt{m(m-1)}}, \quad \therefore r = .8453\,\frac{\Sigma v}{\sqrt{m\,\overline{m-1}}}$$

Thus to find accuracy of observations may either compute Σv^2 & hence μ or by approx above may compute Σv & hence r. Latter easier of course.

$$A.D. = \frac{ad}{\sqrt{m}} = \frac{\Sigma v}{m\sqrt{m-1}} \qquad r_0 = \frac{.8453\,\Sigma v}{m\sqrt{m-1}} \Bigg\} (33)$$

Example 8 measurements resistance piece wire.

Easiest in tabular form.

M	v	v^2
276.97	.095	.009025
88	.005	.000025
91	.035	.001225
99	.115	.013225
83	-.045	.002025
80	-.075	.005625
81	-.065	.004225
81	-.065	.004225
276.875	.250	.039600
	-.250	
M_0 ∴ v's	0	

Given

$$\therefore \mu = \sqrt{\frac{.0396}{7}} = 0.75$$

$$\mu_0 = \frac{\mu}{\sqrt{8}} = .027$$

by (28) $r = .6745\,\mu = .051$

$r_0 = .6745\,\mu_0 = .018$

by (31) $ad = \frac{.5}{\sqrt{56}} = .067$

$AD = \frac{ad}{\sqrt{8}} = 0.24$

$r = .8453\,ad = .057$

$r_0 = .8453\,AD. = .020$

$M_0 = 276.875 \pm 0.18$

Obs. of diff wts

Giv'n m direct obs. diff. wts. sing. unknown M.
Find most prob. val., mean & prob. errors, a.d.; same for any observation.

$M_1, M_2 \ldots\ldots \quad p_1\, p_2 \cdots \mu_1, \mu_2 \cdots\; v_1\, v_2 \cdots M_0, \mu_0, ; \mu =$ mean error of obs. wt. unity.

$$M_0 = \underline{\text{General Mean}} = \frac{p_1 M_1 + p_2 M_2 + \cdots}{p_1 + p_2 \cdots} = \frac{\Sigma p M}{\Sigma p} \quad (32)$$

$$\mu_0 = \frac{\mu}{\sqrt{\Sigma p}}, \quad \mu_K = \frac{\mu}{\sqrt{p_K}}$$

Since $v_1\, v_2 \cdots =$ resid. errors fr. obs wts $p_1\, p_2 \cdots$, then if $M_1, M_2 \cdots$, reduced to equiv. obs. of same wt 1, the resid. errors would prob. be of magnitude $v_1\sqrt{p_1}, v_2\sqrt{p_2} \cdots$ (35)

Then $\mu = \sqrt{\dfrac{\Sigma p v^2}{m-1}}$ by (27) (36)

$r = .6745\mu$

$$\therefore \mu_K = \sqrt{\frac{\Sigma p v^2}{p_K(m-1)}} \qquad r_K = .6745\,\mu_K \qquad (37)$$

$$\mu_0 = \sqrt{\frac{\Sigma p v^2}{\Sigma p(m-1)}} \qquad (38)$$

By 31 $\quad$ a.d. $= \dfrac{\Sigma v\sqrt{p}}{\sqrt{m(m-1)}}$, $\quad AD = \dfrac{\Sigma \sqrt{p}v}{\sqrt{\Sigma p\, m(m-1)}}$ $\quad$ (39) (40)

$r = .8453$ ad $\qquad r_0 = .8453$ A.D

$$\text{a.d.}_K = \frac{\Sigma\sqrt{p}\, v}{\sqrt{p_K(m-1)m}} \qquad (41)$$

[Ex. 15

Given $M_1 = 49.53$ $a.d._1 = .59$

$M_2 = 50.38$ $a.d._2 = .93$

$M_3 = 49.64$ $a.d._3 = .27$

find the average deviations of

$M_1 - M_2 + M_3$ Ans. 1.13

$.5 M_3$ 1.35

$\frac{7}{2} M_1$ 2.06

Ex. 16.

Zenith distance ζ of star observed n_1 times at upper culmination, ζ_1 at lower culm. obs n_2 times. Latitude place $= 90 - \frac{1}{2}(\zeta + \zeta')$, pe single obs $= r$. Find pe of the latitude.]

Example $M_1 = 48.81$ $\mu_1 = .87$

$M_2 = 48.76$ $\mu_2 = .97$

$M_3 = 51.56$ $\mu_3 = 1.12$

find E for $M = M_1 - M_2 - M_3$

By (A3) $E^2 = .7569 + .9405 + 1.2544$

$E = 1.72$

Dec. 7th XVII

To find error of any function of the quantities measured.

E R. D = µ ρ e ad of required function

(a) Let function $M_1 \pm M_2 = M$.

Let no obs. on M_1 M_2 etc be large, approx. same, call m. Real errors obs. for $\mu_1 = \Delta_1'\ \Delta_1'' \cdots \Delta_1^m$ etc

$\mu_2 = \Delta_2'\ \Delta_2'' \cdots \Delta_2^m$

Real errors $M = X', X''$ etc

Then $X' = \Delta_1' \pm \Delta_2'$ $X'' = \Delta_1'' \pm \Delta_2''$ etc.

$$E^2 = \frac{\Sigma X^2}{m} = \frac{\Sigma\Delta_1^2 \pm 2\Sigma\Delta_1\Delta_2 + \Sigma\Delta_2^2}{m} = \mu_1^2 + \mu_2^2 \quad (42)$$

Since $\pm\Sigma\Delta_1\Delta_2$ vanishes in most general case, because as many $-$ve as $+$ve products of same magnitude, more probable that reduce to zero than to any other quantity.

(b) Let $M = M_1 \pm M_2 \pm M_3$, & E' = mean error $M_1 \pm M_2$

& (42) $E^2 = E'^2 + \mu_3^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$

i.e. in general

$$E^2 = \Sigma\mu^2 \quad (43)$$

Since R & D are multiples (fract.) of E

$$R^2 = \Sigma r^2 \qquad D^2 = \Sigma(ad)^2 \quad (44)(45)$$

Let $M = aM_1$ $X_1' = a\Delta_1'$ $X'' = a\Delta_1''$

$$E^2 = \frac{\Sigma X^2}{m} = \frac{a^2\Sigma\Delta_1^2}{m} = a^2\mu_1^2, \quad \varepsilon = a\mu, \quad R = ar \quad (46$$

$D = a(ad)$

* ie 180 − (A+B+C)

Example

r = pe angle triangle. Find pe of the triangle error,*
all angles being equally well measured.

By (14) $R^2 = 3r^2$, $R = R\sqrt{3}$

Example

Expansion bar for 1° C = 9a ± 9r. Find for 1° F.

$1°F = \frac{5}{9}°C$ $\quad$ $Exp. = \frac{5}{9}9a = 5a$

$\therefore \quad R = \frac{5}{9}(9r) = 5r$

$\therefore$ Exp. 1° F = 5a ± 5r.

Example

Telegraphic longitude results, Cambridge w. of
Greenich $4^h 44' 30.''99 \pm .23''$
Omaha w Cambridge $1^h 39\ 15.04 \pm .06$
Springfield e Omaha $25'\ 8.''69 \pm .11$
find longitude Springfield & its p.e.

$L = l_1 + l_2 - l_3$ $\quad$ $R^2 = \Sigma pe^2 = .23^2 + .06^2 + .11^2$
$= .26$

$\therefore$ Long Spring $= 5^h\ 58^m\ 37.^s34 \pm .26.$

Let $M = a_1 M_1 \pm a_2 M_2 \pm \cdots a_m M_m$

By (46) & (43).

$E^2 = a_1^2 \mu_1^2 + a_2 \mu_2^2 \cdots a_m^2 \mu_m^2 = \Sigma a^2 \mu^2$

$R^2 = \Sigma a^2 r^2, \quad D^2 = \Sigma a^2 (ad)^2$ (47)

Dec. 10th XVIII

Let $M = f(M_1, M_2 \cdots)$

find E R D. Assume $a_1 \cdots a_m$, nearly $M_1 M_2$ etc such that

$M_1 = a_1 + m_1$ etc. $M = f(a_1 + m_1, a_2 + m_2$ etc$)$

$\mu_1 \cdot \mu_2 \cdot \mu_3$ etc may be considered as mean errors

of $m_1 m_2$ etc

Expanding by Taylors, $M = f(a_1 a_2 \cdots) + \Big[m_1 \frac{d}{da_1} f(a_1 a_2 \cdots)$

$+ m_2 \frac{d}{da_2} f(a \cdots) + \cdots \Big] +$ terms in m^2 & higher powers,

Last terms may be neglected since m's chosen

very small.

$\therefore$ (by (47)) for the mean error of M, $E^2 = \mu_1^2 \left[\frac{dfa}{da_1}\right]^2 + \mu_2^2 \left[\frac{dfa}{da_2}\right]^2$ etc

& this is approximately

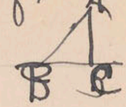
$E^2 = \mu_1^2 \left(\frac{dM}{dM_1}\right)^2 +$ etc

$R^2 = r_1^2 \left(\frac{dM}{dM_1}\right)^2 +$ etc (48)

$D^2 = ad_1^2 \left(\frac{dM}{dM_1}\right)^2 +$ etc.

[Ex. 17. If $r =$ pe $\log_{10} x$, find pe of x.

Ex. 18 Sides rt. triangle measured



$AC = 49.53 \pm 0.59$

$BC = 50.38 \pm 0.93$

Find AB & its pe. Ans 70.65 ± 0.78

Ex. 19.

Radius circle $= 1000 \pm 0.20$. Find pe of circumference & area. Ans. 1.26 & 1260]

Example If pe of $x = r$, what pe of $\log_m x$?

$$R^2 = r^2\left(\frac{d\log x}{dx}\right)^2 = r^2\frac{m^2}{x^2}, \quad R = \frac{m}{x}r$$

Example

Given $z_1\, z_2\, z_3$, pe's $r_1\, r_2\, r_3$, find pe of

$$\Sigma = z_1^2 + z_2^2 + z_3^2, \quad R^2 = r_1^2(2z_1)^2 + r_2^2(2z_2)^2 + r_3^2(2z_2)^2$$

$$= \quad = 4\left(r_1^2 z_1^2 + r_2^2 z_2^2 + r_3^2 z_3^2\right)$$

$$R = 2\sqrt{\Sigma r z}$$

Suppose $M_1\, M_2\, M_3 \dots$ = obs on M, r = pe sing obs.

Find M_0 & r_0

$$M_0 = \frac{1}{m}\left[M_1 + M_2 \dots\right], \quad r_0^2 = \frac{1}{m^2}\left[r^2 + r^2 - \dots\right]$$

$$= \frac{r^2}{m}, \quad r_0 = \frac{r}{\sqrt{m}}$$

which is same result as (25) where considered relation between weights & M_0 -- etc.

<u>Problem</u> Current & tangent-galvanometer.

$C = HK \tan\varphi$. Given errors $\delta_1\, \delta_2\, \delta_3$ in H K & $\tan\varphi$ determine Δ of C.

$$\Delta^2 = \delta_1^2\left(\frac{dC}{dH}\right)^2 + \delta_2^2\left(\frac{dC}{dK}\right)^2 + \delta_3^2\left(\frac{dC}{d\tan\varphi}\right)^2$$

$$= \delta_1^2 K^2 \tan^2\varphi + \delta_2^2 H^2 \tan^2\varphi + \delta_3^2 H^2 K^2$$

If $\delta_1\, \delta_2\, \delta_3$ not given, but the percentage errors only, then ~~as follows~~ for percentage error in C, $\div\, C^2$

$$\left(\frac{\Delta}{C}\right)^2 = \left(\frac{\delta_1}{H}\right)^2 + \left(\frac{\delta_2}{K}\right)^2 + \left(\frac{\delta_3}{\tan\varphi}\right)^2$$

Percentage error = root sum squares percentage errors.

If H to	4 in 1000	.004
K	2	.002
$\tan\varphi$	1	.001

$\frac{\Delta}{C} = \sqrt{.0042\dots}$ $= .0046$ & ½ percent.

<u>Required</u> certain accuracy in result, to find accuracy nec. in H K & $\tan\varphi$. ~~must be of~~

Total within .1 of 1 percent

if each is to have same influence on total error.

~~We have found~~ Nec. $(.001)^2 = 3\left(\frac{\delta_1}{H}\right)^2 = 3\left(\frac{d_2}{K}\right)^2 = 3\left(\frac{d_3}{\tan\varphi}\right)^2$

$\therefore \frac{\delta_1}{H}$ = percent error $H = \frac{.001}{\sqrt{3}} = .000577$

$\delta_1 = .000577\,H$

$\delta_2 = .000577\,K$ difficult for H in practice

$\delta_3 = .000577\tan\varphi$ simple for K & $\tan\varphi$.

Special laws error, when usual assumpt. void

I   All errors equally prob. - limits ~~+~~ a

$y = \varphi\Delta$ is law. Find form $\varphi\Delta$

Since all errors equally prob., $\varphi\Delta$ is a const., curve a horizontal line. $2\int_0^a \varphi\Delta\, d\Delta = 1$, hence

$\varphi\Delta = \frac{1}{2a}$

$$ad = \frac{1}{a}\int_0^a \Delta\, d\Delta = \frac{a}{2} \qquad (50)$$

$$\mu^2 = 2\int_0^a \Delta^2 \varphi\Delta\, d\Delta = \frac{1}{a}\left[\frac{\Delta^3}{3}\right]_0^a = \frac{a^2}{3} \qquad (51)$$

for pe, $2\int_0^r \varphi\Delta\, d\Delta = \frac{1}{2} = \frac{1}{a}r$, $\therefore r = \frac{a}{2}$

ad being abscissa centre gravity, & r bisecting area on one side, they must of course coincide.

Example – In estimation tenths, all errors between .05 & −.05 equally prob., & .05 is max.

Then $\frac{ad}{2} = r = \frac{.05}{2} = 0.25$

Case II. Error Δ due to two sources, each can with same prob. assume all values between a & −a

$\Delta = x + y \qquad \frac{2a}{d\Delta}$ = no diff values x or y

$\frac{4a^2}{(d\Delta)^2}$ = no ways in which error Δ caused

values x & y that produce given error Δ

for x	$\Delta - a$	$\Delta - a + d\Delta$	....	a	x or y
y	a	$a - d\Delta$		$\Delta - a$	neg but Δ +ve

Since a = maximum error & Δ − a min. for x or y

and interval dΔ, the no of causes is

$$\frac{a-(\Delta-a)}{d\Delta} = \frac{2a-\Delta}{d\Delta}$$

* Extreme error for x, = a or neg. Δ
y, = a
x + y or Δ, = 2a

no causes = $\frac{2a+\Delta}{d\Delta}$

Hence no causes occurrence sing error, ÷ by ways occurr. any error = prob. occur. any error = $\frac{2a \mp \Delta}{d\Delta} \div \frac{4a^2}{(d\Delta)^2}$

$= \frac{2a \mp \Delta}{4a^2} d\Delta$

$$r = 2a \pm a\sqrt{2} = a(2 \pm \sqrt{2})$$

Since 2a is the superior error, only the negative sign is applicable.

~~= (absolutely)~~

Case II. Error from 2 sources ($\Delta = x+y$) and x & y can with equal prob. have any val. between a & $-a$.

$\frac{2a}{d\Delta}$ = nº diff. vals x or y

$\frac{4a^2}{(d\Delta)^2}$ = nº ways in which some error produced.

Values x & y for given error Δ are

$$\begin{array}{llll} \Delta - a & \Delta - a + d\Delta & \Delta - a + 2d\Delta & \text{etc for } x \\ a & a - d\Delta & a - 2d\Delta & \text{etc for } y \end{array}$$

Max & min a & $\Delta - a$, nº diff. pairs vals $= \frac{2a-\Delta}{d\Delta}$ = nº diff ~~cases~~ ways getting Δ

Mut. mut. for neg. error, $\frac{2a+\Delta}{d\Delta}$

Prob. of occurrence of any Δ = nº ways getting that Δ divided by nº ways getting some $\Delta = \frac{2a \mp \Delta}{d\Delta} \div \frac{4a^2}{(d\Delta)^2}$

$= \frac{2a \mp \Delta}{4a^2} d\Delta$

$$* \; ad = \int_0^{2a} \frac{2a-\Delta}{4a^2} \Delta d\Delta + \int_{-2a}^0 \frac{2a+\Delta}{4a^2} \Delta d\Delta = \int_0^{2a} \left[\frac{\Delta}{2a} \mp \frac{\Delta^2}{4a^2}\right] d\Delta$$

$$+ \int_{-2a}^0 \left[\frac{\Delta}{2a} + \frac{\Delta^2}{4a^2}\right] d\Delta = \frac{4a^2}{4a} - \frac{8a^3}{12a^2} - \left[\frac{4a^2}{4a} \mp \frac{8a^3}{12a^2}\right]$$

$$= a - \frac{2}{3}a - \left(a - \frac{2}{3}a\right)$$

adding absolute values sin regard signs

$$ad = 2\left[a - \frac{2}{3}a\right] = \frac{2}{3}a$$

For pe, $\int_{-r}^{r} \varphi\Delta d\Delta = \frac{1}{2} = \int_0^r \frac{2a-\Delta}{4a^2} d\Delta + \int_{-r}^0 \frac{2a+\Delta}{4a^2} d\Delta$

$$= \frac{r}{2a} - \frac{r^2}{8a^2} - \left[\frac{-r}{2a} + \frac{r^2}{8a^2}\right] = \frac{r}{a} - \frac{r^2}{4a^2} = \frac{1}{2}, \quad r^2 - 4ar = -2a^2$$

[Ex 20

Observations on angle - determine whether largest residuals to be rejected

12′ 51″.75

48.45 Laws of Error

50.60 Wright § 31, 32, 33

47.85 Rejection 75-81

51.05 Constant errors 51-55

47.75

47.40

48.85

49.20

48.90

50.95

50.55

44.45]

13)637.75

49.06

Dec. 17th

XX.

When some obs. suspiciously different from the rest. Several methods deciding whether to reject. Peirce's & others.

Single obs. – prob. error less than a = Θ funct.

greater $= 1 - (\Theta f)$

m obs, no. grter $a = m(1 - \Theta f))$

if this is less than $\frac{1}{2}$, prob. no whole error beyond a, should reject such an obs. If grter than $\frac{1}{2}$ must retain.

Criterion – $m(1 - \Theta f) = \frac{1}{2}$, $(\Theta f) = \frac{2m-1}{2m}$

Example Given residuals

.30	–24	–1.40
–.44	+06	–.22
1.01	+63	–.05
.48	–13	+.20
+18	+10	
+39		

Determine whether –1.40 should be rejected

$r = .386$

$m = 15$

$\frac{2m-1}{2m} = \frac{29}{30} = .9667 = \Theta\left(\frac{a}{r}\right)$ See Table B

gives $\frac{a}{r} = 3.17 \times .386 = 1.22$

∴ 1.40 should be rejected

~~If given measurements & not residuals~~

[
Ex 21.

Velocity of light. Find most prob. val & pe.

298 000 ± 1000
298 500 ± 1000
299 990 ± 200
300 100 ± 1000
299 930 ± 100.

Ans.
$V = 299\,917 \pm 88$

Ex 22.

Weights of measured angles $\angle_1 \ \angle_2 \ \angle_3$
3 3 1

Find wt. sum angles. $p_0 = .6$

Ex. 23. $\text{Wt } x = p.$ Ans $p\left(\frac{x}{m}\right)^2$
$\text{wt} \log x = ?$

Ex 24 $x = \frac{y}{c}$ & wt $y = p$. What is the wt of x. Ans. $c^2 p$.

In Ex. 12. find most prob. area & pe of field.]

Wright § 68 69 70 – end p 121, 71–5.

~~then~~" After one rejection must find new mean & new residuals. Hence not gen used for more than one or two observations.

Another way

The Huge Error = one of such magnitude that 999 in 1000 errors are smaller & 1 greater.

$\therefore$ prob. ~~an~~ error less = .999, B gives $\frac{a}{r} = 4.9$

$a = 4.9\ r = 3.3\mu = 4.4$ ad.

Does not give such close limit rejection, but very safe. Should not have rejected in last example. Same objection, that valid for only one set of rejections ~~obs~~, must then recompute if another residual lies <u>near</u> limit found."

<u>Constant Errors</u>

Marksmen at target, prob. each symm. about his own centre, diff centres prob. symm. about true if under varying circumstances.

Other circumstances when causes known effects can be calculated.

Instrumental Errors

Personal Equations.

Dec 21st XXI
Recitation
Dec. 27th XXII

To compute wts of unknowns in the normal eqs. All obs of same wt 1.

Three methods.

Obs
$$\left\{\begin{array}{l} a_1x + b_1y + c_1z \; - - - \quad l_1 = M_1 \\ a_2x + b_2y + c_2z \; - - - \quad l_2 = M_2 \\ a_3x + b_3y \; - - - - \quad l_3 = M_3 \end{array}\right.$$ obs

or transposing absolute term

let $M_1 - l_1 = n_1 \qquad M_2 - l_2 = n_2$ etc.

then
$$a_1x + b_1y + c_1z \cdots \quad - n_1 = v_1$$
$$a_2x + b_2y \; - - - \quad - n_2 = v_2$$

= 0 as ord written

The more exact the more nearly true.

Normal equations

$$a_1(a_1x + b_1y \cdots \; - n_1) + a_2(a_2x + \cdots n_2) \cdots\cdots = 0$$
$$b_1(a_1x \cdots \quad) \qquad \text{etc}$$

etc.

id est

$$x\,\Sigma a^2 + y\,\Sigma(ab) + z\,\Sigma(ac) - - - - - - \Sigma(an) = 0$$
$$x\,\Sigma(ab) + y\,\Sigma b^2 + z\,\Sigma(bc) - - - - \;\Sigma(bn) = 0 \qquad (B)$$
$$x\,\Sigma(ac) + y(\Sigma bc) + z\,\Sigma c^2 - - - - \Sigma(cn) = 0$$

etc

Let

$$\alpha_1 = Q a_1 + Q' b_1 + \cdots \qquad (F)$$

$$\alpha_2 = Q a_2 + \cdots$$

Use indeterminate multipliers and add

$x\left(Q\,\Sigma a^2 + Q'\Sigma(ab) + Q''\Sigma(ac) + \cdots\right)$

$+\,y\left(Q\Sigma(ab) + Q'\Sigma b^2 + Q''\Sigma(bc) + \quad\right)$ (C)

$+ \;---$

$+\,Q\Sigma(an) + Q'\Sigma(bn) + Q''\Sigma(cn) \;---$

$= 0$.

Let Q's be so determined that coeffs. x ~~y z etc~~ ~~is~~ unity, y & z have coeffs 0.

~~Then~~ ie $Q\Sigma a^2 + Q'\Sigma(ab) + \cdots \quad -1 = 0$

$Q\Sigma(ab) + Q'\Sigma b^2 + \;---\; = 0$. (D)

and (C) becomes

$\cancel{x + \Sigma q \Sigma}\; -x + Q\Sigma an + Q'\Sigma(bn) + Q''\Sigma(cn) = 0$ (E)

(D) are like normal eqs B if $x\,y\,z$ .. be changed for $Q\,Q'\,Q''$---, $\Sigma(an)$ for -1 & $\Sigma(bn)$. $\Sigma(cn)$ ... by 0

Expanding E.

$x + Q(a_1n_1 + a_2n_2 \cdots) + Q'(\Sigma b_1n_1 + \cdots\cdots)$

$+$ etc $= 0$

or $x + n_1(Qa_1 + Q'b_1 + Q''c_1 + \cdots)$

$+ n_2(Qa_2 + Q'b_2 + Q''c_2 + \cdots)$ (E')

$+ n_3(Qa_3 + Q'b_3 + Q''c_3 + \cdots) = 0$

ie $x + \alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3 + \;--\; = 0$ (G)

Multiplying F by $a_1 a_2 \ldots$ & adding

$$\Sigma a\alpha = Q\Sigma a^2 + Q'\Sigma(ab) + Q''\Sigma(ac) + \ldots = 1 \text{ by eq } D$$

Multiply F by b's

$$\Sigma b\alpha = Q\,\Sigma(ab) + Q'\Sigma b^2 + Q''\Sigma(bc) \ldots = 0 \text{ by } D,$$

$$\Sigma c\alpha = 0 \qquad \Sigma d\alpha = 0 \qquad (H)$$

Now multiply F by α's.

$$\Sigma\alpha^2 = Q\Sigma(a\alpha) + Q'\Sigma(b\alpha) + Q''\Sigma(c\alpha) + \ldots = Q \text{ by } (H) \qquad (I)$$

Let μ_x be the mean error of value x in solution normal equations, p_x = wt, μ = me of obs wt unity.

Mean errors $\mu_1, \mu_2 \ldots$ are those of $n_1\ n_2\ n_3 \ldots$ (since absolute terms l are correct)

$$\therefore \text{ by } (G) \,\&\, (47) \quad \mu_x^2 = \alpha_1^2\mu^2 + \alpha_2^2\mu^2 + \alpha_3^2\mu^2 \ldots = \mu^2(\Sigma\alpha^2)$$

$$= Q\mu^2 \quad (J)$$

$$\text{Now, } \mu_x^2 = \frac{\mu^2}{p_x}, \quad \therefore \quad Q = \frac{1}{p_x} \qquad (K)$$

Therefore, for det. wts, ~~subst~~ for abs. term 1st norm. $\Sigma(an)$ in subst -1, ~~Σan~~, in other norms. subst for Σbn $\Sigma cn \ldots$, ~~Zero~~. Value x found from sol. these eqs is reciprocal of wt. of the x found from normal eqs.

To finds wts y & z proceed same way other normal eqs.

Dec. 28th XXIII

Second method

Normals (B) write with A for 0 etc

$$x\Sigma a^2 + y\Sigma ab + z\Sigma ac + \cdots \Sigma an = A$$
$$x\Sigma ab + y\Sigma b^2 + z\Sigma bc + \cdots \Sigma bn = B \quad (L)$$
$$x\Sigma ac + y\Sigma bc + z\Sigma c^2, \cdots \quad \Sigma cn = C$$
etc

Effect solution as above by indet. mult.
Then (E) becomes

$$x + Q\Sigma an + Q'\Sigma bn + Q''\Sigma cn \cdots = QA + Q'B + Q''C \quad (M)$$

where as shown in K, Q is reciprocal of wt of x. Whatever elimination used ~~the~~ coeff. of A must be same. ‡ I.e. write ABC for 0 in 2d members normals and solve for xyz, their most prob. values will be given by those terms independent of ABC. Wt of x will be recipr. of coeff A in value of x. Wt y is recipr. of coeff. B in value of y. Etc.

* Without clearing of fractions or reducing in any way.

Third method.

From 2d 3d etc of (L) normals with A B C find y z etc in terms of x & *subst. in 1st eq. of (L). Then get $Rx = T + A +$ terms in BC etc. (N)

T = sum absolute terms from subst.

$$x = \frac{T}{R} + \frac{A}{R} + \text{terms in } B \& C \text{ etc.}$$

where $x = \frac{T}{R}$ = val. of x required. But found by second method $\frac{1}{R} = Q \quad R = p_x$ (O)

~~Ie Subst. in no~~

Proceed in same way for y & z.

....

Case II Obs. unequal wt.

If multiply by sq. rt. wt. get equiv. eqs of wt. unity. Then as before.

Dec. 31st

XXIV

Find mean error single obs from series eq. of equal wt.

Given $a_1x + b_1y + c_1z \cdots n_1 = 0$ (A)

$a_2x + b_2y + c_2z \cdots n_2 = 0$

etc

Most prob vals found to be $x'\ y'\ z' \ldots$

Subst $\quad a_1x' + b_1y' + c_1z' \cdots\cdots n_1 = v_1$

$\quad\quad\quad a_2x' + b_2y' + c_2z' \cdots\cdots n_2 = v_2 \quad\quad (B)$

etc.

Real values $x' + \delta x$, $y' + \delta y$, $z' + \delta z$

Subst in A

$$a_1(x' + \delta x) + b_1(y' + \delta y) + c_1(z' + \delta z) + \cdots = \Delta_1$$

etc. (c)

Multiplying (C) by $a_1\ a_2\ a_3 \cdots$ & adding

$$x'\Sigma a^2 + y'\Sigma(ab) + z'\Sigma(ac) \cdots\cdots + + \Sigma(an)$$
$$+ \delta x\,\Sigma a^2 + \delta y\,\Sigma(ab) + \delta z\,\Sigma(ac) \cdots\cdots + \quad = \Sigma(a\Delta)$$
$$\cancel{+ \Sigma(an) = \Sigma(a\Delta)}$$

But by first of (B) in last proof, the first line vanishes.

$$\delta x\,\Sigma a^2 + \delta y\,\Sigma(ab) + \delta z\,\Sigma(ac) \cdots \Sigma a\Delta = 0 \quad (D)$$

Similarly $\delta x\,\Sigma(ab) + \delta y\,\Sigma b^2 + \delta z\,\Sigma(bc) \cdots \Sigma b\Delta = 0$

etc

These D are same form as B in last proof, δx stands for $x_{\text{-}\Delta\text{ for }n}$ Solution therefore same form.

$$\therefore \delta x - \alpha_1\Delta_1 - \alpha_2\Delta_2 - \alpha_3\Delta_3 \cdots = 0 \quad (E)$$

Now mult. (C) by v_1 v_2 v_3 --- & add.

$$\overline{\Sigma(av)}(x'+\delta x) + \overline{\Sigma(bv)}(y'+\delta y) \cdots\cdots + \Sigma(nv) = \Sigma(\Delta v)$$

~~Treat B in same manner~~, get

Mult (B) by a_1 a_2 a_3 --- & add

$$\Sigma(av) = x'\Sigma a^2 + y'\Sigma(ab) + \cdots\cdots \quad \Sigma an$$

$$= 0 \text{ by (B) last proof}$$

Similarly $\Sigma(bv) = 0$ $\Sigma(cv) = 0$ etc.

$$\therefore \quad \Sigma(nv) = \Sigma(\Delta v) \qquad \text{(F)}$$

Mult. (B) by v_1 v_2 v_3 --- & add

$$x'\Sigma(av) + y'\Sigma(bv) + z'\Sigma(cv) \cdots + \Sigma(vn) = \Sigma v^2$$

But $\Sigma(av) = \Sigma(bv) = \Sigma(cv) \cdots\cdots = 0$

$$\therefore \Sigma(nv) = \Sigma(v^2) = \Sigma(\Delta v) \qquad \text{(G)}$$

Mult. (B) by Δ_1 Δ_2 Δ_3 .. & add

$$x'\Sigma(a\Delta) + y'\Sigma(b\Delta) + z'\Sigma(c\Delta) \cdots \Sigma(n\Delta) = \Sigma(\Delta v)$$

$$= \Sigma(v^2) \qquad \text{(H)}$$

Mult (C) by Δ_1 Δ_2 ---- & add.

$$x'\Sigma(a\Delta) + y'\Sigma(b\Delta) + z'\Sigma(c\Delta) + \cdots + \Sigma a\Delta$$

$$+ \delta x\,\Sigma(a\Delta) + \delta y\,\Sigma(b\Delta) + \delta z\,\Sigma(c\Delta) \cdots\cdots = \Sigma\Delta^2$$

1st line by (H) ~~equal~~ to Σv^2

$$\therefore \Sigma v^2 + \delta x\,\Sigma(a\Delta) + \delta y\,\Sigma(b\Delta) + \delta z\,\Sigma(c\Delta) = \Sigma\Delta^2 \qquad \text{(I)}$$

Now if m obs., $\mu^2 = \dfrac{\Sigma\Delta^2}{m}$

$$\therefore \mu^2 = \frac{\Sigma v^2}{m} + \frac{\delta x\,\Sigma(a\Delta) + \delta y\,\Sigma(b\Delta) + -}{m}$$

As already found for $n'=1$

When obs are many the error in most prob. grows small, & approximately $\mu^2 = \frac{\Sigma v^2}{m}$

For approx. better – find mean val. $\delta x \, \Sigma(a\Delta)$

$$\Sigma(a\Delta) = a_1\Delta_1 + a_2\Delta_2 + \dots$$

by (Σ) $\delta x = \alpha_1\Delta_1 + \alpha_2\Delta_2 + \dots$

$\therefore \quad \delta x \, \Sigma(a\Delta) = a_1\alpha_1\Delta_1^2 + a_2\alpha_2\Delta_2^2 +$ + terms in products $\Delta_1\Delta_2 \quad \Delta_2\Delta_3 \dots$ etc. These last products cancel in most prob. case, same no. +ve & –ve prod. same absolute magnitude

Now for approx. write μ^2 for Δ_1^2, Δ_2^2 etc

$\therefore \quad \delta x \, \Sigma(a\Delta) = a_1\alpha_1\mu^2 + \dots = \mu^2 \Sigma(a\alpha)$

and by III last proof $\Sigma(a\alpha) = 1$

$\therefore \quad \delta x \, \Sigma(a\Delta) = \mu^2$

Similarly $\delta y \, \Sigma(b\Delta) = \delta z \, \Sigma(c\Delta) =$ etc $= \mu^2$

$\therefore$ if n' unknowns $\quad \mu^2 = \frac{\Sigma v^2}{m} + \frac{n'\mu^2}{m}$ (K)

$$\mu^2(m - n') = \Sigma v^2$$

$$\mu = \sqrt{\frac{\Sigma v^2}{m - n'}} \qquad (57)$$

by (25)

$$\mu_x = \frac{\mu}{\sqrt{p_x}} = \sqrt{\frac{\Sigma v^2}{p_x(m - n')}} \qquad (58)$$

Ex. 25

Obs. eq. $x - 2y + z - 3 = 0$

$2x + 3y - 4z - 2 = 0$

$3x + y + 2z - 17 = 0$

$-x + 4y + 3z - 10 = 0$

Find most prob. val. $x\,y\,z$ together with wts & means prob. errors.

Ans.

$x = 3.541$	$y = 1.546$	$z = 2.439$
$p_x = 14$	$p_y = 29$	$p_z = 29$
$\mu_x = .036$	$\mu_y = .025$	$\mu_z = .025$
$r_x = .024$	$r_y = .017$	$r_z = .017$

Jan 4th XXV.

Problem.

Given obs. eq.

$$\begin{cases} x-y+2z-3=0 \\ 3x+2y-5z-5=0 \\ 4x+y+4z-21=0 \\ -x+3y+3z-14=0 \end{cases}$$

Find most prob. vals $x\,y\,z$ their wts and errors.

Normals

$$\begin{cases} 27x+6y-88=0 & (1) \\ 6x+15y+z-70=0 & (2) \\ y+54z-107=0 & (3) \end{cases}$$

Solve for most prob. vals.

$\underline{x=2.470}$ $\underline{y=3.551}$ $\underline{z=1.916}$

To find wts.

1st method. Subst. $-1\ 0\ 0$ in normals

For p_x $\begin{cases} 27x'+6y'-1=0 \\ 6x'+15y'+z'=0 \\ y'+54z'=0 \end{cases}$

Solving

$x' = \frac{809}{19899}$ $\therefore p_x = \frac{19899}{809}$ $\underline{=24.60}$

For p_y $\begin{cases} 27x''+6y''=0 \\ 6x''+15y''+\cancel{z''-1}=0 \\ y''+54z''=0 \end{cases}$

$y'' = \frac{54}{737}$, $p_y = \frac{737}{54}$ $\underline{=13.65}$

For p_z $\begin{cases} 27x'''+6y'''=0 \\ 6x'''+15y'''+z'''=0 \\ y'''+54z'''-1=0 \end{cases}$

$z''' = \frac{41}{2211}$, $p_z = \frac{2211}{41}$ $\underline{=53.93}$

Method preferred depends on individual problem. Not much difference in long run.

---

y & p_y in same fashion. Subst in 2d normal, get

$$\frac{737}{54}y - \frac{2617}{54} = 0$$

$$p_y = \frac{737}{54}, \quad y = \frac{2617}{737} \quad \text{as before}$$

z & p_z

Eliminate correspondingly

$$\frac{6633}{123}z - \frac{12707}{123} = 0$$

$$p_z = \frac{6633}{123}, \quad z = \frac{12707}{6633} \quad \text{as before.}$$

Second Method

$$27x + 6y - 88 = A$$
$$6x + 15y + z - 70 = B \qquad \text{~~Solving~~}$$
$$y + 54z - 107 = C$$

Solving $\quad 19899x - \cancel{49159}\,49154 = 809A - 324B \qquad (24)$

$$737y = 2617 + 54B - 12A - C \qquad (25)$$

$$\cancel{369y + 27z - 1362 = 27B - 6A}$$

$$19899z = 38121 + 369C - 27B + 6A \qquad (27)$$

$\therefore$ from (24) $\quad \cancel{p_x}\, x = \frac{49154}{19899} \quad p_x = \frac{19899}{809}$

(25) $\quad y = \frac{2617}{737}, \quad p_y = \frac{737}{54}$ $\quad$ as before.

(27) $\quad z = \frac{38121}{19899}, \quad p_z = \frac{19899}{369}$

Third Method

$$\left.\begin{array}{l} 27x + 6y - 88 = 0 \\ 6x + 15y + z - 70 = 0 \\ y + 54z - 107 = 0 \end{array}\right\} \text{Normals}$$

Find z from 3d eq, subst in 2d whence find y & subst in 1st

$$\cancel{27x - \frac{1944}{809}}$$

$$\frac{19899}{809}x - \frac{49154}{809} = 0$$

$$\therefore p_x = \frac{19899}{809} \qquad x = \frac{49154}{19899}$$

as before

To find residual errors obs., subst values found from normals in obs eq, have

$$2.470 - 3.551 + 3.832 - 3 = v_1 = -.249$$

$$7.410 + 7.102 - 9.580 - 5 = v_2 = -.068$$

$$9.880 + 3.551 + 7.664 - 21 = v_3 = +.095$$

$$-2.470 - 10.653 + 5.788 - 14 = v_4 = -.069$$

$v_1^2 = .0620$

$v_2^2 = .0046$ $\qquad m = 4$ obs.

$v_3^2 = .0090$ $\qquad n = 3$ no unknowns

$v_4^2 = .0048$

$.0804 = \Sigma v^2$

By (57) $\quad \mu = \sqrt{\dfrac{\Sigma v^2}{m-n}} = \sqrt{.0804} = .284 \quad (34)$

$$\mu_x = \frac{\mu}{\sqrt{p_x}} = \frac{.284}{\sqrt{24.60}} = .057 \qquad (35)$$

$$\mu_y = \frac{.284}{\sqrt{13.65}} = .077$$

$$\mu_z = \frac{.284}{\sqrt{53.93}} = .039$$

Summary.

Probability.

Definition. Combinations, permutations & arrangements

$\underline{|n}$ $\quad \dfrac{\underline{|n}}{\underline{|n-r}}$ $\quad \dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$ Dependent & independent events. Look at examples.

Extension Taylor's. Condit. max & min. several variabls

$$e^{\left(h\frac{d}{dx}+k\frac{d}{dy}\right)} f(x,y) = f(x+h, y+k)$$

Application principles of probability. N° obs > n° unknowns.

Errors, constant & ~~actual~~ casual, real & residual. Gauss's assumption, $\Sigma v = 0$. Curve of error, three charact. two imperfections.

Equation of curve of error.

Prob. any Δ, any system Δ's, make max. as regards the mean values. n eqs for n unknowns in $\psi\Delta$. Det. ψ for case direct obs sing quantity, then $\frac{d\Delta}{dz} = 1$, $\therefore \Sigma\psi\Delta = 0$ & we know $\Sigma\Delta = 0$. ψ = a const multiplier. Get φ for ψ by integration. $\varphi\Delta = ke^{-h^2\Delta^2}$. k graphically, in terms h by $\int e^{-t^2}dt$. Total area curve $= 1$, $\frac{h}{2k} = \int_0^\infty e^{-t^2}dt$

Integration $\int_0^\infty e^{-t^2}dt$ (which is tabulated) $\frac{1}{2}\left(\tan^{-1}x\right)_0^\infty = \frac{\pi}{4}$

$k = \frac{h}{\sqrt{\pi}}$. $\quad y = \frac{h}{\sqrt{\pi}}e^{-h^2\Delta^2}$

Pts of inflect. Impossible asymptotic. Measure of precision, steepness curve.

Least Squares, applies to funct. wh. can be put in linear form

$\Delta_1^2 + \Delta_2^2 \ldots$ $\Sigma\Delta^2$ a min. Normal eqs (2 ways). Practical rule.

Wts. of obs., any cause, repetition, 2 rules.

How shorten work when large n^os concerned. Corrections method.

Conditioned obs, Eliminate unknowns. Single condit, divide discrepancy inversely wts, sans Sqs néc. Care not comb. eqs. diff wts.

When coeff of series to be det., easiest use Σ's & tabulate. See examples.

Special method, when all coeffs one unknown are 1. Form mean eq. & subtract. Remember wts. Then solve as usual. Care not 2d mean when coeffs not 1. At beginning cannot divide to make coeffs 1 sans altering wts, use when other coeffs large & not very diff.

Error functions

Av. Dev., Mean Sq., & Prob. Error. Definition, determination & inter-relation. $\frac{1}{h} = \mu\sqrt{2} = a\sqrt{\pi} = \frac{r}{\rho}$.
$\rho = \cdot 4769$ Graphic represent. Pt inflect & centre gravity.

Theta function Θ, ways tabulating, how gives prob, how r, how ρ. Remember coeff $\frac{2}{\sqrt{\pi}}$.

Comparison theory & obs. Two ways – given r of an obs. find fraction giving prob. ($\rho\Theta$) ~~below certain~~ & mult. by no to find no errors each size; compare. Given the errors, find μ & ad and compare ratio found & calculated.

~~Relation~~ Properties ~~among~~ error-functions.

Wts obs. as measures precision $p \propto h^2 \propto \frac{1}{r^2} \propto \frac{1}{\mu^2} \propto \frac{1}{ad^2}$
For when h varies, req. $\Sigma h^2\Delta^2$ min, gives normals in which h's obviously have place wts. with const. h

The Error functions of mean.

Same wt. all obs. ⊥ arithmetical mean
Mean has wt. m, $r^2 : r_0^2 = \frac{1}{1} : \frac{1}{m}$ by above. $r_0 = \frac{r}{\sqrt{m}}$
Same for other errors

Diff. wts, general mean.
$m = \Sigma p$, $\mu_0 = \frac{\mu}{\sqrt{\Sigma p}}$, $\mu_K = \frac{\mu}{\sqrt{p_K}}$. Consider wt. as no obs., errors not v but $v\sqrt{p}$, $\mu^2 = \frac{\Sigma(v^2p)}{m-1}$, $\mu_0^2 = \frac{\Sigma(v^2p)}{m(m-1)}$

Further approximation, v ~~for Δ~~ instead of Δ. $ad = \frac{\Sigma v\sqrt{p}}{m(m-1)}$
$\mu^2 = \frac{\Sigma v^2}{m-1} = \frac{\Sigma\Delta^2}{m}$. Approx. therefore $v = \sqrt{\frac{m-1}{m}}\,\Delta$

May thus calc. Σv & hence ad, instead of Σv^2 & thence μ. Easier. $\frac{\Sigma v}{m} = \sqrt{\frac{m-1}{m}}\,\frac{\Sigma\Delta}{m} = \sqrt{\frac{m-1}{m}}\,ad$

Error functions of a function

The ~~Error~~ of a sum, or difference, of a multiple, sum of multiples
Any function – expand by Taylor's to get series, but make the $\propto^{ls}$ the variables owning the errors, i.e. consider given the upper functions & assume the lower fixed. Strike higher terms & treat as sum multiples. $\Sigma^2 = \Sigma\mu_1^2\left(\frac{d\mu}{d\mu_1}\right)^2$

Accuracy required in results. See example.

Special laws of error, ~~usual~~ assumptions void.

All errors equally prob. $\varphi\Delta = \frac{1}{2a}$ ad = pe. $\mu = \frac{a}{\sqrt{3}}$

Error 2 sources each anywhere $-a$ to a ~~(sum of errors sources)~~ in error. $\Delta = x + y$. No ways in wh. some error produced, no any given error, +ve & −ve, ∴ prob. of Δ. $\varphi\Delta = \frac{2a \mp \Delta}{4a^2}$
Extreme of compound error = $2a$
ad & r.

Rejection of Observations.

1st method. Prob. that less than half an error lies beyond a limit. Reject all beyond & redetermine residual errors.

2d method. Huge Error whose prob $= \frac{1}{100}$. Find $\frac{a}{r}$ fr table. $a = 4.9r = 3.3\mu = 4.4$ ad. Must redetermine residuals.

Constant errors — lead to causes, eg instrumental, personal, circumstantial.

To compute wts of unknowns in normal eqs.

1st rule. Subst, −1 0 0 .. for abs. terms normals. Reciprocal of x thus found is wt. original x. Same for y z .. etc

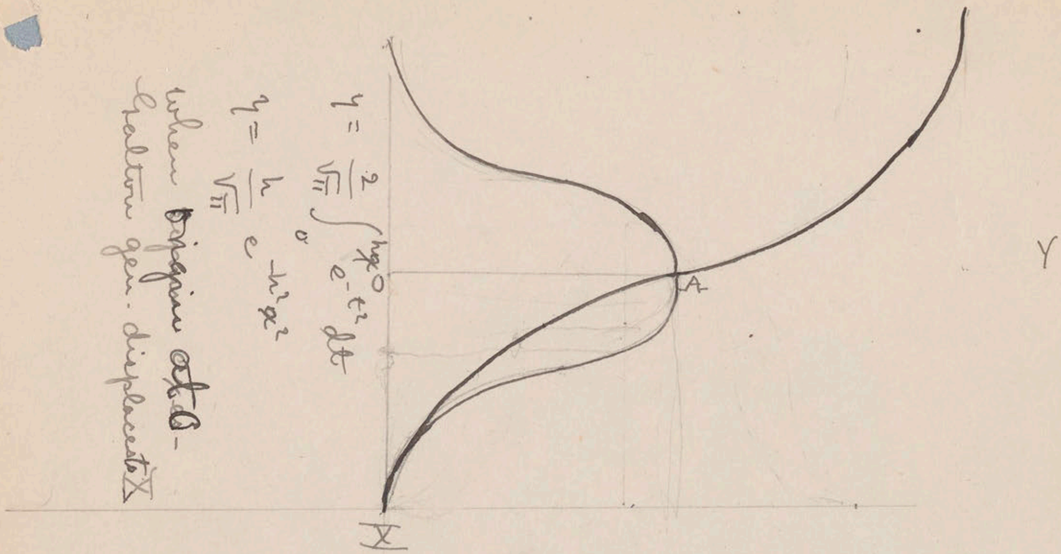
2d Rule Write ABC .. for 2d members normals. Most prob. vals will be terms independent of ABC. Wt x will be reciprocal coff A in value x, wt y recipr. coeff B in value y, ~~Etc~~.

3d rule. Write ABC .. as above (if want most prob vals). In 1st normal for x subst for y z etc their x values without clearing or reducing in any way. Coeff x = its wt.

If obs. unequal wt. mult by $\sqrt{p}$ for equivalent eqs.

Series eqs, find mean error single obs. m obs. n unknowns

$$\mu^2 = \frac{\Sigma v^2}{m-n} \qquad \mu_x^2 = \frac{\Sigma v^2}{p_x^2(m-n)}$$



Galton uses curve from probability integral, shows not prob. error ~~that~~ size, but ~~total~~ prob. that size or smaller. Can divide ordinates instead of areas.
Ordinary curve, prob. gets greater toward summit, Galton's rate change prob. get greater, pt of inflexion in middle. ~~Midpt gives pe~~. Calls curve of distribution v that of frequency.
Distribution - rate change ~~varies~~ symm, curve symm.
Middle ordinate = mean, most common value.
To find pe. instead of dividing area one half into halves divide y, diff. between ⊥ here and at A will ~~be~~ pe, for this curve call Q. Galton always allows for non-symm. cases in his subject. (takes $\frac{1}{2}(Q' - Q'')$, AO not nec. ord. even though always most prob. val.) AO gives k, is K ~~frequency~~, distrib. G calls ~~M~~. M & Q hence det. curve.

Problem. Whole action in strt line.

P varies about 2 pts L & M, dist between = a, pe = b & c. Find most prob. result dist. ~~fr~~ ~~M~~ (for instance) & its pe.

~~Required~~ probable z is function of prob. x & prob. a−x, depends on wts ascribed to x & a−x. Wt of z will be sum their wts, ie $\frac{1}{b^2}+\frac{1}{c^2}$. Hence

$$pe^2 = \frac{1}{\frac{1}{b^2}+\frac{1}{c^2}} = \frac{b^2c^2}{b^2+c^2}, \quad pe = \frac{bc}{\sqrt{b^2+c^2}}$$

Required max. prob. two simult. errors, $\frac{x^2}{c^2}+\frac{y^2}{b^2}$ a minimum, $x+y=a$. Gives $\frac{x_1}{c^2}=\frac{y'}{b^2}=\frac{a}{b^2+c^2}$ and most prob. corrected val fr. M = $\frac{ac^2}{b^2+c^2}$, ie this is central pt new curve.

Surface of frequency.

Variation about varying pt, measure frequency along z.

$$z = \frac{hh'}{\pi}e^{-(h^2x^2+h'^2y^2)} = \frac{1}{\pi r r'}e^{-\left(\frac{x^2}{r^2}+\frac{y^2}{r'^2}\right)}$$

Sections ⊥ z give ellipses, ⊥ y or ⊥ x give curves frequency.

Notice arithmetical mean not always proper mean value. Ordinary height 5 ft, a woman 11 ft is more prob than a dwarf −1 ft. Stimulus & effect on nerve vary as log. Geometric mean would often give better law.