UNITED STATES EXPERIMENTAL MODEL BASIN

NAVY YARD, WASHINGTON, D.C.

THE EFFECTIVE WIDTH IN BOX GIRDERS AND IN THE DOUBLE BOTTOM

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EXPERIMENTAL MODEL BASIN
ERECTED 1898
BUREAU OF
CONSTRUCTION AND REPAIR
NAVY DEPARTMENT

JANUARY 1938

TRANSLATIONS 50

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(Werft, Reederei, Hafen, 1928)

Translated by M. C. Roemer and annotated
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U.S. Experimental Model Basin
Navy Yard, Washington, D.C.

January 1938 Trans. No. 50

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I. INTRODUCTION

The stress distribution in wide girder flanges is among the problems to which increased attention is being devoted in recent times in the various branches of engineering. In naval architecture, Biles, Bruhn and Pietzker[1] were the first to occupy themselves with this question. A critical appreciation of their point of view is contained in my paper to the Schiffbautechnische Gesellschaft*. Their theories are in partial contradiction to experience, and therefore are not generally valid.

Hoffmann has presented an essentially new concept to the Institute of Naval Architects. He assumes that the plating still carries load after the buckling corresponding to Euler's buckling load for a completely fixed rod. On this assumption he carried out a calculation with Biles' data on the destroyer WOLF, and thought he had obtained good agreement between theory and test.

Doubtless the assumption of the finite deflection of buckled rods agrees with experience, particularly when these rods are prevented from further buckling by a connection with parts having greater strength. Müller-Breslau[2] himself supported this view and confirmed it mathematically. However, it must be noted that Hoffmann made errors in his calculations which it is necessary to point out in the interests of scientific clarity.

First, he made use of incorrect external moments. Since he lacked the exact measurements, he attempted to compute the moments back from the measured strains. He found an external moment of 3,120 ft-tons. In reality, however, the WOLF[3] resisted a maximum moment of 4,672 ft-tons. This number also agrees with the moments plotted in the charts. From these there follows a moment of about 3,900 ft-tons at Frame 84 in the boiler room. Agreement between external and internal moments, therefore, has been far from achieved, although Hoffmann assumes that the buckled plating over the frame is completely fixed. This complete fixation does not occur.

Hoffmann's calculation must therefore contain a radical error. Strange to say, he adopts Pietzker's view that buckling (!) occurs on the side of the plating under tensile stress also[4].

Experience has demonstrated, rather, that nearly the entire tension side is effective[5].

Hoffmann made another error in failing to take into account the increased resistance of plates to buckling. The buckling strength of plates is many times

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*See EMB Translation No. 28. 1) Numbers refer to notes on p 36.

as great as the strength of rods. It can be calculated by Timoshenko's method, whose theory has unfortunately been accorded too little attention in Germany. Therefore, in order to obtain correct results, it is necessary to start with the distribution of stresses before buckling. The writer has already given an approximate solution[6] for a box girder on two supports. In order to obtain a completely general solution for a box girder, the principle of minimum strain energy has been used in the following for computing the stresses.

In addition to box girders on two supports, fixed girders and girders fixed at one end are discussed. Good agreement with my older calculation is obtained for the girder on two supports with uniformly distributed load. On the other hand, there are considerable differences evident in the case of girders with concentrated loads. For girders fixed at one end and for fixed girders the following theory only is applicable.

It is further demonstrated how conditions change when a series of parallel girders have a common flange plate like the mid-portion of a long double bottom at some distance from the bulkheads. I should here like to begin with the simple case, in order to establish a basis for calculating the double bottom in general.

The paper of Dr. Schilling did not present a solution, since the individual elements, the box girders of the double bottom, have great torsional strength while Schilling assumes that they have none. The entire double bottom should be regarded as a plate which can be calculated solely by means of the modern plate theory, and whose stress distribution is entirely different from that for a network of girders. For instance, it may be estimated that a square area of double bottom between two bulkheads is relieved of 40% of its load by its torsional rigidity.

II. METHOD OF CALCULATION.

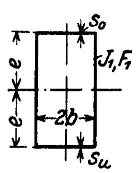


Fig. 1.

The moment curve is developed in a Fourier series such that each term will satisfy the boundary conditions. If we then establish a system of coordinates in the middle of the girder, the moment for the freely supported girder with a length 2a will be

$$M = M_1 \cos \frac{\pi}{2a} x + M_3 \cos \frac{3\pi}{2a} x + \ldots + M_m \cos \frac{m\pi}{2a} x$$
$$(m = 1, 3, 5, \ldots)$$

We start with the narrow box girder whose flanges can be taken as fully effective over the entire cross section (Fig 1). The section modulus of such a girder is given by

$$2 W_o = 2 e b s_o \left(1 + \frac{J_1}{e^2 b s_o} \frac{F_1 + b s_o + b s_u}{F_1 + 2 b s_u} + \frac{F_1 + 2 b s_o}{F_1 + 2 b s_u} \frac{s_u}{s_o}\right)$$

$$2 W_u = 2 e b s_u \left(1 + \frac{J_1}{e^2 b s_u} \frac{F_1 + b s_o + b s_u}{F_1 + 2 b s_o} + \frac{F_1 + 2 b s_u}{F_1 + 2 b s_o} \frac{s_o}{s_u}\right)$$

where the width of the flange is represented by 2b, the thickness of upper flange by s_o, of lower flange by s_u. [Translator's note: These subscripts are from "ober" (upper) and "unter" (lower)]. In addition, let the moment of inertia of one web be J_1, its thickness s_1, the area F_1 and the distance of its center of gravity from the upper and lower edges e. The stresses at the levels of the flanges are designated by σ_o and σ_u. Then

$$\sigma_o = \frac{M}{2\,W_o} \text{ und } \sigma_u = \frac{M}{2\,W_u}$$

The factor ½ takes into account that only half of the girder is under consideration.

Then if we designate the force on the half flanges by:

$$b\,s_o\,\sigma_o = X_o \quad \text{und} \quad b\,s_u\,\sigma_u = X_u$$

we get:

$$X_o = \frac{M}{2e} \cdot \frac{1}{1 + \frac{J_1}{e^2 b\, s_o} \cdot \frac{F_1 + b\,s_o + b\,s_u}{F_1 + 2\,b\,s_u} + \frac{F_1 + 2\,b\,s_o}{F_1 + 2\,b\,s_u}\,\frac{s_u}{s_o}} \tag{1a}$$

$$X_u = \frac{M}{2e} \cdot \frac{1}{1 + \frac{J_1}{e^2 b\, s_u} \cdot \frac{F_1 + b\,s_o + b\,s_u}{F_1 + 2\,b\,s_o} + \frac{F_1 + 2\,b\,s_u}{F_1 + 2\,b\,s_o}\,\frac{s_o}{s_u}} \tag{1b}$$

and for $s_u = 0$

$$X_o = \frac{M}{2e}\,\frac{1}{1 + \frac{J_1}{e^2 b\, s_o} + \frac{J_1}{e^2 F_1}} \tag{1c}$$

and $s_u = s_o$

$$X = \frac{M}{2e}\,\frac{1}{2 + \frac{J}{e^2 b\, s_o}} \tag{1d}$$

In order to determine the effectiveness of the broad flanges, we now calculate the statically indeterminate values X_o and X_u by means of the strain energy and compare the formulas thus obtained with those above.

III. THE AIRY STRESS FUNCTION.

First we assume that the bending strength of the plate is small in comparison with the strength of the girder, so that as in the previous case it can be neglected. [Translator's note: See E.M.B. Translation No. 28].

The flange plate can then be regarded as a flat plate whose long edge is loaded by shearing stresses. When bulkheads are present, shearing stresses also occur on the short edge, the distribution of which we will not discuss for the present.

It is known that the deformations and stresses in a flat plate must satisfy the partial differential equation

$$\Delta \Delta F = \frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0$$

Here F is the Airy stress function, and the stresses are obtained from it by partial differentiation, as follows:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}\,;\ \sigma_y = \frac{\partial^2 F}{\partial x^2} \text{ und } \tau = -\frac{\partial^2 F}{\partial x \partial y}$$

If we now assume that the stresses at the edges can be expressed as a Fourier series, we can write

$$F = F(y) \cos \frac{m\pi}{2a} x$$

The longitudinal stresses at the end of the girder should vanish, and therefore m will be an odd number.

If we introduce this expression into Eq $\Delta\Delta F = 0$, we get the total differential equation

$$\frac{d^4 F(y)}{d y^4} - 2 k^2 \frac{d^2 F(y)}{d y^2} + k^4 F(y) = 0$$

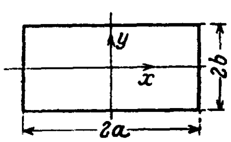


Fig. 2.

the solution of which will be, for each cosine term, provided that the flanges are symmetrical:

$$F(y) = B_m \left(A_m \mathfrak{Cos} \frac{m\pi}{2a} y + y \mathfrak{Sin} \frac{m\pi}{2a} y\right)$$

where $\mathfrak{Cos}$ and $\mathfrak{Sin}$ are the hyperbolic functions, the coordinate system being taken as in Fig 2. Then the stress function will be

$$F = \sum B_m \left(A_m \mathfrak{Cos} \frac{m\pi}{2a} y + y \mathfrak{Sin} \frac{m\pi}{2a} y\right) \cos \frac{m\pi}{2a} x$$

Here A_m can in each case be determined from the boundary conditions for y = b, while B_m is to be calculated from the minimum of the strain energy. [Translator's note: Note that A_m and B_m have values corresponding to the particular term of a series, whereas in the previous paper (EMB Trans. No. 28) the constants in the stress function were taken as true constants. This is the fundamental reason for the greater accuracy of this treatment, as the various stresses and forces are determined as the sum of the individual values corresponding to the separate terms in a convergent series].

By differentiating, the following stresses are obtained:

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} = -\sum \left(\frac{m\pi}{2a}\right)^2 B_m \left(A_m \mathfrak{Cof} \frac{m\pi}{2a} y + y \mathfrak{Sin} \frac{m\pi}{2a} y\right) \cos \frac{m\pi}{2a} x \tag{2a}$$

$$\tau = -\frac{\partial^2 F}{\partial x \partial y}$$

$$= \sum \left(\frac{m\pi}{2a}\right)^2 B_m \left(A_m \mathfrak{Sin} \frac{m\pi}{2a} y + y \mathfrak{Cof} \frac{m\pi}{2a} y + \frac{2a}{m\pi} \mathfrak{Sin} \frac{m\pi}{2a} y\right) \sin \frac{m\pi}{2a} x \tag{2b}$$

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}$$

$$= \sum \left(\frac{m\pi}{2a}\right)^2 B_m \left(A_m \mathfrak{Cof} \frac{m\pi}{2a} y + y \mathfrak{Sin} \frac{m\pi}{2a} y + \frac{4a}{m\pi} \mathfrak{Cof} \frac{m\pi}{2a} y\right) \cos \frac{m\pi}{2a} x \tag{2c}$$

IV. THE STRAIN ENERGY OF THE GIRDER.

(a) The Strain Energy of the Flanges.

The strain energy of a flat plate is represented by the formula

$$L = \frac{s}{2E} \int_0^b \int_{-a}^{+a} (\sigma_x + \sigma_y)^2 \, dx \, dy + \frac{s}{2G} \int_0^b \int_{-a}^{+a} (\tau^2 - \sigma_x \sigma_y) \, dx \, dy$$

By introducing the stress function, we get for the upper flange

$$L_o = \frac{s_o}{2E} \int_0^b \int_{-a}^{+a} \left(\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x^2}\right)^2 dx \, dy + \frac{s_o}{2G} \int_0^b \int_{-a}^{+a} \left[\left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 - \frac{\partial^2 F}{\partial y^2} \cdot \frac{\partial^2 F}{\partial x^2}\right] dx \, dy$$

But from (2a) and (2c),

$$\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x^2} = \sum \frac{m\pi}{a} B_m \mathfrak{Cof} \frac{m\pi}{2a} y \cos \frac{m\pi}{2a} x$$

From this is obtained the first integral

$$L_{10} = \frac{s_o}{2E} \int_{-a}^{+a} \int_0^b \left[\sum \left(\frac{m\pi}{a}\right) B_m \mathfrak{Cof} \frac{m\pi}{2a} y \cos \frac{m\pi}{2a} x \right]^2 dx \, dy$$

or

$$L_{10} = \frac{s_o}{2E} B_m^2 \sum \left(\frac{m\pi}{2a}\right)^2 \int_0^b \frac{1}{2}\left(1 + \mathfrak{Cof} \frac{m\pi}{a} y\right) dy \int_{-a}^{+a} \frac{1}{2}\left(1 + \cos \frac{m\pi}{a} x\right) dx$$

for one half of the girder. Therefore

$$L_{10} = \frac{s_o}{2E} \cdot \frac{B_m^2}{2} \sum \left(\frac{m\pi}{a}\right)^2 \left(b + \frac{a}{m\pi} \mathfrak{Sin} \frac{m\pi}{a} b\right) a$$

The second integral is changed into a line integral by Stokes' theorem. [Translator's note: The mathematical expression of Stokes' theorem in two dimensions is:

$$\oint (u\,dx + v\,dy) = \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA$$

Any surface integral can be transformed into a line integral provided it can be expressed in this form. See "An Introduction to Mathematical Physics" by R. A. Houston, p 59 ff]. Therefore

$$\int_0^b \int_{-a}^{+a} \left[\left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 - \frac{\partial^2 F}{\partial y^2} \cdot \frac{\partial^2 F}{\partial x^2}\right] dx\,dy = \oint \frac{\partial F}{\partial x} \left[\frac{\partial^2 F}{\partial x \partial y} dx + \frac{\partial^2 F}{\partial y^2} dy\right]$$

The line integral along the boundary lines $y = b$ and $x = \pm a$ and $y = 0$ is to be taken in clockwise direction.

In the case of the box girder we have (for reasons of symmetry) when $y = 0$

$$\tau = -\frac{\partial^2 F}{\partial x \partial y} = 0$$

and when $x = \pm a$

$$\frac{\partial^2 F}{\partial y^2} = \sigma_x = 0$$

Furthermore, at the edge $y = b$ the normal stress σy must vanish. From this last condition it follows that

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} = 0 = \sum \left(\frac{m\pi}{2a}\right)^2 \left[F(y)\right]_{y=b} \cos \frac{m\pi}{2a} x$$

where

$$\left[F(y)\right]_{y=b} = 0$$

and

$$\left(\frac{\partial F}{\partial x}\right)_{y=b} = -\frac{m\pi}{2a} \sum F(y) \sin \frac{m\pi}{2a} x = 0$$

The entire line integral is consequently zero.

Therefore

$$\boxed{L_o = \frac{s_o}{E} \sum B_m^2 \left(\frac{m\pi}{2a}\right)^2 a\,b \left(1 + \frac{a}{b\,m\pi} \mathfrak{Sin} \frac{m\pi}{a} b\right)} \qquad (3a)$$

and for the lower flange

$$\boxed{L_u = \frac{s_u}{E} \sum B_m^2 \left(\frac{m\pi}{2a}\right)^2 a\,b \left(1 + \frac{a}{b\,m\pi} \mathfrak{Sin} \frac{m\pi}{a} b\right)} \qquad (3b)$$

From $\left[F(y)\right]_{y=b} = 0 = A_m \mathfrak{Cof} \frac{m\pi}{2a} b + b \mathfrak{Sin} \frac{m\pi}{2a} b$

it follows further that

$$\boxed{A_m = -b \mathfrak{Tg} \frac{m\pi b}{2a}} \qquad (4)$$

(b) The Strain Energy of the Web.

The energy of the web[7] is the sum of the work of the normal force exerted by the flanges and the work of the bending moment M_1 of the web.

The normal forces exerted by the two flanges are in opposite directions. If we denote the force of the upper flange by X_o and that of the lower flange by X_u, we can express the work as

$$L_3 = \frac{1}{2EF_1} \int_{-a}^{+a} (X_o - X_u)^2 \, dx$$

But

$$X_o = s_o \int_0^b \sigma_x \, dy$$

and

$$X_u = s_u \int_0^b \sigma_x \, dy$$

or

$$X_o = s_o \int_0^b \frac{\partial^2 F}{\partial y^2} \, dy = s_o \left[\frac{\partial F}{\partial y}\right]_0^b$$

But according to the basic equations of the stress function

$$\left[\frac{\partial F}{\partial y}\right]_0^b = \sum \frac{m\pi}{2a} B_m \left(A_m \mathfrak{Sin} \frac{m\pi}{2a} b + b \mathfrak{Cof} \frac{m\pi}{2a} b + \frac{2a}{m\pi} \mathfrak{Sin} \frac{m\pi}{2a} b\right) \cos \frac{m\pi}{2a} x$$

and for

$$A_m = -b \mathfrak{Tg} \frac{m\pi}{2a} b$$

$$X_o = s_o \sum \frac{m\pi}{2a} B_m \, b \, \frac{1 + \frac{a}{b m\pi} \mathfrak{Sin} \frac{m\pi b}{a}}{\mathfrak{Cof} \frac{m\pi}{2a} b} \cos \frac{m\pi}{2a} x$$

Then, letting

$$X_{om} = s_o \frac{m\pi}{2a} B_m \, b \, \frac{1 + \frac{a}{b m\pi} \mathfrak{Sin} \frac{m\pi b}{a}}{\mathfrak{Cof} \frac{m\pi b}{2a}}$$

where X_{om} is a constant, and B_m is given by:

$$B_m = \frac{X_{om}}{s_o} \frac{2a}{b m \pi} \cdot \frac{\mathfrak{Cof}\, \frac{m \pi b}{2a}}{1 + \frac{a}{b m \pi} \mathfrak{Sin}\, \frac{m \pi b}{a}} \tag{5a}$$

we get then:

$$X_o = \sum X_{om} \cos \frac{m \pi}{2a} x \tag{5b}$$

Similarly, when

$$X_{um} = s_u B_m \frac{b m \pi}{2a} \cdot \frac{1 + \frac{a}{b m \pi} \mathfrak{Sin}\, \frac{m \pi b}{a}}{\mathfrak{Cof}\, \frac{m \pi b}{2a}} \tag{5c}$$

we get

$$X_u = \sum X_{um} \cos \frac{m \pi}{2a} x \tag{5d}$$

From this the strain energy of the normal forces in the web is found to be

$$L_3 = \frac{1}{2 E F_1} \int_{-a}^{+a} (X_o - X_u)^2 \, dx$$

$$= \frac{1}{2 E F_1} \int_{-a}^{+a} \left[\sum (X_{om} - X_{um}) \cos \frac{m \pi}{2a} x \right]^2 dx$$

i.e.

$$L_3 = \frac{1}{2 E F_1} \sum (X_{om} - X_{um})^2 a \tag{6}$$

If we introduce the terms X_{om} and X_{um} in the strain energy of the flanges also, the results are:

$$X_{om}^2 = s_o^2 \left(\frac{m \pi}{2a}\right)^2 B_m^2 b^2 \frac{\left(1 + \frac{a}{b m \pi} \mathfrak{Sin}\, \frac{m \pi b}{a}\right)^2}{\mathfrak{Cof}^2\, \frac{m \pi b}{2a}}$$

$$L_o = \frac{1}{s_o E} \sum X_{om}^2 \frac{a}{b} \cdot \frac{\mathfrak{Cof}^2\, \frac{m \pi}{2a} b}{1 + \frac{a}{b m \pi} \mathfrak{Sin}\, \frac{m \pi b}{a}}$$

and

$$L_u = \frac{1}{s_u E} \sum X_{um}^2 \frac{a}{b} \cdot \frac{\mathfrak{Cof}^2\, \frac{m \pi}{2a} b}{1 + \frac{a}{b m \pi} \mathfrak{Sin}\, \frac{m \pi b}{a}}$$

These two expressions can also be written in the form

$$L_o = \frac{1}{2 E} \sum X_{om}^2 \frac{a}{b_m s_o} \tag{7a}$$

and

$$L_u = \frac{1}{2E} \sum X_{um}^2 \frac{a}{b_m s_u} \tag{7b}$$

where a flange of unequal stress distribution is replaced by a flange of uniform stress with the width b_m.

[Translator's note: Imagine X to produce a uniform stress in a flange width b_m. Let the value of X be distributed along the length of the flange according to $X = X_m \cos\frac{m\pi x}{2a}$; the strain energy will then be simply:

$$L_m = \int_{-a}^{+a} \frac{X^2 dx}{2AE} = \frac{X_m^2}{2 b_m s E} \int_{-a}^{+a} \cos^2 \frac{m\pi x}{2a}\, dx$$
$$= \frac{X_m^2 a}{2 b_m s E}$$

for one term of the series. The total strain energy becomes:

$$L = \frac{1}{2E} \sum X_m^2 \frac{a}{b_m s}$$

whency by comparison with the above expressions the necessary value of the "effective width" for each term is obtained].

Then

$$b_m = \int_0^b \frac{\sigma_x\, dy}{\sigma_{ox}} = \frac{b}{2} \cdot \frac{1 + \frac{a}{b m \pi} \operatorname{Sin} \frac{m\pi b}{a}}{\operatorname{Cos}^2 \frac{m\pi b}{2a}}$$
$$= b\, \frac{1 + \frac{a}{b m \pi} \operatorname{Sin} \frac{m\pi b}{a}}{1 + \operatorname{Cos} \frac{m\pi b}{a}} \tag{8}$$

is the effective width for one term of the Fourier series, when σ_{ox} represents the stress at the edge.

This formula is identical with

$$b_m = \frac{b}{2} \operatorname{Tg} \frac{m\pi}{2a} b \left(-\operatorname{Tg} \frac{m\pi}{2a} b + \frac{1}{\operatorname{Tg} \frac{m\pi b}{2a}} + \frac{2a}{b m \pi} \right) \tag{8'}$$

which I derived previously[8)].

The strain energy of the moment in the web can be expressed by

$$L_u = \frac{1}{2EJ_1} \int_{-a}^{+a} M_1^2\, dx$$

Here M_1 is the moment taken up by the web alone.

But $M_1 = M - eX_o - eX_u$, where M is the external moment and e the distance of the center of gravity of the web from the edges.

If we here introduce the Fourier series, we get

$$M_1 = \sum \left(\frac{M_m}{2} - e\,X_{om} - e\,X_{um} \right) \cos \frac{m\pi}{2a} x$$

Then

$$L_4 = \frac{1}{2EJ_1} \int_{-a}^{+a} \left[\sum \left(\frac{M_m}{2} - e\,X_{om} - e\,X_{um} \right) \cos \frac{m\pi}{2a} x \right]^2 dx$$

or

$$L_4 = \frac{1}{2EJ_1} \sum \left(\frac{M_m}{2} - e\,X_{om} - e\,X_{um} \right)^2 a \qquad (9)$$

Adding Eq (6), (7) and (9), the total strain energy of the half-girder becomes:

$$\begin{aligned} L &= L_o + L_u + L_3 + L_4 \\ &= \frac{a}{2E} \sum \left(\frac{X_{om}^2}{s_o\,b_m} \right) + \frac{a}{2E} \sum \left(\frac{X_{um}^2}{s_u\,b_m} \right) + \frac{a}{2EF_1} \sum (X_{om} - X_{um})^2 \\ &\qquad + \frac{a}{2EJ_1} \sum \left(\frac{M_m}{2} - e\,X_{om} - e\,X_{um} \right)^2 \end{aligned}$$

V. THE MINIMUM STRAIN ENERGY AND DETERMINATION OF THE CONSTANT B_m.

From this sum of the energies of the various parts we obtain the statically indeterminate values of the flange forces by partial differentiation, or

$$\frac{\partial L}{\partial X_{om}} = 0 \quad \text{und} \quad \frac{\partial L}{\partial X_{um}} = 0$$

[Translator's Note: This follows from the condition that the strain energy of the system represented as a function of the statically indeterminant quantities X_{om} and X_{um} must be a minimum. See Timoshenko, Theory of Elasticity, p 51].

Then we have

$$X_{om} \left(\frac{1}{s_o\,b_m} + \frac{1}{F_1} + \frac{e^2}{J_1} \right) - X_{um} \left(\frac{1}{F_1} - \frac{e^2}{J_1} \right) = \frac{e\,M_m}{2J_1}$$

and

$$X_{um} \left(\frac{1}{s_u\,b_m} + \frac{1}{F_1} + \frac{e^2}{J_1} \right) - X_{om} \left(\frac{1}{F_1} - \frac{e^2}{J_1} \right) = \frac{e\,M_m}{2J_1}$$

Solution of these equations gives:

$$X_{om} = \frac{M_m}{2e} \cdot \frac{1}{1 + \frac{F_1 + 2b_m s_o}{F_1 + 2b_m s_u} \cdot \frac{s_u}{s_o} + \frac{J_1}{e^2 b_m s_o} \cdot \frac{F_1 + b_m s_u + b_m s_o}{F_1 + 2b_m s_u}} \qquad (10)$$

and

$$X_{om} = \frac{M_m}{2\,W_o}\,b_m\,s_o \tag{10a}$$

Here W_o is the section modulus of a girder with the flange width b_m. By changing the indices we get

$$X_{um} = \frac{M_m}{2\,W_u}\,b_m\,s_u \tag{10b}$$

Formula (10) can be considerably simplified by letting $s_u = s_o$, which gives:

$$W_o = W_m = 2\,e\,b_m\,s_o + \frac{J_1}{e} \tag{11}$$

and for $s_u = 0$

$$W_o = W_m = e\,b_m\,s_o + \frac{J_1}{e} + \frac{J_1}{F\,e}\,b_m\,s_o \tag{12}$$

Formula (10) corresponds to Formula (1) with the difference that the effective width b_m of the m^{th} term takes the place of the half-width of the flange b. The coefficients B_m of our stress function are obtained from Eq (5a):

$$X_{om} = \frac{m\pi}{2a}\,B_m\,b\,s_o\,\frac{1 + \frac{a}{b\,m\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a}}{\mathfrak{Cof}\,\frac{m\pi b}{2a}} = \frac{m\pi}{a}\,B_m\,s_o\,b_m\,\mathfrak{Cof}\,\frac{m\pi b}{2a} \tag{13}$$

or

$$B_m = \frac{a}{m\pi}\cdot\frac{X_{om}}{b_m\,s_o\,\mathfrak{Cof}\,\frac{m\pi b}{2a}} = \frac{a}{m\pi}\cdot\frac{M_m}{2\,W_{om}}\cdot\frac{1}{\mathfrak{Cof}\,\frac{m\pi b}{2a}} \tag{14}$$

Here 2 W_{om} signifies the full section modulus of a box girder with a flange width of $2b_m$.

The agreement between strains and shear stresses at the point of junction of flange and web can be shown, as well as the fulfillment of the conditions of support.

VI. FORMULAS.

From

$$A_m = -\,b\,\mathfrak{Tg}\frac{m\pi}{2a}\,b$$

and

$$B_m = \frac{a}{m\pi}\cdot\frac{M_m}{W_{om}}\cdot\frac{1}{\mathfrak{Cof}\,\frac{m\pi b}{2a}}$$

where W_{om} represents the section modulus of a girder whose flange width is b_m, the equations for the stress function and the stresses follow. The stress function will be

$$F = -\sum \frac{a\,b}{m\,\pi} \cdot \frac{M_m}{2\,W_{om}} \mathfrak{Tg}\,\frac{m\,\pi\,b}{2\,a} \cdot \left[\frac{\mathfrak{Cof}\,\frac{m\,\pi}{2\,a}\,y}{\mathfrak{Cof}\,\frac{m\,\pi}{2\,a}\,b} - \frac{y}{b} \cdot \frac{\mathfrak{Sin}\,\frac{m\,\pi}{2\,a}\,y}{\mathfrak{Sin}\,\frac{m\,\pi\,b}{2\,a}} \right] \cos\frac{m\,\pi}{2\,a}\,x \tag{15a}$$

By partial differentiation, the stresses follow:

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} = \sum \frac{m\,\pi\,b}{4\,a} \cdot \frac{M_m}{2\,W_{om}} \mathfrak{Tg}\,\frac{m\,\pi\,b}{2\,a} \times \left[\frac{\mathfrak{Cof}\,\frac{m\,\pi}{2\,a}\,y}{\mathfrak{Cof}\,\frac{m\,\pi\,b}{2\,a}} - \frac{y}{b} \cdot \frac{\mathfrak{Sin}\,\frac{m\,\pi}{2\,a}\,y}{\mathfrak{Sin}\,\frac{m\,\pi\,b}{2\,a}} \right] \cos\frac{m\,\pi}{2\,a}\,x \tag{15b}$$

$$\tau = -\frac{\partial^2 F}{\partial x\,\partial y} = -\sum \frac{m\,\pi\,b}{4\,a} \cdot \frac{M_m}{2\,W_{om}} \mathfrak{Tg}\,\frac{m\,\pi\,b}{2\,a} \times \left[\frac{\mathfrak{Sin}\,\frac{m\,\pi}{2\,a}\,y}{\mathfrak{Cof}\,\frac{m\,\pi\,b}{2\,a}} - \frac{y}{b} \cdot \frac{\mathfrak{Cof}\,\frac{m\,\pi}{2\,a}\,y}{\mathfrak{Sin}\,\frac{m\,\pi\,b}{2\,a}} - \frac{2\,a}{b\,m\,\pi} \cdot \frac{\mathfrak{Sin}\,\frac{m\,\pi}{2\,a}\,y}{\mathfrak{Sin}\,\frac{m\,\pi\,b}{2\,a}} \right] \sin\frac{m\,\pi}{2\,a}\,x \tag{15c}$$

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} = -\sum \frac{m\,\pi\,b}{4\,a} \cdot \frac{M_m}{2\,W_{om}} \mathfrak{Tg}\,\frac{m\,\pi\,b}{2\,a} \times \left[\frac{\mathfrak{Cof}\,\frac{m\,\pi}{2\,a}\,y}{\mathfrak{Cof}\,\frac{m\,\pi\,b}{2\,a}} - \frac{y}{b} \cdot \frac{\mathfrak{Sin}\,\frac{m\,\pi}{2\,a}\,y}{\mathfrak{Sin}\,\frac{m\,\pi\,b}{2\,a}} - \frac{4\,a}{b\,m\,\pi} \cdot \frac{\mathfrak{Cof}\,\frac{m\,\pi}{2\,a}\,y}{\mathfrak{Sin}\,\frac{m\,\pi\,b}{2\,a}} \right] \cos\frac{m\,\pi}{2\,a}\,x \tag{15d}$$

For y = b we get

$$\sigma_{o\,x} = \sum \frac{M_m}{2\,W_{om}} \cos\frac{m\,\pi}{2\,a}\,x = \sum \frac{X_{om}}{b_m\,s_o} \cos\frac{m\,\pi}{2\,a}\,x$$

and

$$\tau = -\sum \frac{m\,\pi}{2\,a} \cdot \frac{M_m}{2\,W_{om}}\,b_m \sin\frac{m\,\pi\,x}{2\,a}$$

The effectiveness of the flange at the point x = 0 is found from the effective width:

$$\boxed{b'_m = \frac{X_o}{s\,\sigma_{o\,x}} = \frac{\sum X_{om}\left(\cos\frac{m\,\pi}{2\,a}\,x\right)_{x=0}}{s_o \sum \frac{X_{om}}{b_m\,s_o}\left(\cos\frac{m\,\pi}{2\,a}\,x\right)_{x=0}} = \frac{\sum X_{om}}{\sum \frac{1}{b_m} X_{om}}} \tag{16}$$

[Translator's note: Equations (15a–d) are identical with Eq (33–36) of the earlier paper if $\frac{M_m}{2\,W_{om}}$ is replaced by σ_m. In the earlier paper, σ_m is not included in the summation. It is merely taken as the maximum value of M/W; i.e. $\sigma_m = (\sigma_x)_{\substack{x=o\\y=b}} = (\sigma_r)_{x=o}$. Consequently, in the expression for effective width,

σ_m cancels σ_r and the resulting value of effective width becomes $\Sigma\ b_m$. In the present method, effective width = $\frac{\Sigma\ s b_m \sigma_m}{\Sigma\ s \sigma_{rm}} = \frac{\Sigma\ s b_m \frac{M_m}{W_m}}{\Sigma\ s \frac{M_m}{W_m}}$. Obviously, the quotient of the sums differs from the sum of the quotients of the individual terms and gives a more accurate expression for the effective width. The difference in the two methods can be clearly seen by comparing the examples].

VII. SAMPLE CALCULATIONS.

A box girder without special angles in the corners, with the length 2a, the width 2b and wall thickness s, is loaded by:

(a) a concentrated load in the middle of the girder;

(b) a uniformly distributed load;

(c) two symmetrical concentrated loads.

The items to be calculated are: the effectiveness of the flanges at the section having the greatest bending moment, the effective width at this section, and the maximum longitudinal stress at the edge.

(a) Concentrated load at the middle. $a/b = \pi$

The moment line is a triangle with the maximum moment

$$M = P \frac{a}{2}$$

Resolving this into a Fourier series gives:

$$\frac{M}{2} = \frac{8}{\pi^2} \cdot P \frac{a}{4} \left[\cos \frac{\pi x}{2a} + \frac{1}{9} \cos \frac{3\pi x}{2a} + \frac{1}{25} \cos \frac{5\pi}{2a} x + \dots \right.$$

$$m = 1, 3, 5, \dots$$

or the coefficients

$$M_m = \frac{2}{\pi^2} \frac{Pa}{m^2} = \frac{2}{\pi} \frac{Pb}{m^2} \qquad \text{since } a = \pi\ b.$$

Then the effective widths for the individual terms will be

$$b_m = b \frac{1 + \frac{a}{b m \pi} \operatorname{Sin} \frac{m \pi b}{a}}{1 + \operatorname{Cos} \frac{m \pi b}{a}}$$

and for a = b and m = 1, 2, 3, 4 . . . .

$$b_1 = 0{,}857 b;\ b_2 = 0{,}591 b;\ b_3 = 0{,}391 b;\ b_4 = 0{,}276 b;\ b_5 = 0{,}211 b;$$
$$b_6 = 0{,}168 b;\ b_7 = 0{,}144 b;\ b_8 = 0{,}125 b;\ b_9 = 0{,}112 b;\ b_m = \frac{b}{m} = \frac{a}{m\pi}$$

and the section moduli

$$W_m = \frac{s h^2}{6} + h\ s\ b_m = \frac{s b}{2} \left(\frac{b}{12} + b_m \right)$$

[Translator's note: Here the depth of the girder is taken as b/2. Any value of h can be chosen as the value of b_m is independent of the girder depth].

Then:

$$W_1 = 0{,}940\,\frac{s\,b^2}{2};\quad W_3 = 0{,}474\,\frac{s\,b^2}{a};\quad W_5 = 0{,}294\,\frac{s\,b^2}{2};\quad W_7 = 0{,}227\,\frac{s\,b^2}{2};$$
$$W_9 = 0{,}195\,\frac{s\,b^2}{2};\quad W_{11} = 0{,}174\,\frac{s\,b^2}{2}.$$

The effectiveness of the flanges is shown by the term X. When x = 0 it is

$$X = \sum X_m = \sum s\,b_m\,\sigma_m = \sum s\,b_m\,\frac{M_m}{2\,W_m} = \frac{4}{\pi^2}\,P\,a \sum \frac{1}{m^2}\cdot\frac{s\,b_m}{2\,W_m}$$

thus

$$X = \frac{4}{\pi}\,P\left[\frac{0{,}857}{0{,}940} + \frac{1}{9}\cdot\frac{0{,}391}{0{,}474} + \frac{1}{25}\cdot\frac{0{,}211}{0{,}294} + \frac{1}{49}\cdot\frac{0{,}144}{0{,}227} + \frac{1}{81}\cdot\frac{0{,}112}{0{,}195} + \frac{1}{121}\cdot\frac{0{,}091}{0{,}174} + \ldots\right]$$

$$X = \frac{4}{\pi}\,P\,[0{,}910 + 0{,}092 + 0{,}029 + 0{,}013 + 0{,}007 + 0{,}004 + \ldots]$$

$$X = \frac{4}{\pi}\,P\cdot 1{,}055 = 1{,}352\,P.$$

It is evident that the series for X converges rapidly. The stress at the edge can be calculated approximately from

$$\sigma_r = \sum \sigma_m = \sum \frac{M_m}{2\,W_m} = \frac{4}{\pi^2}\,P\,a \sum \frac{1}{m^2}\cdot\frac{1}{2\,W_m} = \frac{4}{\pi}\,P\,b \sum \frac{1}{m^2}\cdot\frac{1}{2\,W_m}$$

$$\sigma_r = \frac{4}{\pi}\cdot\frac{P}{b\,s}\left[\frac{1}{0{,}940} + \frac{1}{9}\cdot\frac{1}{0{,}474} + \frac{1}{25}\cdot\frac{1}{0{,}294} + \frac{1}{49}\cdot\frac{1}{0{,}227} + \frac{1}{81}\cdot\frac{1}{0{,}195} + \frac{1}{121}\cdot\frac{1}{0{,}174} + \ldots\right]$$

$$\sigma_r = \frac{4}{\pi}\cdot\frac{P}{b\,s}\,[1{,}062 + 0{,}235 + 0{,}136 + 0{,}090 + 0{,}063 + 0{,}047 + \ldots]$$
$$= 2{,}080\,\frac{P}{b\,s}.$$

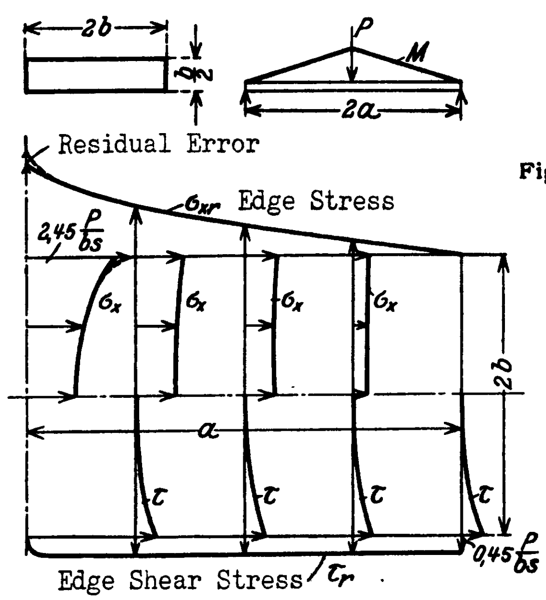


Fig. 3.

The stress distribution in the flange plate is shown in Fig 3. [Translator's note: It is to be noted that here as in the earlier solution, the boundary conditions are not strictly fulfilled since the shear stresses do not vanish at the ends of the girder, x = a]. The convergence of the series for edge stresses is not as good. The error is calculated by finding the difference between the external and internal moments.

$$\frac{M_i}{2} = h\,X + \frac{s\,h^2}{6}\sigma_r = \frac{b}{2}X + \frac{s\,b^2}{24}\sigma_r$$

or

$$\frac{M_i}{2} = \frac{b}{2}\,1{,}352\,P + \frac{2{,}08}{24}P\,b = 0{,}763\,P\,b$$

Half of the external moment at x = 0 is

$$\frac{M_a}{2} = \frac{P\,a}{4} = P\,b\frac{\pi}{4} = 0{,}785\,P\,b$$

There is left a residual moment of $(0.785 - 0.763)P_b = 0.022\ P\ b = M_R/2$. This moment must be taken up by the web and a narrow zone of the flange, the width of which is assumed with sufficient accuracy as half the effective width of the next highest unconsidered term. In the present case, this is

$$b_m = \frac{a}{2\,m\,\pi} = \frac{1}{2}\cdot\frac{b}{m} = \frac{b}{26}$$

The section modulus of this part is

$$W_R = \frac{s\,b^2}{24} + \frac{s\,b^2}{52} = (0{,}042 + 0{,}019)\,s\,b^2 = 0{,}061\,s\,b^2$$

Thus the supplementary stress is

$$\sigma_R = \frac{M_R}{2\,W_R} = \frac{0{,}022}{0{,}061}\cdot\frac{P}{b\,s} = 0.36\,\frac{P}{b\,s}$$

and the actual stress at the edge

$$\sigma_{max} = \sigma_r + \sigma_R = (2{,}080 + 0{,}360)\,\frac{P}{b\,s} = 2{,}44\,\frac{P}{b\,s}$$

and therefore the effective width

$$b_m = \frac{X}{s\,\sigma_{max}} = \frac{1{,}352}{2.44}\,b = 0{,}553\,b$$

is materially decreased.

[Translator's note: The value calculated by the formulas developed by Schnadel in 1926 is $b_m = 0.796$. See E.M.B. Translation No. 28, p 19].

(b) Uniformly Distributed Load. ($a/b = \pi$)

The moment curve is a parabola with the maximum moment Pa/4; its Fourier equation is

$$M = \frac{32}{\pi^3}\cdot\frac{P\,a}{4}\sum \pm \frac{1}{m^3}\cos\frac{m\,\pi\,x}{2\,a} \qquad m = 1,\ 3,\ 5.$$

The effective width for one term of the series and the corresponding section modulus are the same as in the preceding example. Therefore

$$X_m = \pm \frac{M_m}{2\,W_m} \cdot \frac{b_m\,s}{m^3} = \pm \frac{8}{\pi^3} \cdot \frac{P\,a}{2\,W_m} \cdot \frac{b_m\,s}{m^3} = \pm \frac{8}{\pi^2} \cdot \frac{P\,b}{2\,W_m} \cdot \frac{b_m\,s}{m^3}$$

$$X_m = \pm \frac{8}{\pi^2}\,P\,\frac{1}{m^3} \cdot \frac{b_m}{0{,}083\,b + b_m}$$

$$X = \sum X_m = \frac{8}{\pi^2}\,P\left[\frac{0{,}857}{0{,}940} - \frac{1}{27}\cdot\frac{0{,}391}{0{,}474} + \frac{1}{125}\cdot\frac{0{,}211}{0{,}294} - \frac{1}{343}\cdot\frac{0{,}144}{0{,}227} + \frac{1}{729}\cdot\frac{0{,}112}{0{,}195} - \cdots\right]$$

or

$$X = \frac{8}{\pi^2}\,P\,[0{,}910 - 0{,}031 + 0{,}006 - 0{,}002 + 0{,}001 - \cdots]$$

wherefore

$$X = \frac{8}{\pi^2} \cdot 0.884\,P = 0.715\;P.$$

The edge stress is

$$\sigma_r = \frac{8}{\pi^2}\cdot\frac{P}{b\,s}\left[\frac{1}{0{,}940} - \frac{1}{27}\cdot\frac{1}{0{,}474} + \frac{1}{125}\cdot\frac{1}{0{,}294} - \frac{1}{343}\cdot\frac{1}{0{,}227} + \frac{1}{729}\cdot\frac{1}{0{,}195} - \frac{1}{1331}\cdot\frac{1}{0{,}174} + \cdots\right]$$

$$= \frac{8}{\pi^2}\cdot\frac{P}{b\,s}\,[1{,}062 - 0{,}078 + 0.027 - 0{,}013 + 0{,}007 - 0{,}004 + \ldots]$$

or

$$\sigma_r = \frac{8}{\pi^2}\cdot\frac{P}{b\,s}\,[1{,}096 - 0{,}095] = \frac{8}{\pi^2}\cdot 1{,}001\,\frac{P}{sb} = 0{,}81\;\frac{P}{b\,s}$$

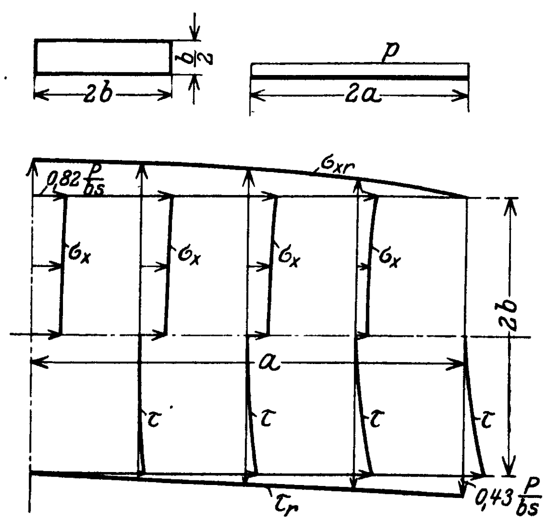


Fig. 4.

The stresses are plotted in Fig 4.

From this follows the effective width at x = 0:

$$(b_m)_{x=0} = \quad b_0 = \frac{\sum X_m}{\sum \frac{1}{b_m} X_m} = \frac{0{,}720}{0{,}817} b = 0{,}882\, b$$

[Translator's note: i.e.

$$b_m = \frac{X}{s\,\sigma_r} = \frac{0.715\,P}{0.810\,\frac{Ps}{bs}} = 0.883\,b].$$

Then the section modulus of the girder at this point is

$$2\,W_o = s\,b\,(0{,}083\,b + b_o) = s\,b^2\,(0{,}083 + 0{,}882) = 0{,}965\ s\,b^2$$

and

$$\sigma_r = \frac{M}{2\,W_o} = \frac{Pa}{8\,W_o} = \frac{P}{b\,s} \cdot \frac{\pi}{4 \cdot 0{,}965} = 0{,}813\,\frac{P}{b\,s}$$

i.e. the stresses agree exactly. The moment of the internal forces is equal to the external moment.

(c) Loading Due to Two Symmetrical Concentrated Loads (Girder with Overhang). (a/b = π)

When the loads are placed at distances a/2 from the ends, we get a trapezoidal distribution of moment, the Fourier form of which is:

$$M = \frac{16}{\pi^2} \cdot \frac{Pa}{2} \sum \pm \sin \frac{m\,\pi}{4} \cos \frac{m\,\pi\,x}{2\,a}$$

As in the two preceding examples

$$X = \sum X_m = \frac{8}{\pi} P \sin \frac{\pi}{4} (0{,}910 - 0{,}092 - 0{,}029 + 0{,}013 + 0{,}007 - 0{,}004)$$

$$X = \frac{8}{\pi} 0{,}707\ (0{,}930 - 0{,}125)\,P = 1{,}445\,P$$

and

$$\sigma_r = \sum \frac{X_m}{b_m\,s} = \frac{8}{\pi}\,0{,}707\,\frac{P}{bs}\ [1{,}062 - 0{,}234 - 0{,}136 + 0{,}090 + 0{,}063 - 0{,}047]$$

or

$$\sigma_r = 1{,}80\,\frac{P}{b\,s}\,(1{,}215 - 0{,}417) = 1{,}80 \cdot 0{,}798\,\frac{P}{b\,s} = 1{,}44\,\frac{P}{b\,s}$$

The stresses are plotted in Fig 5.

Then the effective width at x = 0 becomes:

$$b_0 = \frac{X}{s\,\sigma_r} = \frac{1{,}44}{1{,}44} = 1{,}00\,b$$

The cross section is fully effective. This result agrees with my previous tests (see Fig 5).

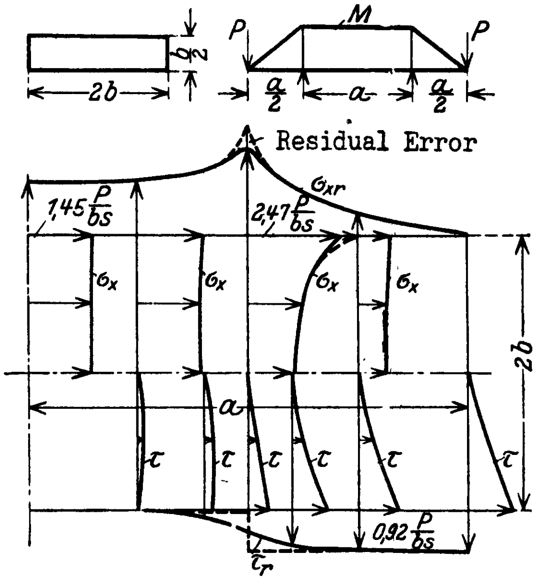


Fig. 5.

The external moment for x = 0 is Pa/2; the section modulus at this point is $2W = sb(0.083b + b_m) = 1.083sb^2$

and
$$\sigma_r = \frac{P\,a}{4W} = \frac{P}{s\,b} \cdot \frac{\pi}{2{,}166} = 1.45\ \frac{P}{b\,s}$$

The stresses have thus been calculated correctly. If we compare this result with the approximate formulas previously given by me, we find an error in this case of 8 per cent, and with a uniformly distributed load, of 2 per cent, for the effective width at the point x = 0. The error is due to the fact that the approximate solution is not exact for the coefficients of the higher terms. In cases in which the higher terms play an important part, the approximate solution is not therefore reliable. This is especially true at the points of attack of concentrated loads. Our example gives for x = a/2:

$$\sigma_r = 2{,}47\ \frac{P}{b\,s}\ \text{und}\ b_m = 0{,}547\,b$$

[Translator's note: The stress condition and effective width at the point $x = \pm a/2$ in this loading condition are identical with those at x = 0 for a concentrated load].

VIII. INTRODUCTION OF CORNER REINFORCEMENTS.

In the figures the stresses have been plotted as obtained by developing the series, and later the corrected values were added. It is evident that there is an increase in stress at the points of attack of the concentrated loads. In the web, moreover, as a result of the concentrated load, local stresses accumulate which are not considered in the present calculation, and the vertical transverse

forces (shear) add their stresses to the load[9]. The increase in stress first mentioned can be decreased by installing heavy flange angles in the corners which increase the natural moment of inertia of the web, while the local stress can be decreased by doubling plates. In general, it may be said that a large natural moment of inertia in the web increases the effective width. In the following it will be shown how conditions change when corner stiffeners are used.
[Translator's note: Evidently refers to angle bars usually used to connect web and flange in riveted built-up girders].

We base our calculations on a box girder with the same dimensions as heretofore, which is provided in each corner with reinforcements having an area of $f = sb/2$. Then the section modulus for one term of the Fourier series will be

$$W_m = h\,f + \frac{s\,h^2}{6} + h\,s\,b_m$$

and for $h = b/2$

$$W_m = \frac{s\,b}{2}(0{,}50\,b + 0{,}083\,b + b_m) = \frac{s\,b}{2}\,(0{,}583\,b + b_m)$$

Thus

$$W_1 = 1{,}44\,\frac{s\,b^2}{2};\quad W_3 = 0{,}974\,\frac{s\,b^2}{2};\quad W_5 = 0{,}793\,\frac{s\,b^2}{2};$$

$$W_7 = 0{,}727\,\frac{s\,b^2}{2};\quad W_9 = 0{,}695\,\frac{s\,b^2}{2};\quad W_{11} = 0{,}674\,\frac{s\,b^2}{2}.$$

Then, for the concentrated load at the point $x = 0$, the force in the half flange will be

$$X = \frac{4}{\pi^2}\,P\,a\sum\frac{1}{m^2}\cdot\frac{s\,b_m}{2\,W_m} = 0{,}840\,P$$

and the edge stress for $x = 0$

$$\sigma_r = 1{,}168\,\frac{P}{b\,s}$$

considering six terms of the series. The moment of internal forces for the half girder is

$$\frac{M_i}{2} = 0{,}762\,P\,b$$

while the external moment amounts to

$$M_a = 0{,}785\,P\,b$$

The residual moment will then be

$$M_R = 0{,}023\,P\,b$$

which is taken up by the web, corner plates and a narrow zone of the flange plate. The residual section modulus becomes

$$W_R = 0{,}312\,s\,b^2$$

and the corrective stress

$$\sigma_R = \frac{0{,}023}{0{,}312} = 0{,}074\,\frac{P}{b\,s}$$

which gives a maximum stress of

$$\sigma_m = \sigma_r + \sigma_R = 1{,}24\,\frac{P}{b\,s}$$

and an effective width of

$$b_m = \frac{X}{s\,\sigma_m} = 0{,}68\,b$$

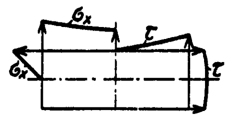


Fig. 6.

as compared with $b_m = 0.55b$ without the corner reinforcement, an improvement of 23%.

These values have not been plotted, in order to avoid complicating the figure. Fig 6 shows the stress distribution over the cross section.

IX. THE CONTINUOUS GIRDER ON MULTIPLE SUPPORTS.
(Completely fixed girder)

(a) General.

This case can be easily solved from the foregoing theory. The only change is in the boundary conditions at the point x = a. When the frame spacing is uniform, there will naturally be two conditions to be satisfied:

First, for x = a, the displacement in the X-direction is zero, and second, the rotation there is zero.

As Karman has shown in his paper the required condition is

$$\left(\frac{\partial^2 F}{\partial x\,\partial y}\right)_{x=a} = 0 \quad \text{und} \quad \cdot\,\frac{\partial^3 F}{\partial x^3} - \frac{1}{m}\cdot\frac{\partial^3 F}{\partial x\,\partial y^2} = 0$$

The expression $F = F(y)\cos\frac{m\pi}{a}y$ satisfies this condition.

[Translator's note: The reference appears to be to Festschrift August Föppls, 1923, p 114; von Karman "Die Mittragende Breite". The conditions are derived as follows:

Since there must be no change in displacement in the X-direction at x = a, $\frac{\partial u}{\partial y}$ = constant or $\frac{\partial^2 u}{\partial y^2} = 0$. The condition of "no rotation" means that there must be no shear in the plane of the plate at x = a, or:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{\partial^2 F}{\partial x\,\partial y} = 0 \qquad \text{(a)}$$

It follows that $\frac{\partial \gamma}{\partial y} = 0$, or:

$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\,\partial y} = 0,$$

or

$$\frac{\partial^2 v}{\partial x \partial y} = 0 \quad \text{since} \quad \frac{\partial^2 u}{\partial y^2} = 0$$

This may be written:

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial y}\right] = \frac{\partial}{\partial x}\,\varepsilon_y = \frac{\partial}{\partial x}\left[\sigma_y - \nu\sigma_x\right]\frac{1}{E}$$

$$\therefore \qquad \frac{\partial^3 F}{\partial x^3} - \nu\frac{\partial^3 F}{\partial x\,\partial y^2} = 0 \tag{b}$$

which correspond to the above conditions].

The solution will be the same as that for the freely supported girder, with the exception that the term a is substituted for 2a.

It follows mathematically that the effective width is subject to a sharp decrease. This is most undesirable in the case of very wide girders, since there the effective width decreases by one half.

For the continuous girder we have

$$b_m = \frac{b}{2}\;\frac{1 + \frac{a}{b m\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a}\,\mathfrak{Cos}\,\frac{m\pi b}{a}}{\mathfrak{Cos}^2\,\frac{m\pi b}{a}}$$

For a large $\frac{m\pi b}{a}$, i.e. for wide girders, $b'_m = \frac{a}{2m\pi}$.

[Translator's note: This is readily seen when bm is expressed in the form of Eq (8') since the hyperbolic tangent approaches unity for large arguments].

It would lead too far afield to take up the general case of a continuous girder with varying frame spacing. A sufficiently accurate solution is obtained by regarding the part with the positive moment area as a girder which is simply supported at the zero moment points.

(b) Sample Calculation. ($a/b = 2\pi$)

In the case of a fixed girder with a concentrated load it may be concluded that over the supports the effective width is the same as in the middle of the positive part of the moment distribution. In the case of the fixed girder with uniformly distributed load, the moment curve is a parabola, intersected at two thirds of its height by the zero axis. Its equation in a Fourier series is

$$M = \frac{4}{\pi^2}\cdot\frac{Pa}{4}\left(\cos\frac{\pi}{a}\,x - \frac{1}{4}\cos\frac{2\pi}{a}\,x + \frac{1}{9}\cos\frac{3\pi}{a}\,x - \cdots\right)$$

If we select a ratio of $\frac{a}{b} = 2\pi$, we get the same effective width for each term, as in the previous examples.

From this follow the section moduli for one term of the series in the case of the girder with a height of h = b

$$W_m = \frac{s\,b^2}{6} + b\,b_m\,s$$

i.e.

$$W_1 = 1{,}024\,s\,b^2;\quad W_2 = 0{,}758\,s\,b^2;\quad W_3 = 0{,}558\,s\,b^2;\quad W_4 = 0{,}443\,s\,b^2;$$
$$W_5 = 0{,}378\,s\,b^2;\quad W_6 = 0{,}335\,s\,b^2;\quad W_7 = 0{,}311\,s\,b^2;\quad W_8 = 0{,}293\,s\,b^2;$$
$$W_9 = 0{,}279\,s\,b^2 \text{ usw.}$$

and for x = 0

$$X = \sum X_m = \sum \frac{M_m}{2\,W_m}\,s\,b_m = \frac{P}{\pi} \sum \pm \frac{b_m}{m^2} \cdot \frac{1}{\frac{b}{6} + b_m}$$

or

$$X = 0{,}220\,P$$

and

$$\sigma_r = \sum \frac{M_m}{2\,W_m} = \frac{1}{\pi^2} \cdot \frac{P\,a}{2\,s\,b} \sum \pm \frac{1}{m^2} \cdot \frac{1}{\frac{b}{6} + b_m}$$

thus, for x = 0

$$\sigma_r = 0{,}250\,\frac{P}{b\,s}$$

Then the moment of the internal forces will be

$$M_i = \frac{s\,h^2}{6}\,\sigma_r + h\,X = 0{,}262\,P\,b$$

which coincides exactly with the external moment. The effective width will be

$$b_m = \frac{X}{s\,\sigma_r} = 0{,}88\,b$$

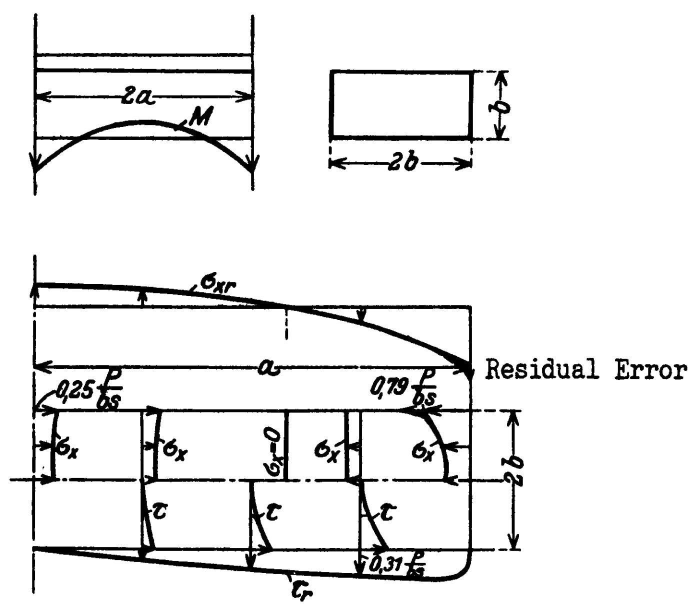


Fig. 7.

It is in good agreement with the effective width of the simply supported girder of half the length.

Over the supports, i.e. for x = a

$$X_{x=a} = 0{,}386\,P \quad \text{und} \quad \sigma_r = -\,0{,}651\,\frac{P}{b\,s}$$

if we confine ourselves to the same number of terms as for x = 0. Then the moment of the internal forces is M_i = 0.495 Pb as compared to an external moment of M_a = 0.524 Pb, leaving an error of M_R = 0.029 Pb, which is taken up by the web and a zone of the flange of width b/26. From this we get W_R = 0.205 sb^2.

$$\sigma_r = 0{,}142\,\frac{P}{s\,b} \quad \text{und} \quad \sigma_{max} = \sigma_r + \sigma_R = 0{,}793\,\frac{P}{s\,b}$$

Then the effective width for x = a is

$$b_m = \frac{X_a}{s\,\sigma} = 0{,}486\,b$$

The effective width over the supports, then, decreases very greatly. The stresses are plotted in Fig 7.

X. THE CANTILEVER GIRDER (a/b = π)

(a) Concentrated load.

With respect to the effective width, the cantilever girder obeys the same laws as the girder fixed at the supports. In the case of a cantilever girder, under a concentrated load at the free end, the effective width will obviously be the same as for a freely supported girder of twice its length. [Translator's note: This follows from the fact that the moment curves are identical in the two cases].

(b) Uniformly distributed load.

As an example we will consider a box girder having the flange ratio a:b = π , a web height of h = b/2, and a uniform thickness of s. Let the load be uniformly distributed. The moment curve will be a parabola, with a maximum value of $pa^2/2$. The moment curve in Fourier form has the equation

$$M = \frac{16}{\pi^2}\,p\,\frac{a^2}{2}\,\sum \frac{1}{m^2}\left(1 \mp \frac{2}{m\,\pi}\right)\cos\frac{m\,\pi}{2\,a}\,x \quad (m = 1, 3, 5, \ldots)$$

$$M = \frac{8}{\pi^2}\,P\,a\,\Big(0{,}362\cos\frac{\pi}{2\,a}\,x + 0{,}135\cos\frac{3\,\pi}{2\,a}\,x + 0{,}035\cos\frac{5\,\pi}{2\,a}\,x$$
$$+ 0{,}022\cos\frac{7\,\pi}{2\,a}\,x + 0{,}0114\cos\frac{9\,\pi}{2\,a}\,x + 0{,}0098\cos\frac{11\,\pi}{2\,a}\,x + \ldots\Big)$$

and

$$W_m = \frac{s\,h^2}{6} + h\,s\,b_m = \frac{s\,b}{2}\left(\frac{b}{12} + b_m\right)$$

The section moduli are the same as in the first three examples. Therefore

$$X = \sum X_m = \sum M_m\,\frac{s\,b_m}{2\,W_m} = \frac{8}{\pi}\,P\,\sum \frac{k\,b_m}{0{,}083\,b + b_m}$$

and for x = 0

$$X = \frac{8}{\pi} P \left[0{,}362 \frac{0{,}857}{0{,}940} + 0{,}135 \frac{0{,}391}{0{,}474} + 0{,}035 \frac{0{,}211}{0{,}294} + 0{,}022 \frac{0{,}144}{0{,}227} \right.$$
$$\left. + 0{,}0114 \frac{0{,}112}{0{,}195} + 0{,}0088 \frac{0{,}091}{0{,}174} + \ldots \right]$$

or
$$X = \frac{8}{\pi} P \left[0{,}330 + 0{,}112 + 0{,}025 + 0{,}014 + 0{,}007 + 0{,}005 + \ldots\right]$$
$$= \frac{8}{\pi} 0{,}493\, P = 1{,}26\, P$$

Furthermore when $x = 0$

$$\sigma_r = \frac{8}{\pi} \cdot \frac{P}{b\,s} \left[\frac{0{,}362}{0{,}940} + \frac{0{,}135}{0{,}474} + \frac{0{,}035}{0{,}294} + \frac{0{,}022}{0{,}227} + \frac{0{,}011}{0{,}195} + \frac{0{,}088}{0{,}174} + \ldots \right]$$
$$= 2{,}54 \frac{P}{b\,s}$$

Then the moment of the internal forces will be

$$\frac{M_i}{2} = h X + \frac{s\,h^2}{6} \sigma_r = 0{,}736\ P\,b$$

and the external moment amounts to

$$\frac{M_a}{2} = P \frac{a}{4} = 0{,}785\, P\, b$$

Therefore the error is $M_R = 0.0049$ Pb. From this there results a corrective stress on the upper edge of the web

$$\sigma_R = \frac{0{,}049}{0{,}061} \cdot \frac{P}{b\,s} = 0.82 \frac{P}{b\,s}$$

The total stress will be $\sigma_{max} = 3{,}36 \frac{P}{b\,s}$ and the effective width $b_m = 0.375b$ (see Fig 8).

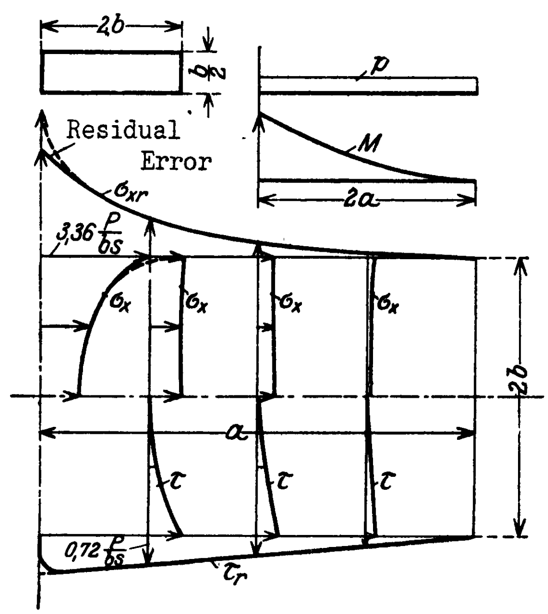


Fig. 8.

If corner reinforcements are installed in the two last discussed girders, having an area f = sb/2, then in the case of the fixed girder over the supports

$W_m = \frac{s\,b^2}{6} + \frac{s\,b^2}{2} + s\,b\,b_m$ and

$$X = \sum \frac{M_m}{2\,W_m}\,b_m\,s = \frac{P\,a}{b \cdot 2\,\pi^2} \sum \pm \frac{1}{m^2} \cdot \frac{b_m}{0{,}667\,b + b_m}$$

and with $a = b\,\pi$:

$$X = 0{,}238\,P \quad \text{und} \quad \sigma_r = \sum \frac{M}{2\,W_m} = 0{,}381\,\frac{P}{s\,b}$$

The moment of the internal forces will be M_i = 0.492Pb as compared to an external moment M_a = 0.524 Pb. The residual moment yields an additional stress of $\sigma_R = 0{,}046\,\frac{P}{s\,b}$ on the upper edge of the web and a maximum stress $\sigma_{max} = 0{,}427\,\frac{P}{s\,b}$ at an effective width for x = a of b_m = 0.56b as compared to b_m = 0.486b without corner reinforcements.

Similarly we get for the girder fixed at one end and with corner reinforcements, a flange force X = 0.727P, and a stress for x = 0 of $\sigma_r = 1{,}26\,\frac{P}{b\,s}$, a corrective stress $\sigma_R = 0{,}090\,\frac{P}{s\,b}$, and therefore a maximum stress $\sigma_{max} = 1{,}35\,\frac{P}{s\,b}$ and an effective width of b_m = 0.54 b as compared to 0.38 b without corner reinforcement, i.e. an improvement of 42%.

XI. CALCULATION OF THE FLANGE PLATES OF THE DOUBLE BOTTOM.

(a) We base our calculation on an I-girder with narrow flanges, which we can consider as having been cut out of the double bottom. If the double bottom itself is to be calculated, the shearing stresses and their opposing forces can be added by superposition. Tne effective width will be thereby affected little or not at all.

For the I-beam the section modulus for the upper flange will be

$$W_o = 2\,e\,b\,s_o \left[1 + \frac{J_1}{2\,e^2\,b\,s_o} \cdot \frac{F_1 + 2\,b\,(s_o + s_u)}{F_1 + 4\,b\,s_u} + \frac{F_1 + 4\,b\,s_o}{F_1 + 4\,b\,s_u} \cdot \frac{s_u}{s_o}\right] \qquad (17)$$

The section modulus for the lower flange follows from this by changing the indices of s. Here s_o and s_u represent the thicknesses of the flanges, 2b their width, J_1 the moment of inertia and F the area of the web. If we write $s_o = s_u = s$, we get

$$W = 2\,e\,b\,s\left(2 + \frac{J_1}{2\,e^2\,b\,s}\right)$$

and for $s_u = 0$

$$W_o = 2\,e\,b\,s\left(1 + \frac{J_1}{2\,e^2\,b\,s} + \frac{J_1}{e^2\,F_1}\right)$$

(b) The stresses in the flange plate, as in the case of every flat plate, can now be determined with the Airy stress function. This, as we know, must satisfy the partial differential equation

$$\frac{\partial^4 F}{\partial x^4} + \frac{2\,\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0$$

We once more write the series function

$$F = \sum F(y) \cos \frac{m\pi}{2a} x$$

choosing the m so that the most important boundary conditions for x = a will be automatically satisfied.

From this equation follows the function

$$F(y) = B_m \left(A_m \operatorname{Cos} \frac{m\pi}{2a} y + y \operatorname{Sin} \frac{m\pi}{2a} y \right) \tag{18}$$

The boundary condition in X-direction is so taken that for x = a the stresses are zero. In Y-direction the following reasoning applies: The floors as well as the centerlines between two floors are axes of symmetry. The lines must therefore remain straight. Now since for x = a, σ_x and σ_y are zero, the width b is maintained there and is constant for the entire length of the girder. Therefore, at every point the sum of all displacements in the transverse direction must vanish. If we call the displacement in Y-direction η, the extension ε, and the modulus of elasticity E, we have

$$E \frac{\partial \eta}{\partial y} = \sigma_y - \frac{\sigma_x}{\mu} = \frac{\partial^2 F}{\partial x^2} - \frac{1}{\mu} \cdot \frac{\partial^2 F}{\partial y^2}$$

and therefore

$$E\eta = 0 = \int_0^b \frac{\partial^2 F}{\partial x^2}\, dy - \frac{1}{\mu} \left(\frac{\partial F}{\partial y} \right)_0^b$$

where μ is the coefficient of transverse contraction (Poisson's Ratio).

But we also have

$$\frac{\partial^2 F}{\partial x^2} = -\left(\frac{m\pi}{2a}\right)^2 B_m \cos \frac{m\pi}{2a} x \left[A_m \operatorname{Cos} \frac{m\pi}{2a} y + y \operatorname{Sin} \frac{m\pi}{2a} y \right]$$

$$\begin{aligned}
\int_0^b \frac{\partial^2 F}{\partial x^2}\, dy = &- \left(\frac{m\pi}{2a}\right) B_m \cos \frac{m\pi}{2a} x \\
&\times \left[A_m \operatorname{Sin} \frac{m\pi}{2a} y + y \operatorname{Cos} \frac{m\pi}{2a} y - \frac{2a}{m\pi} \operatorname{Sin} \frac{m\pi}{2a} y \right]_0^b \\
= &- \left(\frac{m\pi}{2a}\right) B_m \cos \frac{m\pi}{2a} x \\
&\times \left[A_m \operatorname{Sin} \frac{m\pi b}{2a} + b \operatorname{Cos} \frac{m\pi b}{2a} - \frac{2a}{m\pi} \operatorname{Sin} \frac{m\pi b}{2a} \right]
\end{aligned}$$

and

$$\left[\frac{\partial F}{\partial y}\right]_0^b = \frac{m\pi}{2a} B_m \cos \frac{m\pi}{2a} x \times \left[A_m \operatorname{Sin} \frac{m\pi b}{2a} + b \operatorname{Cos} \frac{m\pi b}{2a} + \frac{2a}{m\pi} \operatorname{Sin} \frac{m\pi b}{2a} \right]$$

Therefore

$$E\eta = 0 = \frac{m\pi}{2a} B_m \cos\frac{m\pi}{2a} x \left[A_m \frac{\mu+1}{\mu} \mathfrak{Sin}\frac{m\pi b}{2a} + b\frac{\mu+1}{\mu} \mathfrak{Cof}\frac{m\pi b}{2a} - \frac{\mu-1}{\mu} \cdot \frac{2a}{m\pi} \mathfrak{Sin}\frac{m\pi b}{2a} \right]$$

The condition will be satisfied when the term in brackets becomes zero. Therefore

$$\boxed{A_m = -\, b\,\frac{\mathfrak{Cof}\frac{m\pi b}{2a}}{\mathfrak{Sin}\frac{m\pi b}{2a}} + \frac{\mu-1}{\mu+1}\cdot\frac{2a}{m\pi}} \tag{19}$$

(c) Taking into account that the flange has two sides, the statically indeterminate value X_0 will be

$$X_0 = 2 s_0 \int_0^b \sigma_x \, dy = 2 s_0 \int_0^b \left(\frac{\partial^2 F}{\partial y^2}\right) dy = 2 s_0 \left(\frac{\partial F}{\partial y}\right)_0^b$$

therefore

$$X_0 = \frac{2 m\pi}{2a} s_0 B_m \left[A_m \mathfrak{Sin}\frac{m\pi b}{2a} + b\, \mathfrak{Cof}\frac{m\pi b}{2a} + \frac{2a}{m\pi} \mathfrak{Sin}\frac{m\pi b}{2a} \right] \cos\frac{m\pi x}{2a}$$

$$= X_{om} \cos\frac{m\pi x}{2a}$$

where

$$X_{om} = \frac{2 m\pi}{2a} s_0 B_m \left[-b\, \mathfrak{Cof}\frac{m\pi b}{2a} + \frac{\mu-1}{\mu+1}\cdot\frac{2a}{m\pi} \mathfrak{Sin}\frac{m\pi b}{2a} + b\, \mathfrak{Cof}\frac{m\pi b}{2a} + \frac{2a}{m\pi} \mathfrak{Sin}\frac{m\pi b}{2a} \right]$$

Therefore

$$\boxed{X_{om} = 2 s_0 B_m \frac{2\mu}{\mu+1} \mathfrak{Sin}\frac{m\pi b}{2a}} \tag{20a}$$

or

$$B_m = \frac{X_{om}}{2 s_0} \cdot \frac{\mu+1}{2\mu} \cdot \frac{1}{\mathfrak{Sin}\frac{m\pi b}{2a}} \tag{20b}$$

(d) The Strain Energy in the Flanges.

The strain energy in the flange plates, as before, is

$$L_0 = \frac{2s}{2E} \iint (\sigma_x + \sigma_y)^2 \, dx\, dy + \frac{2s}{2G} \iint (\tau^2 - \sigma_x \sigma_y) \, dx\, dy$$

The factor 2 is due to the fact that in this case there is a working flange on both sides of the web.

The first integral is again found to be

$$L_1 = \frac{2\,s_0}{2\,E}\int_0^b\int_{-a}^{+a}\left[\sum \frac{m\pi}{a} B_m \,\mathfrak{Cof}\, \frac{m\pi}{2a} y \cos\frac{m\pi}{2a} x\right]^2 dx\,dy$$

i.e.

$$L_1 = \frac{2\,s_0}{E}\sum B_m^2 \left(\frac{m\pi}{2a}\right)^2 a\,b\left(\frac{a}{b\,m\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a} + 1\right) \qquad (3a)$$

The second integral, on the other hand, does not vanish.

In changing to a line integral we get, as formerly

$$L_2 = \frac{2\,s_0}{2\,G}\oint \frac{\partial F}{\partial x}\left(\frac{\partial^2 F}{\partial x \partial y} dx + \frac{\partial^2 F}{\partial y^2} dy\right)$$

When x = $\pm$a we here get $\frac{\partial^2 F}{\partial y^2} = c$, , and when y = 0, $\frac{\partial^2 F}{\partial x \partial y} = 0$. There remains, then, only the integral

$$\int_{-a}^{+a}\left(\frac{\partial F}{\partial x}\right)_{y=b}\left(\frac{\partial^2 F}{\partial x \partial y}\right)_{y=b} dx$$

However

$$\left(\frac{\partial F}{\partial x}\right)_{y=b} = -\sum B_m \frac{m\pi}{2a}\sin\frac{m\pi}{2a} x\left[A_m\,\mathfrak{Cof}\,\frac{m\pi}{2a} y + y\,\mathfrak{Sin}\,\frac{m\pi}{2a} y\right]_{y=b}$$

Therefore

$$\left(\frac{\partial F}{\partial x}\right)_b = -\sum B_m \frac{m\pi}{2a}\sin\frac{m\pi}{2a} x \times\left[-b\,\frac{\mathfrak{Cof}^2\frac{m\pi b}{2a} - \mathfrak{Sin}^2\frac{m\pi b}{2a}}{\mathfrak{Sin}\,\frac{m\pi b}{2a}} + \frac{\mu-1}{\mu+1}\cdot\frac{2a}{m\pi}\,\mathfrak{Cof}\,\frac{m\pi b}{2a}\right]$$

or

$$\left(\frac{\partial F}{\partial x}\right)_b = -\sum B_m \frac{m\pi}{2a}\sin\frac{m\pi}{2a} x \times \frac{b}{\mathfrak{Sin}\,\frac{m\pi b}{2a}}\left[\frac{\mu-1}{\mu+1}\cdot\frac{a}{b\,m\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a} - 1\right] \qquad (21a)$$

Likewise

$$\left(\frac{\partial^2 F}{\partial x \partial y}\right)_{y=b} = -\sum B_m\left(\frac{m\pi}{2a}\right)^2\sin\frac{m\pi}{2a} x \times\left[A_m\,\mathfrak{Sin}\,\frac{m\pi b}{2a} + b\,\mathfrak{Cof}\,\frac{m\pi b}{2a} + \frac{2a}{m\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{2a}\right]$$

$$= -\sum B_m\left(\frac{m\pi}{2a}\right)\sin\frac{m\pi}{2a} x\cdot\frac{2\mu}{\mu+1}\,\mathfrak{Sin}\,\frac{m\pi b}{a} \qquad (21b)$$

If we further write G $=\frac{\mu}{2(\mu+1)}E$, we get, since the trigonometric function is orthogonal,

$$L_2 = \frac{s_o}{E} \cdot \frac{2(\mu+1)}{\mu} \sum B_m^2 \left(\frac{m\pi}{2a}\right)^2 \cdot \frac{2\mu}{\mu+1} \times \left[\frac{\mu-1}{\mu+1} \cdot \frac{a}{b\,m\pi} \mathfrak{Sin}\,\frac{m\pi b}{a} - 1\right] b \int_{-a}^{+a} \sin^2 \frac{m\pi}{2a} x\,dx$$

or

$$L_2 = \frac{4\,s_o}{E}\,a\,b \sum B^2_m \left(\frac{m\pi}{2a}\right)^2 \left[\frac{\mu-1}{\mu+1} \cdot \frac{a}{b\,m\pi} \mathfrak{Sin}\,\frac{m\pi b}{a} - 1\right] \tag{22}$$

Then

$$L_0 = L_1 + L_2$$

$$L_o = \frac{2\,s_o}{E}\,a \sum B^2_m \left(\frac{m\pi}{2a}\right)^2 b \times \left[\frac{a}{b\,m\pi} \mathfrak{Sin}\,\frac{m\pi b}{a} + 2\,\frac{\mu-1}{\mu+1} \cdot \frac{a}{b\,m\pi} \mathfrak{Sin}\,\frac{m\pi b}{a} - 1\right]$$

$$L_o = \frac{2\,s_o}{E}\,a \sum B^2_m \left(\frac{m\pi}{2a}\right)^2 b \left[\frac{3\mu-1}{\mu+1} \cdot \frac{a}{b\,m\pi} \mathfrak{Sin}\,\frac{m\pi b}{a} - 1\right] \tag{23}$$

If in addition we introduce into the expression for the strain energy the statically indeterminate values X_{om} and X_{um}, we will have

$$L_o = \frac{a}{2\,s_o\,E} \left(\frac{\mu+1}{2\mu}\right)^2 \sum X^2_{om} \left(\frac{m\pi}{2a}\right)^2 b\, \frac{\frac{3\mu-1}{\mu+1} \cdot \frac{a}{b\,m\pi} \mathfrak{Sin}\,\frac{m\pi b}{a} - 1}{\mathfrak{Sin}^2 \frac{m\pi b}{2a}}$$

or

$$L_o = \frac{a}{2\,s_o\,E} \sum \frac{X^2_{om}}{2\,b'_m} \tag{24a}$$

where

$$b_m' = \frac{\left(\frac{2\mu}{\mu+1}\right)^2 \mathfrak{Sin}^2 \frac{m\pi b}{2a}}{2\left(\frac{m\pi}{2a}\right)^2 b \left(\frac{3\mu-1}{\mu+1} \cdot \frac{a}{b\,m\pi} \mathfrak{Sin}\,\frac{m\pi b}{a} - 1\right)} \tag{25}$$

and

$$L_u = \frac{a}{2\,s_u\,E} \sum \frac{X^2_{um}}{2\,b'_m} \tag{24b}$$

The strain energy of the web, as in the first part, is expressed by X_{om} or X_{um}. The strain energy of the web will then be

$$L_s = \frac{a}{2EF_1} \sum (X_{om} - X_{um})^2 + \frac{a}{2EJ_1} \sum (M_m - e\,X_{om} - e\,X_{um})^2$$

The total strain energy becomes:

$$L = L_o + L_u + L_s$$

or

$$L = \frac{a}{2E} \sum \left[\left(\frac{X^2_{om}}{2\,s_o\,b'_m} + \frac{X^2_{um}}{2\,s_u\,b'_m}\right) + \frac{(X_{om} - X_{um})^2}{F_1} + \frac{1}{J_1} (M_m - e\,X_{om} - e\,X_{um})^2\right]$$

(e) The minimum condition yields, from $\frac{\partial L}{\partial X_{om}} = 0$ and $\frac{\partial L}{\partial X_{um}} = 0$, the two equations for X_{om} and X_{um}

$$X_{om}\left(\frac{1}{2\,s_o\,b'_m} + \frac{1}{F_1} + \frac{e^2}{J_1}\right) - X_{um}\left(\frac{1}{F_1} - \frac{e^2}{J_1}\right) = \frac{e}{J_1} M_m$$

and

$$X_{um}\left(\frac{1}{2\,s_u\,b'_m} + \frac{1}{F_1} + \frac{e^2}{J_1}\right) - X_{om}\left(\frac{1}{F_1} - \frac{e^2}{J_1}\right) = \frac{e}{J_1} M_m$$

the solution of which gives:

$$X_{om} = \frac{M_m}{e} \times \frac{1}{1 + \frac{F_1 + 4\,b'_m\,s_o}{F_1 + 4\,b'_m\,s_u} \cdot \frac{s_u}{s_o} + \frac{J_1}{2\,e^2\,b'_m\,s_o} \cdot \frac{F_1 + 2\,b'_m\,s_o + 2\,b'_m\,s_u}{F_1 + 4\,b'_m\,s_u}} \tag{27a}$$

or

$$X_{om} = \frac{M_m}{W_m} 2\,b'_m\,s_o \tag{27b}$$

where Wm is the section modulus of a girder having an effective width of 2b'm as flange and a web with the cross section F_1 and moment of inertia J_1.

For $s_u = s_o = s$

$$X'_{om} = \frac{M_m}{e} \frac{1}{2 + \frac{J_1}{2\,e^2\,b'_m\,s}} \tag{27c}$$

and for $s_u = 0$

$$X''_{om} = \frac{M_m}{e} \frac{1}{1 + \frac{J_1}{2\,e^2\,b'_m\,s} + \frac{J_1}{e^2\,F_1}} \tag{27d}$$

In the same way we get X_{um} by substituting s_u for s_o.

(f) The constant B_m, according to Eq (20b) will be

$$B_m = \frac{X_{om}}{2\,s_o} \frac{\mu + 1}{2\,\mu} \frac{1}{\mathfrak{Sin}\,\frac{m\,\pi\,b}{2\,a}} = \frac{M_m}{W_m} b'_m \frac{\mu + 1}{2\,\mu} \frac{1}{\mathfrak{Sin}\,\frac{m\,\pi\,b}{2\,a}} \tag{20b}$$

or

$$B_m = \frac{M_m}{W_m} \cdot \frac{\frac{\mu}{\mu + 1}\,\mathfrak{Sin}\,\frac{m\,\pi\,b}{2\,a}}{\left(\frac{m\,\pi}{2\,a}\right)^2 b \left(\frac{3\,\mu - 1}{\mu + 1} \cdot \frac{a}{b\,m\,\pi}\,\mathfrak{Sin}\,\frac{m\,\pi\,b}{a} - 1\right)} \tag{28}$$

Then the stress function will be

$$F = \frac{M_m}{W_m}\cdot\frac{\left\{\frac{\mu}{\mu+1}\left[\left(\frac{\mathfrak{Cof}\,\frac{m\pi b}{2a}}{\mathfrak{Sin}\,\frac{m\pi b}{2a}}+\frac{\mu-1}{\mu+1}\cdot\frac{2a}{bm\pi}\right)\times\mathfrak{Cof}\,\frac{m\pi y}{2a}+\frac{y}{b}\,\mathfrak{Sin}\,\frac{m\pi y}{2a}\right]\mathfrak{Sin}\,\frac{m\pi b}{2a}\right\}}{\left(\frac{m\pi}{2a}\right)^2\cdot\left(\frac{3\mu-1}{\mu+1}\cdot\frac{a}{bm\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a}-1\right)}\cos\frac{m\pi}{2a}x \tag{29a}$$

The stresses follow by differentiation. We will have

$$\sigma_y=\frac{\partial^2 F}{\partial x^2}=-\sum\frac{M_m}{W_m}\cdot\frac{\mu\,\mathfrak{Sin}\,\frac{m\pi b}{2a}}{\mu+1}\times\frac{\left[\left(-\mathfrak{Ctg}\,\frac{m\pi b}{2a}+\frac{\mu-1}{\mu+1}\cdot\frac{2a}{bm\pi}\right)\mathfrak{Cof}\,\frac{m\pi}{2a}y+\frac{y}{b}\,\mathfrak{Sin}\,\frac{m\pi}{2a}y\right]}{\frac{3\mu-1}{\mu+1}\cdot\frac{a}{bm\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a}-1}\times\cos\frac{m\pi}{2a}x \tag{29b}$$

$$\tau=-\frac{\partial^2 F}{\partial x\,\partial y}=\sum\frac{M_m}{W_m}\cdot\frac{\mu\,\mathfrak{Sin}\,\frac{m\pi b}{2a}}{\mu+1}\times\frac{\left[\left(-\mathfrak{Ctg}\,\frac{m\pi b}{2a}+\frac{2\mu}{\mu+1}\cdot\frac{2a}{bm\pi}\right)\mathfrak{Sin}\,\frac{m\pi}{2a}y+\frac{y}{b}\,\mathfrak{Cof}\,\frac{m\pi}{2a}y\right]}{\frac{3\mu-1}{\mu+1}\cdot\frac{a}{bm\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a}-1}\times\sin\frac{m\pi}{2a}x \tag{29c}$$

$$\sigma_x=\frac{\partial^2 F}{\partial y^2}=\sum\frac{M_m}{W_m}\cdot\frac{\mu\,\mathfrak{Sin}\,\frac{m\pi b}{2a}}{\mu+1}\times\frac{\left[\left(-\mathfrak{Ctg}\,\frac{m\pi b}{2a}+\frac{3\mu+1}{\mu+1}\cdot\frac{2a}{bm\pi}\right)\mathfrak{Cof}\,\frac{m\pi}{2a}y+\frac{y}{b}\,\mathfrak{Sin}\,\frac{m\pi}{2a}y\right]}{\frac{3\mu-1}{\mu+1}\cdot\frac{a}{bm\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a}-1}\times\cos\frac{m\pi}{2a}x \tag{29d}$$

From this follow the special cases: for y = b

$$\sigma_{y_0}=-\sum\frac{M_m}{W_m}\cdot\frac{\mu}{\mu+1}\cdot\frac{\frac{\mu-1}{\mu+1}\cdot\frac{a}{bm\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a}}{\frac{3\mu-1}{\mu+1}\cdot\frac{a}{bm\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a}-1}\cos\frac{m\pi}{2a}x \tag{30a}$$

$$\tau_0=\sum\frac{M_m}{W_m}\cdot\frac{\mu}{\mu+1}\cdot\frac{\frac{2\mu}{\mu+1}\cdot\frac{a}{bm\pi}\left(\mathfrak{Cof}\,\frac{m\pi b}{a}-1\right)}{\frac{3\mu-1}{\mu+1}\cdot\frac{a}{bm\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a}-1}\sin\frac{m\pi}{2a}x \tag{30b}$$

$$\sigma_{x_0}=\sum\frac{M_m}{W_m}\cdot\frac{\mu}{\mu+1}\cdot\frac{\frac{3\mu+1}{\mu+1}\cdot\frac{a}{bm\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a}-1}{\frac{3\mu-1}{\mu+1}\cdot\frac{a}{bm\pi}\,\mathfrak{Sin}\,\frac{m\pi b}{a}-1}\cos\frac{m\pi}{2a}x \tag{30c}$$

and for wide flanges we get:

[Translator's note: For large values of b, unity can be neglected in comparison with the term containing the hyperbolic function. Also the hyperbolic sine can be considered equal to the hyperbolic cosine].

$$\sigma_{x_0} = \sum \frac{M_m}{W_m} \cdot \frac{3\mu^2 + \mu}{3\mu^2 + 2u - 1} \cos \frac{m\pi}{2a} x$$

$$\tau_0 = \sum \frac{M_m}{W_m} \cdot \frac{2\mu^2}{3\mu^2 + 2\mu - 1} \sin \frac{m\pi}{2a} x$$

$$\sigma_{y_0} = -\sum \frac{M_m}{W_m} \cdot \frac{\mu^2 - \mu}{3\mu^2 + 2\mu - 1} \cos \frac{m\pi}{2a} x$$

It can be demonstrated that at the juncture of flange and web, strains and shear forces agree and the conditions of support are properly fulfilled.

XIII. SAMPLE CALCULATION.

A freely supported girder grid with the aspect ratio a:b = π, a web height of h = b/2, and a uniform thickness s is to be investigated, it being assumed that the grid has a continuous flange plate.

Let each girder be subjected to a concentrated load in the middle.

Solution: For a harmonic stress distribution in the flange we will have

$$b_m = \frac{\left(\frac{\mu}{\mu+1}\right)^2 \left[\mathfrak{Cof} \frac{m\pi b}{a} - 1 \right]}{b \left(\frac{m\pi}{2a}\right)^2 \left(\frac{3\mu - 1}{\mu + 1} \cdot \frac{a}{b m \pi} \mathfrak{Sin} \frac{m\pi b}{a} - 1 \right)}$$

$$\frac{\pi b}{a} = 1; \quad \frac{3\pi b}{a} = 3; \quad \frac{5\pi b}{a} = 5; \quad \frac{7\pi b}{a} = 7;$$

$$\mu = {}^{10}/_{3}; \quad \text{also} \quad \left(\frac{\mu}{\mu+1}\right)^2 = \frac{100}{169} = 0{,}59;$$

$$\frac{3\mu - 1}{\mu + 1} = \frac{27}{13} = 2{,}07.$$

$$\mathfrak{Cof}\, 1 = 1{,}543; \quad \mathfrak{Cof}\, 3 = 10{,}07; \quad \mathfrak{Cof}\, 5 = 74{,}21;$$

$$\mathfrak{Sin}\, 1 = 1{,}175; \quad \mathfrak{Sin}\, 3 = 10{,}02; \quad \mathfrak{Sin}\, 5 = 74{,}20.$$

Then

$$b_1 = \frac{0{,}59 \cdot 0{,}543\, b}{\frac{1}{4}(2{,}07 \cdot 1{,}175 - 1)} = \frac{0{,}32\, b}{\frac{1{,}43}{4}} = 0{,}90\, b$$

Furthermore, for m = 3

$$b_3 = \frac{0{,}59 \cdot 9{,}07}{\frac{9}{4b}\left(\frac{2{,}07}{3} \cdot 10{,}02 - 1\right)} = \frac{4 \cdot 5{,}35\, b}{9 \cdot 5{,}92} = 0{,}40\, b$$

For m = 5, $b'm = \frac{4a\mu^2}{m\pi(3\mu^2 + 2\mu - 1)}$, when 1 is neglected in the numerator and denominator in comparison with

$$\mathfrak{Cof} \frac{m\pi b}{a} \text{ bzw } \frac{a}{b m \pi} \mathfrak{Sin} \frac{m\pi b}{a}$$

Then

$$b_5 = \frac{4 \cdot 100\,a}{351 \cdot 5\,\pi} = 0{,}228\frac{a}{\pi} = 0.228\,b$$

The accurate calculation gives:

$$b_5 = \frac{4 \cdot b\,0{,}59\,(74{,}2 - 1)}{25\left(\frac{2{,}07}{5} \cdot 74{,}2 - 1\right)}\,b = 0.232\,b$$

$$b_7 = \frac{4\,a}{m\,\pi}\,0{,}28 = 0{,}16\,\frac{a}{\pi} = 0{,}16\,b$$

$$b_9 = \frac{4\,a}{9\,\pi}\,0{,}28 = 0{,}124\,\frac{a}{\pi} = 0{,}124\,b$$

$$b_{11} = \frac{4\,a}{11\,\pi}\,0{,}28 = 0{,}102\,\frac{a}{\pi} = 0{,}102\,b$$

We see that in these girders the effective width differs only by a negligible amount from the effective width of a box girder and is 5–11% greater than in the latter.

From this we obtain the W_m values from the formula:

$$W_m = \frac{s\,b^2}{24} + 2\,\frac{s\,b}{2}\,b_m$$

where the factor 2 is due to the flange having two sides.
Then we have

$$W_1 = 0{,}944\,s\,b^2, \qquad W_3 = 0{,}447\,s\,b^2, \qquad W_5 = 0{,}274\,s\,b^2,$$
$$W_7 = 0{,}202\,s\,b^2, \qquad W_9 = 0{,}166\,s\,b^2, \qquad W_{11} = 0{,}144\,s\,b^2.$$

From this we get

$$X = \sum X_m = \sum \frac{M_m}{W_m}\,b_m\,s$$

where
$$M_m = \frac{1}{m^2} \cdot \frac{8}{\pi^2} \cdot \frac{P\,a}{2} = \frac{4}{\pi} \cdot \frac{P\,b}{m^2} \quad \text{und} \quad X = \frac{4}{\pi}\,P \sum \frac{b_m}{W_m}$$

whence
$$X = \frac{4}{\pi}\,1{,}122\,P = 1{,}43\,P$$

and
$$\sigma_{max} = \sum \frac{M_m}{W_m} = \frac{4}{\pi}\,1{,}625\,\frac{P}{s\,b} = 2{,}06\,\frac{P}{s\,b}$$

The effective width for x = 0 becomes:

$$b_0 = \frac{1{,}43}{2{,}06}\,b = 0{,}695\,b$$

Check: From b_0 we obtain for x = 0 a section modulus of W_0 = $(0.0492 + 0.695)sb^2 = 0.737\ sb^2$, and from this a maximum stress

$$\sigma_{max} = \frac{M_0}{W_0} = \frac{P\,a}{2 \cdot 0{,}737\,s\,b^2} = 2{,}14\,\frac{P}{b\,s}$$

The error thus amounts to only 4%.

The present calculation can be applied with sufficient accuracy to T-girders, since a study of limits shows that the formulas for T-girders with very wide flanges lead to the formulas given by Karman, who has considered this special case in detail.

Boundary Conditions at the Short Edges.

The boundary conditions at the short edges are satisfied if bulkheads of such strength are provided as to exclude any appreciable change in length in transverse direction. Transverse members are almost always present, so that this condition also is satisfied with sufficient accuracy. No decrease in stress occurs at the ends of the girder in this case, which may be considered as general.

If transverse stiffeners are applied to the girder at short intervals, the transverse contraction is greatly decreased. The resistance of the plate to extension in the longitudinal direction increases. It is not difficult to express this effect mathematically. However, I have refrained from doing so, since the solution given is on the safe side.

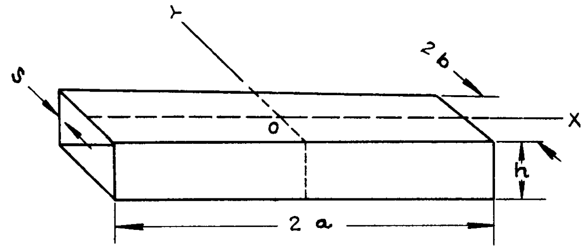
SUMMARY.

The calculation and stress distribution in the flange plates of box girders is demonstrated. The considerable influence of corner reinforcements on the effective width is shown by examples, and the accuracy of the calculation is improved by determining the error in residual moment.

[Translator's Note: The illustrative examples are summarized in tabular form on p 35. This table is inserted not only to summarize the results but to emphasize the fact that the values of effective width calculated in this paper are for special cases only, usually referring to the length-width ratio $a/b = \pi$ and to the point of maximum bending moment. If effective width is required for other a/b ratios or at other sections of the girder, the calculations must be made from the original formulas].

BOX GIRDERS

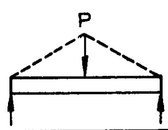
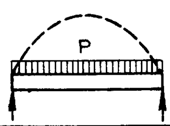
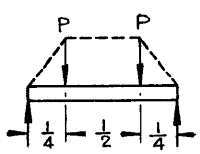
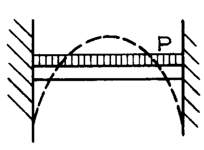
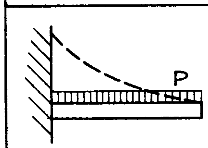
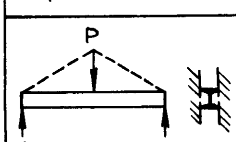
RESULTS OF CALCULATIONS OF EFFECTIVE WIDTH



X = ½ FORCE IN FLANGE

σ_r = VALUE OF σ_x AT EDGE ($y = b$)

β = EFFECTIVE WIDTH AT $x = 0$ (CENTER)

LOADING AND MOMENT CURVE	a/b	h	M	x	$\frac{X}{P}$	$\frac{\sigma_r}{P/bs}$	$\frac{\beta}{b}$
	π	$b/2$	$\frac{4Pb}{\pi}\sum\frac{1}{m^2}\cos\frac{m\pi x}{2a}$ (m = 1, 3, 5 ---)	$x = 0$	1.32	2.44	.553
	π	$b/2$	$\frac{8Pb}{\pi^2}\sum\pm\frac{1}{m^3}\cos\frac{m\pi x}{2a}$ (m = 1, 3, 5 ---)	$x = 0$	.720	.817	.882
	π	$b/2$	$\frac{8Pb}{\pi}\sum\left(\frac{1}{m^2}\sin\frac{m\pi}{4}\right.$ $\left.\times\cos\frac{m\pi x}{2a}\right)(-1)^{\frac{m+1}{2}}$ (m = 1, 3, 5 ---)	$x = 0$ $x = a/2$ $x = a/4$	1.46 1.34 .714	1.47 2.47 .716	1.00 .547 1.00
	2π	b	$\frac{PB}{\pi}\sum\pm\frac{1}{m^2}\cos\frac{m\pi x}{a}$ (m = 1, 2, 3, 4, 5 ---)	$x = 0$ $x = a$	.220 .385	.250 .793	.88 .486
	π	$b/2$	$\frac{8Pb}{\pi}\sum\frac{1}{m^2}\left(1\mp\frac{2}{m\pi}\right)\cos\frac{m\pi x}{2a}$ (m = 1, 3, 5 ---)	$x = 0$	1.26	3.36	.375
 I - GIRDER WITH RESTRAINED FLANGES	π	$b/2$	$\frac{4PB}{\pi}\sum\frac{1}{m^2}\cos\frac{m\pi x}{2a}$ (m = 1, 3, 5 ---)	$x = 0$	1.42	2.11	.675

FOOTNOTES

1) See bibliography.
2) See "Die neueren Methoden der Festigkeitslehre". p 365.
3) See Biles, INA, 1905, p 93 footnote.
4) See Hoffmann, WRH, 1925.
5) See VDI, 1926, Siemann, Ferndehnungsmessungen.
6) See bibliography.
7) The reinforcements at the corners are also to be counted with the web.
8) See STG, 1926, p 207.
9) See K. Huber, Die Schubspannungen im gebogenen T-Balken, Föppl-Festschrift, and a recent paper by Karman.

BIBLIOGRAPHY

Biles, The strength of ships with reference to experiments and calculations made upon H.M.S. WOLF, Transactions of the Institution of Naval Architects 1905, p 80.

Bortsch, Die mitwirkende Plattenbreite, Der Bauingenieur 1921, p 662.

Bruhn, Freebord and strength of ships, Transactions of the Institution of Naval Architects 1920.

Hoffmann, Annalysis of Sir John Biles' experiments upon H.M.S. WOLF in the light of Pietzker's theory, as above 1925, p 41.

Hoffmann, Die Querverbände unter dem Einfluss der Längsbiegung, W.R.H. 1925, p 641.

Huber, Schubspannungen in gebogenen T-Balken, Beiträge zur technischen Mechanik (Föppl-Festschrift).

v. Karman, Die mittragende Breite. As above.

Melan, Die Verteilung der Kraft in einem Streifen von endlicher Breite, Zeitschrift für angewandte Mathematik und Mechanik 1925, p 314.

Nadai, Elastische Platten, Berlin 1924, Julius Springer.

Pietzker, Festigkeit der Schiffe, Berlin 1911, Julius Springer.

Reissner, Über Knicksicherheit ebener Bleche, Zentralblatt der Bauverwaltung 1909, p 93.

Timoshenko, Sur la stabilité des Systemes elastiques, Annales des Ponts et Chaussees.

Timoshenko, Biegung von Stäben, Beitrage zur technischen Mechanik, p 74.