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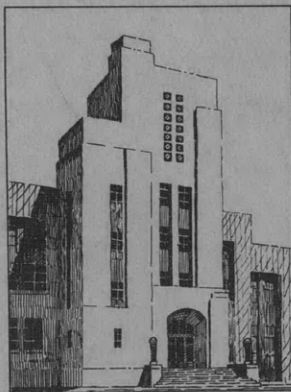
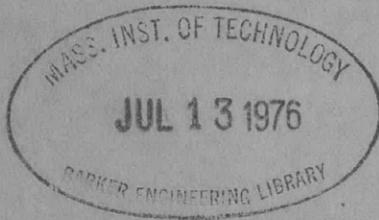
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THE DAVID W. TAYLOR MODEL BASIN

UNITED STATES NAVY

THEORY AND DESIGN OF COMPRESSION MEMBERS

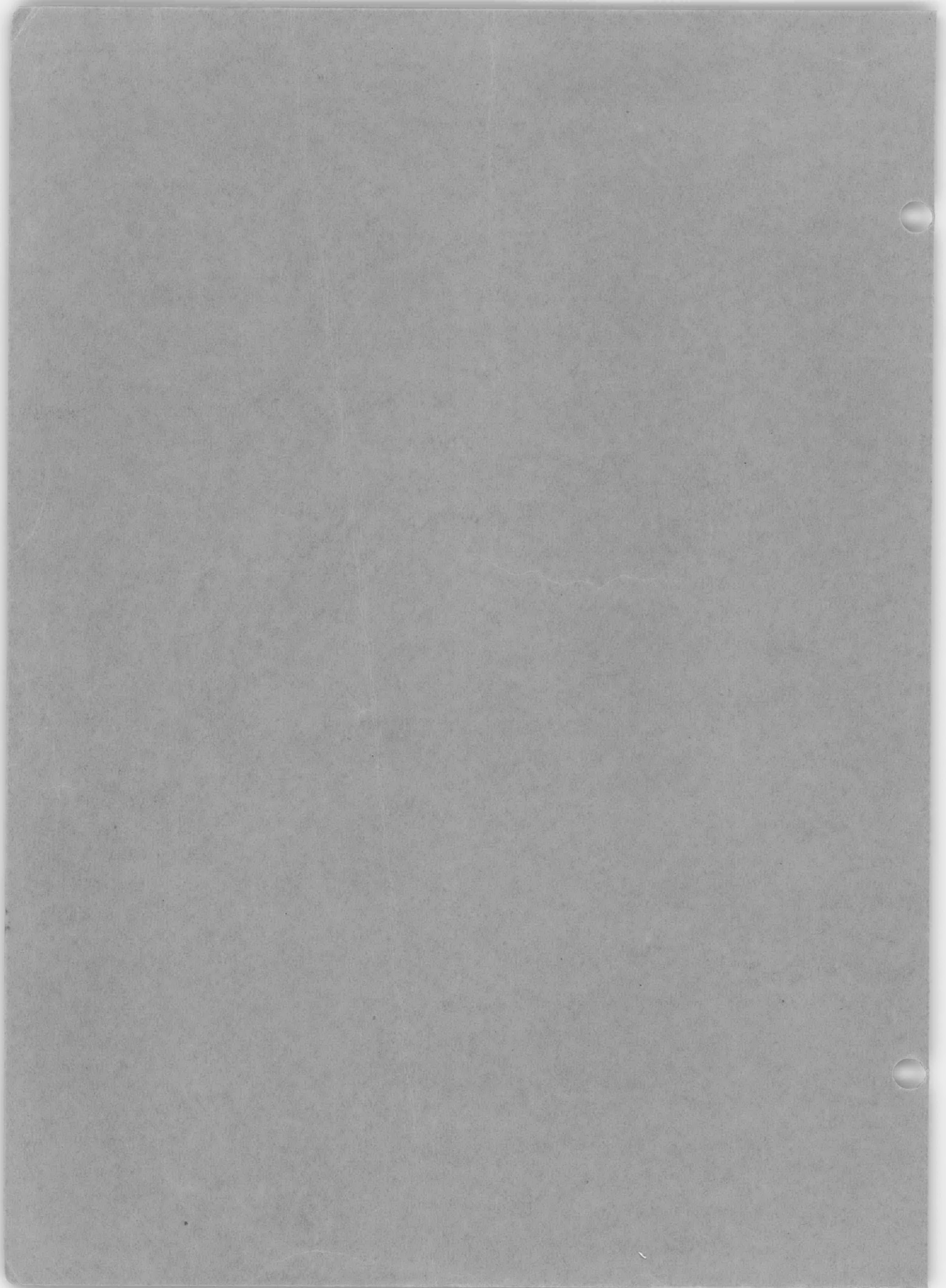
BY FRIEDRICH BLEICH, DR.-ING



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THEORY AND DESIGN OF COMPRESSION MEMBERS

Excerpts from
THEORIE UND BERECHNUNG DER EISERNEN BRÜCKEN

by

Friedrich Bleich, Dr.-Ing

Berlin
Julius Springer
1924

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The David W. Taylor Model Basin
Bureau of Ships
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January 1942

Translation 30

PREFACE

The chapters included in this translation have been excerpted from "Theorie und Berechnung der eisernen Brucken" (Theory and Design of Steel Bridges), by Dr.-Ing. Friedrich Bleich, Julius Springer, Berlin, 1924.

Some justification is needed for the project of making available to naval architects various chapters from a book on bridge building. This is found in the fact that Bleich has written no ordinary book on bridges, nor even an ordinary book on the theory of bridges. The part of the book that refers specifically to bridge design is contained in the last 9 sections, the first 15 sections are devoted to the underlying principles of strength of materials and theory of elasticity. Three of the 15 sections are herewith presented in English, chosen because they are especially pertinent to a problem that has long engaged the attention of the U.S. Experimental Model Basin and the Taylor Model Basin, the behavior of steel structure loaded in compression.

However, this does not exhaust the reasons for the choice of these pages for translation. The book has a flavor not until very recently to be found in the work of American authors on structural design. I quote from the Preface:

"If sometimes the benefit obtained from experiments seemed not to justify the effort expended, this was largely because the tests were not well planned. An experiment not based on careful theoretical preparation is almost sure to be useless. Experiment can lead to only three results: it can confirm theory, indicate how theory can be improved, or lead to its total rejection in favor of another theory. The questions to be answered by experiment must be simply and clearly stated if results are to be satisfactory. Success will await the cooperation of theory and experiment.

"It is necessary to say this because many practical engineers believe that if experiments are sufficiently extended the solutions of all problems can be found. The works of the great investigators, however, prove that a combination of experiment with theory is necessary for progress in knowledge."

The Taylor Model Basin is sympathetic to the author's view in this matter. This sample of his work, especially Section 13, is especially applicable to ship structure, which after all, is just an assembly of a great many plates subjected to edge-compressive load.

W.P. Roop, Commander, USN
Applied Mechanics Section
David W. Taylor Model Basin

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THEORY AND DESIGN OF COMPRESSION MEMBERS

§ 8. BUCKLING FORMULAS FOR STRAIGHT COLUMNS*

25. Introduction

No other field in the study of strength of materials has such a varied history as the theory of the buckling strength of compression members. Even today, in spite of the numerous writings of past decades, research in this specialized field has by no means reached its conclusion. Only the basic problems have been completely solved. Many difficult problems still await a rigorous theoretical solution and a majority of the answers which have been explained theoretically still remain to be verified by experiment so that they may be included among our assured scientific assets.

The causes of the difficulties encountered in research into the problems of buckling strength are based upon the peculiarities of the problem itself. While all other problems of the elastic theory are concerned with the stable equilibrium of internal and external forces, or can be attributed to conditions that indicate equilibrium, the buckling problem presents an entirely new aspect: the investigation of unstable equilibrium between the external forces and the elastic reactions of bodies.

The chief cause of the errors and unsuccessful experiments of investigators was a peculiar misconception of this fact and their desire to permit comparison of the results of buckling strength calculations with admissible loads. For decades this view retarded the solution of the problem of buckling strength, although Euler, who was the first to occupy himself with the buckling problem, pointed out the right path to a mathematical solution. Thus failure marked all attempts which sought to determine buckling strength by consideration of flexural stresses in the condition of stable equilibrium, or mixed up such considerations with investigations of the condition of unstable equilibrium.

In judging buckling strength it is not a question of avoiding a certain stress in a body by an adequate margin, but of preventing the occurrence of a peculiar condition of unstable equilibrium. This condition is characterized in practice by disproportionately large increases, indeterminate as to magnitude, to which deformations and bending stresses are subject at slight increases in load. In this more or less sudden breakdown of internal resistance lies the characteristic feature of the buckling phenomenon, regardless of whether at the moment of buckling the elastic limit is exceeded or not.

The failure of Euler's formula in the case of short columns was the primary cause of its complete abandonment, together with the reasoning by which it was derived. It had not been recognized clearly enough that the observed discrepancies had their origin in the fact that the elastic limit was exceeded before occurrence of buckling. This fact, naturally, was not taken into account by Euler's formula in its original

* Translation of pp. 99-106.

form. Tetmajer,* on the strength of his numerous tests, restored Euler's formula to its place, at least for the elastic buckling range.

Engesser, however, had already expressed belief in the unlimited validity of Euler's formula, though in a generalized form. Moreover, Engesser was the first in Germany to recognize and point out that the reasons for the deviations of the results of this formula from the data of tests with short, thick columns lay in the fact that the elastic limit was exceeded before buckling.** Engesser's labors, however, received scant notice until von Kármán recently proved Engesser's assumptions correct by very precise tests. The importance of the work of these two scientists lies primarily in the fact that they have supplied the fundamentals for theoretical treatment of difficult problems of buckling strength. It is self-evident that the numerical determination of certain empirical values is necessary for the practical application of the results of such theoretical investigations. Fortunately we have at our disposal in Tetmajer's empirical formulas - as the expression of the results of his tests - such empirical coefficients for mild steel, which in his day was the important bridge material. Starting, therefore, with the theoretical reasoning of Engesser-von Kármán, and using Tetmajer's formulas as compact expressions of certain empirical values, it is now possible with a prospect of success to discuss even difficult problems theoretically. In this sense we must consider every buckling problem, both in the elastic range as well as in the inelastic range of buckling, as a problem of unstable equilibrium between internal and external forces. Naturally, this does not preclude calculation, when necessary, of material stresses due to deformation, in addition to investigations of buckling strength. In originally bent columns, or when the load is not applied axially such stresses may even precede buckling. That kind of calculation has nothing to do with the actual buckling problem. They are parallel problems and any attempt to include the buckling question and determination of stress in a single formula must be regarded from the start as a failure.

In the following sections we shall first subject the buckling formulas used in practice to a brief consideration in order to gain an idea of the value of those most frequently applied. We shall defer until later the development of a coherent theory of unstable equilibrium of the elastic systems which have significance in this book. This theory will supply a basis for the explanation of the special buckling problems arising in the construction of steel bridges.

* Tetmajer, L. von: "Die Gesetze der Knickungs- und zusammengesetzten Druckfestigkeit der technisch wichtigsten Baustoffe" (The Laws of Buckling Strength and of Composite Compressive Strength of the Most Important Engineering Materials), Leipzig and Vienna, 1903.

** Engesser's first treatise on this subject dates from 1889. It should be noted that the Belgian investigator E. Lamarle, as early as 1845, had established the elastic limit as the limit of validity of Euler's formula in a voluminous paper "Memoire sur la Flexion du Bois" (Notes on the Flexion of Wood). See Todhunter and Pearson, "A History of Elasticity and Strength of Materials," Volume I, 1253 ff. Cambridge, 1886.

26. Euler's Buckling Formula

The formula

$$P_k = \pi^2 \frac{EJ}{l^2} \quad (1)$$

published by Euler* in 1744, applies to a simply supported column of constant cross section and length l . P_k represents the buckling load, E the modulus of elasticity, and J the smallest moment of inertia. Equation (1) assumes unlimited validity of the law of elasticity.** When the column is not simply supported, Equation (1) still applies, the only change being the introduction of a suitably altered value of the buckling length l . If l_0 represents the length of the column, then the buckling length l will be,

$l = 0.7 l_0$ for a column with one built-in end and the other simply supported

$l = 0.5 l_0$ for a column with both ends built in

$l = 2 l_0$ for a column with one end free and the other built in

Euler's buckling formula permits direct determination of the required sectional moment of inertia for a given safety factor ψ .†

We have††

$$J_{\text{erf}} = \frac{\psi}{\pi^2 E} Pl^2$$

If we assume $\pi^2 = 9.87$, $E = 2,150$ t/cm² (metric tons per square centimeter - Translator) for mild steel, it follows from the safety factor $\psi = 5$ formerly specified for bridges in Prussia that

$$J_{\text{erf}} = 2.33 Pl^2$$

P is to be given in metric tons, l in meters; J_{erf} is obtained in cm⁴.

Converting Equation (1) by dividing by the cross sectional area F , and taking the quotient $P_k/F = \sigma_k$ as buckling stress, we get with $i = \sqrt{J/F}$ (radius of gyration)

$$\sigma_k = \frac{\pi^2 E}{(l/i)^2} \quad (2)$$

l/i is called the slenderness ratio of the column. Equation (2) represents the buckling stress σ_k as a function of the slenderness ratio l/i . Geometrically this relation can be exhibited in the form of a hyperbola. In Figure 83 the l/i values are

* Euler, Leonhard: "De curvis elasticis" (The Elastic Curve). Lausanne and Geneva, 1744; German translation in Ostwald's: "Klassiker der exakten Wissenschaften" (Classicists of the Exact Sciences), No. 175. Leipzig, W. Engelmann.

** The derivation of this formula follows in Section 34, page 12.

† The simplicity of the computation probably was the principal reason why, in spite of the deficiencies of Formula (1), its use until recently was specified for stress analysis in Prussia.

†† Translator's Note: J_{erf} is the required moment of inertia.

plotted as abscissas, and the σ_k/ψ values with a safety factor ψ are plotted as the ordinates. When l/i decreases uniformly σ_k approaches infinity, but since the structural strength of the material is limited, it is necessary in addition in the case of short columns to investigate whether the allowable compressive stress σ_{zul} will be exceeded. l_g/i is called the limiting slenderness ratio and designates that point in which

$$\frac{\sigma_k}{\psi} = \sigma_{zul}$$

With smaller l/i , accordingly, the allowable compressive stress governs dimensions. The limiting slenderness ratio, as one may easily verify, is given by the relation

$$\frac{l_g}{i} = \pi \sqrt{\frac{E}{\psi \sigma_{zul}}}$$

For various values of σ_{zul} , the l_g/i ratio for $\psi = 5$ and the same E and π^2 values as above, are found to be as follows:

$\sigma_{zul} = 850$	950	950	1,000	1,050	1,100	1,150	1,200	1,250
$l_g/i = 71.2$	69.2	67.4	65.6	64.0	62.6	61.2	59.9	58.7

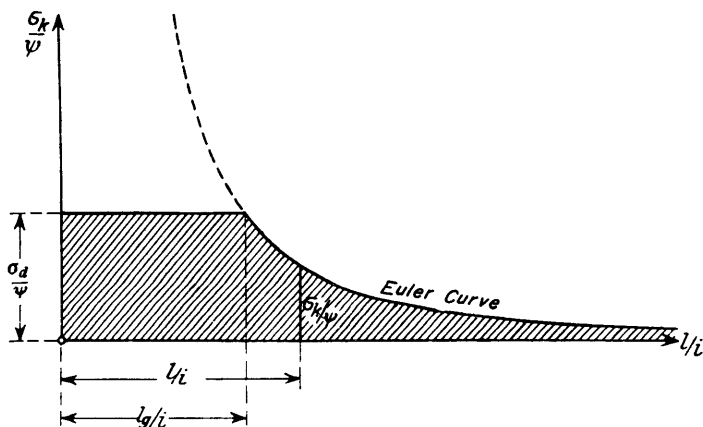


Figure 83

According to the former Prussian "Specifications for the Design of Bridges with Steel Superstructures," as already stated, a factor of safety of at least five was required by Euler's formula. The full length of the column was in all cases to be introduced as the buckling length when the ends of the column were held in space. In view of the more or

less rigid connection with other structural members, any decrease in the buckling length is inadvisable within a range of, say $l/i < 100$, because of the uncertainty of Formula (1), and moreover was never put into practice. The defect of Formula (1), that is to say, lies in the fact that it becomes invalid for small slenderness ratios (for mild steel about $l/i < 100$). In the range of $l/i = 30$ to 100, the most important in bridge building, this formula fails completely. The cause of this discrepancy lies in the circumstance that in shorter columns, even before the buckling limit is reached, i.e., while the columns are still straight or nearly straight and the compressive stress is therefore still uniformly distributed over the cross-section, this cross-sectional stress exceeds the elastic limit. The assumptions upon which Euler's formula are based are no longer just, since Hooke's law becomes invalid when the

elastic limit is exceeded. The modulus of elasticity E , which in Euler's formula is assumed to be independent of the cross-sectional stress after the elastic limit is exceeded, becomes a function of σ_k and decreases more and more as the buckling stress increases, and the material becomes more yielding, so to speak. The column buckles at an earlier stage than would be expected from Formula (1). Consequently we may differentiate two ranges:

1. The range of elastic buckling, in which Euler's formula applies, and
2. The range of plastic buckling, where this is no longer the case.

27. The Formulas of Tetmajer

In order to clarify the laws of buckling strength, Prof. von Tetmajer, at the Polytechnic Institute in Zürich, undertook extensive buckling tests which were the most extensive hitherto carried out.*

For mild steel having a strength of about 3.8 t/cm², Tetmajer found the relations

$$\text{for } l/i = 10 \text{ to } 105 \quad \sigma_k = 3.1 - 0.0114 \frac{l}{i} \text{ t/cm}^2 \quad (3)$$

$$\text{for } l/i > 105 \quad \sigma_k = 21220 \left(\frac{i}{l}\right)^2 \text{ t/cm}^2 \quad (4)$$

For the zone of elastic buckling, i.e., where $l/i > 105$, Tetmajer adhered to Euler's formula. In Equation (4), $\pi^2 E = 21,220 \text{ t/cm}^2$. For less slender bars Tetmajer proposed the straight-line Formula (3) which has the advantage of simplicity. The relation of the buckling stress σ_k to the slenderness ratio l/i is shown in Figure 84, in which the numerous experimental results of Tetmajer are plotted. The centers of gravity of test groups, noted by the solid points, show good agreement with Euler's hyperbola and Tetmajer's straight line. Equation (3) is an experimental formula, and Equation (4) is a relationship developed on theoretical principles, which has been borne out by experiment. According to Tetmajer, there is no noticeable decrease in strength due to reduction of the cross section when the deduction for rivet holes amounts to less than 12 per cent of the full cross section. Therefore it is customary to use the full area of the cross section in determining J and F .

In addition to mild steel columns, Tetmajer subjected medium and high carbon steel columns to tests, as well as wrought iron and wood columns. The results for this material (medium and high carbon steel) are worthy of note by bridge builders. For mild steel having a strength of 6 t/cm², Tetmajer found

$$\left. \begin{array}{l} \text{for } l/i = 10 \text{ to } 90 \quad \sigma_k = 3.35 - 0.0062 \frac{l}{i} \text{ t/cm}^2 \\ \text{for } l/i > 90 \quad \sigma_k = 22210 \left(\frac{i}{l}\right)^2 \text{ t/cm}^2 \end{array} \right\} \quad (5)$$

* See footnote on page 2.

For Siemens-Martin (open-hearth) steel with a strength of 6.8 t/cm², Müller-Breslau,* basing his computations on the results of von Karman's careful experiments, obtained

$$\left. \begin{array}{l} \text{for } l/i = 10 \text{ to } 91 \quad \sigma_k = 3.84 - 0.0136 \frac{l}{i} \text{ t/cm}^2 \\ \text{for } l/i > 91 \quad \sigma_k = 21420 \left(\frac{i}{l}\right)^2 \text{ t/cm}^2 \end{array} \right\} \quad (6)$$

(Translator's Note: These data appear to be those plotted in Figure 89, page 574 of Part II of Timoshenko's "Strength of Materials." It is clear there that von Kármán interpreted these data in quite another manner than Müller-Breslau.)

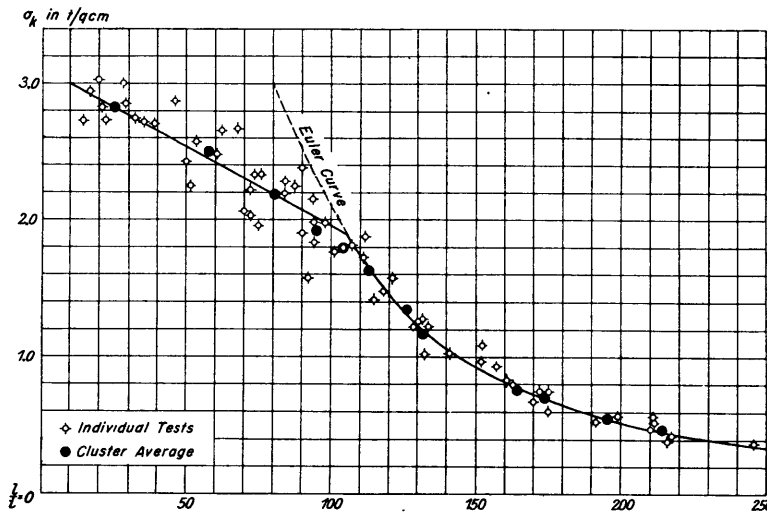


Figure 84

Calculation of the strength of columns under compression by Tetmajer's method may be carried out in two ways, depending upon whether it is desired to check a certain safety factor ψ as is often customary in Germany, or maximum stress, as is done in Austria and more recently in Germany.

a) In Table I of the Appendix are tabulated the buckling stresses σ_k for mild steel columns having slenderness ratios $l/i = 10$ to 210, computed according to Equation (3) and Equation (4). When ψ is the desired safety factor, the compressive stress over the cross section may not be greater than σ_k/ψ .

It will be best to select a safety factor ψ in agreement with the factor of safety of the members under tensile stress. It is recommended:**

* Dr. Eng. Müller-Breslau: "Über exzentrisch gedrückte Stäbe und über Knickfestigkeit," (On Eccentrically Loaded Columns and Buckling Strength). Eisenbau, 1911, p. 339.

** Taking into consideration the greater sources of danger in compression as well as the dynamic effects, the safety factors here given correspond to an actual safety factor from 1.8 to 2 times as great, such as we have also sought to attain in cases of tensile and bending stress.

1. When allowance is made for the dynamic augment of the live load by means of an addition for impact:
 - under ordinary loads $\psi = 3$
 - under unusual loads $\psi = 2.5$
2. When no provision is made for the dynamic effect of the live load:

For σ_{zul} = 800 - 900	900 - 1,100	>1,100
$\psi = 4$	3.5	3

EXAMPLE: For a load of 55 t (metric tons), a mild steel column with a buckling length of 5 meters, consisting of two channel irons with a distance of 20 mm between webs will have to be designed for fourfold safety. We take II NP 30. With $F = 117.6 \text{ cm}^2$ and $i_{\min} = 4.70 \text{ cm}$, we first compute

$$\frac{l}{i} = \frac{500}{4.70} = 106.4$$

and from this we find in the Appendix (Table I) that $\sigma_k = 1.876 \text{ t/cm}^2$. Consequently the buckling force will be

$$P_k = 1.876 \cdot 117.6 = 220.5 \text{ t}$$

and the factor of safety

$$\psi = \frac{220.5}{55} = 4$$

b) If we let σ_d represent the compressive strength of the structural material and σ_{zul} the allowable stress in pure tension or compression, then $\sigma_d : \sigma_{zul} = \psi$ is the factor of safety against rupture of the members of the structure designed for tension or pure compression, assuming that $\sigma_d = \sigma_z$ is sufficiently accurate for the structural material. If we now wish to design the members under buckling stress for the same factor of safety, then it is essential that

$$\frac{P}{F} = \frac{\sigma_k}{\psi} = \frac{\sigma_k}{\sigma_d} \sigma_{zul}$$

Designating $\sigma_d/\sigma_k = \omega$ as the buckling factor, it follows that

$$\omega \frac{P}{F} \leq \sigma_{zul}$$

The buckling coefficients ω for mild steel are given in Table II of the Appendix. In computing ω , it was assumed that $\sigma_d = 3.8 \text{ t/cm}^2$, in accordance with the average tensile strength of the test material as determined by Tetmajer.

EXAMPLE: Let a truss member consisting of four angles be under a load $P = 80 \text{ t}$. Buckling length $l = 4.00 \text{ meters}$, $\sigma_{zul} = 1 \text{ t/cm}^2$. Choose $\angle 120 \times 120 \times 15$, with a distance of 1.4 centimeters between legs. $F = 135.6 \text{ cm}^2$, $i_{\min} = 5.60 \text{ centimeters}$; therefore

$$\frac{l}{i} = \frac{400}{5.60} = 71.5$$

for which value we get $\omega = 1.664$ from Table II. Herewith we ultimately find

$$\sigma = 1.663 \frac{80}{135.6} = 0.981 \text{ t/cm}^2$$

where, as prescribed, $\sigma < 1 \text{ t/cm}^2$ was obtained. Naturally it must not be overlooked that the computed value $\omega \frac{P}{F}$ is not an actual stress, but merely has the significance of a comparative value, and consequently takes the place of the safety factor ψ computed by the first process. This factor of safety is here governed by the choice of σ_{zul} .

Buckling Length. The comparatively good agreement between Tetmajer's formulas and actual buckling stresses (with well-centered columns there is even a slight margin of safety) renders it permissible from case to case to make allowance for the type of fastening of the compression member (degree of fixation) by suitable choice of the buckling length. If l_0 signifies the distance between the fixed end points (panel points) of the compression member, the buckling length* l_k to be used in the computation may be assumed as follows:**

1. Truss Chords

(a) When the panel points are fixed in space:

α) for buckling in the plane of the truss $l_k = l_0$

β) for buckling normal to the plane of the truss $l_k = l_0$

(b) When the panel points are elastically supported perpendicularly to the plane of the truss:

α) for buckling in the plane of the truss $l_k = l_0$

β) for buckling perpendicular to the plane of the truss the formulas in Division 50 apply.

(c) When the compressive force S_1 prevails in one half of the member and S_2 in the other, S_1 being greater than S_2 † $l_k = \left(0.75 + 0.25 \frac{S_1}{S_2}\right) l_0$

2. Truss Web Members

(a) Members not crossed by other members:

α) for buckling in the plane of the truss $l_k = 0.8 l_0$

β) for buckling normal to the plane of the truss,

when the member lies within the plane of a

portal frame and is connected rigidly with at

least one rigid horizontal member of the latter $l_k = 0.75 l_0$

when both ends of the member are rigidly joined $l_k = 0.6 l_0$

* Translator's Note: The terms "reduced length" or "effective length" are more common in American technical usage. This is the distance between inflection points in the member after buckling.

** For the theoretical principles on which the following rules are based and for examples, see Section 11, of Bleich pp. 161 ff.

† If the two halves of the column have different moments of inertia, use Formula 15, Division 45, p. 185.

$$[\gamma = (0.63 + 0.13k_1^2) + (0.11 + 0.13k_1^2)k_2^2]$$

to stiff cross frames

when there is no rigid connection with a stiff cross frame

$$l_k = l_0$$

γ) in verticals of K-web systems where S_1 is the compressive force in one half of the column and S_2 is the tensile or minor compressive force in the other half

$$l_k = \left(0.75 \mp 0.25 \frac{S_2}{S_1} \right) l_0$$

The minus (-) sign is used when S_2 is tension, the plus (+) sign when it is compression

(b) Members crossed at the middle by other members:

α) for buckling in the plane of the truss

$$l_k = 0.5 l_0$$

β) for buckling perpendicular to the plane of the truss

$$l_k = l_0 \sqrt{1 - \frac{3 S_z l_d}{4 S_d l_z}}$$

where S_z is the load and l_z the length of the tension member, and S_d is the load and l_d the length of the compression member.

If the coefficient of l_0 is less than 0.5, a value of 0.5 should be substituted.

In Austria, use of Tetmajer's formulas in designing compression members is required by all authorities concerned. In this case

$$\omega \frac{P}{F} \leq \sigma_{zul}$$

where σ_{zul} signifies the admissible compressive stress in the structural member. The buckling length to be taken for computing the buckling coefficients is given specifically for the cases most frequently encountered.*

The Baden State Railroads likewise prescribe the use of Tetmajer's formulas, specifying that the factor of safety ψ against buckling shall be:

	Main Lines	Secondary Lines
for $l/i \leq 100$	$n = 3 + 0.01 l/i$	$n = 2.3 + 0.007 l/i$
for $l/i > 100$	$n = 4$	$n = 3$

The buckling length to be used in calculation is prescribed exactly for the most important cases.**

The Swiss Railroads also base their regulations regarding the design of compression members on Tetmajer's formulas. The maximum permissible loads on compression members of mild steel are:

In railway bridges:

$$\begin{aligned} \text{for } l/i \text{ from } 10 \text{ to } 110 & \quad s_k = 1.000 - 0.005 \frac{l}{i} \text{ t/cm}^2 \\ \text{for } l/i > 110 & \quad s_k = 5500 \left(\frac{i}{l} \right)^2 \text{ t/cm}^2 \end{aligned}$$

* "Vorschriften betreffend die Berechnung gedrückter Konstruktionsteile aus Eisen oder Holz mit Rücksicht auf Knickung" (Regulations Regarding Computation of Compression Members of Steel or Wood with respect to Buckling), Issue of Feb. 16, 1907.

** "Vorschriften der Badischen Staatsbahnen" (Regulations of the State Railways of Baden for February 1903).

In highway bridges:

$$\begin{aligned} \text{for } l/i \text{ from } 10 \text{ to } 110 & \quad s_k = 1.1 \left(1.000 - 0.005 \frac{l}{i} \right) \text{ t/cm}^2 \\ \text{for } l/i > 110 & \quad s_k = 6000 \left(\frac{i}{l} \right)^2 \end{aligned}$$

The Swedish standards recommend the Euler-Tetmajer design with a factor of safety of 4 for ordinary loads, and a factor of 3.2 for unusual loads.

In addition to several other more or less unjustified criticisms it was also said of Tetmajer's straight line formula that for a slenderness ratio $l/i = 0$ it would yield a strength different from the block compressive strength of the material. This fact is to be attributed to the deficiencies of Tetmajer's test set-up, in which the effects of small initial eccentricities were not adequately eliminated in the case of short columns. In the neighborhood of the yield point, even small initial eccentricities in themselves are capable of so impairing the strength of the member, that, converted to a unit of area, it can not increase greatly above the yield point in compression. The more accurate tests of Von Kármán, which will be discussed later (Division 30), prove, moreover, that with constantly decreasing l/i the buckling stress approaches the block strength; but since in practical cases small axial deflections must always be taken into account, Tetmajer's straight line formula may after all express correctly the actual behavior of this type of member.

The approval gained by Tetmajer's formulas naturally does not preclude the possibility that more exact numerical values will be required, which, obtained by means of more recent tests, appear to be free of those sources of error which obscure the regularities of Tetmajer's data. It will not be discussed here whether it is better directly to derive these data from buckling tests with the most accurately centered columns of simplest cross section - essentially as in Tetmajer's method - or to derive them from the compressive stress-strain curve by von Kármán's method, which will be shown later. It is likewise immaterial whether or not the numerical values of the column strength σ_k thus obtained can be expressed in an analytical formula, since in practical use of the test data, it is only the tabulations such as Tables I and II that are required. Such fresh investigations should include mild steel, carbon steel, and nickel steel with the most frequently encountered physical properties, but since no such tests have yet been carried out, we are obliged to make use of Tetmajer's results, which, as we shall see, are slightly conservative, and to base our investigations upon these empirical values.

§ 9. THEORY OF THE BUCKLING STRENGTH OF STRAIGHT COLUMNS*

33. Introduction

In the preceding section we discussed the buckling formulas used at present

* Translation of pp. 122-126.

in practical calculations, their significance and the methods of applying them. Even though these formulas are frequently adequate in specifications to determine the buckling strength of compression members of steel bridges, cases nevertheless frequently occur in which the cited equations, patterned only to the simplest possible conditions of support of a straight simple column, can no longer be considered sufficient. We shall mention only the problem of buckling strength in pin-ended columns with elastic cross-bracing, the calculation of the buckling strength of columns of built-up section, and investigation of the local buckling strength of thin walls of members of large bridges. Since no adequate test data are as yet available, the usual rules for applying the formulas treated in the foregoing also require a critical examination, which can be undertaken only on the basis of careful theoretical discussion with as few assumptions as possible.

The experimental data hitherto published form a narrow but adequate basis for our present needs, on which to build up a reliable theory. If we succeed in isolated instances in finding adequate agreement between calculated results and available test data, the developed formulas and rules may then be applied with some confidence to design the members of steel bridges. They will then deserve greater trust than the usual "rules of thumb" based on opinions and estimates, or on formulas derived from highly questionable assumptions. In order to be able to penetrate sufficiently into the nature of the buckling problem, i. e., into the theory of unstable equilibrium of elastic systems, we begin, first, with a detailed discussion of the simplest case: the case of unstable equilibrium of a straight column.

34. Elastic Buckling of Straight Columns

Consider a completely elastic slender bar of constant cross section, originally perfectly straight, which is acted upon by a longitudinal force S applied along the axis of the member, as shown in Figure 91. Let there be also a transverse loading of any kind whatsoever, which, acting alone, would produce the moment \mathfrak{M}_x at the point x . The total bending moment at x will then be

$$M_x = Sy + \mathfrak{M}_x$$

Let the plane of bending fall in an axis of symmetry of the section perpendicular to the axis about which the moment is smallest. Assuming small deflections y , the differential equation of the elastic line is

$$EJ \frac{d^2 y}{dx^2} + Sy + \mathfrak{M}_x = 0 \quad (1)$$

We now assume that there is a particular case in which the column is simply supported at the ends, and that \mathfrak{M}_x is caused by a concentrated load acting on the middle of the column. Then the solution

$$y = \frac{\mathfrak{M}_x}{S} \left[\frac{\sin \alpha x}{\alpha x \cdot \cos \frac{\alpha l}{2}} - 1 \right] \quad (2)$$

will satisfy the differential Equation (1), where $\alpha = \sqrt{\frac{S}{EJ}}$, so long as the disturbing

moment \mathfrak{M}_x is other than zero. For small values of S , the factor $\cos \alpha l/2$ in Equation (2) is only slightly different from unity. But as S (and consequently α) increases, the value of the fraction in Equation (2) will likewise increase, growing in magnitude very rapidly when $\alpha l/2$ approaches $\pi/2$. Thus y will also increase continually, until, assuming for the moment that Equation (2) has unlimited validity* when $\alpha l = \pi$, it increases beyond limit. The column has already buckled. From $\alpha l = l\sqrt{\frac{S}{EJ}}$ follows the Euler formula for S_k as the upper limit of the buckling load

$$S_k = \frac{\pi^2 EJ}{l^2} \quad (3)$$

Euler's buckling load S_k is independent of the moment \mathfrak{M}_x . We therefore assert:

Bending moments acting upon the column in addition to the longitudinal compressive forces that cause buckling are without effect on Euler's buckling force defined by Equation (3).

If we let \mathfrak{M}_x decrease more and more, becoming very small but still finite, y will still become infinitely large when $\alpha l = \pi$. Only when \mathfrak{M}_x vanishes does y assume the indeterminate form $0:\infty$. As we can easily ascertain by substituting, when $\mathfrak{M}_x = 0$ there will be the following solutions of the differential Equation (1):

for any given values of S , $y = 0$

for the special values $S = n^2 \pi^2 \frac{EJ}{l^2}$, $y = C \sin n\pi \frac{x}{l}$, (4)

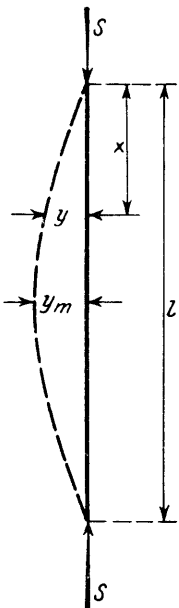


Figure 91

where n is any whole number and C an arbitrary constant. Of these solutions, only the smallest with $n = 1$, therefore, $S = S_k$, and the solution $y = 0$ have any present significance for $S < S_k$, since the forms of equilibrium of the column corresponding to other values of n can arise only under certain special circumstances, which, however, shall not be considered as prevailing. The column, in the absence of a disturbing moment, continues straight at any value of $S < S_k$, and when $S = S_k$, bends in a sine curve of indeterminate amplitude C . Thus to the critical load there corresponds an infinite number of equally probable states of equilibrium.

We can therefore state that the column will behave as follows:

If the column is under an axial load S , which is smaller than S_k , it will remain straight. If a disturbing moment \mathfrak{M}_x is added, it will deflect, and the magnitude of this deflection is given by Equation (2). If the moment \mathfrak{M}_x ceases to act, the column will resume its straight condition. When the axial force S reaches the value S_k , the column may assume any possible condition of equilibrium (sine curve with any amplitude), as long as $\mathfrak{M}_x = 0$. Still, any moment \mathfrak{M}_x , however small, disturbs this condition of equilibrium, resulting in an irresistible deflection of the column since, for finite values

* As the deflections increase the approximate Equation (1) from which (2) is derived ceases to be valid.

of \mathfrak{M}_x , Equation (2) will yield the value $y = \infty$.

Summing up, we may therefore state: As long as $S < S_k$, the column will deflect under the influence of a disturbing moment, but will return to its original condition when the disturbance has ceased. External and internal forces are in stable equilibrium. If S becomes equal to S_k in the absence of a transverse load, an infinite number of states of equilibrium is possible, but the slightest disturbing moment will suffice to cause an irresistible deflection of the column, and it will not return to its original condition. The case is one of unstable equilibrium between the internal and external forces. The critical load S_k thus characterizes a point of instability in the behavior of the compressed bar.*

These remarkable findings, however, merely are the results of a mathematical fiction, since the differential Equation (1), from which the foregoing conclusions were drawn, describes the behavior of a deflected column only approximately. Nature knows no discontinuities; the completely straight column would actually behave somewhat differently. If we start the investigation with the true equation for the elastic line

$$\frac{EJ}{\rho} = -M_x$$

where ρ represents the radius of curvature of the deflected axis of the column, we will find, as Grashof** has already proved, that the value y for the deflection at the middle of the column with vanishing transverse load is

$$y_m = \frac{2l}{\pi} \sqrt{\frac{S_k}{S}} \sqrt{\sqrt{\frac{S}{S_k}} - 1} \quad (5)$$

S and S_k having the same meaning as in the foregoing.

As long as $S < S_k$, y_m will be imaginary; there will be no deflection; and the column remains straight. Even when $S = S_k$, and $y_m = 0$, the column undergoes no deflection. Only when $S > S_k$ does there occur a finite deflection y_m determinate as to magnitude. However, as may easily be computed, a very slight increase of S beyond S_k will be sufficient to cause an appreciable deflection. Although S_k may be only slightly exceeded, deflections occur which are dangerous to the stability of the column.†

* Considering the fact that an infinite number of states of equilibrium may correspond to a loading by S_k with $\mathfrak{M}_x = 0$, which, however, can not withstand any disturbance, we actually have here a remarkable combination of neutral and unstable equilibrium.

** Grashof, Dr. F.: "Theorie der Elastizität und Festigkeit" (Theory of Elasticity and Strength), II. Ed., Berlin 1878, p. 168. Also Schneider, A.: "Zur Theorie der Knickfestigkeit" (On the Theory of Buckling Strength), Zeitschrift des österreichischen Ingenieur und Architekten Vereines, 1901, p. 633.

† For example, if we let S be to S_k as 1.001 is to 1, it follows that $y_m = 0.0142l$. For a column, say of $l/i = 110$ and $e/i = 2$ (e is the distance of the extreme fiber from the axis of the column), the maximum fiber stress will be $\sigma = 4.13 \sigma_k$. Accordingly an increase over the buckling load S_k of 1/1000 of its value will be sufficient to raise the extreme fiber stress to 4.13 times the buckling stress $\sigma_k = S_k/F$, i.e., in the present instance, 7.24 t/cm², a load which a mild steel column, say, is unable to support. Therefore it will fail somewhat earlier.

Likewise we find that when $S = S_k$ a sufficiently small disturbing moment can still be carried, the column, when $S = S_k$, being merely very sensitive to such disturbances, reacting to them with finite but very large deflections.

Practically speaking, therefore, the result of a more exact study is the same as that obtained in the foregoing. After Euler's buckling load S has been reached, the behavior of the column becomes uncertain, and extremely small excesses over this load or very slight disturbances may cause failure. The more accurate calculation shows only that between S_k and the actual failing load there is a zone of transition, however narrow, through which any unnatural discontinuity seems to be avoided in the behavior of the column near the buckling limit. In the formulas initially derived, this transitional zone, due to the neglect of certain terms of second order in the basic equation, shrinks to a point of discontinuity. We are concerned only with determining the critical load S_k at which the behavior of the column changes more or less suddenly. More precisely, we wish to establish the conditions for the occurrence of the buckling process, but not with the development of the buckling action itself. In order to simplify the analysis, we employ the fiction of a column which fails abruptly. Thus, we base our investigations on the differential Equation (1), which, as comparison shows, suffices to represent correctly the conditions for inception of buckling. The further agreement of the conclusions developed from this fiction with the results of tests confirms the admissibility of the simplified method of calculation.

We are therefore justified in regarding the buckling condition as a state of unstable equilibrium, which is characterized by an infinite number of possible equilibrium configurations of the column axis.

For the sake of simplicity, we have hitherto considered only a column simply supported at each end. The column with elastically restrained, or with fixed ends, exhibits essentially the same behavior, so that it is permissible to extend the foregoing conclusion to the general case. If M_1 and M_2 are the end-fixing moments in the condition of unstable equilibrium due to the deflection y , the differential equation of the elastic line of the unstable condition of equilibrium in the most general instance will read

$$EJ \frac{d^2 y}{dx^2} + Sy + M_1 + \frac{M_2 - M_1}{l} x = 0 \quad (6)$$

M_1 and M_2 will vanish when y vanishes.

The value of the buckling load S_k determined from the Euler formula (3) naturally can never be reached in actual conditions, since no absolutely straight columns exist and since no loads are ever applied exactly on the centroidal axis in the manner assumed by the theory. Equation (3) is independent of the strength properties of the material from which the column is made, and thus its unrestricted application demands a material of unlimited strength.

The Euler buckling load represents an upper limit for the critical load, which is more nearly approached when the disturbing moment \mathfrak{M}_x is smaller and the

strength of the material is greater.

If we represent the deflection expressed by Formula (2) as a function of the average cross-sectional stress σ_m for various values of \mathfrak{M}_x , we get the system of hyperbolic curves plotted in Figure 92, whose ascending branches have the straight line $x = \sigma_k$ as asymptote. In the figure, letting μ represent a small deflecting moment, the deflection curves for $\mathfrak{M} = 0, \mu, 2\mu, 3\mu, 4\mu$ have been plotted. If we then assume that corresponding to a given deflection y_m for a given column section there is a given compressive stress at the extreme fiber which may not be exceeded (for example, the strength at fracture), then Figure 92 shows plainly how the value of the failure load constantly

decreases as the value of \mathfrak{M}_x increases. The differences between the actual stress at failure σ'_k , and the buckling stress σ_k are comparatively slight, at least in the case of slender columns with the initial moments \mathfrak{M}_x occurring in practice, and thus the theoretical buckling load S_k may be considered

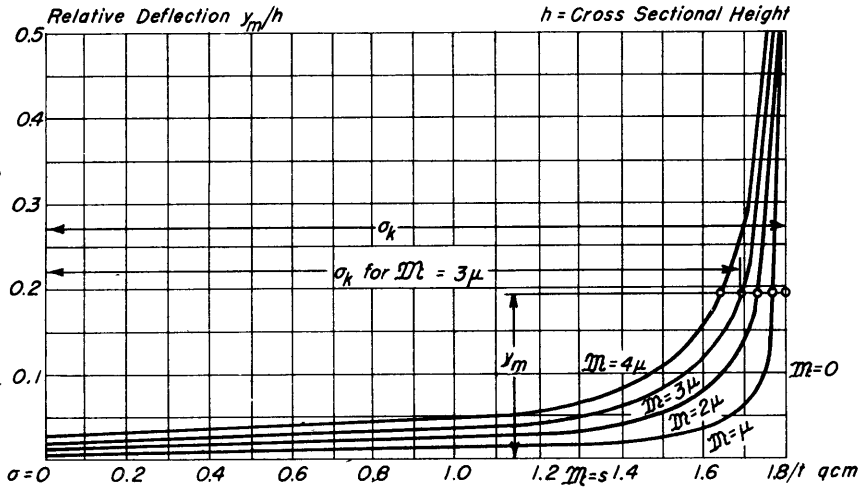


Figure 92

as the ultimate load. The tests of Bauschinger, Tetmajer and von Kármán have established that in the range of elastic buckling, i.e., with sufficiently slender columns, Euler's formula shows very good agreement with test data. This agreement is better when the theoretical assumption, i.e., the condition $\mathfrak{M}_x = 0$, is satisfied more precisely.

35. Inelastic Buckling of Straight Columns

The considerations expressed in the preceding section were based on the assumption that the buckling stress, i.e., the compressive stress uniformly distributed over the cross section, would still be below the elastic limit at the moment when the equilibrium becomes unstable. This, however, is true only of slender columns. In shorter columns the elastic limit is exceeded before inception of buckling. Permanent deformation occurs and the modulus of elasticity E , hitherto constant, becomes a function of the buckling stress σ_k .

We shall now attempt to follow, as in the analysis of the perfectly elastic column beyond the elastic limit, the process of bending as equilibrium becomes unstable. To do this we make the following assumptions:

1. The deformations are small in comparison to the dimensions of the column;

2. Plane cross sections remain plane after bending;
3. The same relationship between stress and strain exists for bending that holds for pure tensile or compressive stress;
4. The plane of bending is a plane of symmetry of the cross section.

With respect to these assumptions the following is to be noted: Since our study deals only with determination of the conditions at the appearance of unstable equilibrium, we may, without loss of generality, assume the deformations to be small with respect to the dimensions of the cross section of the column. Furthermore, Prof. Eugen Meyer of Charlottenburg has proved by tests* that the deflection of mild steel columns computed from the stress-strain chart on the assumption that cross sections will remain plane is in good agreement with observed deflections up to failure, even beyond the elastic limit. Therefore, Assumptions 2 and 3 also appear to be sufficiently justifiable. Assumption 4 was made in order to simplify calculation. The results obtained may then be rationally extended to more general cases, which, however, are rare in bridge practice.

Let us now imagine that a sufficiently short column is compressed by an axially applied load S , so that $\sigma_m = S/F$ will exceed the elastic limit (F = area of cross section - Translator). Then let the load S be further increased until the

column reaches the condition of unstable equilibrium similar to the condition of unstable equilibrium of the perfectly elastic

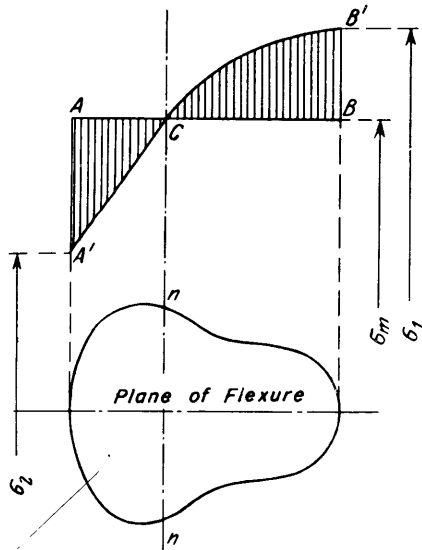


Figure 93

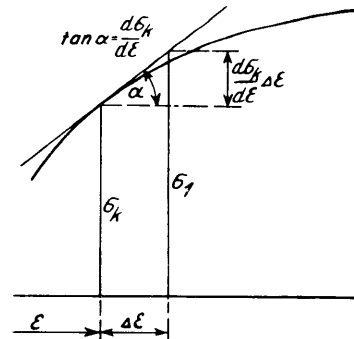


Figure 94

column. In this condition let the column have a small deflection, which, however, is to be indeterminate as to magnitude. In every cross section there will then be a line n - n perpendicular to the plane of bending, in which the cross sectional stress σ_m

* Meyer, Eugen: "Die Berechnung der Durchbiegung von Stäben, deren Material dem Hookeschen Gesetze nicht folgt" (Calculation of the Deflection of Columns whose Material does not obey Hooke's Law), Zeitschrift des Vereines deutscher Ingenieure, 1908, p. 167.

developed previous to deflection has remained unchanged, (Figure 93). We designate this line as the σ_m line. On one side of the straight line $n-n$ the longitudinal compressive stresses will be increased by bending, and the stress diagram will be bounded by a curved line CB' which may be derived from the proper portion of the stress-strain diagram for the structural material. On the other side there will occur a reduction of longitudinal stresses (unloading), due to the superimposed bending stresses, and since this unloading relieves only the elastic portion of the strain, the law of proportionality of stress and strain with the constant modulus E remains in effect for this portion. The stress diagram on the compression side is bounded by the straight line CA' .

For a study of the buckling process, as we have explained in the foregoing, it will suffice to assume very slight deflection after reaching a mean stress $\sigma_m = \sigma_k$ (buckling stress). The extreme fiber stresses σ_1 and σ_2 on the compression or tension side, respectively, will then differ only slightly from the mean stress σ_k . Figure 94 represents a portion of the stress-strain diagram. Since the fiber strain $\Delta\epsilon$ due to bending is very small, the extreme fiber stress on the compression side (see Figure 94), may be expressed with sufficient accuracy by the linear expression*

$$\sigma_1 = \sigma_k + \frac{d\sigma_k}{d\epsilon} \Delta\epsilon$$

where $\frac{d\sigma_k}{d\epsilon}$ is the variable modulus of elasticity in the plastic range. von Kármán designates it as the modulus of total strain.** If we take $\frac{d\sigma_k}{d\epsilon} = E'$, then the simple linear law

$$\sigma = \sigma_k + E' \Delta\epsilon$$

will apply for the stress in any given fiber of the compression side, while for stresses on the tension side

$$\sigma = \sigma_k + E \Delta\epsilon$$

as explained previously.

Figure 95 shows the linear stress distribution in the cross-section just discussed. With this in mind, it is not difficult now to develop the fundamental relations determinative of bending in the unstable condition.

Using the symbols in Figure 95, equilibrium between the elastic forces and the external loads requires that

$$\int_0^{h_1} s_1 df - \int_0^{h_2} s_2 df = 0 \quad (7)$$

* If we imagine σ_1 developed for powers of $\Delta\epsilon$, the higher powers of this term may be neglected because of the smallness of $\Delta\epsilon$.

** von Kármán, Theodor: "Untersuchungen über Knickfestigkeit" (Investigations on Buckling Strength), "Mitteilungen über Forschungsarbeiten aus dem Gebiete des Ingenieurwesens" (Reports on Research in the Field of Engineering Practice), Published by Verein deutscher Ingenieure, Berlin, 1910. The present presentation agrees materially with von Kármán's.

Introducing

$$\xi_1 = \eta_1 - e \quad \text{and} \quad \xi_2 = \eta_2 + e$$

where e designates the distance of the axis through the center of gravity from the σ_m -line, we get

$$\frac{E'}{\rho} \int_0^{h_1} \eta_1^2 df + \frac{E}{\rho} \int_0^{h_2} \eta_2^2 df - \frac{e}{\rho} \left[E' \int_0^{h_1} \eta_1 df - E \int_0^{h_2} \eta_2 df \right] = M$$

In view of the condition of equilibrium (9), the expression in brackets is zero, and we obtain

$$\frac{1}{\rho} (E' J_1 + E J_2) = M \quad (10)$$

where J_1 and J_2 represent the moments of inertia of the cross-sectional areas separated by the σ_m -line about this line as an axis.

Since, for small deflections,

$$\frac{1}{\rho} = -\frac{d^2 y}{dx^2}$$

is approximately true, it follows that

$$(E' J_1 + E J_2) \frac{d^2 y}{dx^2} + M = 0$$

or

$$T J \frac{d^2 y}{dx^2} + M = 0 \quad (11)$$

where

$$T = E' \frac{J_1}{J} + E \frac{J_2}{J} \quad (12)$$

This is the differential equation for the elastic line in the condition of unstable equilibrium. Following Engesser (see 111)*, we designate T as the buckling modulus.** It has a value determined by the proportions of the cross-section as well as by the elastic properties of the structural material. If the form of the cross-section is known, it can be determined by means of Equation (12) from the stress-strain diagram as a function of the buckling stress σ_k .

The differential Equation (11) has the same general form as the differential equation for the elastic line in the condition of unstable equilibrium for the case of the perfectly elastic column, since T as well as E is a constant† independent of the abscissa x . Thus we establish the important fact that the differential equation for the elastic line in the condition of unstable equilibrium in the form of Equation (11)

* Not translated.

** von Karman calls this value the "Resultant Modulus." (Translator's Note: The term "Effective Modulus" is at present almost universally used by American and English authors.)

Equation (12) reduces to the well-known form $T = \frac{4E'E}{(\sqrt{E} + \sqrt{E'})^2}$ for rectangular cross section since $E'h_1^2 = E'h_2^2$, $h_1 + h_2 = h$. See Timoshenko.)

† T depends only upon $\sigma_m = \sigma_k$, which, however, has the same value at every point of the prismatic rod. [Translator's Note: It is extremely important to realize that T is obtained from the average stress ($\sigma_m = P/F = \sigma_k$) of the column at the inception of buckling.]

is valid in the elastic as well as in the inelastic range. In the inelastic range T is variable and independent of σ_k , while in the elastic range T becomes equal to E . Thus we have obtained a basis valid under all conditions for study of the condition of unstable equilibrium and for theoretical explanation of buckling conditions.

If we consider again the axially-loaded pin-ended column, the solution

$$y = C \sin n\pi \frac{x}{l}$$

will again apply as in the foregoing section, for the deflection y in the absence of disturbing moments. This solution, however, can only apply when

$$S = n^2 \pi^2 \frac{TJ}{l^2}$$

with $n = 1$ we get a general equation

$$S_k = \pi^2 \frac{TJ}{l^2} \quad (13)$$

for the buckling load.

From Equation (13) it follows that

$$T = S_k \frac{l^2}{\pi^2 J} = \left(\frac{l}{i}\right)^2 \frac{\sigma_k}{\pi^2}$$

Thus the regular relation of T and σ_k for various forms of cross-sections can also be derived from buckling tests. (Translator's Note: This, in fact, is the simplest and most direct method of determining T , and should be used whenever reliable column data are available.)

For convenience we introduce the ratio, or buckling factor*, $\tau = T/E$. Then the fundamental equations assume the forms

$$EJ\tau \frac{d^2 y}{dx^2} + M = 0 \quad (11')$$

$$\tau = \frac{E'}{E} \frac{J_1}{J} + \frac{J_2}{J} \quad (12')$$

$$S_k = \pi^2 \frac{EJ\tau}{l^2} \quad (13')$$

(Translator's Note: J is the ordinary moment of inertia of the section for bending in this plane).

We designate S_k as the *natural buckling load* of the column.

In order to present the effect of the form of cross section on the magnitude of τ and thus upon the ultimate strength S_k , the buckling coefficient τ is plotted in Figure 96 as a function of the buckling stress σ_k for two forms of cross section differing greatly in their behavior. The figure contains the curve E'/E for mild steel with an elastic limit of 2.05 t/cm², and a yield point of 2.63 t/cm², as well as the τ -curves derived from it for the + and H section, the latter curve applying for the case of the column buckling in the plane of the web. The differences begin to be appreciable in the neighborhood of the yield point. In general it is true that, for equal moments of inertia, compact cross sections have greater strength than

* See statements regarding the buckling factor on p.110 (not translated).

extended forms. The influence of the form of the cross-section increases with increase of σ_k , that is to say, at lower values of the slenderness ratio.

In columns having only one axis of symmetry which is contained in the plane of buckling, the value of the buckling factor τ varies according to whether we assume the σ_m -curve to be to the right or the left of the center of gravity of the cross section. For the buckling direction a of Figure 97, τ , as the calculation shows, will be smaller than for the buckling direction b . The column will buckle, accordingly,

toward the center of gravity side of the section when the load is centered with sufficient accuracy. In this respect the behavior of a column which fails above the elastic limit differs from that of a slender column which buckles below the elastic limit. In the elastic case it is impossible to predetermine the direction of buckling in an unsymmetrical column.

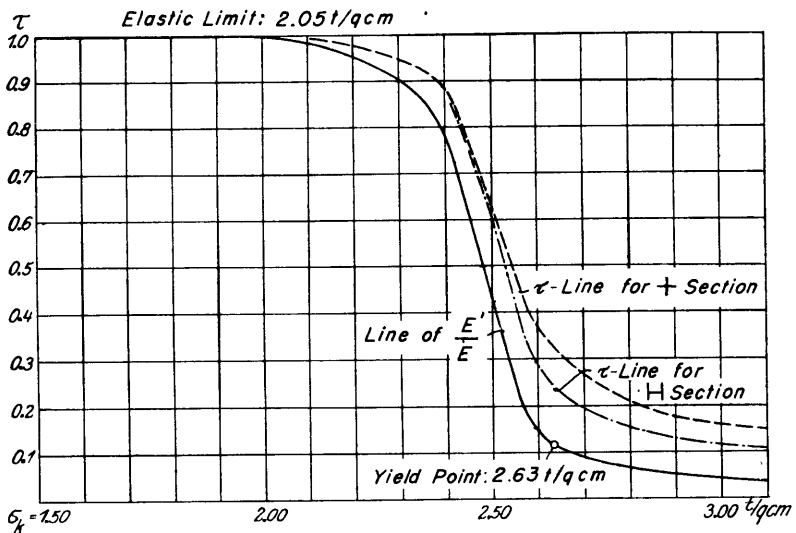


Figure 96

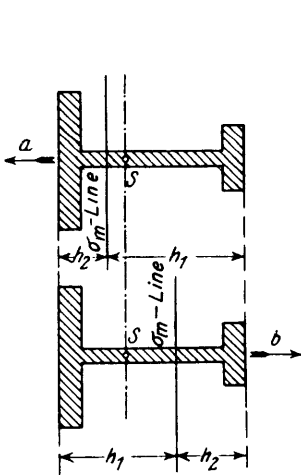


Figure 97

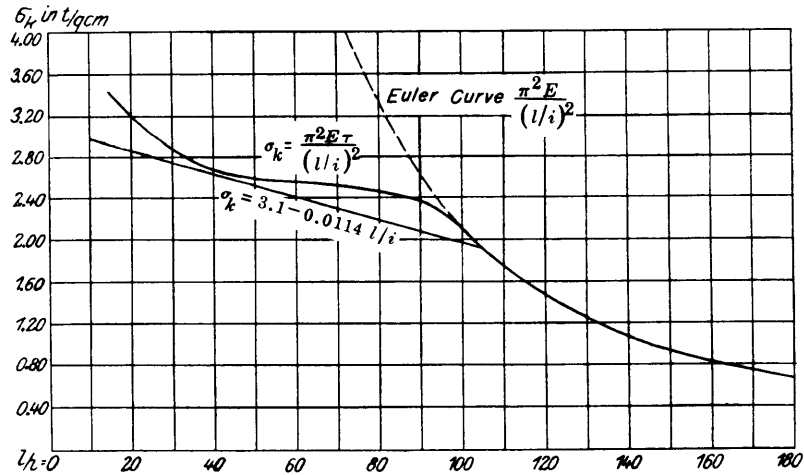


Figure 98

The differences in the buckling factor τ caused by the form of the cross section, at least for average slenderness ratios, are not so large as to justify consideration of the form of cross section in practical usage. The differences are about of the same order of magnitude as the unavoidable discrepancies between the

buckling factor τ and a mean value due to the varying strength characteristics of the material. Therefore we may content ourselves with a single series of τ -values, preferably determined by buckling tests so that they will establish a lower limit for the probable buckling factor values. In Figure 98 there is represented with the aid of the τ -curve for the H-section shown in Figure 96, the curve of buckling stress as a function of the slenderness ratio, according to the Engesser-Kármán formula,

$$\sigma_k = \frac{\pi^2 E}{(l/i)^2} \tau \quad (14)$$

The Tetmajer formula

$$\sigma_k = 3.100 - 0.0114 l/i$$

is shown also for comparison. The value of $\pi^2 E$ was taken to be 21,220 t/cm². As may be noted, Tetmajer's curve lies everywhere below the values from the Engesser-Kármán formula. For columns which buckle above the yield point, the strength increases very rapidly, according to Formula (14), with decreasing slenderness up to the compressive strength at $l/i = 0$. For columns of intermediate slenderness, beginning at $l/i = 30$, the τ values computed by Engesser's method from Tetmajer's formulas (See Appendix, Table III) may be regarded as satisfactory temporary substitutes, which still present an excess of safety.

If, in addition to the load S , there is an initial eccentric moment, and if the elastic limit is exceeded before buckling, the column behavior will differ in certain important respects from the behavior of the perfectly elastic column. In the graph shown in Figure 92, the deflection curves y_m , corresponding to the various disturbing moments (initial eccentricities) all approach the same maximum load asymptotically, i.e., the buckling load; they all tend towards one and the same maximum value. As long as the disturbing moments remain small, the ultimate loads are not greatly different from the buckling loads. This behavior is based on the characteristic circumstance that deflections in the neighborhood of the maximum load is reached at a relatively small deflection. The magnitude of the load is greatly influenced by the initial eccentricity. From this point on the deflections increase very rapidly while the load decreases. From this we conclude, and von Kármán's tests confirm the fact: The influence of initial moments on short columns which buckle above the elastic limit is considerably greater than on slender columns.

Within the elastic range ($\tau = 1$) the strength decreases in proportion to the square of l/i . Slender columns, accordingly, are very sensitive to variations in buckling length. In columns with $\tau < 1$, the diminishing effect of the increasing $(l/i)^2$ on the strength is partially offset by the influence of the very rapid increase of τ with l/i . The result is a considerably slower decrease in the strength of short columns with increasing slenderness. A glance at Figure 98 confirms this reasoning. In the range of l/i from 40 to 90 the curve of buckling stresses is approximately parallel to the abscissa. According to this, restraining the ends of the column, which is taken account of in the calculation by a corresponding reduction of buckling lengths,

is of great influence on the strength within the elastic range and of slight influence in the inelastic range.

36. Effect of Shear*

We shall discuss briefly the effect of shearing forces on the magnitude of the buckling load. We have hitherto ignored this effect.

If Q_x represents the shear at point x , and G the shear modulus in the range where the deformation is partially inelastic, and

χ a factor dependent upon the form of cross section then the complete differential equation for the elastic line will be

$$\frac{d^2 y}{dx^2} = -\frac{M_x}{TJ} + \frac{\chi}{G'F} \frac{dQ_x}{dx}$$

Here, in order to take care of the inelastic range as well, T has been substituted for E as the buckling modulus. In case of unstable equilibrium $M_x = Sy$, and hence

$$\frac{dQ_x}{dx} = S \frac{d^2 y}{dx^2}$$

The differential equation consequently becomes

$$\frac{d^2 y}{dx^2} \left(1 - \frac{S\chi}{G'F}\right) + \frac{S}{TJ} y = 0 \tag{15}$$

Equation (15) has the same form as the differential equation of the preceding discussions. If we now write

$$\alpha' = \sqrt{\frac{S}{TJ \left(1 - \frac{S\chi}{G'F}\right)}}$$

there follows, as previously, from the condition for the occurrence of unstable equilibrium

$$\alpha' l = \pi$$

the equation for the buckling load

$$S'_k - \frac{\pi^2 TJ}{l^2} \left(1 - \frac{S'_k \chi}{G'F}\right) = 0$$

With $T/G' \sim E'/G' = v$, we get

$$S'_k \left[1 + \pi^2 \chi v \left(\frac{i}{l}\right)^2\right] = \frac{\pi^2 TJ}{l^2}$$

* Translation of pp. 132-134.

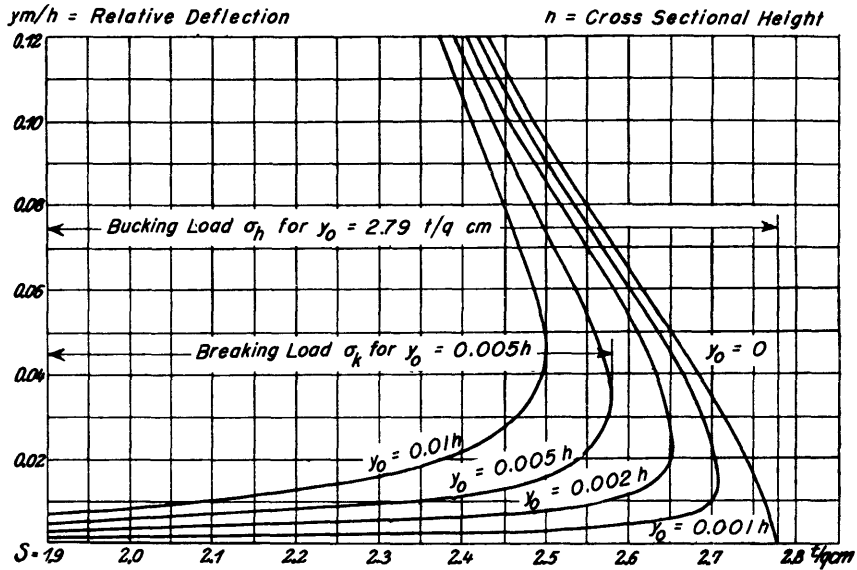


Figure 99

whence

$$S'_k = \frac{\pi^2 T J}{l^2 \left[1 + \pi^2 \chi v \left(\frac{i}{l} \right)^2 \right]} = \frac{\pi^2 T J}{(\gamma l)^2} \quad (16)$$

in which

$$\gamma = \sqrt{1 + \pi^2 \chi v \left(\frac{i}{l} \right)^2}$$

If we consider the effect of shear stresses on the deformations, the buckling force will therefore be somewhat smaller, since we must allow for the effect of the transverse forces by increasing the buckling length in the calculations from l to γl .

Poisson's ratio m for solids varies between 3 and 4, and therefore $v = 2(m+1)/m$ is between 8/3 and 5/2. We take the ratio $v = 8/3$ which is more unfavorable in the present instance, and let $\gamma = 2$ (for the H-section). Thus we get

for $l/i = 20$	30	40	50	100	150
$\gamma = 1.063$	1.028	1.016	1.010	1.003	1.002
$S'_k/S_k = 99.5$	99.7	99.8	99.8	99.8	99.8

S_k is the strength without consideration of shear forces.

The decrease in strength for all practical slenderness ratios is consequently extremely slight and need not be considered in practice. In the literature we frequently find it stated that with short columns the effect of the shear forces is considerable. This is false, and can be explained by the fact that without much thought, Equation (16) is written in the form $S'_k = S_k \frac{1}{1 + \pi^2 \chi v (i/l)^2}$. This would be correct if the buckling modulus T were associated with a buckling stress S_k/F . But instead it is associated with the buckling stress S'_k/F , and therefore S_k can not be substituted for $\frac{\pi^2 T J}{l^2}$. For short columns the coefficient $1 + \pi^2 \chi v (i/l)^2$ actually differs widely from unity, but its influence is nearly counterbalanced by the accompanying increase in the buckling modulus T .

§ 13. BUCKLING OF THE WALLS OF COMPRESSION MEMBERS*

54. Stability of Plates

In the investigations of conditions of instability hitherto made, we have usually considered the column as a whole. However, as a rule, the compression members of steel bridges consist of plate elements. It is therefore conceivable that, even before the inception of the condition of unstable equilibrium which we have just considered, and which involves failure of the column as a whole, the thin plates of which the columns are built up will reach a state of unstable equilibrium and buckle locally under the action of compressive forces, so that premature failure of the entire column will occur. Naturally it will not be necessary in each individual case to

* Translation of pp. 216-231.

undertake a tedious investigation of the conditions for the occurrence of local buckling. The problems of this section will be limited to setting up rules for proportioning the plate thickness to the other dimensions of the column for practical use by means of the general results of the following discussion. In close theoretical relationship with the problem here mentioned is that of the buckling strength of the web plates of plate girders. This question, however, will be discussed later in Section 60.

Bryan was first to treat the problem of the buckling of rectangular plates.* He presented the solution for a plate, simply supported on all its edges, which is stressed on two opposite edges by uniformly distributed compressive forces acting in the plane of the plate. Additional cases have been treated by Timoshenko,** and by Reissner.† H. Rode†† has given a very detailed description of the problems to be considered in this field. The publications just noted on the subject of the buckling of plates are of limited practical significance, however, since their theories are based on the assumption of unlimited elasticity and thus fail to include consideration of stresses above the elastic limit, such as occur in short columns before buckling. In Timoshenko's paper, it is true, there are several expressions relating to stability beyond the elastic limit. But because in bridge building short columns or those of intermediate slenderness are most important, the numerical results of the investigations cited in the foregoing do not have conclusive significance in bridge construction. However, it is not difficult in the problems under discussion here to take into account the variability of the modulus of elasticity and thus obtain results which will apply in the ranges of elastic and inelastic buckling. The following investigations will take into account this consideration.

(a) The Two Fundamental Cases of the Buckling of Rectangular Plates.

We consider a flat plate which is loaded on two edges b parallel to the y -axis by the uniformly distributed load $\sigma\delta$, where δ is the plate thickness, Figure 169. We assume these edges to be simply supported so that the plate can rotate freely about them.+ The edges parallel to the x -axis (edges a) may be supported in various ways, namely:

* London Mathematical Society Proceedings, 1891, p. 54.

** Timoshenko, S.: "Einige Stabilitätsprobleme der Elastizitätstheorie" (Some Stability Problems of the Theory of Elasticity), *Zeitschrift für Mathematik und Physik*, 1910, p. 337, and "Sur la Stabilité des Systemes elastiques" (On the Stability of Elastic Systems), *Annales des Ponts et Chaussées*, 1913.

† Reissner, H.: "Über die Knicksicherheit ebener Bleche" (On the Buckling Strength of Flat Plates), *Zentralblatt der Bauverwaltung*, 1909, p. 93.

†† Rode, H.: "Beitrag zur Theorie der Knickerscheinungen" (Contribution to the Theory of Buckling Phenomena), *Eisenbau*, 1916, p. 121 ff.

+ Clamping the loaded edge has little effect on the critical load of long plates such as occur in columns, since the influence of the type of support of the edges parallel to the direction of loading is decisive.

Case I: The plate is elastically fixed at both edges a . This case includes the cases of free rotation about the edges a and of complete fixation of these edges as limiting cases.

Case II: The one edge a is elastically fixed, the other is free. This case likewise includes the two limiting cases in which the supported edge is free to rotate, or is completely fixed.

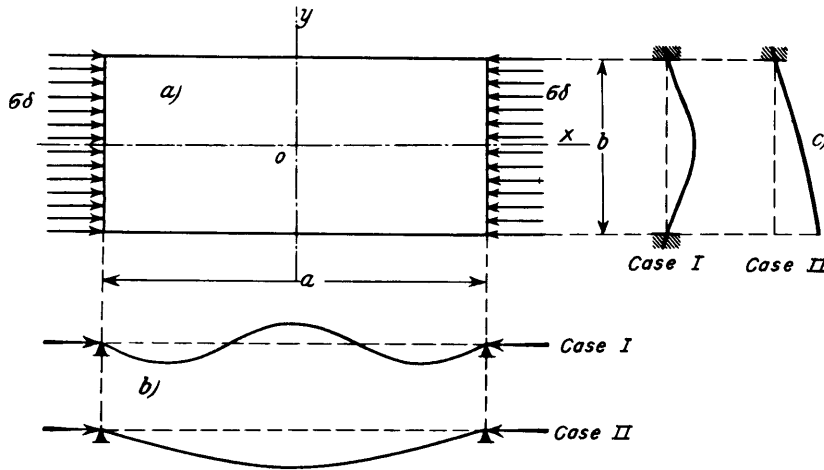


Figure 169

Figure 169b shows the longitudinal section of the buckled plate and Figure 169c the cross-section for each case of support. In plates supported at both edges (Case I) buckling occurs in one or more waves, depending upon the length a of the plate; and with one free edge (Case II) the plate

usually buckled in one wave when, for example, it is free to rotate at the supported edge, (Figure 169b).

(b) The Differential Equation for the Deflection of Thin Plates.

The investigation of the buckled state of columns was based on the differential equation of the elastic line. Similarly we start here also with a differential equation from the theory of elasticity, which describes the deflection of thin plates perpendicular to the plane of the plate. This equation reads:*

$$\frac{EJ}{1-m^2} \left[\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] + \delta \left[\sigma_x \frac{\partial^2 w}{\partial x^2} + \sigma_y \frac{\partial^2 w}{\partial y^2} + 2 \tau_{xy} \frac{\partial^2 w}{\partial x \partial y} \right] = 0$$

where w is the deflection perpendicular to the plane of the plate (xy -plane),

σ_x and σ_y are the normal stresses in the direction of the x or y axis respectively. These stresses do not vary over the height of the plate and are to be considered as the effect of the edge loads in the plane of the plate. The variations which the stresses σ undergo when the plate is bent may be neglected if we assume only small displacements w .

τ_{xy} signifies the shear stresses in a section perpendicular to the plane of the plate and cut parallel to the x or y axis, as the case may be.

$J = (1/12)\delta^3$ signifies the moment of inertia of a strip of plating of unit width, and thickness δ .

$m = 0.3$ is Poisson's ratio

* Enzyklopädie der Mathematischen Wissenschaften, Vol. IV, Mechanik, 4, Teilband, p. 377.

Since we shall consider only a uniformly distributed loading $\sigma\delta$ at the edges b , $\sigma_y = 0$ and $\tau_{xy} = 0$, the differential equation for w assumes the simplified form

$$\frac{EJ}{1 - m^2} \left[\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] + \sigma_x \delta \frac{\partial^2 w}{\partial x^2} = 0 \quad (1)$$

This equation, which in its present form is valid only within the range of Hooke's law, has yet to be adapted for the general case where the modulus of elasticity E varies for the directions x and y when σ_x exceeds the elastic limit. In order, then, to find the significance of the individual terms of Equation (1) so that the buckling factor τ may be correctly introduced, we proceed as follows:

If we let the second and third terms in the brackets in Equation (1) equal zero, there will remain

$$\frac{EJ}{1 - m^2} \frac{\partial^4 w}{\partial x^4} + \sigma_x \delta \frac{\partial^2 w}{\partial x^2} = 0$$

We obtain a similar relationship by double differentiation of the equation of the elastic line of a bar. (Translator's Note: See Equation (1) above.)

$$EJ \frac{\partial^2 w}{\partial x^2} + Pw = 0$$

From which we get

$$EJ \frac{\partial^4 w}{\partial x^4} + P \frac{\partial^2 w}{\partial x^2} = 0$$

The first term, then, denotes the bending of the strips of plating of unit width, parallel to the x -axis. However, since these strips of plating are stressed by the longitudinal force σ_x , the factor $EJ\tau^*$ must be substituted for EJ when σ_x exceeds the elastic limit, so that the first term will read

$$EJ\tau \frac{\partial^4 w}{\partial x^4}$$

In the same manner, the third term in the brackets may be taken as the bending term, arising from the bending of the strips of plating running parallel to the y -axis. Since these strips are free of stresses ($\sigma_y = 0$) with the exception of small normal stresses due to bending, E retains its value. Thus the third term will read

$$EJ \frac{\partial^4 w}{\partial y^4}$$

The middle term in the brackets, finally, describes the distortion of a square plate element. Since both directions of the plate affect this value, we take this circumstance into account by introducing a coefficient representing the dependence upon the longitudinal stresses, which will have a mean value - only approximate,

* For the buckling factor τ see Section 30.

it is true - between 1 and τ . It will be found expedient to select $\sqrt{\tau}$ as coefficient, and thus we will get for the second term in the brackets

$$EJ\sqrt{\tau}\frac{\partial^4 w}{\partial x^2 \partial y^2}$$

Equation (1) then will assume the general form

$$\frac{EJ}{1-\bar{m}^2}\left[\frac{\partial^4 w}{\partial x^4}\tau + 2\frac{\partial^4 w}{\partial x^2 \partial y^2}\sqrt{\tau} + \frac{\partial^4 w}{\partial y^4}\right] + \sigma_x \delta \frac{\partial^2 w}{\partial x^2} = 0 \quad (2)$$

The uncertainty involved in this equation lies in the arbitrary choice of the coefficient $\sqrt{\tau}$ of the middle term in the brackets, as well as in the assumption of a value of Poisson's ratio \bar{m} independent of σ_x . The influence of \bar{m} is very slight since this value can fluctuate very little. The influence of an error in the middle term, likewise, is relatively small, as numerical checking has shown, even when the hypothetical coefficient differs considerably from the correct value.

(c) The General Solution of Differential Equation (2).

The solution of partial differential Equation (2) must first satisfy the following boundary conditions at the edges b : The displacements w and the moments M_x (Translator's Note: About the edges $x = \pm \frac{a}{2}$) vanish; then for $x = \pm \frac{a}{2}$

$$w = 0 \quad (3)$$

and

$$M_x = \left(\frac{\partial^2 w}{\partial x^2} + m \frac{\partial^2 w}{\partial y^2}\right) \frac{EJ}{1-\bar{m}^2} = 0$$

Since along the edges $x = \pm \frac{a}{2}$, $\frac{\partial^2 w}{\partial y^2} = 0$ (the edges remain straight by assumption) the second boundary condition becomes

$$\frac{\partial^2 w}{\partial x^2} = 0 \quad (4)$$

Equation (2) and the boundary conditions in (3) and (4) are satisfied by the expression

$$w = Y \cos \frac{n\pi x}{a} \quad (n=1, 2, \dots) \quad (5)$$

where Y is a function of y yet to be determined. Equation (5), when y is held constant, represents a cosine curve with n half-waves on the length a . Then if we introduce the expression (5) into the partial differential Equation (2) and cancel $\cos \frac{n\pi x}{a}$ we obtain the ordinary differential equation of the fourth order

$$\frac{d^4 Y}{dy^4} - 2\sqrt{\tau} \left(\frac{n\pi}{a}\right)^2 \frac{d^2 Y}{dy^2} + \left[\tau \left(\frac{n\pi}{a}\right)^4 - \frac{\sigma_k \delta}{D} \left(\frac{n\pi}{a}\right)^2\right] Y = 0 \quad (6)$$

where

$$\frac{EJ}{1-\bar{m}^2} = D \quad \text{and} \quad \sigma_x = \sigma_k$$

σ_k is the desired critical longitudinal stress at which the plate buckles. Integration of Equation (6) is not difficult. With the particular solution

$$Y = e^{ky}$$

We obtain by substitution in Equation (6)

$$k^4 - 2\sqrt{\tau} \left(\frac{n\pi}{a}\right)^2 k^2 + \left[\tau \left(\frac{n\pi}{a}\right)^4 - \frac{\sigma_k \delta}{D} \left(\frac{n\pi}{a}\right)^2 \right] = 0$$

an equation determining the coefficient k , from which are derived four roots $\pm k_1$ and $\pm k_2$, namely

$$\left. \begin{aligned} \pm k_1 &= \pm \sqrt{\sqrt{\left(\frac{n\pi}{a}\right)^2 \frac{\sigma_k \delta}{D} + \sqrt{\tau} \left(\frac{n\pi}{a}\right)^2}} \\ \pm k_2 &= \pm i \sqrt{\sqrt{\left(\frac{n\pi}{a}\right)^2 \frac{\sigma_k \delta}{D} - \sqrt{\tau} \left(\frac{n\pi}{a}\right)^2}} \end{aligned} \right\} \quad (7)$$

It is now easily demonstrated that the expression under the radical in k_2 is always real, and therefore k_2 is always imaginary. This is true because, in consequence of the greater constraint experienced by a plate supported at the edges a as compared to a plate whose edges a are free, the buckling stress σ_k in the plate supported at all its edges will be greater than the buckling stress on the plate with its edges free. If σ'_k represents the buckling stress on the plate whose edges are entirely free (Euler's case), we then have for σ'_k , when we express the lateral restraint in the wide plate by the factor $\frac{1}{1-\bar{m}^2}$

$$\sigma'_k = \frac{(n\pi)^2 E J \tau}{(1-\bar{m}^2) a^2} \quad \left\{ \begin{array}{l} \text{(Translator's Note: This is Euler's} \\ \text{formula for a column of length } a \\ \text{which buckles in } n \text{ bulges)} \end{array} \right.$$

From this follows the inequality

$$\delta \sigma_k > \frac{(n\pi)^2 E J \tau}{(1-\bar{m}^2) a^2}$$

Accordingly

$$\frac{\delta \sigma_k}{D} > \left(\frac{n\pi}{a}\right)^2 \tau$$

Multiplying both sides of the inequality by $\left(\frac{n\pi}{a}\right)^2$ and extracting the square root, it follows that

$$\sqrt{\left(\frac{n\pi}{a}\right)^2 \frac{\delta \sigma_k}{D}} > \left(\frac{n\pi}{a}\right)^2 \sqrt{\tau}$$

which is what we desired to prove.

Then the general solution of the differential Equation (6) will read:

$$Y = C_1 e^{k_1 y} + C_2 e^{-k_1 y} + C_3 e^{i k_2 y} + C_4 e^{-i k_2 y}$$

where k_1 and k_2 now represent real positive numbers.

If we replace the exponential functions with trigonometric and hyperbolic functions, the desired general solution of differential Equation (2) will take the form

$$w = \cos \frac{n\pi x}{a} [A \cosh k_1 y + B \sinh k_1 y + C \cos k_2 y + D \sin k_2 y] \quad (8)$$

The constants A , B , C , and D are in any case to be so determined that the boundary conditions at the sides a , characteristic of the case in question, will be satisfied.

- (d) Case I. The Plate is Free to Rotate about the Edges b and the Edges a are Elastically Built-in.

We assume the origin of the xy axes to be in the midpoint of the plate, as indicated in Figure 169. The deflection w , because of identical conditions at the two edges a is then obviously a symmetrical function in y , and the terms $B \sinh k_1 y$ and $D \sin k_2 y$ vanish from Equation (8), which then assumes the shorter form

$$w = \cos \frac{n\pi x}{a} [A \cosh k_1 y + C \cos k_2 y] \quad (9)$$

The equations which determine the constants A and B follow from the boundary conditions at the edges a , so that for $y = \pm \frac{b}{2}$:

$$w = 0 \quad \text{and} \quad M_y = \nu \frac{\partial w}{\partial y}$$

The first boundary condition states that the deflection at the fixed sides a must vanish, and the second condition states that the restraining moment about the edges a is everywhere proportional to the rotation of the tangent parallel to the y -axis at the plate edge.

Case I is realized in sections having two axes of symmetry, for example, as shown in Figure 170. The lighter webs being elastically built-in, are hindered from free deformation by the heavier flange plates. Considering a strip parallel to the y -axis, the web is subject to the loading and deformation conditions indicated in Figure 170b. However, through the rigid connection between the web and cover plates, the flange plates likewise are bent when the webs bulge. The deformation of these flange plates is represented in Figure 170c.

Between the edge moment M_y and the rotation ϕ' of the end tangent of the flange plate there exists the relationship

$$\tan \phi' = \frac{1 - \bar{m}^2}{EI} \frac{c}{2} M_y$$

easily derived from the equation of the elastic line, I being the moment of inertia of a strip of the flange with unit width. Now since

$$\tan \phi = - \tan \phi'$$

we have

$$\left(\frac{\partial w}{\partial y} \right)_{y=\frac{b}{2}} = - \frac{1 - \bar{m}^2}{EI} \frac{c}{2} M_y$$

from which it follows that

$$M_y = - \frac{EI}{1 - \bar{m}^2} \frac{2}{c} \left(\frac{\partial w}{\partial y} \right)_{y=\frac{b}{2}}$$

This establishes the second boundary condition. But edge moment M_y is given in an alternative form by the previously used expression

$$M_y = \frac{EJ}{1-m^2} \left(\frac{\partial^2 w}{\partial y^2} + m \frac{\partial^2 w}{\partial x^2} \right)$$

Since w and $\partial^2 w / \partial x^2$ are zero everywhere along the edge, this simplifies to

$$M_y = \frac{EJ}{1-m^2} \left(\frac{\partial^2 w}{\partial y^2} \right)_{y=\frac{b}{2}}$$

Combination of the two equations found for M_y finally produces the boundary conditions sought, in the form

$$\frac{\partial w}{\partial y} + \zeta \frac{b}{2} \frac{\partial^2 w}{\partial y^2} = 0 \text{ for } y = \pm b/2 \quad (10)$$

where the fixation coefficient ζ is defined by

$$\zeta = \frac{Jc}{Ib} = \left(\frac{\delta}{\delta'} \right)^3 \frac{c}{b} \quad (11)$$

We note here that ζ , theoretically at least, can assume any value from 0 to ∞ . When $\zeta = 0$ ($I = \infty$), the plate is completely fixed at the edges a , and when $\zeta = \infty$ ($I = 0$) it is free to rotate at the edges a .

Introducing solution (9) into the boundary conditions

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial y} + \zeta \frac{b}{2} \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{for } y = \pm b/2$$

yields the two determinative equations

$$A \cosh k_1 \frac{b}{2} + C \cos k_2 \frac{b}{2} = 0$$

$$\left(A k_1 \sinh k_1 \frac{b}{2} - C k_2 \sin k_2 \frac{b}{2} \right) + \zeta \frac{b}{2} \left(A k_1^2 \cosh k_1 \frac{b}{2} - C k_2^2 \cos k_2 \frac{b}{2} \right) = 0$$

from which equations, non-zero values for A and B result only when the determinant of this system of equations vanishes. Therefore $\Delta = 0$ is the buckling criterion. Carrying out the calculation for Δ yields the equation

$$\left(\tan k_2 \frac{b}{2} + \zeta k_2 \frac{b}{2} \right) k_2 + \left(\tanh k_1 \frac{b}{2} + \zeta k_1 \frac{b}{2} \right) k_1 = 0$$

For the $k_1 b/2$ values here considered, $\tanh k_1 b/2$ differs only slightly from unity,* so if we write $\tanh k_1 b/2 = 1$ the buckling criterion is simplified considerably. Finally we get the relation

* According to Equation (7)

$$\frac{k_1 b}{2} = \frac{b}{2} \sqrt{\sqrt{\left(\frac{n\pi}{a} \right)^2 \frac{\sigma_k \delta}{D}} + \left(\frac{n\pi}{a} \right)^2 \sqrt{\tau}}$$

We have already proved in Division (c) that the first term under the radical sign is greater than the

(Continued)

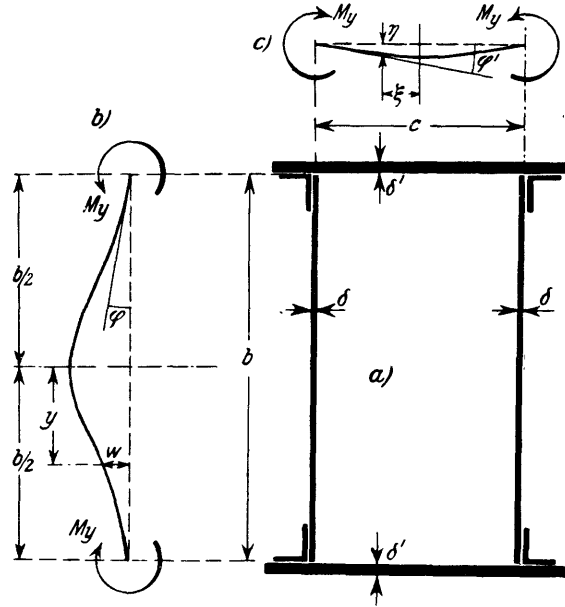


Figure 170

$$\tan k_2 \frac{b}{2} = -\zeta k_2 \frac{b}{2} - \frac{k_1}{k_2} \left(1 + \zeta k_1 \frac{b}{2}\right) \quad (12)$$

If $\frac{a}{b} = \alpha$ in the first Equation (7), it follows that

$$\frac{k_1 b}{2} = \frac{1}{2} \sqrt{\sqrt{\left(\frac{n\pi}{\alpha}\right)^2 \frac{\sigma_k \delta b^2}{D} + \left(\frac{n\pi}{\alpha}\right)^2 V\tau}} = \frac{n\pi}{2\alpha} \sqrt{\sqrt{\frac{\sigma_k \delta}{D} b^2 \left(\frac{\alpha}{n\pi}\right)^2 + V\tau}}$$

If we write

$$\sigma_k = \frac{(n\pi)^2 E J \tau}{a^2 \delta (1 - m^2)} \mu^2 = \left(\frac{n\pi}{a}\right)^2 \frac{\tau}{D \delta} \mu^2, \quad (13)$$

i.e., if we compare the buckling stress when the edges a are elastically built-in with the buckling stress in Euler's case (free edges a), where $1/\mu$ is a reduction factor applied to the plate length a , the equation for $k_1 b/2$ becomes

$$\frac{k_1 b}{2} = \frac{n\pi}{2\alpha} \sqrt[4]{V\tau} \sqrt{\mu + 1}$$

In the same way we find

$$\frac{k_2 b}{2} = \frac{n\pi}{2\alpha} \sqrt[4]{V\tau} \sqrt{\mu - 1}$$

With these relationships the buckling condition (12) assumes the form

$$\tan \left[\frac{\pi}{2} \sqrt{\mu - 1} \frac{n \sqrt[4]{V\tau}}{\alpha} \right] = -\frac{\pi \mu \zeta}{\sqrt{\mu - 1}} \frac{n \sqrt[4]{V\tau}}{\alpha} - \frac{\sqrt{\mu + 1}}{\sqrt{\mu - 1}} \quad (14)$$

This transcendental equation represents a combination of the reduction coefficient $\frac{1}{\mu}$ with the $\frac{n \sqrt[4]{V\tau}}{\alpha}$ values. If we compute μ from this equation with a given $\frac{n \sqrt[4]{V\tau}}{\alpha}$, determination of σ_k by means of Equation (13) is referred back to Euler's simple case. However, before we approach discussion of the general solution given by Equation (14) we will subject the simplest case to closer study in order to comprehend more easily the nature of the problems to be explained here, i.e., simple support of the edges a . Treatment of the general Equation (14) will then create little difficulty.

If we introduce $\zeta = \infty$ into Equation (14), we obtain the simple relationship

$$\tan \left[\frac{\pi}{2} \sqrt{\mu - 1} \frac{n \sqrt[4]{V\tau}}{\alpha} \right] = -\infty$$

The smallest root of this equation permissible in the present case is

$$\frac{\pi}{2} \sqrt{\mu - 1} \frac{n \sqrt[4]{V\tau}}{\alpha} = -\frac{\pi}{2}$$

second. If we substitute the second for the first, we have the inequality

$$\frac{k_1 b}{2} > \frac{b n \pi}{2 a} \sqrt[2]{V\tau}$$

As will appear upon carrying the calculation further, $\frac{a}{nb} \leq \sqrt[4]{V\tau}$

The inequality will still hold if we substitute $\sqrt[4]{V\tau}$ for $\frac{a}{nb}$, which ultimately yields $\frac{k_1 b}{2} > \frac{\pi}{\sqrt{2}}$ or $\frac{k_1 b}{2} > 2.22$. Therefore $\tanh k_1 \frac{b}{2}$ lies between 0.977 and 1.

Solving for μ yields

$$\mu = \left(\frac{\alpha}{n\sqrt[4]{\tau}} \right)^2 + 1$$

and thus, referring back to Equation (13), we get

$$\sigma_k = \frac{(n\pi)^2 E J \tau}{a^2 \delta (1 - \bar{m}^2)} \left[\left(\frac{\alpha}{n\sqrt[4]{\tau}} \right)^2 + 1 \right]^2$$

Substituting $1/12\delta^3$ for J and multiplying the numerator and denominator by b^2 we finally obtain

$$\sigma_k = \frac{\pi^2 E \sqrt{\tau}}{12(1 - \bar{m}^2)} \left(\frac{\delta}{b} \right)^2 \left(\frac{\alpha}{n\sqrt[4]{\tau}} + \frac{n\sqrt[4]{\tau}}{\alpha} \right)^2 \quad (15)$$

The only value in Equation (15) yet remaining to be found is n , which indicates the number of half-waves into which the plate buckles in its longitudinal section. In order to find the number of half-waves for a given aspect ratio $\alpha = a/b$, we proceed as follows: With sufficiently short plates, that is, with smaller α , buckling will occur in one half-wave. Above a certain ratio α' , two half-waves will be formed in buckling. For this limiting ratio at which there is a sudden transition from one state of equilibrium to the other, i.e., when both conditions are equally possible at the same buckling stress σ_k , Equation (15) must yield the same value for σ_k whether we introduce $n = 1$ in one case or $n = 2$ in the other. In the same way it will be possible to determine the value of α for buckling in two or three half-waves at which σ_k will yield the same value for $n = 2$ and $n = 3$. In general, we find the limiting ratio α at which either n or $n + 1$ half-waves can occur, from the equation

$$\left(\frac{\alpha'}{n\sqrt[4]{\tau}} + \frac{n\sqrt[4]{\tau}}{\alpha'} \right)^2 = \left(\frac{\alpha'}{(n+1)\sqrt[4]{\tau}} + \frac{(n+1)\sqrt[4]{\tau}}{\alpha'} \right)^2$$

from which it follows that

$$\frac{\alpha'}{\sqrt[4]{\tau}} = \sqrt{n(n+1)} \quad (16)$$

If we write

$$n = 1 \quad 2 \quad 3 \quad 4 \quad \text{etc.},$$

we get

$$\frac{\alpha'}{\sqrt[4]{\tau}} = \sqrt{2} \quad \sqrt{6} \quad \sqrt{12} \quad \sqrt{20} \quad \text{etc.}$$

For $\tau = 1$, when the law of elasticity applies, buckling occurs in one half-wave up to $a = 1.414b$, and from $a = 1.414b$ to $a = 2.449b$ in two half-waves. If the elastic limit is exceeded the ratios approach each other more closely, since $\tau < 1$, and the waves become shorter the higher the buckling stress σ_k .

In order to be able to form a concept of the curve of the buckling stresses for various aspect ratios α , the expression

$$z = \left(\frac{\alpha}{n} + \frac{n}{\alpha} \right)^2$$

is graphically depicted in Figure 171, i.e., for $\tau = 1$ with the argument α . Since σ_k is proportional to z , the z -curve also gives the curve of the buckling stresses σ_k as a function of α . As shown in Figure 171, the z -curve is made up of individual irregularly connected segments, which, correspond to buckling in $n = 1, 2,$ and $3 \dots$ half-waves, depending upon the aspect ratios.

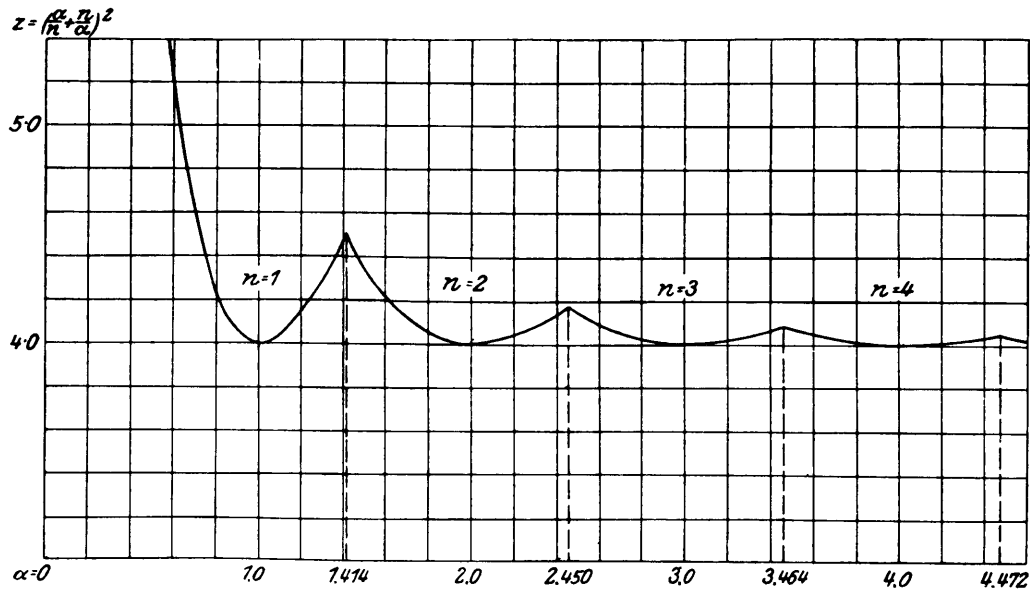


Figure 171

For each number of half-waves there is an aspect ratio α_0 at which σ_k assumes a minimum value. As a matter of fact, the figure shows, and we will furnish additional proof farther on, that this minimum σ_k , which is chiefly important as a criterion of design, occurs when a is an integral multiple of the width b . The lowest points of the z -curve lie at $\sigma = 1, 2, 3$. An additional noteworthy fact may be observed: the minimum values of the individual branches of the curve are equal throughout, and with increasing plate length a the difference between the value of z (or σ_k) corresponding to the actual plate length a and the minimum value $\min \sigma_k$ constantly decreases. Thus, for example, when $\alpha = 3.464$, $z_{\max} = 4.08$, while $z_{\min} = 4$. The difference here is 2 per cent and decreases very rapidly as α increases. Since only long, narrow plates need be considered in connection with columns, we can always base our calculations on the minimum value of σ_k which is constant for all length/width ratios α , which greatly simplifies the rest of the computation. Similar conditions apply when $\tau > 1$, or when the elastic limit is exceeded

From Equation (15) for σ_k , based on the condition

$$\frac{\partial \sigma_k}{\partial \alpha} = 0$$

we find that value α_0 which results in a minimum σ_k , and which is

$$\alpha_0 = n\sqrt[4]{\tau} \quad (17)$$

When $\tau = 1$, $\alpha_0 = n$, as has been stated before. If we introduce α_0 in Equation (15), we get

$$\min \sigma_k = \frac{\pi^2 E}{3(1 - \bar{m}^2)} \left(\frac{\delta}{b}\right)^2 \sqrt{\tau} \quad (18)$$

or, with $E = 2150 \text{ t/cm}^2$ and $\bar{m} = 0.3$

$$\min \sigma_k = 7772 \left(\frac{\delta}{b}\right)^2 \sqrt{\tau} \quad (\text{in tons per sq. cm.}) \quad (18')$$

[Note: With $E = 29 \times 10^6 \text{ lb. per sq. in.}$ and $\bar{m} = 0.3$

$$\min \sigma_k = 1.05 \times 10^8 \left(\frac{\delta}{b}\right)^2 \sqrt{\tau} \quad (\text{in lb. per sq. in.)}]$$

This expression is independent of the number of half-waves n , and thus also of the aspect ratio α . In practice, then, we will have to reckon with only the one most unfavorable value of σ_k at all α -ratios, even after the elastic limit has been exceeded. It is remarkable here, in contrast to the case with columns, that σ_k is proportional to $\sqrt{\tau}$. (Translator's Note: Compare Equation (14), page 33, for columns, where σ_k is proportional to τ .)

τ is 1 or less than 1 and in this case dependent upon σ_k . If in the latter case we introduce τ as in Chapter 30, Equation (9), into Equation (18'), we get a quadratic equation for σ_k , namely

$$\sigma_k = \frac{r}{2} - \sqrt{\frac{r^2}{4} - 9.61} \quad \text{with} \quad r = \frac{(b/\delta)^4}{2190 \cdot 10^4} + 6.2 \quad (\text{in tons per sq. cm.}) \quad (18'')$$

We now return to the discussion of the general buckling condition Equation (14). The transcendental form in which the quantities depend upon $\alpha/n\sqrt[4]{\tau}$ renders it very difficult to apply Equation (14). In the case just considered in detail where $\zeta = \infty$ we have found the rigorously correct expression for μ^2 to be

$$\mu^2 = 1 + 2 \left(\frac{\alpha}{n\sqrt[4]{\tau}}\right)^2 + \left(\frac{\alpha}{n\sqrt[4]{\tau}}\right)^4 \quad (19)$$

The obvious thought occurs to investigate whether the relationship between μ and $\alpha/n\sqrt[4]{\tau}$ given by Equation (14) may be represented, at least approximately, by a similar expression. This is actually the case since Equation (14) may be replaced with sufficient accuracy (error less than 1.5 per cent) by functions of the form

$$\mu^2 = 1 + p \left(\frac{\alpha}{n\sqrt[4]{\tau}}\right)^2 + q \left(\frac{\alpha}{n\sqrt[4]{\tau}}\right)^4 \quad (20)$$

*In determining α_0 we assumed τ to be independent of σ_k , which, strictly speaking, is not correct. However, when $n > 1$, the influence of the independence of the factor τ from σ_k is extremely small in determining α_0 .

where p and q are factors dependent upon the fixation ratio ζ . For p and q the following values were obtained:

$\zeta = 0$	0.1	0.2	0.5	1
$p = 2.50$	2.30	2.18	2.10	2.00
$q = 5.00$	3.55	2.81	2.16	1.00

The last column pertaining to $\zeta = 1$ requires an explanation. At the beginning of this section we assumed proportionality between the edge moments M_y and the angle of rotation ϕ . This assumption may be correctly made only when the supporting structure is itself free of compressive stresses having an influence on its deformations. This, however, is not true of columns with box-shaped sections, such as we are discussing here. The bending of the flanges which represent the supporting structure, in our example, Figure 170, is determined not only by the deflections of the webs in unstable equilibrium, but also by the longitudinal compressive forces acting on the flanges themselves. The heavier the flange, however, (the smaller ζ), the less will be the effect of the compressive stresses. This effect takes on practical significance when ζ approaches unity, so that when $\zeta = 1$ the coefficients p and q attain those values which they should reach only when $\zeta = \infty$, compressive forces in the supporting part being neglected.

This is true because $\zeta = 1$ corresponds to equal stiffness in all the four walls, as represented in the section shown in Figure 172. If only the one pair were stressed by longitudinal compressive forces, the second pair would be able to assume part of the load of the first. The value of the buckling stress involved would be given by the buckling condition, Equation (14), derived under this assumption, or, as the case may be, μ would be given by an expression of the form of Equation (20) adapted to this condition. But if all four sides are under uniform compression, as is the case with columns, both pairs of plates enter the state of unstable equilibrium simultaneously, and the one pair is no longer able to support the other. The plates behave as though they were supported without restraining moments at the edges. This case was treated separately in the foregoing summary, where $\zeta = \infty$.* Therefore we have corrected it by substituting the values $p = 2$, $q = 1$, corresponding to the case $\zeta = \infty$, Equation (19), for the somewhat greater values for p and q , corresponding to $\zeta = 1$. (The correct values would be $p = 2.01$ and $q = 1.68$)

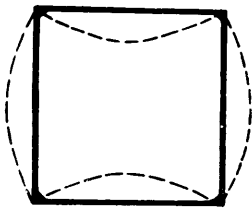


Figure 172

For intermediate values of ζ , interpolation can be made between the values of the foregoing table.

Using the approximative formula Equation (20) for μ , we get from Equation 13 for σ_k the following expression, if we take $J = (1/12)\delta^3$, and multiply the numerator

* This recalls the analogous case of a truss structure, when the buckling limit is reached simultaneously in all members.

and denominator by b^2 :

$$\sigma_k = \frac{\pi^2 E \sqrt{\tau}}{12(1-\bar{m}^2)} \left(\frac{\delta}{b}\right)^2 \left[\left(\frac{n\sqrt[4]{\tau}}{\alpha}\right)^2 + p + q \left(\frac{\alpha}{n\sqrt[4]{\tau}}\right)^2 \right] \quad (21)$$

Based on the same reasoning as the foregoing, we find the limiting ratio α' at which either n or $n+1$ half-waves can exist is

$$\alpha' = \sqrt[4]{\frac{\tau}{q}} \sqrt{n(n+1)} \quad (22)$$

Here q lies between 1 and 5. For $q = 1$ (simple support at the edges a) we find the value of α already derived in the foregoing, namely

$$\alpha' = \sqrt[4]{\tau} \sqrt{n(n+1)}$$

and for $q = 5$ (complete fixation at the edges)

$$\alpha' = 0.668 \sqrt[4]{\tau} \sqrt{n(n+1)}$$

The half-waves, therefore, are appreciably shortened by fixation.

We further find that value α_0 , for which σ_k reaches a minimum, since we use this number as a basis upon which to continue our calculation. The condition

$$\frac{\partial \sigma_k}{\partial \alpha} = 0$$

yields

$$\alpha_0 = n \sqrt[4]{\frac{\tau}{q}} \quad (23)$$

Introducing this value into Equation (21) we finally get

$$\min \sigma_k = \frac{\pi^2 E \sqrt{\tau}}{12(1-\bar{m}^2)} \left(\frac{\delta}{b}\right)^2 (p + 2\sqrt{q}) \quad (24)$$

or with $E = 2150$ t/cm² and $\bar{m} = 0.3^*$

$$\sigma_k = 7772 \left(\frac{\delta}{b}\right)^2 (p + 2\sqrt{q}) \sqrt{\tau} \text{ (tons per sq. cm.)} \quad (24')$$

an expression independent of n , which is thus valid for columns of all lengths.

Our attention must be directed principally to attaining equal security against failure of the structural members involved in all parts of a structure. Therefore the compression members should be so dimensioned that the individual plates present the same resistance to local buckling as the whole member presents to column buckling. Therefore the critical stress σ_k which results in local buckling must be equal to the critical stress at which the column with the slenderness ratio l/i buckles as a whole. This is expressed by the equation

$$\frac{\pi^2 E \tau}{\left(\frac{i}{l}\right)^2} = \frac{\pi^2 E \sqrt{\tau}}{12(1-\bar{m}^2)} \left(\frac{\delta}{b}\right)^2 (p + 2\sqrt{q})$$

* Or with $E = 29 \times 10^6$ lb. per sq. in. and $\bar{m} = 0.3$, $\sigma_k = 1.05 \times 10^8 \left(\frac{\delta}{b}\right)^2 (p + \sqrt{q}) \sqrt{\tau}$ lb. per sq. in.)

from which follows the design criterion

$$\frac{b}{\delta} = \frac{\sqrt{p + 2\sqrt{q}}}{2\sqrt{3(1 - m^2)\sqrt{\tau}}} \left(\frac{l}{i}\right) = \frac{0.303}{\sqrt{\tau}} \sqrt{p + 2\sqrt{q}} \left(\frac{l}{i}\right) \quad (25)$$

Equation (25) indicates the important fact that the admissible ratio of plate width to plate thickness increases with the slenderness ratio of the column as a whole. In slender columns, therefore, it will be possible to use thinner walls than in short columns.

The curve $\frac{0.303}{\sqrt{\tau}} \left(\frac{l}{i}\right)$ was calculated as a function of $\left(\frac{l}{i}\right)$ by means of Tables I and III, and plotted in Figure 173. The slightly curved first branch may be replaced

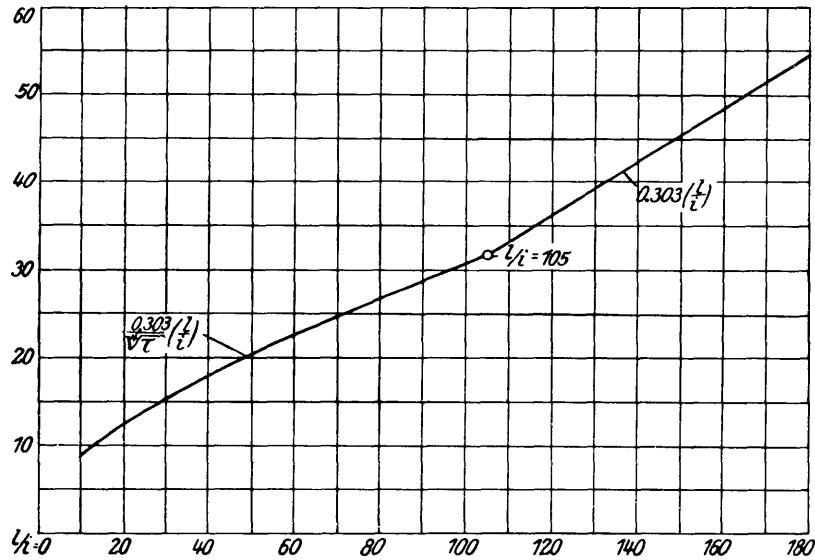


Figure 173

by a parabola with sufficient accuracy. The radical dependent upon ζ may be approximated by the expression $2.64 - 0.64\sqrt{\zeta}$, and thus we get the following simple formulas:

for l/i from 10 to 105

$$\left. \begin{aligned} \frac{b}{\delta} &\leq \left(3.42\sqrt{\frac{l}{i}} - 3.30\right)(2.64 - 0.64\sqrt{\zeta}) \\ \text{for } l/i > 105 \\ \frac{b}{\delta} &\leq 0.303\frac{l}{i}(2.64 - 0.64\sqrt{\zeta}) \end{aligned} \right\} \quad (26)$$

for ζ is to be used the value $\frac{\delta^3 b'}{\delta' b^3}$, where δ and b signify the thickness and width of the supported plate and δ' and b' the thickness and width of the supporting plate.

The method of applying these simple formulas to columns used in bridge building will be shown in Chapter 55.

- (e) The Edges a are Elastically Built-in, the other Edges Completely Free.
Here again, to further the general investigation, we shall consider a

special case in which the most general conditions of support are realized. Figure 174a shows a symmetrical cross-section, the webs of which are plates, completely free on the one side and elastically built into the flange plates on the other side. The character of the deformation of web and flange involved is indicated in Figure 174b and c. We imagine the y -axis as drawn through the middle of the plate, and the x -axis as coinciding with the supported edge. Here we must return to the general solution of the differential equation upon which the problem is based, i. e., Equation (8). This equation reads

$$w = \cos \frac{n\pi x}{a} [A \cosh k_1 y + B \sinh k_1 y + C \cos k_2 y + D \sin k_2 y]$$

where k_1 and k_2 are given by the absolute values of the expressions in Equation (7).

The boundary conditions from which to determine the four constants A to D are

$$\begin{aligned} \text{for } y = 0, w = 0, \text{ and } M_y = \nu \frac{\partial w}{\partial y} \\ \text{for } y = b, M_y = 0, \text{ and } Q_y = 0 \end{aligned}$$

[Translator's Note: Q_y is the transverse shear along the free edge.] At the elastically built-in edge (assuming the supporting part to be free from compressive forces) the edge moment must be proportional to the rotation of the end tangent parallel to the y -axis, as in Case I. Developing this edge condition leads further to the relation (10) obtained previously. For M_y and Q_y the theory of elasticity yields the expressions

$$M_y = \frac{EJ}{1 - m^2} \left(\frac{\partial^2 w}{\partial y^2} + m \frac{\partial^2 w}{\partial x^2} \sqrt{\tau} \right), \quad Q_y = \frac{EJ}{1 - m^2} \frac{\partial}{\partial y} \left(\frac{\partial^2 w}{\partial y^2} + (2 - m) \frac{\partial^2 w}{\partial x^2} \sqrt{\tau} \right)$$

By reasoning similar to that used in setting up Equation (2), the buckling coefficient τ is introduced here as before. Substitution of the solution into the first two boundary conditions yields the relationships

$$\begin{aligned} A + C &= 0 \\ Bk_1 + Dk_2 - \zeta \frac{b}{2} (Ak_1^2 - Ck_2^2) &= 0^* \end{aligned}$$

from which we get

$$\begin{aligned} A &= -C \\ B &= -D \frac{k_2}{k_1} - C \zeta \frac{b}{2} \frac{k_1^2 + k_2^2}{k_1} \end{aligned}$$

* Since here we do not measure the ordinate y from the middle, as in Case I, but from the edge (and thus in the opposite direction than formerly), Equation (10) assumes the form

$$\left(\frac{\partial w}{\partial y} \right) - \zeta \frac{b}{2} \frac{\partial^2 w}{\partial y^2} = 0$$

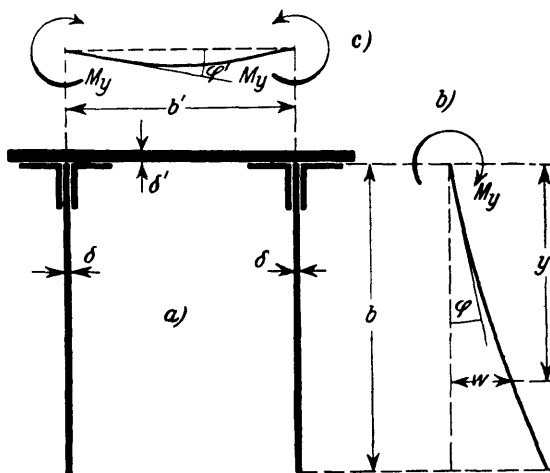


Figure 174

so that the general solution assumes the form

$$w = \cos \frac{n\pi x}{a} \left[C \left(\cos k_2 y - \cosh k_1 y - \zeta \frac{b}{2} \frac{k_1^2 + k_2^2}{k_1} \sinh k_1 y \right) + D \left(\sin k_2 y - \frac{k_2}{k_1} \sinh k_1 y \right) \right]$$

With this value of w , the two other boundary conditions result in the equations

$$\begin{aligned} & C \left\{ \left[k_2^2 + \bar{m} \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 \right] \cos k_2 b + \left[k_1^2 - \bar{m} \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 \right] \cosh k_1 b \right. \\ & \quad \left. + \zeta \frac{b}{2} \frac{k_1^2 + k_2^2}{k_1} \left[k_1^2 - \bar{m} \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 \right] \sinh k_1 b \right\} \\ + D \left\{ \left[k_2^2 + \bar{m} \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 \right] \sin k_2 b + \left[k_1^2 - \bar{m} \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 \right] \frac{k_2}{k_1} \sinh k_1 b \right\} = 0, \end{aligned}$$

$$\begin{aligned} & C \left\{ \left[k_2^2 + (2 - \bar{m}) \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 \right] k_2 \sin k_2 b - \left[k_1^2 - (2 - \bar{m}) \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 \right] k_1 \sinh k_1 b \right. \\ & \quad \left. - \zeta \frac{b}{2} \frac{k_1^2 + k_2^2}{k_1} \left[k_1^2 - (2 - \bar{m}) \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 \right] k_1 \cosh k_1 b \right\} \\ - D \left\{ \left[k_2^2 + (2 - \bar{m}) \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 \right] k_2 \cos k_2 b + \left[k_1^2 - (2 - \bar{m}) \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 \right] k_2 \cosh k_1 b \right\} = 0 \end{aligned}$$

Then, as may easily be checked by means of Equation (7)

$$\begin{aligned} r &= k_2^2 + \bar{m} \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 = k_1^2 - (2 - \bar{m}) \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2, \\ t &= k_1^2 - \bar{m} \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 = k_2^2 + (2 - \bar{m}) \sqrt{\tau} \left(\frac{n\pi}{a} \right)^2 \end{aligned}$$

If in addition we write

$$\zeta \frac{b}{2} \cdot \frac{k_1^2 + k_2^2}{k_1} = h$$

then the foregoing equations assume the form

$$\begin{aligned} & C [r \cos k_2 b + t \cosh k_1 b + ht \sinh k_1 b] + D \left[r \sin k_2 b + t \frac{k_2}{k_1} \sinh k_1 b \right] = 0 \\ & C \left[t \frac{k_2}{k_1} \sin k_2 b - r \sinh k_1 b - hr \cosh k_1 b \right] - D \left[t \frac{k_2}{k_1} \cos k_2 b + r \frac{k_2}{k_1} \cosh k_1 b \right] = 0 \end{aligned}$$

The condition that the determinant $\Delta = 0$, as in Case I, yields the buckling condition, namely

$$\begin{aligned} & \frac{2rt}{\cosh k_1 b \cdot \cos k_2 b} + (r^2 + t^2) - \\ & \frac{r^2 k_1^2 - t^2 k_2^2}{k_1 k_2} \tanh k_1 b \cdot \tan k_2 b + h \left[t^2 \tanh k_1 b - r^2 \frac{k_1}{k_2} \tan k_2 b \right] = 0 \end{aligned} \quad (27)$$

The relation between μ and $\frac{\alpha}{n\sqrt{\tau}}$ established by the buckling condition may be represented with sufficient accuracy, as in Case I, by a rational expression of

the form of (20).* Consequently the critical stress may be represented in this case too, by an expression of the form of Equation (21):

$$\sigma_k = \frac{\pi^2 E \sqrt{\tau}}{12(1-\bar{m}^2)} \left(\frac{\delta}{b}\right)^2 \left[\left(\frac{n\sqrt[4]{\tau}}{\alpha}\right)^2 + p + q \left(\frac{\alpha}{n\sqrt[4]{\tau}}\right)^2 \right]$$

The coefficients p and q in this equation are dependent only upon the fixation factor ζ and are computed as follows by means of the exact Equation (27):

$\zeta = 0$	0.1	0.2	1
$p = 0.570$	0.475	0.452	0.425
$q = 0.125$	0.115	0.098	0

Intermediate values of ζ can be obtained by direct interpolation. Proceeding from considerations similar to those in Case I, for $\zeta = 1$, those values of p and q are inserted which correspond to moment-free support along one edge ($\zeta = \infty$). Moreover, these differ very little from the calculated values. (The correct values for $\zeta = 1$ would be $p = 0.427$ and $q = 0.010$.)

From the validity of Equation (21) also follows the validity of the simple relation determined in Case I, Equations (22) to (24), namely:

The limiting aspect ratio α' is

$$\alpha' = \sqrt[4]{\frac{\tau}{q}} \sqrt{n(n+1)} \quad (22)$$

The aspect ratio α_0 corresponding to the minimum value of σ_k is

$$\alpha_0 = \sqrt[4]{\frac{\tau}{q}} \quad (23)$$

and thus

$$\min \sigma_k = \frac{\pi^2 E \sqrt{\tau}}{12(1-\bar{m}^2)} \left(\frac{\delta}{b}\right)^2 (p + 2\sqrt{q}) \quad (24)$$

It follows from Equation (23) that with decreasing fixation, since q will decrease very rapidly, α_0 will constantly increase, reaching infinity with moment-free support (simple support - Translator), along the one edge a , because $q = 0$. With increasing α , on the other hand, $\min \sigma_k$ constantly decreases and asymptotically approaches the limit

$$\min_{n=\infty} \sigma_k = \frac{\pi^2 E \sqrt{\tau}}{12(1-\bar{m}^2)} \left(\frac{\delta}{b}\right)^2 p \quad (28)$$

* If, as in Section (d), we substitute the coefficient μ , we will have

$$k_1 b = \frac{n\pi}{\alpha} \sqrt[4]{\tau} \sqrt{\mu+1}, \quad k_2 b = \frac{n\pi}{\alpha} \sqrt[4]{\tau} \sqrt{\mu-1}$$

$$r = \frac{1}{b^2} \left[\left(\frac{n\pi}{\alpha}\right)^2 \sqrt{\tau} (\mu-1+m) \right]$$

$$t = \frac{1}{b^2} \left[\left(\frac{n\pi}{\alpha}\right)^2 \sqrt{\tau} (\mu+1-m) \right]$$

If the one edge is free to rotate, the plate will bulge in one half-wave, regardless of what its length a may be. However, no matter how great the length of the plate may be, the value of the buckling stress will not increase above the upper limit value, Equation (28). When one side of the plate is elastically built-in, several half-waves will form if the length a is sufficiently great.

If it is desired to have equal factors of safety on local buckling of the plates and on general buckling of the whole column, then in analogy with Case I, the following condition must hold:

$$\frac{b}{\delta} = \frac{\sqrt{p + 2\sqrt{q}}}{2\sqrt{3(1-\bar{m})^2\sqrt{\tau}}} \left(\frac{l}{i}\right) = \frac{0.303}{\sqrt{\tau}} \sqrt{p + 2\sqrt{q}} \left(\frac{l}{i}\right) \quad (29)$$

If we take into consideration, as in the foregoing, the relation between the buckling coefficient τ and the slenderness ratio l/i , and if we further replace the radical $\sqrt{p + 2\sqrt{q}}$ with $1.13 - 0.48\zeta$, which simple expression approximately satisfies the series of p and q values given in the foregoing, we obtain the following formulas, appropriate for practical use:

For l/i from 10 to 105

$$\frac{b}{\delta} \leq \left(3.42\sqrt{\frac{l}{i}} - 3.30\right)(1.13 - 0.48\zeta)$$

For $l/i > 105$

$$\frac{b}{\delta} \leq 0.303\frac{l}{i}(1.13 - 0.48\zeta)$$

(30)

55. Application to Compression Sections Common in Bridge Practice*

(a) Closed Box Sections

Compression members of the section shown in Figure 175 occur in bridge construction usually with slenderness ratios l/i less than 105. The dimensions of their walls are to be designed individually by Equation (26), i.e., by the formula

$$^{**} \left(\frac{b}{\delta}\right)_{\text{erf}} = \left(3.42\sqrt{\frac{l}{i}} - 3.30\right)(2.64 - 0.64\sqrt{\zeta})$$

ζ is defined by the relation

$$\zeta = \frac{\delta^3 b'}{\delta'^3 b}$$

With thin web plates and heavy flange plates (cross section, Figure 175a,) δ and b refer to the dimensions of the web plate. If the webs are more rigid than the flange plates then the dimensions of the latter are to be used for δ and b in the calculation (cross section, Figure 175b). The fixation factor ζ , therefore, must always be

* Translation of Chapter 55, pp. 232-239.

** Translator's Note: b/δ_{erf} is the value required by the previous considerations.

less than, or at most equal to unity. The ratio b/δ , depending upon the slenderness ratio and the fixation factor ζ , varies between 28 and 61.

The installation of transverse diaphragms to lessen the danger of buckling is useless with sections of this type. If the diaphragms are widely spaced (for example at distances several times the width of the plates b) as is customary, several half-waves will be formed between every two bulkheads when buckling occurs, the length of which will differ only slightly from that unfavorable length which corresponds to the value σ_k upon which our design formulas are based. It would be possible to achieve a slight increase in strength, 6 per cent

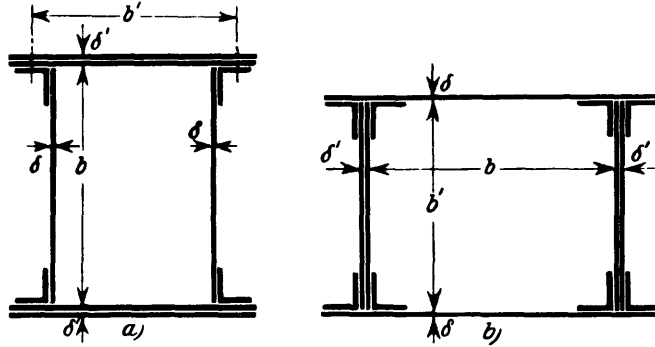


Figure 175

at most, if the bulkheads were arranged so closely with spacings of $a = 1.414 \sqrt[4]{\tau}$ as to permit the formation of one half-wave having the greatest possible length. With columns of $l/i = 25$, for example, this would be accomplished with $a = 0.76b$, and with $l/i = 60$, $a = 1.13b$. To space the transverse plates so closely for so slight an increase in strength would be extremely uneconomical.

We therefore state that with full-walled box cross sections, transverse diaphragms spaced at intervals two or more times the width b of the plate are without noticeable effect on the strength of the column. Therefore diaphragms should be avoided.

(b) Latticed Box Sections and I-Sections

The webs of the sections represented in Figure 176 act as plates simply supported along the longitudinal edges, since there is only slight resistance to torsion of the webs except near the ends of the member. The size of the b/δ ratio is then determined with $\zeta = 1$ from the Formulas (26).

$$\left. \begin{array}{l} \text{for } l/i \text{ from } 10 \text{ to } 105, \\ \text{for } l/i > 105 \end{array} \right\} \begin{array}{l} \left(\frac{b}{\delta}\right)_{\text{erf}} \leq 6.84 \sqrt{\frac{l}{i}} - 6.60 \\ \left(\frac{b}{\delta}\right)_{\text{erf}} \leq 0.606 \left(\frac{l}{i}\right) \end{array} \quad (31)$$

With section forms as shown in Figure 176a, whose l/i may be between 25 and 60, b/δ will be between 28 and 46.

With rolled I-beams and channel irons the b/δ ratio will be,
 for [- beams N P 8 - 60 between 30 and 28
 for] - irons N P 8 - 30 between 13 and 30
 for Differding I-beams 18B to 60B between 21 and 29

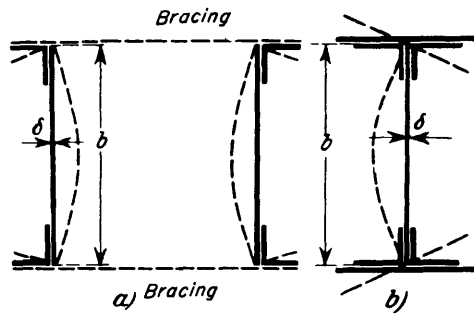


Figure 176

Local buckling is in these cases precluded at slenderness ratios greater than 25.

The remarks of the previous Section regarding the installation of transverse diaphragms in members with box sections apply likewise to sections of the type of Figure 176a.

Example. In the course of tests of mild steel members for the new Quebec bridge at Phoenixville, Pa., in 1912, six columns of the section shown in Figure 177 were tested to failure under compression.* They consisted of hard mild steel having a yield point of 3.0 t/cm² and a tensile strength of 4.55 t/cm². Their ends were mounted in cylindrical clamps so that with the given column dimensions they were free to buckle only in the direction of the free axis.

With an area $F = 296.3 \text{ cm}^2$, radius of gyration $i_x = 20.55 \text{ cm}$, and length $l = 286 \text{ cm}$, it is found that $l/i_x = 13.92$, and by the first of the two formulas (31).

$$\left(\frac{b}{\delta}\right)_{\text{erf}} = 6.84 \sqrt{\frac{l}{i}} - 6.60 = 18.9$$

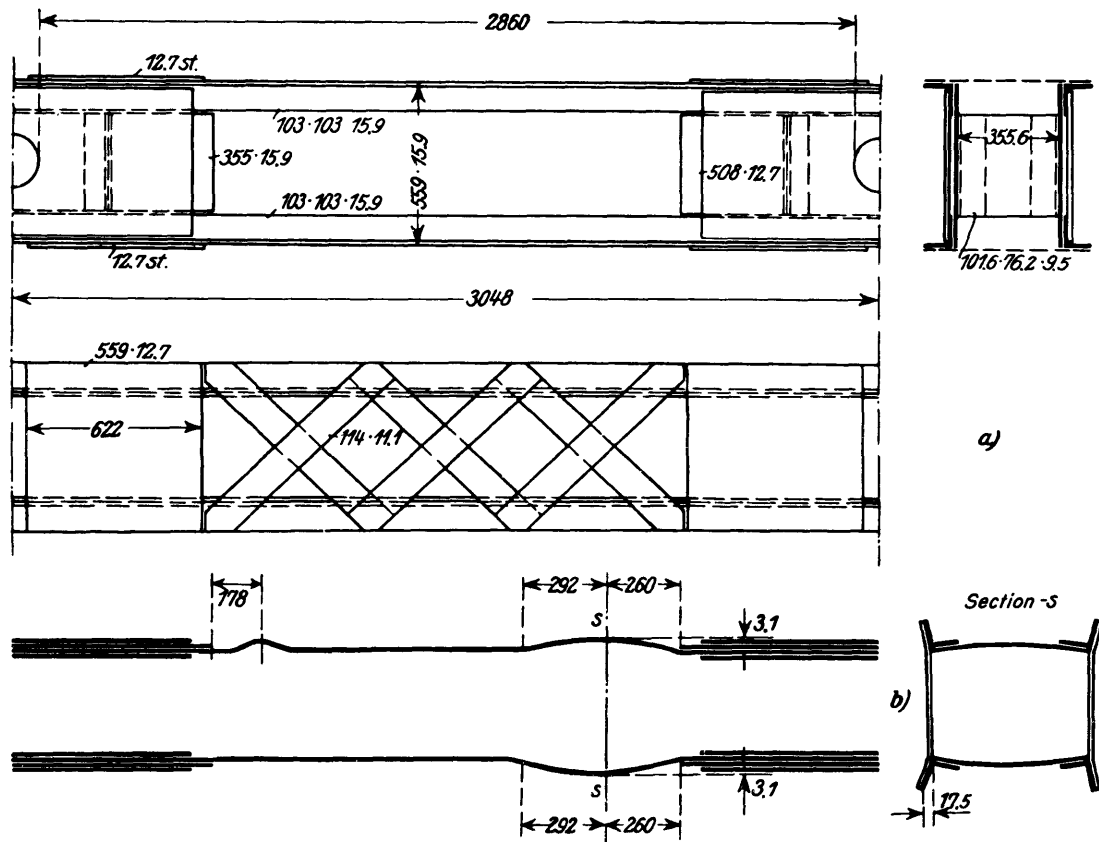


Figure 177

* See "Die Knickfestigkeit" (Buckling Strength), Mayer, Dr. Ing. R.: p. 424

The actual ratio, however, was

$$\frac{55.9}{1.59} = 35.2$$

It is therefore conceivable that all six of these columns, differing only as to the stiffening plates at the ends, might have failed prematurely due to buckling of the web plates. In spite of the high yield point, which indicated a somewhat greater strength than that obtained from Tetmajer's formula ($\sigma_k = 2.94 \text{ t/cm}^2$) the column illustrated in Figure 177 buckled at 2.66 t/cm^2 .

From Equation (18") we find the lower limiting value σ_k of the column stress at which the plating begins to buckle, using

$$r = \frac{\left(\frac{b}{\delta}\right)^4}{2190 \cdot 10^4} + 6.2 = 6.2702$$

It is

$$\min \sigma_k = \frac{r}{2} - \sqrt{\frac{r^2}{4} - 9.61} = 2.67 \text{ t/cm}^2$$

in very close agreement with the observed buckling stress of 2.66 t/cm^2 .

Figure 177b shows test Column 44 in axial section and in cross section after testing.

Second Example. Two further test columns of material similar to that in the foregoing example are shown in cross-section in Figure 178. This test also was conducted in connection with the construction of the Quebec bridge in 1913. The columns were so mounted in clamps that buckling was possible only in the direction of the y-axis.

The length of the column was 1036 cm and the distance between the points of contact of the clamps, which we shall take as the buckling length was 1018 cm. The lattices consisted of crossed flat bars 108 x 15.9. The ends of the columns were reinforced with diaphragms and with doubler plates on the webs.

The area $F = 227 \text{ cm}^2$, and the radius of gyration $i_x = 20.08 \text{ cm}$, whence $l/i_x = 50.7$, and the buckling stress is $\sigma_k = 2.520$ (from the Tetmajer formula).

As before, we calculate the required ratio b/δ from Equation (31), and get

$$\left(\frac{b}{\delta}\right)_{\text{ert}} = 6.84 \sqrt{50.7} - 6.60 = 4.21$$

The actual width-thickness ratio was $55.9/1.27 = 44$, which is somewhat too high, suggesting that the strength of the column may be less than that determined from the Tetmajer formula. This is borne out by test. The average of both test values is $\sigma_k = 2.44 \text{ t/cm}^2$

From Equation (18") we now find the fiber stress $\min \sigma_k$ at which the plates

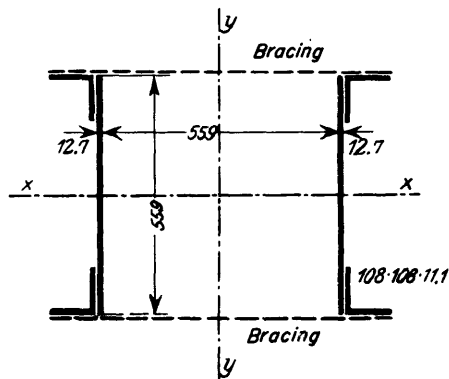


Figure 178

buckle when the slenderness ratio is $b/\delta = 44$. We discover

$$r = \frac{\left(\frac{b}{\delta}\right)^4}{2190 \cdot 10^4} + 6.2 = 6.370$$

and

$$\min \sigma_k = \frac{r}{2} - \sqrt{\frac{r^2}{4} - 9.61} = 2.458 \text{ t/cm}^2$$

in good agreement with the results of the tests in which the failure of the column apparently started in the web plates, since at some points they had separated from the flange angles.

(c) H-Shaped Cross Sections

The web of the H-section, as a rule, may be regarded as completely built-in, since the strong side walls which are restrained from deflecting by the flange stiffening angles and the latticing, can deflect only slightly (Figure 179). If the columns are very long it is essential to reinforce the lateral rigidity of the flange angles by installing battens or lattices, connecting the flange angles with each other at intervals of 10 to 15 times the width of the let of the flange stiffening angle. With $\zeta = 0$ (complete fixation), Formula (26) gives the following design rule of

for l/i from 10 to 105

$$\left(\frac{b}{\delta}\right)_{\text{erf}} = 9 \sqrt{\frac{l}{i}} - 8.7. \quad (32)$$

With the usual slenderness ratios $l/i = 25$ to 60 , b/δ will then be between 36 and 61.

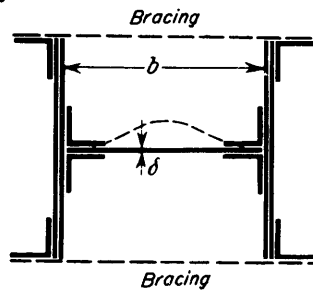


Figure 179

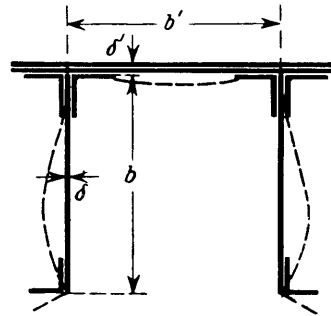


Figure 180

(d) Π -Section with Angles Stiffening the Free Edges.

Another important form of section to be discussed here is illustrated in Figure 180. The tops of the webs appear to be elastically fixed to the flange plates, and the bottoms attached to the freely rotatable longitudinal stiffening angles, as these angles are unable to offer any resistance to twist. Naturally precautions must be taken so that the stiffening angles will have sufficient lateral rigidity to prevent lateral buckling of the webs. This is achieved by providing battens or lattices at proper intervals. This cross-bracing is to be spaced

according to the rules of Section 10, part 40*, in such a manner that the angles, by themselves, will have the same general buckling strength, in terms of stress, as possessed by the section as a whole.

In calculating the strength of the column as a whole, then, the stiffening angles are to be included as fully effective components of the cross section. In order to allow for the fact that the web is built-in on one side only, we find an average value for the b/δ ratio involved, by striking the arithmetical mean for b/δ from the value of b/δ with $\zeta = 1$ (simply supported) and from the value of b/δ for a ζ corresponding to the stiffness of the flange plate and the webs. Thus we get

$$\frac{b}{\delta} = \frac{1}{2} \left[\left(3.42 \sqrt{\frac{l}{i}} - 3.30 \right) (2.64 - 0.64 \sqrt{\zeta} + 2) \right]$$

or

for l/i from 10 to 105

$$\left(\frac{b}{\delta} \right)_{\text{eff}} = \left(3.42 \sqrt{\frac{l}{i}} - 3.30 \right) (2.32 - 0.32 \sqrt{\zeta}) \quad (33)$$

For the commonly encountered slenderness ratios of l/i between 25 and 60, b/δ varies between 28 and 54, depending upon the thickness and width of the flange plates.

Installation of diaphragms between the ends of the member to increase the buckling strength of the webs is useless for the reasons given under Paragraph a, page 43. Such bulkheads at best serve only to develop the buckling lengths of the marginal angles. On the other hand, it is advisable to install transverse diaphragms at the ends of the member near the joints in order to render the tube of the member a three-dimensional and rigid whole.

(e) Cruciform and Angle Sections

These two forms of sections are shown in Figure 181. Since one leg of the equilateral angle can not prevent deformation of the other, both being in a state of unstable equilibrium simultaneously, the cross section will twist in the middle of the member as shown in Figure 181a. The same applies to the cruciform section (Figure 181b). With $\zeta = 1$ (one edge simply supported), we get from the Equations (30)

for l/i from 10 to 105

$$\left(\frac{b}{\delta} \right)_{\text{eff}} = 2.22 \sqrt{\frac{l}{i}} - 2.15 \quad (34)$$

for $l/i > 105$

$$\left(\frac{b}{\delta} \right)_{\text{eff}} = 0.197 \left(\frac{l}{i} \right)$$

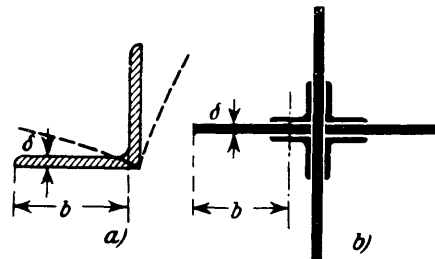


Figure 181

Since in the case of simple angles the slenderness ratio will hardly be less than 30 (usually it is much higher), the most unfavorable b/δ value, according to Equation

* Not included in this translation.

(34) will be

$$\frac{b}{\delta} = 10$$

where δ is the inside height of the legs. The angle sections used in bridge construction, with the exception of some angles with unequal legs, satisfy this requirement.

With cruciform sections of the type illustrated in Figure 181b, it is permissible, in order to allow for the reinforcement of the angles, to measure the height b from the center of the angle to the edge of the plate (Figure 181b).

(f) T- Sections

In this category belong the sections depicted in Figure 182a and 182b. Since the torsional resistance of the two legs riveted together (Figure 182a) is extremely slight, the section in the middle of the member may twist as indicated in Figure 181a. This is also true of T-shaped chords as shown in Figure 182b. Here,

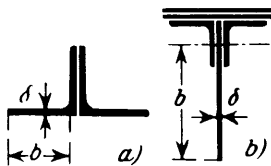


Figure 182

likewise, the flange plates twist when the web reaches a state of unstable equilibrium. Since ζ must be taken as unity, the formulas of Equation (34) apply, in which case b is to be measured from the center of the rivets to the bottom edge of the plate in order to allow for the reinforcement of the flange angles. For short members Equation (34) calls for relatively low values of b/δ . For example, at $l/i = 25$, $b/\delta = 9$; but this ratio increases rapidly with increasing slenderness of the flange. With $l/i = 50$, for instance, it is 13.5. Bridge practice often employs higher b/δ ratios, which should be avoided, however. If the free edge of the web plate is stiffened with angles, it is permissible to take a b/δ ratio about twice as high as that permitted by the formulas of Equation (34). This was learned from an investigation of the subject, but its discussion here would lead too far afield.

(g) Π - Shaped Sections

Sections with two webs, as shown in Figure 183, are far more efficient than the T-shaped flange sections considered under paragraph f, since the fixation to the flange plate is exceedingly effective in supporting the webs. With the fixation coefficient

$$\zeta = \frac{\delta^2 b'}{\delta'^3 b}$$

Equation (30) applies here as the design formula:

$$\left(\frac{\bar{b}}{\delta}\right)_{\text{erf}} = \left(3.42 \sqrt{\frac{l}{i}} - 3.30\right) (1.13 - 0.48 \zeta)$$

Should ζ become greater than unity, as it may in the case of thick webs and a thin flange plate, it is to be taken as unity and at the same time it must be considered whether the flange plate is sufficiently thick. ζ is to be determined using the full height b of the web plate, but due to the clamping effect of the angles, the ratio \bar{b}/δ may be taken with respect to the free portion of the web plate, from the middle of the leg of the angle to the lower edge of the plate.

In the case of plates supported at one edge the wave-lengths in buckling are considerably greater than in plates with both ends supported (when $\zeta = 1$ the wave-length will be equal to the length of the column. It is possible then to effect a considerable increase in strength by reducing the wave-length through the installation of bulkheads in those cases in which ζ is close to unity, that is, when the flange plates are wide and thin. For this purpose we start with the buckling stress Equation (21), which still includes the ratio $\alpha = a/b$ of the plate length to the plate width. If we now take a as the distance between the diaphragms, this equation will permit us to find the effect of their spacing. For $\zeta = 1$, $q = 0$ and $p = 0.425$, and with $n = 1$ -- since only one wave will be formed between two bulkheads--Equation (21) simplified, becomes

$$\sigma_k = \frac{\pi^2 E \sqrt{\tau}}{12(1-m^2)} \left[\frac{\sqrt[4]{\tau}}{\alpha} + 0.425 \right] \left(\frac{\delta}{b} \right)^2$$

The condition for equality of column strength and local buckling strength is the equation

$$\frac{\pi^2 E \tau}{\left(\frac{l}{i} \right)^2} = \frac{\pi^2 E \sqrt{\tau}}{12(1-m^2)} \left[\frac{\sqrt[4]{\tau}}{\alpha} + 0.425 \right] \left(\frac{\delta}{b} \right)^2$$

from which, with $\alpha = a/b$, it follows that

$$\left(\frac{b}{\delta} \right) = \frac{\sqrt{\frac{b \sqrt[4]{\tau}}{a} + 0.425}}{2 \sqrt{3(1-m^2) \sqrt{\tau}}} \frac{l}{i} = \frac{0.303}{\sqrt[4]{\tau}} \left(\frac{l}{i} \right) \sqrt{\frac{b}{2a} + 0.425}$$

For l/i between 20 and 60, $\sqrt[4]{\tau}$ increases from 0.483 to 0.800. We take the lowest value, expressing it in round numbers as 1/2, and insert it under the radical sign. The coefficient of the radical may be written with sufficient accuracy as

$$\frac{0.303}{\sqrt[4]{\tau}} \frac{l}{i} = 3.42 - 3.30 \frac{l}{i}$$

with the final result,

$$\frac{b}{\delta} = \left(3.42 \sqrt{\frac{l}{i}} - 3.30 \right) \sqrt{\frac{b}{2a} + 0.425} \quad (35)$$

The formulas established in paragraphs 1 to 7 inclusive summarized for clarity in Table 28.

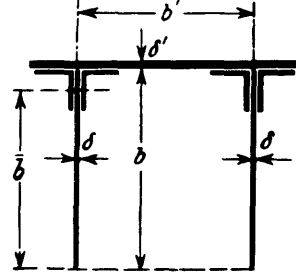
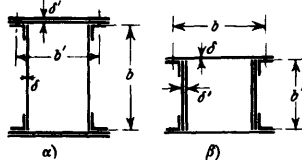
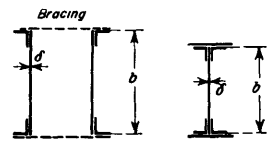
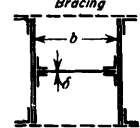
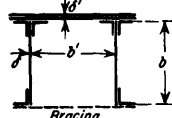
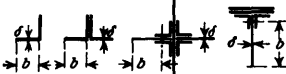
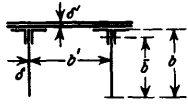


Figure 183

Table 28
The Ratios $(b/\delta)_{\text{erf}}$ for the Surfaces of Compressional Girders

Cross Section of Girder	$(\frac{b}{\delta})_{\text{erf}}$ erf = required	Remarks
	<p>for $l/i = 10$ to 105</p> $(\frac{b}{\delta})_{\text{erf}} = (3.42 \sqrt{\frac{l}{i}} - 3.30) (2.64 - 0.64 \sqrt{\zeta})$ $\zeta = \frac{\delta^3 b'}{\delta'^3 b}$	<p>If each girder in the box beam section is attached to the rider plate by two angle irons, the distance between the girders must be substituted for b' in case α), and similarly with respect to b in case β).</p> <p>Transverse diaphragms only at the ends of the beam; additional transverse plates do not increase the load capacity.</p>
	<p>for $l/i = 10$ to 105</p> $(\frac{b}{\delta})_{\text{erf}} = 6.84 \sqrt{\frac{l}{i}} - 6.60$ <p>for $l/i > 105$</p> $(\frac{b}{\delta})_{\text{erf}} = 0.606 \frac{l}{i}$	<p>Transverse diaphragms only at the ends of the beam. Additional transverse plates do not increase the load capacity. In the case of rolled I-beams and channels (Standard Sections), as well as in the case of Differding I-beams 18B to 60B, buckling is out of the question when $l/i > 25$.</p>
	<p>for $l/i = 10$ to 105</p> $(\frac{b}{\delta})_{\text{erf}} = 9 \sqrt{\frac{l}{i}} - 8.7$	
	<p>for $l/i = 10$ to 105</p> $(\frac{b}{\delta})_{\text{erf}} = (3.42 \sqrt{\frac{l}{i}} - 3.30) (2.32 - 0.32 \zeta)$ $\zeta = \frac{\delta^3 b'}{\delta'^3 b}$	<p>Transverse diaphragms at the ends of the beam only. Additional transverse plates do not increase load capacity.</p>
	<p>for $l/i = 10$ to 105</p> $(\frac{b}{\delta})_{\text{erf}} = 2.22 \sqrt{\frac{l}{i}} - 2.15$ <p>for $l/i > 105$</p> $(\frac{b}{\delta})_{\text{erf}} = 0.197 \frac{l}{i}$	<p>If the T-section is provided with angles then the ratio (b/δ) calculated by the equations in Column 2 will be doubled.</p>
	<p>for $l/i = 10$ to 105</p> $(\frac{b}{\delta})_{\text{erf}} = (3.42 \sqrt{\frac{l}{i}} - 3.30) (1.13 - 0.48 \zeta)$ $\zeta = \frac{\delta^3 b'}{\delta'^3 b}$	<p>In the case of weak sheathing plates $\zeta \sim 1$ it is advisable to provide transverse bulkhead plates. If the distance between them is a then</p> $(\frac{b}{\delta})_{\text{erf}} = (3.42 \sqrt{\frac{l}{i}} - 3.30) \sqrt{\frac{b}{2a} + 0.425}$

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