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PLASTIC DEFORMATION AND ENERGY ABSORPTION OF A THIN RECTANGULAR PLATE UNDER HYDROSTATIC PRESSURE

by

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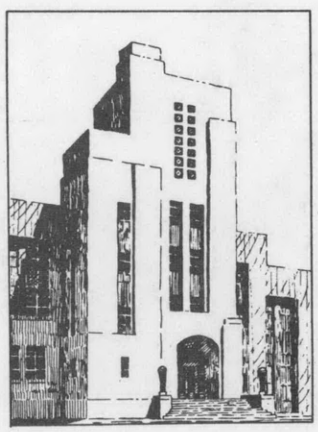

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PERSONNEL

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ABSTRACT

On the basis of two assumptions, one concerning the shape of thin, clamped, circular or rectangular plates under pressure, and the other concerning the invariant character of the curve of energy absorption against areal strain of thin plates, it is shown that the central deflections of two plates of the same material and the pressures which produce these deflections are *related* to each other by *simple algebraic equations* involving their corresponding physical dimensions. Thus any one thin, clamped, circular or rectangular plate serves as a *model* for predicting pressure and deflection for any other such plate of the same material. By use of measured tensile stresses the equations are made to include plates of different materials.

Experimental pressure-deflection data are used to check this theory. Good agreement is obtained.

As a theoretical check the theory is applied to membranes so that one circular or rectangular membrane is used as a model for another. The result is in good agreement with exact membrane theory.

A simple algebraic formula for pressure against deflection for membranes is derived for rectangular membranes, which is found to agree well with results of exact membrane theory. By use of a concept of "average tension" this formula also has validity when applied to bulged thin clamped plates.

INTRODUCTION

There is available for reference (1)* (2) (3) a considerable amount of data showing the relationship of lateral pressure and deflection in clamped circular plates. Some data also exist on certain clamped rectangular plates (4) (5). One may well ask if a new experiment is necessary for each circular or rectangular plate depending on its radius or length and width, and on its thickness and material. Fortunately no such triply or quadruply multiplied set of experiments is required and it is the purpose of this report to establish a mode of transfer of results from one circular or rectangular plate to another such plate.

For this purpose a method is described for correlating central deflection and lateral pressure, in the plastic range, for clamped rectangular

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* Numbers in parentheses indicate references on page 18 of this report.

diaphragms with the corresponding quantities for clamped circular diaphragms so that predictions may be made of pressure, energy, and deflection for a thin rectangular plate from the experimental data on a single thin circular plate. Such correlation also makes possible the prediction of the action of one rectangular plate from that of another rectangular plate, through the intermediary of a hypothetical equivalent circular plate. The theory given here meshes with the theory given in TMB Report R-248 (1), where for circular plates, universal curves of energy and pressure are shown plotted on central deflection. These curves were obtained from experimental data.

The use of "model plates" in this manner makes it possible to bypass the lengthy calculation necessary to solve the equations which describe the action of the clamped rectangular plate under pressure when the material is flowing plastically.*

As a basis for the theory given in this report two assumptions are made.

ASSUMPTION 1

The plates deflect parabolically.

For a rectangular plate

$$z = Z_{\square}\left(1 - \frac{x^2}{a_1^{\,2}}\right)\left(1 - \frac{y^2}{a_2^{\,2}}\right) \qquad [1]$$

where, as in Figure 1, z is the deflection of the point (x, y), $Z_{\square}$ is the central deflection, and a_1 and a_2 are the semi-length and semi-width, respectively, of the rectangular plate.

For a circular plate

$$z = Z_0\left(1 - \frac{r^2}{a^2}\right) \qquad [2]$$

where, as shown in Figure 2, z is the deflection at radial distance r, Z_0 is the central deflection, and a is the radius of the plate.

ASSUMPTION 2

The energy absorbed, W, is the same function of the change in area, ΔA, for all thin clamped plates of the same thickness and material under increasing hydrostatic pressure.

This assumption is based on the following considerations. Experimental results on circular plates show that the tension T, that is, the force per unit length, in a biaxially stressed plate, is approximately

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* Such a solution of the problem for the circular plate is included in TMB Report 532 (2) to be published later. In that report the stresses, strains, and deflection for a circular diaphragm, as functions of the radial distance, are computed for various pressures.

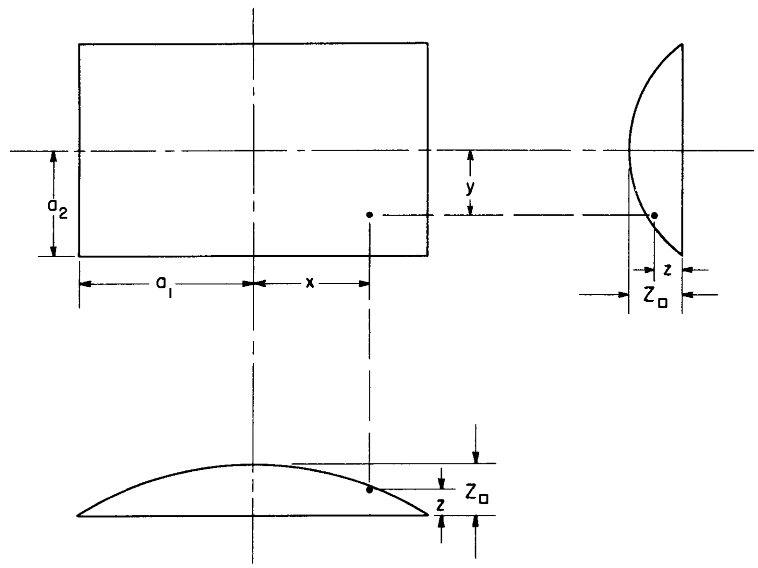

Figure 1 - Diagram Showing Notation for Rectangular Plate

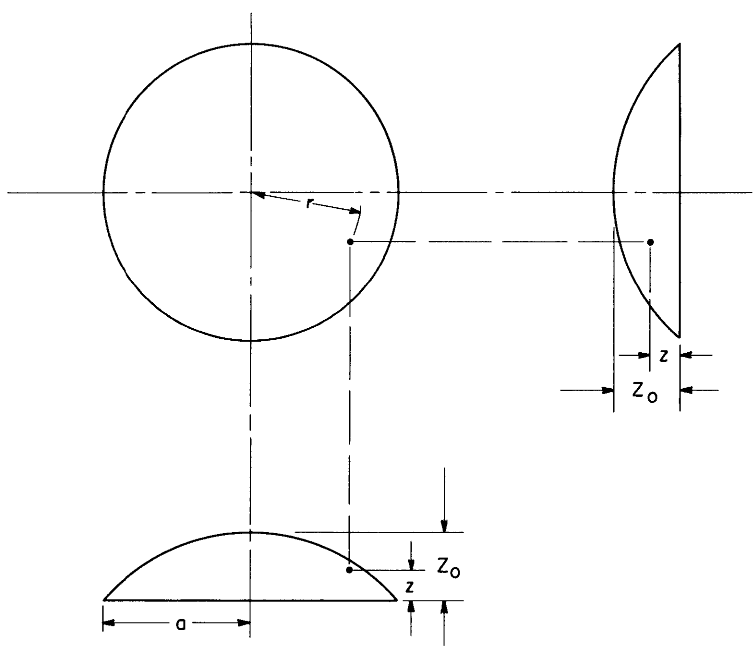

Figure 2 - Diagram Showing Notation for Circular Plate

constant. This constancy is due to the balancing of two effects, the strain-hardening of the material and the reduction in thickness as the stress increases. The assumption that T is constant implies that W, the energy absorbed by a plate as it deflects under increasing hydrostatic pressure, is proportional to ΔA, the total change in area. In fact, for any plate with constant tension, we may write

$$W = T \Delta A$$

This equation might be chosen as an assumption, and pressure-deflection equations for rectangular plates derived from it. However, for the present discussion, the less exacting Assumption 2 is chosen. It should be noted that for two thin clamped plates of the same material with geometrically similar boundaries this assumption should be fulfilled exactly.

To permit comparison of results for plates of different material it is convenient to use, instead of the total energy W, a non-dimensional quantity $W/\overline{W}$, where

$$\overline{W} = \sigma h A \qquad [3]$$

is expressed as the product of a stress σ, the original thickness h, and the original area A. Since the theory given here is often comparative rather than absolute some freedom in the choice of σ exists. With one or two exceptions, σ is taken in the following discussion to be the ultimate stress. In some theoretical derivations it is found convenient to take σ to be the yield stress rather than the ultimate stress. The formal statement of the second assumption modified to include plates of different materials is: **$W/\overline{W}$ is the same function of $\Delta A/A$ for all thin plates.**

In equational form

$$\frac{W}{\overline{W}} = f\left(\frac{\Delta A}{A}\right) \qquad [4]$$

The function f is to be determined by experiment or by theory, depending on the point of view adopted. The consequences of Assumptions 1 and 2 are now considered.

GENERAL PROCEDURE

The method used is essentially as follows: For a clamped plate under pressure, the deflection at a point is known as a function of the central deflection Z and the coordinates of the point by Equation [1]. By integration over the surface of the plate the differential of area $d(\Delta A)$ and the differential of volume dV displaced by the plate as it deforms, as functions of Z and its differential dZ, can be computed. The pressure p is determined by the energy equation

$$W = \int p\,dV$$

which implies that

$$p = \frac{dW}{dV}$$

where, from Equation [4],

$$dW = \frac{\overline{W}}{A}\, f'\left(\frac{\Delta A}{A}\right) d(\Delta A)$$

If f is known, then p may be expressed as a function of Z.

It is deemed desirable, in the present report, to refrain at first from making any assumption concerning the form of the function f. The implications of the statement that f is the same for all diaphragms are studied instead. It is found that values of pressure on different plates are related to each other through the dimensions and central deflections of the plates by algebraic equations.

Later on, to obtain approximate absolute values of pressure and to compare with exact membrane theory it is assumed that Equation [4] becomes simply

$$\frac{W}{\overline{W}} = \frac{\Delta A}{A}$$

The point of view maintained here may be contrasted to one often used in the theory of elasticity. In the theory of elasticity, stress-strain laws are known, so that pressure and energy absorption can be calculated from an assumed deflection function, using conditions of static equilibrium. In this report a relation between energy and areal strain is assumed known, and pressure and energy absorption are calculated from an assumed deflection function, using the principal of conservation of energy.

CORRESPONDING STATES

Suppose the semi-length and semi-width of a thin, clamped, rectangular plate are a_1 and a_2, respectively, and the central deflection is $Z_\square$ as in Figure 1. The numbers a_1, a_2, and $Z_\square$ are considered to describe the *state* of the rectangular plate. Let us consider a deformed circular plate whose radius a_0 and central deflection Z_0 will be determined so that the plate may be said to be in a *corresponding state* with the rectangular plate according to the following definition.

Two plates are considered to be in corresponding states if the areal strain $\Delta A/A$ is the same in both plates.

CORRESPONDING DEFLECTIONS

As a consequence of Assumptions 1 and 2 and the definition of corresponding states, the following statements are valid.

The areal strain for a bulged rectangular plate is

$$\frac{\Delta A_{\square}}{A_{\square}} = \frac{\int_0^{a_1}\int_0^{a_2} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy - a_1 a_2}{a_1 a_2}$$

where z is given by Equation [1]. Therefore, approximately,

$$\frac{\Delta A_{\square}}{A_{\square}} = \frac{\int_0^{a_1}\int_0^{a_2} \frac{1}{2}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] dx\, dy}{a_1 a_2}$$

where powers of the slopes $\partial z/\partial x$ and $\partial z/\partial y$ higher than the second are neglected. The evaluation of this integral yields

$$\frac{\Delta A_{\square}}{A_{\square}} = \frac{16}{45} Z_{\square}^2 \left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right) \qquad [5]$$

Similarly, for a circular plate,

$$\frac{\Delta A_0}{A_0} = \frac{Z_0^2}{a_0^2} \qquad [6]$$

If the two plates are in corresponding states they have equal areal strain and

$$\frac{Z_0^2}{a_0^2} = \frac{16}{45} Z_{\square}^2 \left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right) \qquad [7]$$

Two plates, one rectangular and one circular, which are in corresponding states have central deflections related according to Equation [7]. Central deflections related in this way may be termed *corresponding deflections*.

EQUIVALENT RADIUS

Equation [7] suggests that the equivalent radius $a_{\square}$ of a rectangular plate be defined by the equation

$$\frac{1}{a_{\square}^2} = \frac{16}{45} \left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right) \qquad [8]$$

Equation [5] may then be written

$$\frac{\Delta A_{\square}}{A_{\square}} = \frac{Z_{\square}^{2}}{a_{\square}^{2}} \qquad [9]$$

Equation [7] for corresponding states becomes

$$\frac{Z_0}{a_0} = \frac{Z_{\square}}{a_{\square}} \qquad [10]$$

Equation [10] implies that if two plates in corresponding states have the same *equivalent radius* their central deflections will be equal.

Let Z be the central deflection of a circular or rectangular plate and a its radius or equivalent radius. Since $W/\overline{W}$ is a unique function of $\Delta A/A$ by Assumption 2 and $\Delta A/A$ is a unique function of Z/a by Equations [6] and [9], $W/\overline{W}$ is a unique function, say ϕ, of Z/a. That is, Equation [4] may be replaced by

$$\frac{W}{\overline{W}} = \phi\left(\frac{Z}{a}\right) \qquad [11]$$

The function ϕ may not be known immediately but the importance of Equation [11] lies in the fact that if ϕ is determined experimentally for one clamped thin plate it is the same, according to the theory presented here, for all other clamped thin plates.

EQUIVALENT PRESSURES

The energy W absorbed by a plate under increasing pressure p is related to the volume V, displaced by the plate as it deforms, by the equation

$$dW = p\,dV \qquad [12]$$

The volume V, for the rectangular plate, is

$$V = 4\int_0^{a_1}\int_0^{a_2} z\,dx\,dy$$

where z is given by Equation [1]. Evaluating this integral,

$$V = \frac{16}{9}\,a_1 a_2 Z_{\square} \qquad [13]$$

Similarly, for the circular plate

$$V = \frac{1}{2}\pi a_0^2 Z_0$$

If Equation [11] is differentiated and $Z_\square$ and $a_\square$ are substituted respectively for Z and a

$$\frac{dW}{W} = \phi'\left(\frac{Z_\square}{a_\square}\right) d\left(\frac{Z_\square}{a_\square}\right)$$

or, by Equation [12]

$$\frac{p\,dV}{\overline{W}} = \phi'\left(\frac{Z_\square}{a_\square}\right) d\left(\frac{Z_\square}{a_\square}\right) \qquad [14]$$

But, by Equation [13]

$$dV = \frac{16}{9}\, a_1 a_2\, dZ_\square$$

and, by Equation [3]

$$\overline{W} \doteq \sigma h A = 4\sigma h a_1 a_2$$

The use of these two expressions in Equation [14] yields

$$\frac{4}{9}\,\frac{p_\square\, a_\square}{\sigma_\square\, h_\square} = \phi'\left(\frac{Z_\square}{a_\square}\right) \qquad [15]$$

where rectangular subscripts are added to the letters p, σ, and h, to indicate reference to the rectangular diaphragm. A similar calculation for the circular plate shows that

$$\frac{1}{2}\,\frac{p_0\, a_0}{\sigma_0\, h_0} = \phi'\left(\frac{Z_0}{a_0}\right) \qquad [16]$$

where circular subscripts indicate reference to the circular plate. For corresponding states $Z_0/a_0 = Z_\square/a_\square$, and Equations [15] and [16] imply

$$\frac{8}{9}\,\frac{p_\square\, a_\square}{\sigma_\square\, h_\square} = \frac{p_0\, a_0}{\sigma_0\, h_0} \qquad [17]$$

as the equation for corresponding pressures.

Let P denote the quantity $\frac{8}{9}\frac{p_\square\, a_\square}{\sigma_\square\, h_\square}$ in the case of the rectangular plate, and the quantity $\frac{p_0\, a_0}{\sigma_0\, h_0}$ in the case of the circular plate. We may now write

$$\frac{1}{2} P = \phi'\left(\frac{Z}{a}\right)$$

which is an equation analogous to Equation [11].

The implications of Assumptions 1 and 2 may now be compactly stated as follows:

If $$\frac{Z}{a} = \frac{Z_\square}{a_\square} = \frac{Z_0}{a_0}$$

then

$$\frac{W}{\overline{W}} = \frac{W_{\square}}{4a_1\, a_2\, \sigma_{\square}\, h_{\square}} = \frac{W_0}{\pi a_0^2\, \sigma_0\, h_0} = \phi\left(\frac{Z}{a}\right) \qquad [18]$$

and

$$\frac{1}{2}P = \frac{4}{9}\,\frac{p_{\square}\, a_{\square}}{\sigma_{\square}\, h_{\square}} = \frac{1}{2}\,\frac{p_0\, a_0}{\sigma_0\, h_0} = \phi'\left(\frac{Z}{a}\right) \qquad [19]$$

where ϕ is the same function for all thin clamped rectangular or circular plates.

EXPERIMENTAL RESULTS

Figure 3 shows experimental values of P plotted against Z/a for four circular and four rectangular plates which have been tested at the David Taylor Model Basin.* The plates were of medium steel and of furniture steel. The stress σ was taken to be the nominal ultimate tensile stress in each case, determined by standard tests.

A number of points based on tests made on a rectangular plate (4), as well as several curves derived by membrane theory using a method to be described later in this report are shown in Figure 3.

It is seen that the experimental values of P are reasonably close to each other for all plates with the same Z/a. For example, when Z/a is near 0.25 the values of P differ from their average value by less than 6 per cent for the 8 rectangular and circular plates. Agreement is better among the medium-steel plates than among the furniture-steel plates.

The proximity of the three values of P calculated from membrane theory for three shapes of membranes, circular, square, and rectangular, shown in Figure 3, is a confirmation of the validity of the use of the concepts of equivalent pressures, deflections, and radii for circular and rectangular plates.

The curves of Figure 3 show that for small deflections σ should be taken near the yield stress in order to predict pressures by the membrane theory; for large deflections, σ should be taken near the ultimate stress.

Equations [18] and [19] are now applied to the particular case of circular and rectangular membranes with uniform isotropic tension. The fact that a circular membrane has a circular profile and not a parabolic one as in Assumption 1 introduces a degree of uncertainty as to how to determine ϕ or P satisfactorily. Actually, for a circular plate ϕ is found from known

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* See TMB Report R-142 (3) for a full account of experimental results on circular plates. The complete presentation of experimental results on the rectangular plates will appear in a forthcoming report.

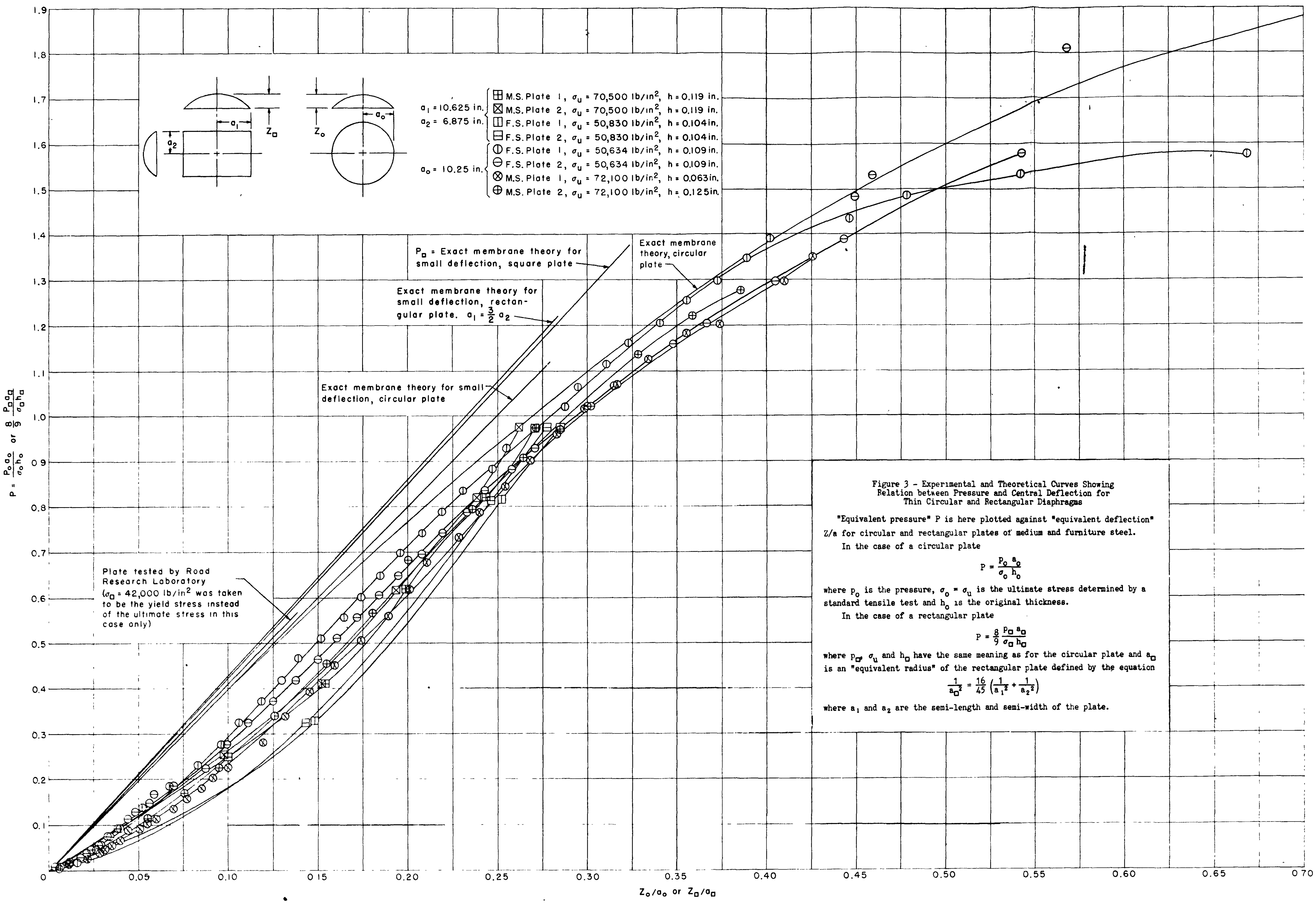

Figure 3 - Experimental and Theoretical Curves Showing Relation between Pressure and Central Deflection for Thin Circular and Rectangular Diaphragms

"Equivalent pressure" P is here plotted against "equivalent deflection" Z/a for circular and rectangular plates of medium and furniture steel.

In the case of a circular plate

$$P = \frac{p_o\, a_o}{\sigma_o\, h_o}$$

where p_o is the pressure, $\sigma_o = \sigma_u$ is the ultimate stress determined by a standard tensile test and h_o is the original thickness.

In the case of a rectangular plate

$$P = \frac{8}{9}\,\frac{p_\square\, a_\square}{\sigma_\square\, h_\square}$$

where $p_\square$, σ_u and $h_\square$ have the same meaning as for the circular plate and $a_\square$ is an "equivalent radius" of the rectangular plate defined by the equation

$$\frac{1}{a_\square^2} = \frac{16}{45}\left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right)$$

where a_1 and a_2 are the semi-length and semi-width of the plate.

formulas to be Z_0^2/a_0^2 and P to be $4\,\dfrac{Z_0/a_0}{1+Z_0^2/a^2}$. Thus $\frac{1}{2}P$ is the derivative of ϕ to the first order in Z_0/a_0 only and not the exact derivative as implied by Equation [19]. This discrepancy is due to the approximate nature of Assumption 1. For additional accuracy at higher deflections we shall take P to be the foregoing expression.

The formulas to be derived in the following by means of Equations [18] and [19] for rectangular membranes, though not exact, have the advantage of simplicity of form. To indicate their validity they will be compared with the results obtained by the exact but more complicated formulas of membrane theory.

APPROXIMATE FORMULAS FOR ESTIMATION OF ENERGY ABSORBED

The exact formula for the energy absorbed by a thin circular plate is, by membrane theory,

$$W_0 = \pi \sigma_0 h_0 Z_0^2$$

where $\sigma_0 h_0$ is an assumed constant tension in the plate. Thus, for a circular membrane deflected parabolically,

$$\phi\left(\frac{Z_0}{a_0}\right) = \frac{W_0}{\overline{W}_0} = \frac{\pi \sigma_0 h_0 Z_0^2}{\pi \sigma_0 h_0 a_0^2} = \frac{Z_0^2}{a_0^2}$$

Therefore by Equation [18], for a thin rectangular membrane deflected parabolically,

$$\phi\left(\frac{Z_\square}{a_\square}\right) = \frac{W_\square}{\overline{W}_\square} = \frac{Z_\square^2}{a_\square^2}$$

or

$$W_\square = \frac{4 a_1 a_2 \sigma_\square h_\square Z_\square^2}{a_\square^2}$$

Consequently,

$$W_\square = \frac{64}{45}\left(\frac{a_2}{a_1} + \frac{a_1}{a_2}\right) \sigma_\square h_\square Z_\square^2$$

In the case of a square membrane

$$W_\square = 2.84\, \sigma_\square h_\square Z_\square^2$$

Thus, at equal deflections the energy absorbed plastically by a square membrane is by this approximate theory independent of its lateral dimensions and is less by about 10 per cent than in the circular membrane where $W_0 = \pi\sigma_0 h_0 Z_0^2$, provided $\sigma_0 h_0 = \sigma_\square h_\square$.

The validity of these formulas for energy absorbed by a rectangular membrane is about as high as the validity of the formulas showing the relation between pressure and deflection which are discussed in the following.

ESTIMATION OF PRESSURE FROM RESULTS ON A SINGLE MODEL PLATE

Equation [19] may be used to make an estimate of the pressure on a given rectangular or circular plate, at a given deflection when the pressure-deflection function for a single rectangular or circular plate is known. To do this P is plotted as a function of Z/a for the model plate. Then Z/a is computed for the given plate and the ordinate P is determined. The pressure on the given plate can then be calculated.

Similar statements are true for energy-deflection curves.

It may be said that, with the theory given here, a single thin circular or rectangular plate becomes a model for any other thin circular or rectangular plate. Of course, it is to be expected that the more nearly the two plates agree in both material and dimensions the more nearly correct will be the predicted results.

APPROXIMATE PRESSURE-DEFLECTION FORMULAS FOR RECTANGULAR MEMBRANES

By Equations [18], [19], and [20]

$$\frac{1}{2} P = \frac{1}{2} \frac{p_0 a_0}{\sigma_0 h_0} = \phi'\left(\frac{Z_0}{a_0}\right) = 2 \frac{Z_0}{a_0}$$

Therefore

$$p_0 = 4 \frac{\sigma_0 h_0 Z_0}{a_0^2}$$

This formula for pressure is correct to the first order in Z_0. Instead of this equation one may begin with the well-known exact equation for pressure on a membrane of tension $\sigma_0 h_0$:

$$p_0 = 4 \frac{\sigma_0 h_0 Z_0}{a_0^2 + Z_0^2}$$

Multiplying both members of this equation by $\frac{1}{2}\frac{a_0}{\sigma_0 h_0}$ there results

$$\frac{1}{2}\frac{p_0 a_0}{\sigma_0 h_0} = 2\frac{\frac{Z_0}{a_0}}{1+\frac{Z_0^2}{a_0^2}}$$

By Equation [19] the right member of this equation is the invariant function $\phi'(Z_0/a_0)$. We now *transfer* these results, by means of Equation [19], to rectangular plates, obtaining

$$\frac{4}{9}\frac{p_\square a_\square}{\sigma_\square h_\square} = 2\frac{\frac{Z_\square}{a_\square}}{1+\frac{Z_\square^2}{a_\square^2}} \qquad [20]$$

or

$$p_\square = 4.5\frac{\sigma_\square h_\square Z_\square}{a_\square^2 + Z_\square^2}$$

This is a simple approximate relation between pressure and deflection for rectangular membranes, derived from the pressure-deflection relation for circular membranes by means of the notion of equivalent radius and pressure. Figure 3 shows values of P plotted against Z/a obtained by the equation

$$P = 2\frac{\frac{Z}{a}}{1+\frac{Z^2}{a^2}}$$

The resulting curve gives the exact relation between pressure and deflection for a circular membrane, and an approximate one for a rectangular membrane, when P, Z, and a are properly interpreted.

SQUARE PLATES

Suppose $a_1 = a_2 = a$. Then

$$a_\square = \sqrt{\frac{45}{32}}\, a$$

Equation [20] becomes, if we drop the term $Z_\square^2$ appearing in the denominator of the right-hand side

$$p_\square = 3.2\frac{\sigma_\square h_\square Z_\square}{a^2} \qquad [21]$$

so that, to a first order, the pressure for the "inscribed" circular plate is 25 per cent greater than that of the square plate at the same central deflection, if $\sigma_0 h_0 = \sigma_\square h_\square$.

EXACT PRESSURE-DEFLECTION FORMULA FOR A RECTANGULAR MEMBRANE

It may be shown (6) that, for a rectangular membrane under tension T, the pressure-deflection relationship given by the exact theory for small deflections is

$$\frac{T^2}{pa_1} = \frac{16}{\pi^3} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^3} (-1)^{\frac{n-1}{2}} \left[1 - \frac{\cosh\left(\frac{n\pi y}{2a_1}\right)}{\cosh\left(\frac{n\pi a_2}{a_1}\right)}\right] \cos\frac{n\pi x}{2a_1}$$

Setting $x = y = 0$, the equation relating pressure and central deflection is found to be

$$\frac{TZ}{pa_1{}^2} = \frac{16}{\pi^3} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^3} (-1)^{\frac{n-1}{2}} \left(1 - \operatorname{sech}\frac{n\pi a_2}{2a_1}\right)$$

or, since

$$\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^3} (-1)^{\frac{n-1}{2}} = \frac{\pi^3}{2^5}$$

it follows that

$$\frac{TZ}{pa_1{}^2} = \frac{1}{2} - \frac{16}{\pi^3} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^3} (-1)^{\frac{n-1}{2}} \operatorname{sech}\frac{n\pi a_2}{2a_1}$$

For a square membrane $a_1 = a_2 = a$ and

$$\frac{TZ}{pa^2} = \frac{1}{2} - \frac{16}{\pi^3} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^3} (-1)^{\frac{n-1}{2}} \operatorname{sech}\frac{n\pi}{2}$$

If the first term only of this series is used

$$\frac{TZ}{pa^2} = 0.294$$

Substitution of σh for T on the assumption that the plate behaves as a membrane implies

$$p_\square = 3.40 \frac{\sigma_\square h_\square Z_\square}{a^2}$$

The coefficient 3.40 in this formula may be compared with the coefficient 3.2 which appears in the approximate Equation [21]. Or, in terms of $a_\square = \sqrt{45/32}\, a$ instead of a,

$$p_\square = 4.78 \frac{\sigma_\square h_\square Z_\square}{a_\square^2}$$

The coefficient 4.78 in this formula may be compared with the coefficient 4.5 which appears in Equation [20]. Thus, for small deflections, exact membrane theory gives a value for pressure in the case of a square plate 6 per cent larger than the value given by the simple formula [20].

PRESSURE-DEFLECTION FORMULA FOR RECTANGULAR MEMBRANES WHEN $a_1/a_2 = 3/2$

A calculation similar to that in the preceeding section shows that,* when $a_1/a_2 = 3/2$,

$$p_\square = \frac{1}{0.179}\,\frac{Z_\square\,\sigma_\square\,h_\square}{a_1^{\,2}} \qquad [22]$$

If $a_1/a_2 = 3/2$ then $a_\square^{\,2} = 45/52\ a_1^{\,2}$. Therefore, in terms of $a_\square$

$$p_\square = 4.84\,\frac{\sigma_\square\,h_\square\,Z_\square}{a_\square^{\,2}}$$

This equation may be compared with Equation [20],

$$p_\square = 4.5\,\frac{\sigma_\square\,h_\square\,Z_\square}{a_\square^{\,2} + Z_\square^{\,2}}$$

Thus, for small deflections, the pressure as predicted by exact membrane theory is again higher by about 7 per cent than the pressure calculated by the simple formulas of the present theory. If the square membrane were used as a model the results would be correct to 1 per cent.

In the case of a rectangular plate tested by Road Research Laboratory (4), $\sigma_y = 42{,}000$ pounds per square inch, $h = 0.066$ inch, $a_1 = 9$ inches, and $a_2 = 6$ inches. If $\sigma_\square = \sigma_y$, Equation [20] becomes to a first order in $Z_\square$,

$$p_\square = 200\,Z_\square$$

For a membrane of the above dimensions under constant tension $T = \sigma_y h = 42{,}000 \times 0.066$ pounds per inch, Equation [22] yields

$$p_\square = 214\,Z_\square$$

By measurement on the test plate, at small deflections

$$p_\square = 207\,Z_\square$$

Thus the value of $p_\square$ calculated by membrane theory, taking σ to be $\sigma_y = 42{,}000$ pounds per square inch, is high by about 3.5 per cent. The value given by the theory of the present report applied to membranes is low by about the same amount.

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* The calculation for this case was carried out by Taylor (5); his result is used here.

COMPARISON WITH EXACT MEMBRANE THEORY WHEN $a_1 = a, \quad a_2 = \infty$

One might believe that the substitution of $a_2 = \infty$ in an equation such as [5] would be mathematically permissible. Investigation shows this to be untrue, due to the fact that the order of the limiting processes used in the calculation of areal strains and volumes may not be interchanged with the limiting process $a_2 \to \infty$. Carrying out anew the integration for the case where $a_2 = \infty$ it is found that

$$\frac{\Delta A_\square}{A_\square} = \frac{2}{3} \frac{Z_\square^2}{a_\square^2}$$

Thus the equivalent radius of the infinitely long rectangular plate, or the plate clamped along two parallel edges and free on the other two, is given by

$$\frac{1}{a_\square^2} = \frac{2}{3} \frac{1}{a^2}$$

Similarly, instead of Equation [17], there is found for this case,

$$\frac{4}{3} \frac{p_\square a_\square}{\sigma_\square h_\square} = \frac{p_0 a_0}{c_0 h_0}$$

Consequently, the pressure-deflection equation, which takes the place of Equation [20] is

$$p_\square = 3 \frac{\sigma_\square h_\square Z_\square}{a_\square^2 + Z_\square^2}$$

Or, since $a_\square^2 = \frac{3}{2} a^2$

$$p_\square = 2 \frac{\sigma_\square h_\square Z_\square}{a^2 + \frac{2}{3} Z_\square^2}$$

which checks the exact equation for a membrane clamped along two parallel edges except for the factor 2/3 which multiplies $Z_\square^2$.

CONCLUSIONS

1. The observed relation between pressure and deflection for a single, thin, clamped, circular or rectangular plate can be used to predict the relation between pressure and deflection of any thin, clamped, circular or rectangular plate. The more nearly the model corresponds to the prototype in dimensions and material, the more accurate the predictions will be.

2. Simple approximate formulas for pressure and energy absorption for circular and rectangular membranes, which are developed here, give results in good agreement with results based on exact membrane theory.

REFERENCES

(1) "Progress Report on Underwater Explosion Research - Bureau of Ships Symbol E139 - Part 14 - Plastic Deformation of and Absorption of Thin Circular Plates under Normal Loading," by A.N. Gleyzal, TMB CONFIDENTIAL Report R-248, May 1944.

(2) "Plastic Deformation of a Thin Circular Plate under Normal Pressure," by A.N. Gleyzal, TMB Report 532, to be published.

(3) "Plastic Strain and Deflection Tests on Clamped Circular Plates," by A.N. Gleyzal, TMB Report R-142, May 1944.

(4) "The Deflection of a Box Model Plate under Static Loads," Road Research Laboratory, Department of Scientific and Industrial Research, CONFIDENTIAL Note ADM 106/GC, Undex 24, March 1944.

(5) "The Distortion under Pressure of a Diaphragm Which Is Clamped along Its Edge and Stressed beyond the Elastic Limit," by G.I. Taylor, Ministry of Home Security, Interdepartmental Committee on Shock Waves, CONFIDENTIAL S.W. 24, September 1942.

(6) "Theory of Elasticity," by S. Timoshenko, McGraw-Hill, New York, N.Y., 1934, p. 246.