NAVY DEPARTMENT
DAVID TAYLOR MODEL BASIN
WASHINGTON, D.C.

PROGRESS REPORT ON UNDERWATER EXPLOSION RESEARCH

Bureau of Ships Symbol E139

PART 21 - VIBRATION OF ELASTICALLY CLAMPED BEAMS

PART 22 - THE EFFECT OF A CONTACT OR NEAR-CONTACT EXPLOSION ON A FREELY SUSPENDED PLANE TARGET

PART 23 - NOTES ON PLASTIC DEFORMATION OF ELLIPTICAL PLATES UNDER HYDROSTATIC PRESSURE

CO[redacted]

44

December 1944 Report R-279

FOREWORD

This progress report deals with various phases of the experimental program at the David Taylor Model Basin which relate to the protection of ships from the effects of underwater explosions. It contains a series of separate reports, each pertaining to one phase of the project. No attempt is made to cover all phases in each progress report, but it is hoped that over a period of several months each subdivision of the program will be adequately represented.

Ultimately the majority of these brief presentations will be included in formal Taylor Model Basin reports. However, it is felt that a definite purpose is served by presenting this information in preliminary form for the use of other technical workers in this field.

The conclusions contained in these reports are to be considered as tentative and subject to possible revision as further data are obtained. However, each part has been carefully reviewed and is believed to be as accurate as possible in the light of current information.

PART 21 - VIBRATION OF ELASTICALLY CLAMPED BEAMS

By R. Specht

INTRODUCTION

The static behavior and the natural frequencies of beams that are either rigidly clamped or simply supported at the ends are well known. In practice, however, the end conditions are often intermediate between the two extremes of zero slope, in the case of rigid clamping, and zero bending moment, as in the case of simple support.

The most general conditions of end fixity of a beam include both *elastic clamping* and *elastic support*. The term *elastic clamping* is used to indicate that the ends of the beam are subjected to a restraining bending moment which may have any value between zero, as for simply supported beams, and the maximum end moment for rigidly clamped beams. By *elastic support* is meant that the ends of the beam are held by springs which allow transverse deflection of the ends. The degree of elastic support may vary from no restraint on transverse motion of the end, as in a free end, to rigid support in which the end experiences no deflection. Although the fixity of actual beams commonly involves both elastic clamping and elastic support, we shall consider here only the case of a beam with rigid support and elastic clamping at each end. Such a beam is defined as one in which the ends do not deflect but do rotate against a restraining end moment which is proportional to the slope at the end.

The degree of end fixity of an elastically clamped rod can be specified in several ways. We can define a dimensionless *clamping factor* C as the ratio of the fundamental frequency f_1 of the beam to the fundamental frequency f_{1H} of a simply supported or hinged beam; that is,

$$C = \frac{f_1}{f_{1H}} \qquad [1]$$

The end clamping can also be characterized by the *stiffness parameter* F which is the ratio of the stiffness S of the beam under concentrated central load to the stiffness S of a simply supported beam. That is

$$F = \frac{S}{S_H} \qquad [2]$$

The stiffness S is defined to be the ratio of a concentrated central load W to the central deflection of the beam. An elastically clamped beam has been defined as one in which the bending moment at the end is proportional to the slope at the end. Since the bending moment is proportional to the curvature of the beam, we can write the end condition as

$$\frac{d^2u}{dx^2} = -\lambda \frac{du}{dx} \qquad [3]$$

where u is the transverse deflection of the beam, λ is the constant of proportionality, and the variable x indicates distance along the axis of the beam. The dimensionless *fixity parameter* is defined as $l\lambda$ where l is the length of the beam. A knowledge of any one of the three indices of fixity, C, F, or $l\lambda$, is sufficient to determine the other two. Relations between these parameters will be derived.

STATIC EQUILIBRIUM OF AN ELASTICALLY CLAMPED BEAM

The static deflection of the elastically clamped beam, Figure 1, can be obtained by superposing the deflection $u_{cl}(x)$ of a rigidly clamped beam and the deflection $u_H(x)$ of a simply supported beam. We write

$$u(x) = Au_{cl}(x) + Bu_H(x) \qquad [4]$$

and determine A and B so as to satisfy the end condition given by Equation [2].

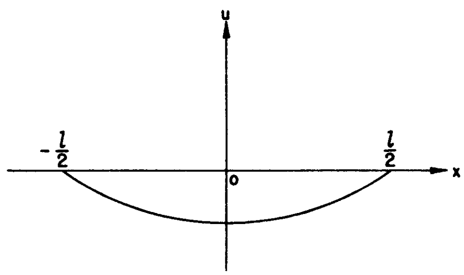

Figure 1 - Deflection Curve of a Beam

Now the deflection curves of a rigidly clamped and of a simply supported beam under the action of a concentrated central load W are given* by

$$u_{cl}(x) = -\frac{W}{192EI}(l^3 - 12lx^2 + 16x^3), \quad 0 \leqq x \leqq \frac{l}{2},$$

$$u_H(x) = -\frac{W}{48EI}(l^3 - 6lx^2 + 4x^3), \quad 0 \leqq x \leqq \frac{l}{2}, \qquad [5]$$

where E is Young's modulus, and I is the moment of inertia of a cross section with respect to the neutral axis. From Equations [5] the stiffness in each case is seen to be

$$S_{cl} = \frac{W}{u_{cl}(0)} = -\frac{192EI}{l^3}, \quad S_H = \frac{W}{u_H(0)} = -\frac{48EI}{l^3} \qquad [6]$$

---

* These formulas can be derived from others given by Roark; see Reference (1),** Chapter 8.
** Numbers in parentheses indicate references on page 19 of this report.

One equation for the determination of the constants A, B comes from Equations [2] and [4]. We get

$$\frac{1}{S} = \frac{1}{FS_H} = \frac{u(0)}{W} = \frac{Au_{cl}(0) + Bu_H(0)}{W} = \frac{A}{S_{cl}} + \frac{B}{S_H}$$

or by use of Equation [6]

$$\frac{1}{F} = A\frac{S_H}{S_{cl}} + B = \frac{1}{4}A + B$$

The second equation for A and B expresses the fact that the total load W is balanced by the shear force at the two ends. We have

$$\frac{d^3u}{dx^3} = -\frac{W}{2EI}$$

or

$$Au_{cl}'''(x) + Bu_H'''(x) = A\left(-\frac{W}{2EI}\right) + B\left(-\frac{W}{2EI}\right) = -\frac{W}{2EI}$$

from which it follows that

$$A + B = 1.$$

From these two equations, namely,

$$A + B = 1,$$

$$\frac{1}{4}A + B = \frac{1}{F},$$

we find that

$$A = \frac{4(F-1)}{3F}, \quad B = \frac{4-F}{3F}$$

The deflection curve of an elastically clamped beam is thus described by

$$u(x) = \frac{4(F-1)}{3F}u_{cl}(x) + \frac{4-F}{3F}u_H(x) \tag{7}$$

The relations between the dimensionless fixity parameters F and $l\lambda$ can now be found. From Equation [3] we have

$$l\lambda = -l\frac{u''\left(\frac{l}{2}\right)}{u'\left(\frac{l}{2}\right)},$$

where primes denote differentiation. From Equations [5] and [7] it follows that

$$l\lambda = \frac{8(F-1)}{4-F}, \quad F = \frac{4l\lambda + 8}{l\lambda + 8}. \qquad [8]$$

The parameters F and $l\lambda$ have the following range of values:

simply supported ends $\quad F = 1, \quad l\lambda = 0,$

rigidly clamped ends $\quad F = 4, \quad l\lambda = \infty.$

THE FREE VIBRATION OF ELASTICALLY CLAMPED BEAMS

SOLUTION OF THE DIFFERENTIAL EQUATION

The differential equation for the free transverse motion of a beam (2) is

$$\frac{\partial^2 u(x,t)}{\partial t^2} + r^2 c^2 \frac{\partial^4 u(x,t)}{\partial x^4} = 0, \qquad [9]$$

where c is the speed of sound in the material, and r is the radius of gyration of a cross section with respect to the neutral axis. To solve Equation [9] we write

$$u(x,t) = X(x)\,T(t),$$

and the differential equation becomes

$$X(x)\,T''(t) + r^2 c^2 X''''(x)\,T(t) = 0,$$

or, upon division by $X(x)T(t)$,

$$\frac{T''(t)}{T(t)} = -r^2 c^2 \frac{X''''(x)}{X(x)}, \qquad [10]$$

where the primes and superscripts denote differentiation. From Equation [10] we see that $T''(t)/T(t)$ can not depend on t for it is equal to $-r^2c^2X''''(x)/X(x)$ which varies only with x. Hence $T''(t)/T(t)$ must be constant and we write

$$\frac{T''(t)}{T(t)} = -p^2$$

from which it follows that $T(t)$ is given by the expression

$$T(t) = C\cos pt + D\sin pt \qquad [11]$$

so that the frequency of the vibrating beam is

$$f = \frac{p}{2\pi}$$

where p is a natural angular frequency of the beam. Similarly, the right-hand side of Equation [10] can now be written as

$$\frac{X''''(x)}{X(x)} = \frac{p^2}{r^2 c^2}. \qquad [12]$$

If we introduce the abbreviation

$$k^4 = \frac{p^2}{r^2 c^2}$$

or $k^2 = \frac{p}{rc} = \frac{2\pi f}{rc}$ and $f = \frac{rc}{2\pi} k^2$, then Equation [12] takes the form

$$\frac{d^4 X(x)}{dx^4} = k^4 X(x)$$

whose general solution is

$$X(x) = A_1 \sin kx + A_2 \cos kx + A_3 \sinh kx + A_4 \cosh kx. \qquad [13]$$

The coefficients A_1, A_2, A_3, A_4 and the constant k from which the frequency is to be obtained will now be determined from the conditions at the ends of the beam.

THE SYMMETRICAL MODES OF VIBRATION

We consider first those modes of vibration that are symmetrical with respect to the centerline, $x = 0$. The end conditions are

$$X''\left(\frac{l}{2}\right) = -\lambda X'\left(\frac{l}{2}\right), \qquad [14]$$

$$X\left(\frac{l}{2}\right) = 0, \qquad [15]$$

$$X(-x) = X(x). \qquad [16]$$

Equation [14] corresponds to Equation [13] and describes the end fixity; Equation [15] states that the ends do not deflect, and Equation [16] represents the symmetry of the modes under discussion.

From the symmetry of the vibrating beam, Equation [16], it is seen that nonsymmetrical functions must not appear in Equation [13]. Accordingly we set $A_1 = A_3 = 0$ in Equation [13] and get

$$X(x) = A_2 \cos kx + A_4 \cosh kx.$$

Equation [15] now yields

$$A_2 \cos\mu + A_4 \cosh\mu = 0,$$

or

$$A_4 = -A_2 \frac{\cos\mu}{\cosh\mu}$$

where we have used the abbreviation

$$\mu = \frac{kl}{2}.$$

The constant μ is related to the natural frequency f by

$$\mu^2 = \frac{k^2 l^2}{4} = \frac{\pi l^2 f}{2rc}$$

From Equation [14] and using the value of A_4 just found we get

$$A_2 [2k \cos\mu \cosh\mu + \lambda(\sin\mu \cosh\mu + \cos\mu \sinh\mu)] = 0.$$

Since A_2 can not vanish the expression within the bracket must be zero, from which it follows that

$$\frac{\sin\mu \cosh\mu + \cos\mu \sinh\mu}{\cos\mu \cosh\mu} = -\frac{2k}{\lambda}$$

or

$$\tan\mu + \tanh\mu = -\frac{2k}{\lambda}$$

This can be rewritten as

$$\frac{1}{\tan\mu + \tanh\mu} = -\frac{\lambda}{2k}$$

or, multiplying by 4μ,

$$\frac{4\mu}{\tan\mu + \tanh\mu} = -\frac{4\mu\lambda}{2k} = -\frac{4\left(k\frac{l}{2}\right)\lambda}{2k}$$

that is,

$$\frac{-4\mu}{\tan\mu + \tanh\mu} = l\lambda \qquad [17]$$

This is the frequency equation for the symmetric modes of vibration of an elastically clamped beam and connects the dimensionless fixity factor $l\lambda$ with the dimensionless quantity μ from which the natural frequency can be calculated according to the relation

$$f = \frac{2rc}{\pi l^2} \mu^2 \tag{18}$$

The fundamental frequency f_{1H} of a beam with hinged ends is found from Equation [17] by setting $l\lambda$ equal to zero:

$$\frac{-4\mu}{\tan\mu + \tanh\mu} = 0$$

From this it follows that

$$\mu = \frac{\pi}{2}$$

or

$$f_{1H} = \frac{2rc}{\pi l^2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi rc}{2l^2}$$

The ratio of the frequency of an elastically clamped beam, Equation [18], to the fundamental frequency of a beam with hinged ends is thus given by

$$\frac{f}{f_{1H}} = \frac{4}{\pi^2}\mu^2$$

The relation between the frequency and the stiffness parameter $F = S/S_H$ is found from Equations [8] and [17] to be

$$F = \frac{2 + l\lambda}{2 + \frac{1}{4} l\lambda} = \frac{2 - \dfrac{4\mu}{\tan\mu + \tanh\mu}}{2 - \dfrac{\mu}{\tan\mu + \tanh\mu}}$$

or

$$F = \frac{2(\tan\mu + \tanh\mu) - 4\mu}{2(\tan\mu + \tanh\mu) - \mu} \tag{19}$$

with

$$\frac{f}{f_{1H}} = \frac{4}{\pi^2}\mu^2 \tag{20}$$

From Equations [19] and [20] curves may be drawn exhibiting the relation between the frequency ratio, f/f_{1H}, and the stiffness ratio, S/S_H. It should be noted that the normal procedure of assigning values to the stiffness ratio F and solving for the frequency parameter μ presents formidable calculational difficulties. If the problem is inverted, on the other hand, and the stiffness is found from prescribed values of frequency, the computational task is a simple one.

THE ANTISYMMETRICAL MODES OF VIBRATION

The modes of vibration that are antisymmetrical with respect to the centerline, $x = 0$, are characterized by end conditions

$$X''\left(\frac{l}{2}\right) = -\lambda X'\left(\frac{l}{2}\right) \qquad [21]$$

$$X\left(\frac{l}{2}\right) = 0 \qquad [22]$$

$$X(-x) = -X(x) \qquad [23]$$

The first two equations are identical with Equations [14] and [15] and express the conditions of elastic clamping and rigid support. Equation [23] represents the skew symmetry of the modes being considered.

The values of the constants A_1, and of the frequency parameter μ are found in the same way as before. From Equation [23] we get

$$A_2 = A_4 = 0$$

and

$$X(x) = A_1 \sin kx + A_3 \sinh kx.$$

Equation [22] yields

$$A_3 = -A_1 \frac{\sin\mu}{\sinh\mu},$$

and from Equation [21] it follows that

$$A_1[-2k \sin\mu \sinh\mu + \lambda(\cos\mu \sinh\mu - \sin\mu \cosh\mu)] = 0$$

thus

$$\frac{\cos\mu \sinh\mu - \sin\mu \cosh\mu}{\sin\mu \sinh\mu} = \frac{2k}{\lambda}$$

or

$$\cot\mu - \coth\mu = \frac{2k}{\lambda}$$

This can be rewritten as

$$\frac{4\mu}{\cot\mu - \coth\mu} = \frac{4\mu\lambda}{2k} = l\lambda$$

from which we get finally

$$F = \frac{S}{S_H} = \frac{2(\cot\mu - \coth\mu) + 4\mu}{2(\cot\mu - \coth\mu) + \mu}$$

with

$$\frac{f}{f_{1H}} = \frac{4}{\pi^2}\mu^2$$

DISCUSSION

Figures 2 and 3 exhibit the relation between frequency and fixity for the first five modes of vibration. In using these curves it should be noted that the stiffness under concentrated central load, S_H, and the fundamental frequency, f_{1H}, of a simply supported or hinged beam are given by

$$S_H = -\frac{48EI}{l^3}, \quad f_{1H} = \frac{\pi r c}{2l^2}$$

If the stiffness of a beam under concentrated central load is found experimentally then these curves yield values for the fundamental frequency and the four higher frequencies. Alternatively, if the value of any one of the frequencies is known, then the other frequencies and the stiffness of the beam may be found.

The fundamental frequency of an elastically clamped beam has been given by Hohenemser and Prager (3) in the form of a graph of frequency as a function of the parameter $l\lambda$; the latter variable takes values from zero for a simply supported beam to infinity for a clamped beam.

The concept of an elastically clamped and rigidly supported beam constitutes a considerable idealization of the physical situation in which a beam may be held, for example, by clamps with rubber inserts. A closer approach to the actual situation could be made by considering the beam to be elastically clamped by end moments and elastically supported by end forces. The frequency equation for this case has been given by Hohenemser and Prager (3). Even this formulation of the problem, however, involves estimating the effective length of the beam and disregards the actual mode of application of the clamping forces which are distributed over the sides of the beam. An alternative approach would be to consider the beam to be supported by an

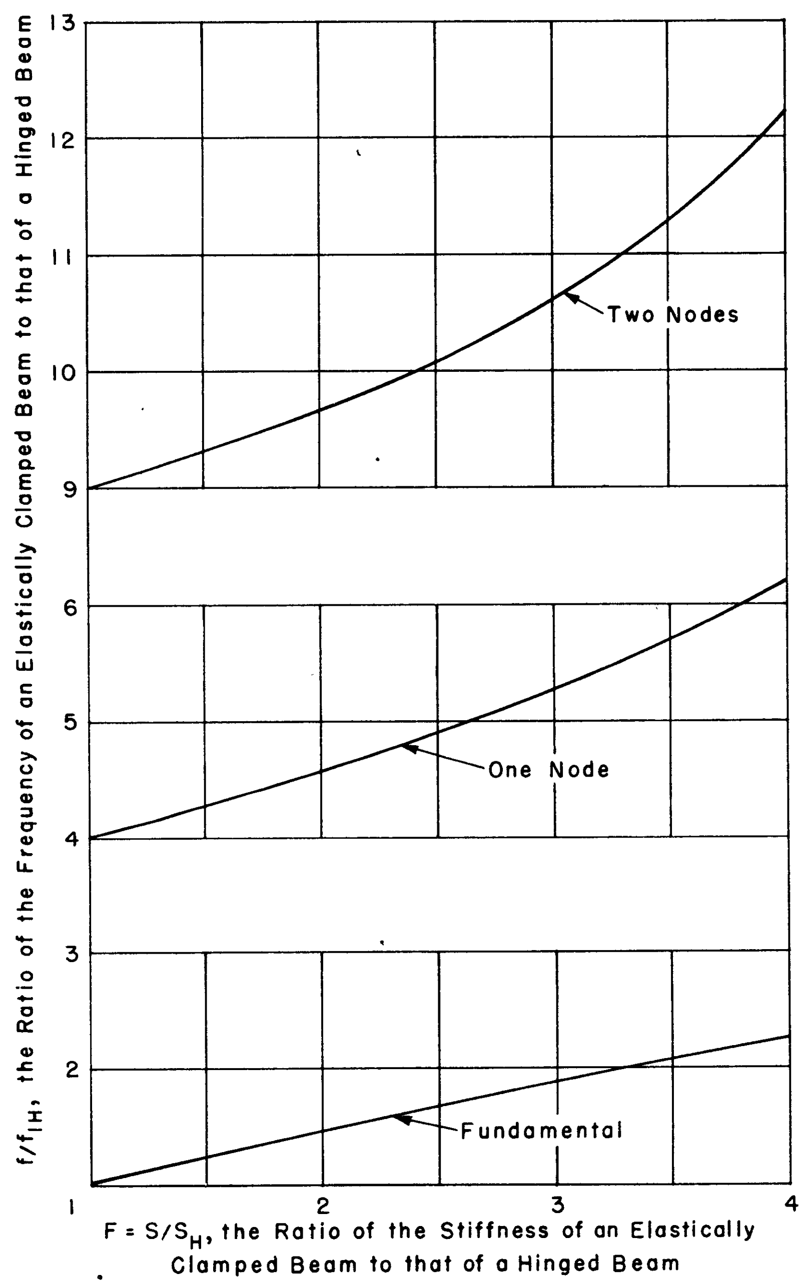

Figure 2 - Frequency and Fixity of an Elastically Clamped Beam

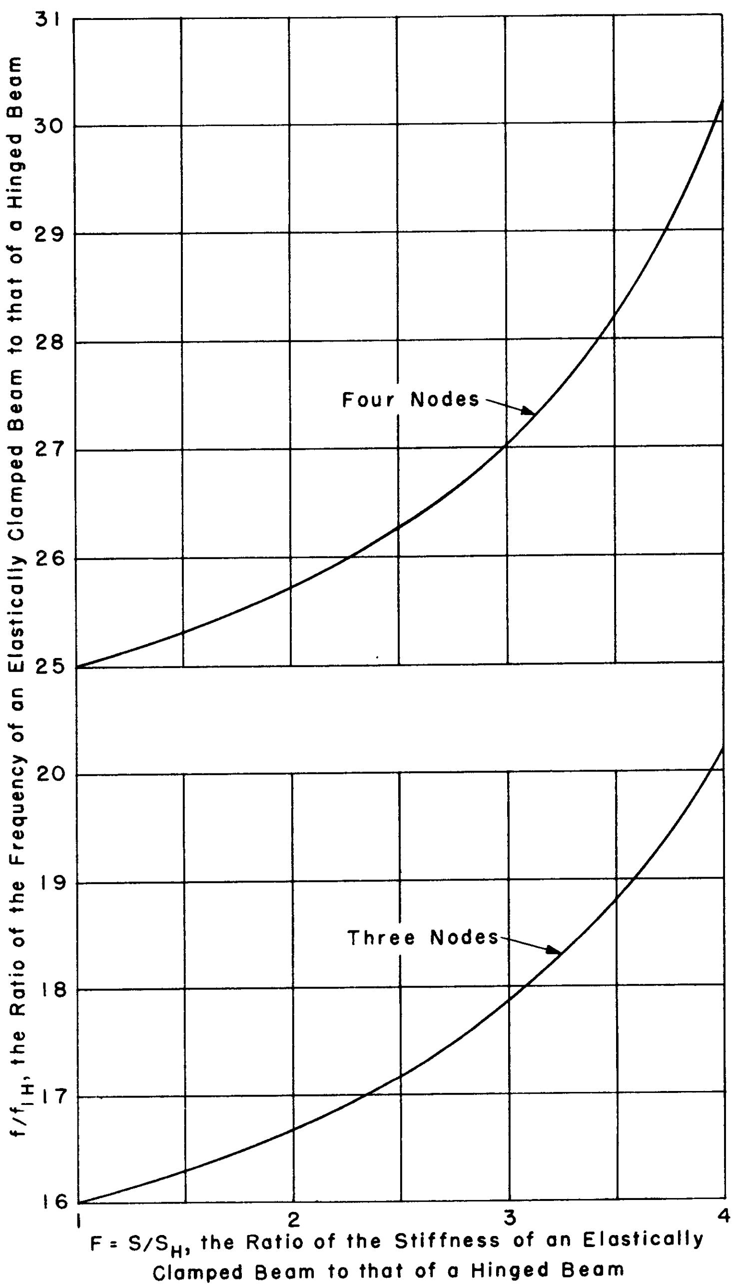

Figure 3 - Frequency and Fixity of an Elastically Clamped Beam

elastic foundation over a given portion of its length at each end. The static behavior and the natural frequencies could then be determined in terms of the length and stiffness of the end supports. This method of attack upon the problem will be considered in a later report.

PART 22 - THE EFFECT OF A CONTACT OR NEAR-CONTACT EXPLOSION ON A FREELY SUSPENDED PLANE TARGET

By C.T. Johnson

INTRODUCTION

A short series of experiments has been performed at the David Taylor Model Basin for the purpose of investigating the amount of damage done to sheet-metal targets freely suspended at a water surface, by contact and near-contact explosions. It was hoped that information might be obtained which would indicate the validity or non-validity of a theory of rupture suggested by Dr. A.B. Focke of the Bureau of Ordnance, Navy Department (4).

According to Dr. Focke, "The basic assumption is that in order to rupture a plate a definite amount of energy per unit area must be available at the plate from the explosion. ... It is assumed that at least this amount of energy must be available for each unit of area so deformed." To a first order of approximation, this minimum value, determined empirically, depends only on the thickness of the plate and the kind of material in it, provided that the plate is somewhat larger than the maximum rupture which might occur. For small charges it very likely depends upon the charge weight as well, since the size of the charge determines the form of the pressure pulses. Thus, for any given weight of charge and thickness of plate, there will be some distance of separation of the charge and plate at which the required energy density for rupture will be reached only at one point on the plate. At this distance, then, the plate should be just barely ruptured. If the charge were to be brought closer to the plate, the requisite energy density would be attained over a larger area, and the extent of rupture would be correspondingly greater. As the charge is brought nearer and nearer to the plate, the area over which rupture should occur will reach a maximum and then decrease because of obliquity of incidence of the energy at the edge of the ruptured area. Mathematical analysis shows that the critical distance at which the maximum rupture should occur is about 0.44 times the distance at which rupture just occurs, at least for large charges. The amount of rupture, however, should not vary appreciably from the maximum if the charge is between 1/2 and 1 1/2 times the critical distance from the target. It should be noted that action by the pressure pulses sent out by oscillations of the gas globe, which causes considerable damage in certain instances, is not explicitly taken account of by this theory.

Due to an unfortunate combination of charge weight and plate thickness in the present experiments, the physical size of the charge prevented placing its center close enough to the plate to pass through the critical distance. The results obtained, therefore, shed little light on the validity of the theory being tested. The results are nevertheless reported here in the hope that they will be of interest and value to other workers in underwater explosion research.

TEST ARRANGEMENT AND PROCEDURE

The metal panels used as targets for the experiment were made of furniture steel. They were 47 inches square, 0.063 inch in nominal thickness, and were provided with a flanged rim 1/2 inch high on each of the four edges. The explosive charge for each experiment consisted of 13.25 grams of granular tetryl in a cylindrical bakelite container 1 inch in diameter and 1 3/4 inch long, which was detonated by a cap on the axis of the charge.

The tests were conducted in a tank 11 feet in diameter and 8 feet deep. The target for each experiment was suspended in the horizontal position as shown in Figure 4 by a sling, attached to 4 eyes, shown in Figure 5. The charge, aligned with its axis parallel to the plane of the panel, was

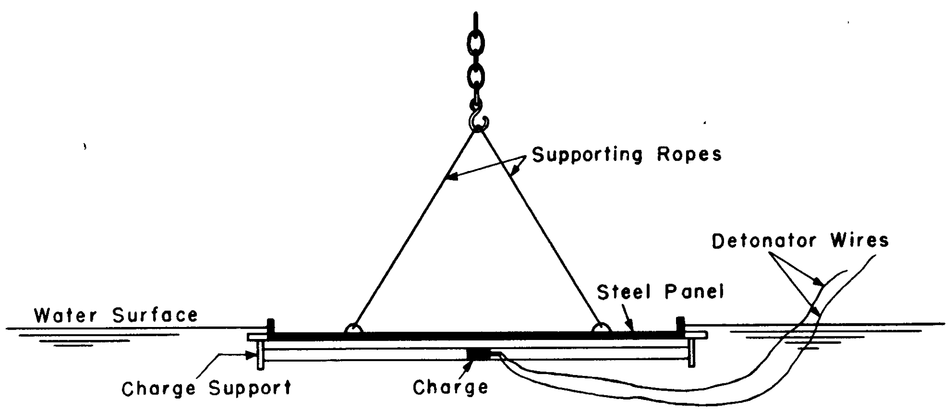

Figure 4 - Schematic Diagram of Experimental Arrangement

held in position in the water underneath the center of the panel by strings stretched horizontally between two small pieces of wood attached to the plate and projecting downward from opposite edges. In addition, for the contact explosions, the charge was held against the target by strips of friction tape. For the experiments in which the charge was 1 inch or 1 1/2 inch from the target, a cardboard spacer was attached between the strings and the diaphragm near the charge to hold the charge at the required distance from the plate. For each test the target was lowered into the water until the water nearly overflowed the upper edge of the rim. In this position, the target was buoyed up by the water so that there was only a slight tension in the supporting lines. Precautions were taken to remove all air from under the plate.

Six tests were made; two with contact charges, that is, with the center of the charge 1/2 inch from the diaphragm, and four with the center of the charge successively 1 inch, 1 1/2 inch, 2 inches, and 4 inches from the diaphragm.

TEST RESULTS

On each of the contact shots the panel ruptured at the center and the metal from the ruptured part was thrown back in the form of six flaps, each roughly triangular in shape. The flaps were bent well back along the non-ruptured portion of the panel, as may be seen in Figure 5. The areas of the holes in the panels were about 290 square inches and 310 square inches respectively.

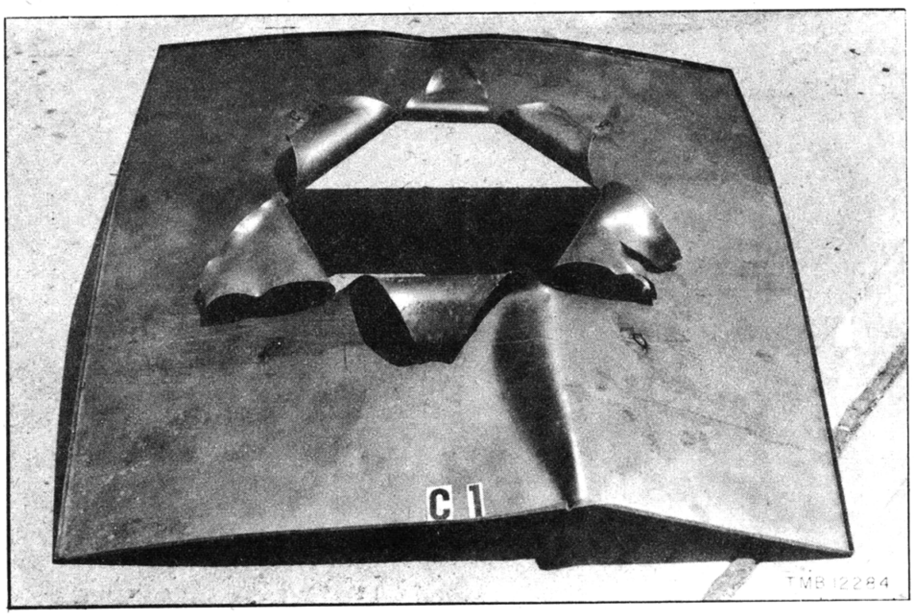

Figure 5 - Panel Ruptured by the Explosion of a Charge in Contact with Its Center

The curvature of the leaves does not reverse.

With the charge center 1 inch from the target a 4-leafed rupture left a hole almost square; its area was approximately 230 square inches. As shown in Figure 6, the leaves of the ruptured part of this diaphragm underwent a reversal of curvature near their tips. This might have been caused by the points striking the outer portions of the diaphragm during deformation.

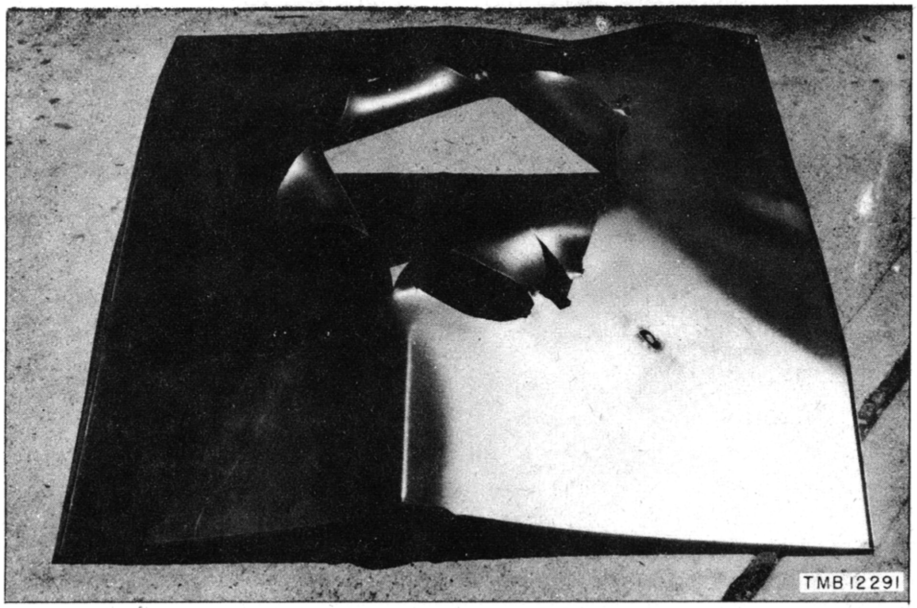

Figure 6 - Panel Ruptured by the Explosion of a Charge
1 Inch below Its Center

The curvature of the leaves reverses near the points.

This form is unlike that resulting from the contact explosions where, for the most part, the leaves were curved in one direction only; that is, their ends were curled under. This is shown in Figure 5.

The three panels attacked by charges more than 1 inch distant were not ruptured. When the charge center was 1 1/2 inch or 2 inches from the plate, all four edges of the plate were sharply buckled by the explosion; when the charge distance was 4 inches, three edges were buckled. In contrast, each of the three ruptured panels had only one edge sharply buckled. The difference may have been caused by the fact that in the case of the ruptured panels, the hole vented the gas globe into the air immediately, preventing the damage which would have accompanied pulsations of the globe. On the other hand, the rupture may have so altered the constraints on the material around the edge of the ruptured portion that it was able to deform in such a manner as to prevent buckling of the rest of the panel. Buckling in all cases consisted of one or more reverse bends in the metal of the plate, as shown in Figure 7. Each of the buckles ran nearly perpendicular to the edge of the panel and was situated near the midpoint of the edge.

When the distance of the charge from the panel was 4 inches, the metal of the panel was stretched very little by the explosion, but the central portions of the two other non-ruptured panels, attacked by charges 1 1/2 and 2 inches away, underwent a considerable amount of stretching.

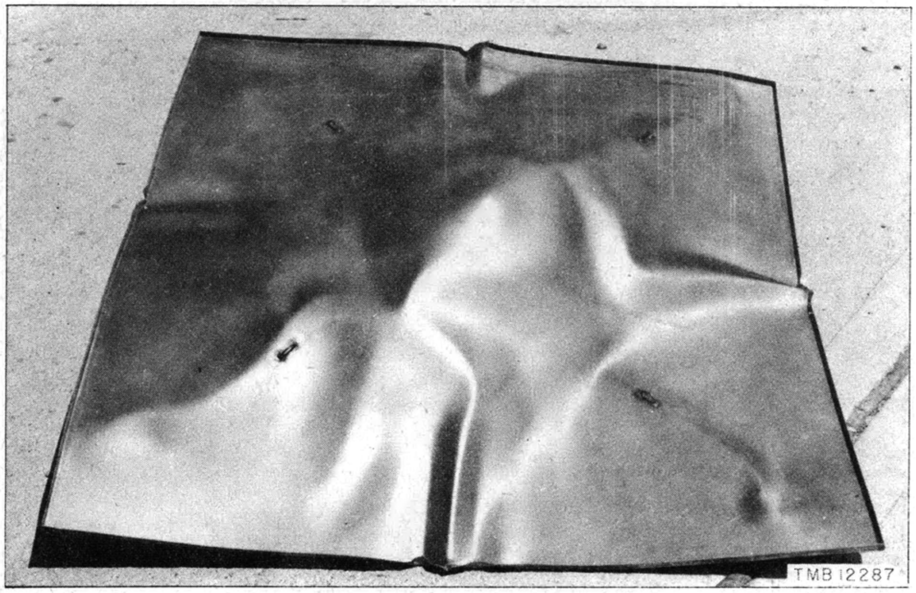

Figure 7 - Panel Attacked by the Explosion of a Charge 1 1/2 Inch Distant from Its Center

The central portion of the panel exhibits a deformation indicative of damage caused by the second pulse of the gas globe.

Note that the amount of buckling here is greater than on the panels of Figures 5 and 6.

Inasmuch as rupture did not occur in any of the experiments in which the charge center was more than 1 inch from the target, the experiments, as was noted previously, yield data insufficient to validate any rule relating the charge distance to the extent or type of rupture produced. Further tests are planned, using larger charges.

PART 23 - NOTES ON PLASTIC DEFORMATION OF ELLIPTICAL PLATES UNDER HYDROSTATIC PRESSURE

By M. Levenson

It is shown in this report that under certain reasonable assumptions, the pressure-deflection and energy-deflection formulas for an elliptical plate deformed plastically under hydrostatic pressure may be obtained from the corresponding formulas for a circular plate having an *equivalent* radius (5). The equivalent radius a of an elliptical plate is defined by the relation

$$\frac{1}{a^2} = \frac{1}{2}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) \qquad [24]$$

where α and β are the semi-minor and semi-major axes of the ellipse.

The following assumptions are made:

a. The deflections are less than one-half of the equivalent radius.

b. The surface after deformation assumes the form of an elliptic paraboloid. If Z_0 is the central deflection, the equation of the surface is

$$\frac{Z}{Z_0} - 1 = -\left(\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2}\right) \qquad [25]$$

c. The tension in the plate is constant, uniform, and isotropic during the deformation.

It can be shown (6) that under these assumptions the energy absorbed by the plate in deforming is approximately

$$U = \sigma h \Delta A \qquad [26]$$

where σ is $\frac{\text{tension}}{\text{thickness}}$, i.e., stress

h is the thickness of the plate after deforming

A is the area of the surface after deforming

ΔA is the increase in area

The area bounded by the surface and the initial plane of the plate is

$$A = 4\int_0^{\beta}\int_0^{\frac{\alpha}{\beta}\sqrt{\beta^2 - y^2}} \sqrt{1 + \left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2}\, dx\, dy$$

or approximately

$$A = 4\int_0^{\beta}\int_0^{\frac{\alpha}{\beta}\sqrt{\beta^2 - y^2}} \left[1 + \frac{1}{2}\left(\frac{\partial Z}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial Z}{\partial y}\right)^2\right] dx\, dy$$

which reduces to

$$A = \pi\left[\alpha\beta + Z_0^2\left(\frac{\alpha}{2\beta} + \frac{\beta}{2\alpha}\right)\right] \qquad [27]$$

Hence Equation [26] becomes

$$U = \sigma h Z_0^2\left(\frac{\alpha}{2\beta} + \frac{\beta}{2\alpha}\right) \qquad [28]$$

since $\pi\alpha\beta$ is the original area of the plate. The energy absorption dU, corresponding to a change in deflection dZ_0, is therefore

$$dU = \sigma h Z_0\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) dZ_0 \qquad [29]$$

The work done on the plate by the pressure during this change in deflection is pdV, where p is the pressure and dV is the change in volume. The volume bounded by the surface and the initial plane of the plate is

$$V = 4\int_0^{\beta}\int_0^{\frac{\alpha}{\beta}\sqrt{\beta^2 - y^2}} Z\, dx\, dy$$

which reduces to

$$V = \frac{\pi\alpha\beta Z_0}{2} \qquad [30]$$

Upon setting $pdV = dU$ there results the pressure-deflection formula

$$p = 2\sigma h Z_0 \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} \qquad [31]$$

or

$$p = \frac{4\sigma h Z_0}{a^2} \qquad [31a]$$

where the quantity $\frac{2\,\alpha^2\,\beta^2}{\alpha^2 + \beta^2}$ has been replaced by a^2, the square of the equivalent radius.

Equation [31a] is an approximate pressure-deflection formula for a circular plate of radius a, when the deflection is less than one-half the radius of the plate (6).

If both sides of Equation [28] are divided by $\pi\alpha\beta$ to calculate the energy absorbed per unit area, the result is

$$u = \frac{U}{\pi \alpha \beta} = \frac{\sigma h Z_0^2}{2}\left(\frac{\alpha^2 + \beta^2}{\alpha^2 \beta^2}\right)$$

or

$$u = \frac{U}{\pi \alpha \beta} = \frac{\sigma h Z_0^2}{a^2} \qquad [32]$$

which is recognized (6) as an approximate formula for plastic energy per unit area absorbed by a circular plate of radius a.

These results are important for the study of the plastic deformation of elliptical plates. Moreover it is of interest to note that most of the formulas describing the elastic action of the plate can be obtained by substitution of the equivalent radius a into the corresponding formulas (6) for a circular plate.

REFERENCES

(1) "Formulas for Stress and Strain," by R.J. Roark, McGraw-Hill, New York, 1938.

(2) "Vibration Problems in Engineering," by S. Timoshenko, Van Nostrand, New York, 1937.

(3) "Dynamik der Stabwerke" (Dynamics of Trusses), by K. Hohenemser and W. Prager, J. Springer, Berlin, 1933.

(4) "Estimation of the Size of Hull Rupture Caused by Nearby Underwater Explosions," by Dr. A.B. Focke, Navy Department, Bureau of Ordnance, CONFIDENTIAL, Acoustical Analysis Report 23, 27 January 1944.

(5) "Plastic Deformation and Energy Absorption of a Thin Rectangular Plate under Hydrostatic Pressure," by Dr. A.N. Gleyzal, TMB Report R-280, December 1944.

(6) "Protection Against Underwater Explosion, Plastic Deformation of a Circular Plate," by Dr. A.N. Gleyzal, TMB Report 490, September 1942.