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NAVY DEPARTMENT

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METHOD OF MECHANICAL IMPEDANCE AND THE ELECTRICAL ANALOGY

by

R.T. McGoldrick

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Notation

In adopting symbols to be used in the electrical analogy and impedance formulas it seems necessary to duplicate certain symbols previously used in the Vibration Manual as these symbols are almost universally used in electrical theory. This should not cause confusion as the same symbol when used as defined previously will not occur in any of the formulas given in this supplement.

L - inductance in henries
R - resistance in ohms
C - capacity in farads
I - current in amperes

q - electrical charge in coulombs
Z - electrical impedance in ohms or mechanical impedance in ips units
E - electromotive force in volts

Introduction

The subject of electrical-mechanical analogies is a very broad one of which only a very small part is considered here. In fact the main emphasis in this supplement is not on the electrical analogy but on the usefulness of the concept of impedance in the solution of mechanical vibration problems. The text is limited to the solution of the steady state vibrations of mechanical systems under sinusoidal excitation by the impedance method.

The broader treatment of analogies by direct comparison of the differential equations representing the electrical and mechanical systems is omitted. This supplement is further limited to "lumped" systems, that is systems in which the various masses, elastic constants, and friction-producing members may be adequately represented by discrete elements, each element having only one of these properties. Problems such as the vibrations of ship hulls having distributed mass and stiffness are not considered.

Most of the vibration formulas given in the Manual are limited to systems of one or two degrees of freedom. Beyond two degrees of freedom the formulas become quite complicated. In electrical theory the impedance method has been used for computing the steady state currents in networks of many degrees of freedom. In mechanical problems the impedance method has not met such wide acceptance largely because of the difficulty of writing down the impedance by inspection as is so readily done in the electrical case. However, while the mechanical relations are more difficult to visualize there are definite rules for writing down the mechanical impedance by means of which the problem may be solved directly.

Definition of impedance

In electrical problems it is usually required to determine the steady state alternating current in the various branches of a network having elements of known resistance, inductance, and capacitance under known impressed sinusoidal voltages. In the mechanical case it is usually required to determine the amplitudes of various mass elements when known sinusoidal driving forces act at certain points in the system. Since the concept of impedance is derived from the solution of the differential equations in complex notation the electrical impedance is invariably defined as that complex quantity by which to divide the sinusoidal voltage applied at the terminals of a network in order to find the steady state current flowing in the network.

As it is not essential to consider the electrical analogy in using the method of mechanical impedance the mechanical impedance is here defined as that complex quantity by which it is necessary to divide the sinusoidal driving force in order to obtain the amplitude at the driving point.

The corresponding electrical and mechanical quantities remain the same regardless of the choice of the definition of the impedances. Thus displacement in the mechanical case corresponds to charge, velocity to current, mass to inductance, and force to voltage. However the mathematical expressions for the impedances as here defined are not similar, as the mechanical impedance is based on displacement which corresponds to charge whilst the electrical impedance is based on current which corresponds to velocity.

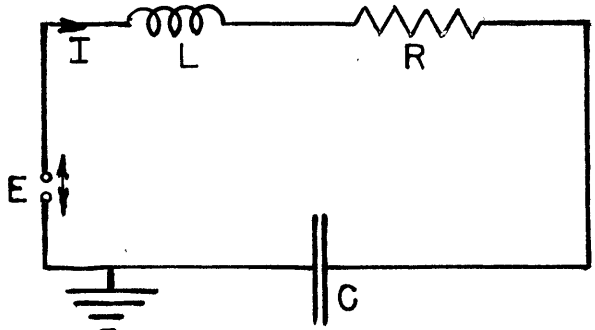
Strictly speaking the particular analogies to be presented here are mathematical rather than physical. A truly physical electrical-mechanical analogy is found in many hydraulic problems. In the mechanical mass-spring-dashpot systems we think of alternating driving forces as acting at certain points in the system whereas in electrical circuits we consider voltages as acting across the terminals of networks to produce currents flowing into and out of the networks. Of course mechanical problems can be solved directly from the differential equations, but by visualizing mechanical systems as made up of "series" and "parallel" elements or groups of elements whereby the mechanical impedance can be written down by inspection a direct expression for the amplitude can be obtained. To recognize combinations of mechanical elements as being either in series or in parallel is much more difficult than the corresponding operation in the electrical case. In fact mechanical elements that appear to be in series are usually in parallel and vice versa. The three fundamental mechanical elements are usually represented schematically by a concentrated mass, a spring, and a dashpot. The reaction of the mass is proportional to its acceleration, that of the dashpot to the relative velocity between its members or end connections and that of the spring to the relative displacement between its end connections. The technique of using the method of mechanical impedance will become clearer in the derivation of the formulas and their application in the examples given.

Derivation of the impedance formulas

The simplest series and parallel electrical circuits and their mechanical equivalents are shown in juxtaposition in Figure 1. The directions indicated by the arrows are the positive directions assumed for the quantities in question. Actually these quantities are all sinusoidal and in general differ from one another in phase. The differential equations applicable in each instance are also set down in the same figure. These equations are fundamental and are derived in each case directly from physical laws without any consideration of the analogy between the electrical and mechanical systems. In fact it is the similarity of the differential equations that justifies the analogies.

It is clear that in both of the examples on the left of Figure 1 the displacements (electrical displacement being synonymous with charge) are the same for all elements, whilst in both of the examples on the right the force (emf) is the same for all elements. Under this convention, by the displacement of a spring or dashpot must be understood the relative displacements between the end connections of the elements. Thus electrical displacements through an element are equivalent to mechanical displacements between the ends of an element if the latter element is a spring or dashpot and to the displacement of the whole element if the latter is a mass.

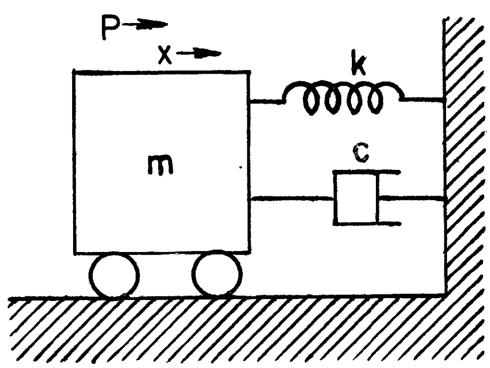
Series Electrical



$$L\frac{dI}{dt} + RI + \frac{\int I\,dt}{C} = E$$

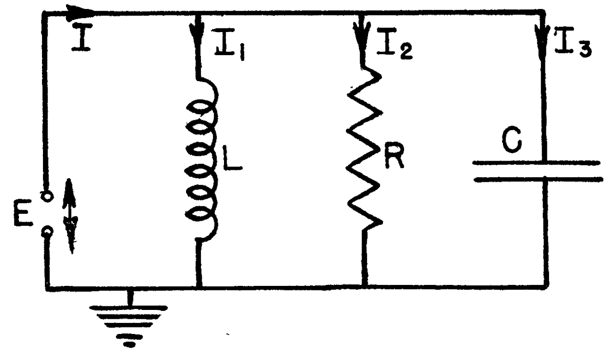
$$\text{or } L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$

Mechanical



$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = P$$

Parallel Electrical



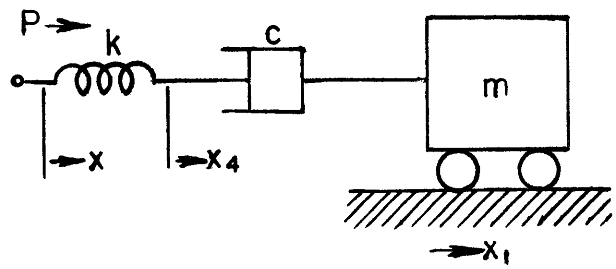
$$L\frac{dI_1}{dt} = RI_2 = \frac{\int I_3\,dt}{C} = E$$

$$\text{or } L\frac{d^2q_1}{dt^2} = R\frac{dq_2}{dt} = \frac{q_3}{C} = E$$

and $I = I_1 + I_2 + I_3$

$q = q_1 + q_2 + q_3$

Mechanical



let $x_2 = (x_4 - x_1)$

$x_3 = (x - x_4)$

$$m\frac{d^2x_1}{dt^2} = c\frac{dx_2}{dt} = kx_3 = P$$

$$x = x_1 + x_2 + x_3$$

Figure 1. Simplest Series and Parallel Electrical Circuits and Their Mechanical Equivalents with the Corresponding Differential Equations (arrows indicate positive directions assumed --- these quantities will in general be alternating).

The differential equation applicable to the series electrical circuit shown in Figure 1 is

$$L\frac{dI}{dt} + RI + \frac{\int I\,dt}{C} = E$$

In operational notation and under the condition that the impressed voltage is sinusoidal this equation takes the form

$$\left(Lp + R + \frac{1}{pC}\right)I = E_0 e^{j\omega t}$$

where the operator p denotes differentiation with respect to time and the operator 1/p denotes integration with respect to time, the exponent of p, if any, indicates the number of times the differentiation or integration is to be performed. In like manner the analogous mechanical equation becomes

$$(mp^2 + cp + k)x = P_0 e^{j\omega t}$$

Under sinusoidal excitation the steady state currents or displacements will also be sinusoidal and of the form $I_0 e^{j(\omega t - \epsilon)}$ and $Xe^{j(\omega t - \epsilon)}$ respectively, ϵ denoting the phase angle by which the voltage leads the current or the force leads the displacement. In the electrical case if ϵ is negative the current leads the voltage. Hence if I has the form $I_0 e^{j(\omega t - \epsilon)}$ then $pI = j\omega I_0 e^{j(\omega t - \epsilon)}$,

$$\left(j\omega L + R - \frac{j}{\omega C}\right)I_0 = E_0 e^{j\omega t}$$

and by the previous definition the electrical impedance

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right).$$

Similarly in the mechanical case

$$(-m\omega^2 + cj\omega + k)X = P_0 e^{j\omega t}$$

and the mechanical impedance

$$Z = k - m\omega^2 + cj\omega.$$

It should be noted that the impedance expressed in the form of a complex number gives not only the magnitude but also the phase of the quantity in question. For instance in the expression

$$I_0 = \frac{E_0 \epsilon^{j\omega t}}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

it is seen that I_0 is the ratio of two complex numbers, the numerator being expressed in the polar form, the denominator in the ordinary form. This division may be performed graphically in accordance with DeMoivre's theorem. Since the polar angle for E_0 namely ωt continually increases with time the line from the origin to the point representing the voltage as a complex number may be thought of as a time vector continually rotating in the counter clockwise direction with angular velocity ω. The length of this vector is the absolute value of E namely E_0, and if the voltage is assumed to be zero when $t = 0$ it is obviously represented in magnitude at any instant by the projection of this vector on the Y-axis. This is equivalent to saying $E = E_0 \sin \omega t$. In Figure 2, ωt is represented for convenience as an acute angle but the relations hold good for any angle. Thus the length of the voltage vector is the absolute value of E, namely E_0 and the polar angle is ωt. The impedance is plotted as a complex number in the usual way, the real part being layed off along the X-axis and the imaginary part along the Y-axis. The length of the line representing the impedance is $\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$. The impedance is a complex scalar, not a vector, and does not rotate. Hence according to DeMoivre's theorem the vector representing the current is obtained by dividing the absolute value E_0 by the absolute value of Z and subtracting the polar angle of Z from that of E_0.

Similarly in the mechanical case $X = \dfrac{P_0 e^{j\omega t}}{k - m\omega^2 + jc\omega}$ the amplitude is $X = \dfrac{P_0}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}}$ and the force leads the displacement by the angle

$$\epsilon = \arctan \frac{c\omega}{k - m\omega^2}$$

as shown in Figure 3. In the mechanical case ϵ is always positive.

In the electrical cases on the right hand side of Figure 1 it is obvious that the same voltage exists across each element and hence that the current in each branch can be found by dividing the voltage vector by the impedance. Thus

$$I_1 = \frac{E_0 e^{j\omega t}}{j\omega L}; \qquad I_2 = \frac{E_0 e^{j\omega t}}{R}; \qquad I_3 = \frac{E_0 e^{j\omega t}}{-\frac{j}{\omega C}}.$$

The vector diagrams covering these cases are given in Figure 4. The current flowing into the entire network in this case is obtained by combining vectorially the current vectors I_1, I_2, and I_3, i.e. $I_0 = I_1 + I_2 + I_3$ all these terms being complex. Hence $I_0 = E_0\left(\frac{1}{j\omega L} + \frac{1}{R} + \frac{1}{-\frac{j}{\omega C}}\right)$

and $Z = \dfrac{E_0}{I_0} = \dfrac{1}{\frac{1}{j\omega L} + \frac{1}{R} + \frac{1}{\frac{-j}{\omega C}}} = \dfrac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}}$

The resultant current vector obtained by combining I_1, I_2, and I_3 will have the correct phase relation to E_0 provided that E_0 is drawn identically in the three vector diagrams shown in Figure 4. The vector diagrams for the mechanical system

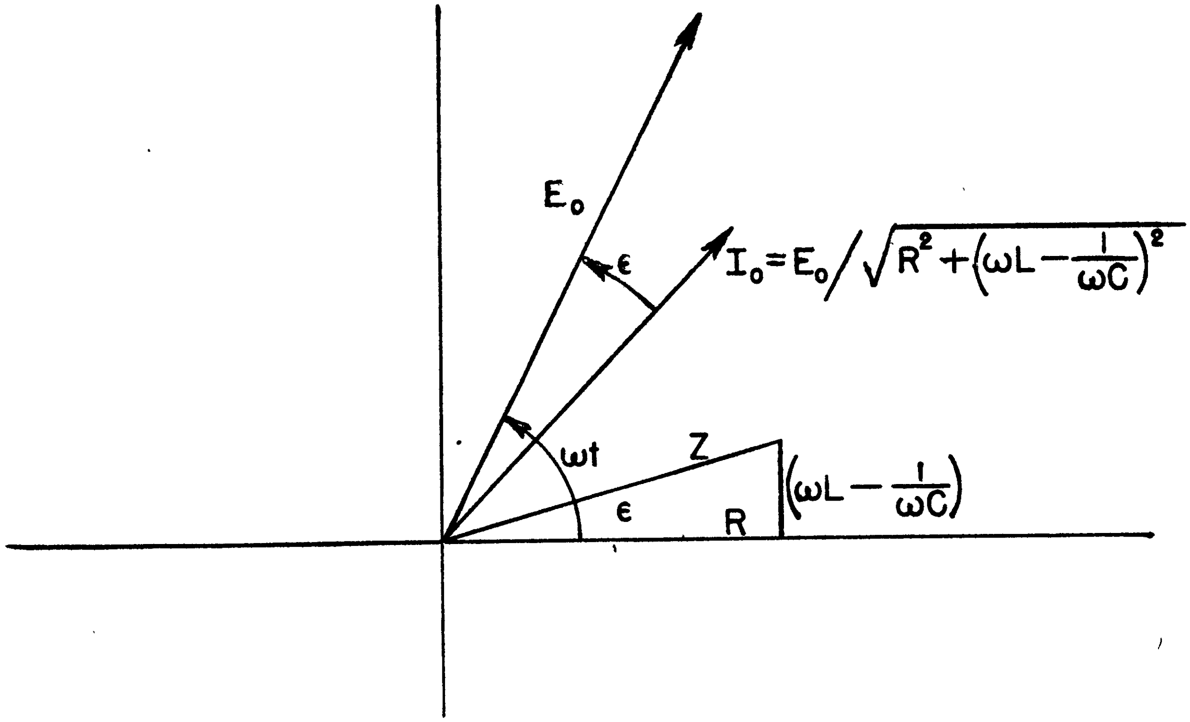
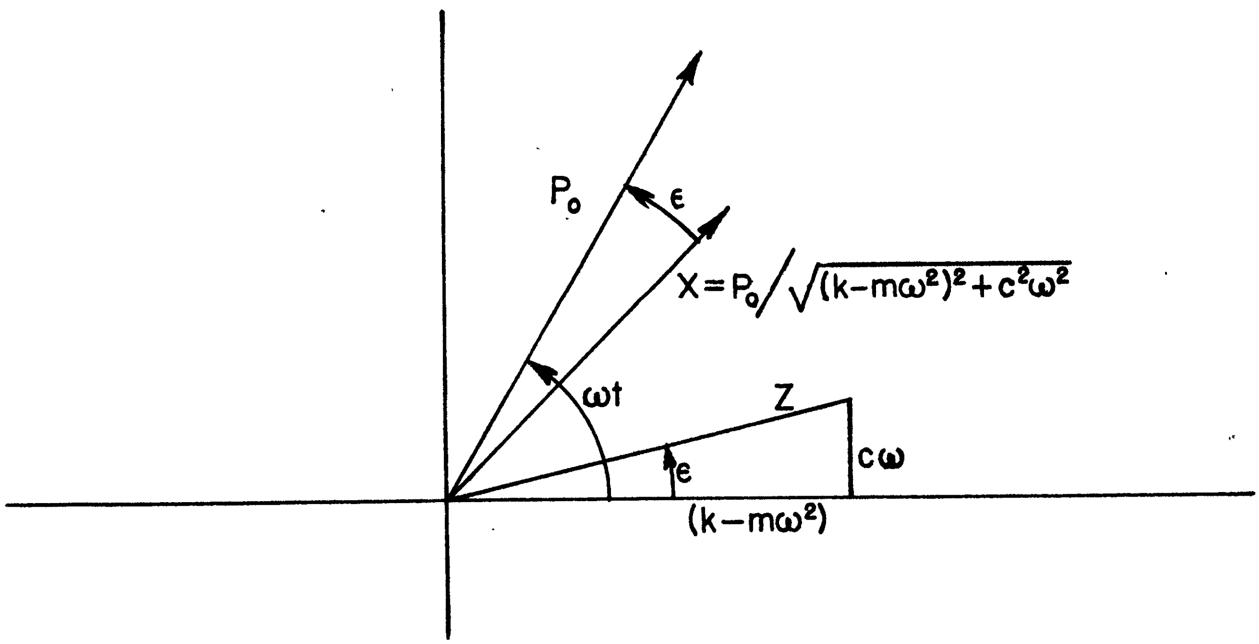


Figure 2 Vector Diagram for Series Electrical Circuit.



Firgure 3 Vector Diagram for Series Mechanical Circuit.

shown on the right hand side of Figure 1 are shown in Figure 5.

Thus $X_3 = \frac{P_o e^{j\omega t}}{k}$, $X_2 = \frac{P_o e^{j\omega t}}{jc\omega}$, $X_1 = \frac{P_o e^{j\omega t}}{-m\omega^2}$

$X = X_1 + X_2 + X_3$ (the addition to be made vectorially)

This will give X in the correct phase relation to P_0 provided P_0 is drawn identically in the three vector diagrams shown in Figure 5. In this case the force will lead the displacement by the phase angle $\epsilon = \arctan \frac{km\omega}{m\omega^2 c - kc}$

Obviously by analogy the mechanical impedance for the parallel system is $Z = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}} = \frac{1}{\frac{1}{k} + \frac{1}{-m\omega^2} + \frac{1}{j\omega c}}$

In finding the steady state oscillations of more complicated systems than those so far discussed by the impedance method the problem is chiefly one of recognizing the series and parallel relationships between the elements. In the electrical case it is not very difficult to pick out the series and parallel groups and subgroups and to write down the complex impedance of the network by inspection. In the mechanical case it is not as simple. It has already been pointed out that by the displacement of a spring or dashpot element must be understood the relative displacement between its end connections. When one end of a spring or dashpot is fixed the element must be considered as in series with the elements attached to the other end. The chief problem in using the method of mechanical impedance is to correctly set up the lumped system representing the actual conditions. Actual dashpot dampers are seldom present and it must be determined whether friction exists between two elements or between these elements and the ground. Furthermore if the friction is of the Coulomb type the equivalent viscous

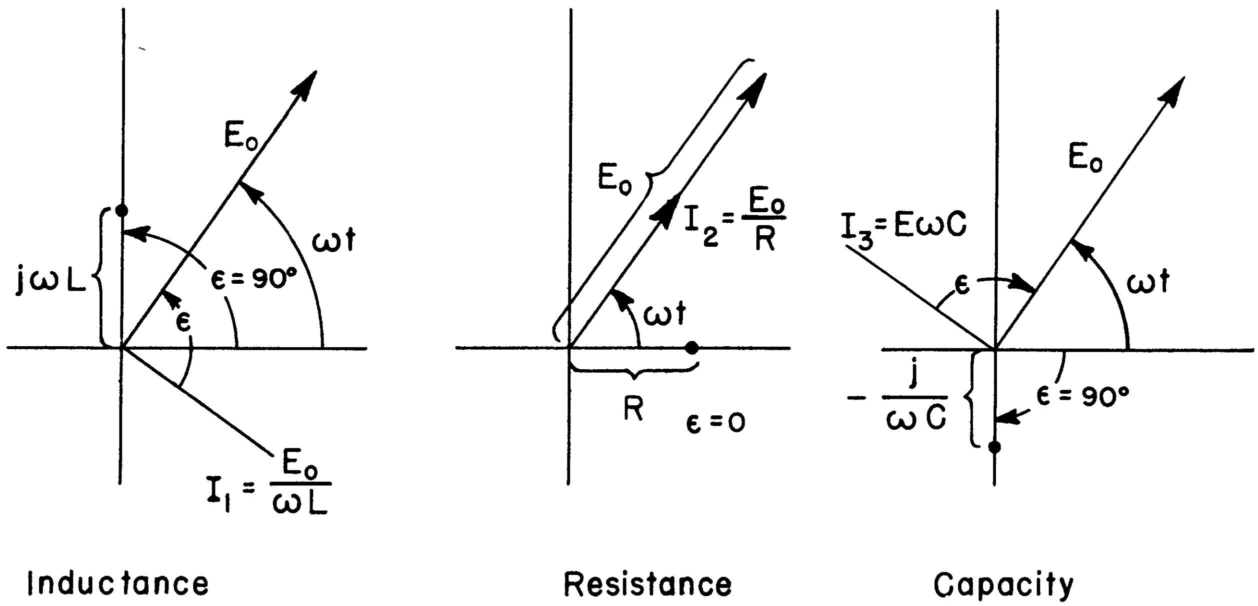


Figure 4: Vector diagrams for the electrical elements in parallel as shown in right hand side of Figure I.

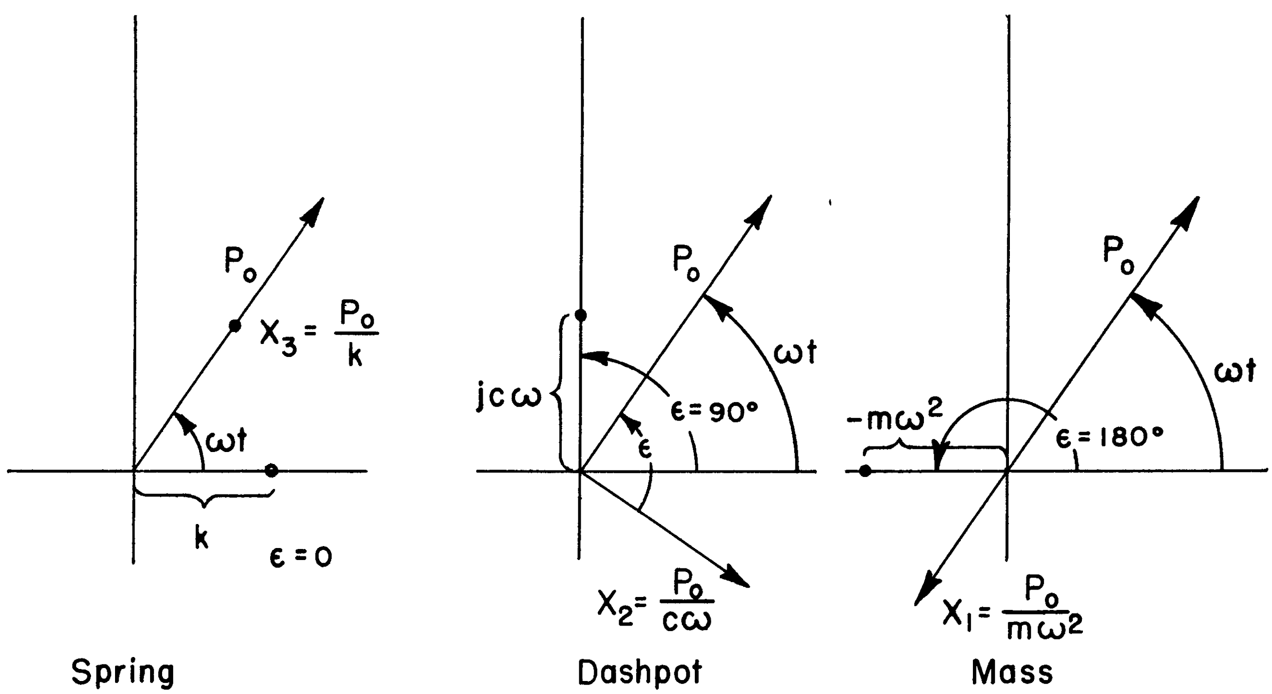


Figure 5: Vector diagram for the mechanical elements in parallel as shown in right hand side of Figure I.

damping constant must be found. If the constant frictional force under Coulomb damping is F the equivalent viscous damping constant is $c = 4F/\pi\omega X$.

It is best to adopt some standard form of representing mechanical systems, if possible, say by placing the element on which the driving force acts at the left. Then by the impedance to the right of any point in the system is obviously meant the ratio of the force that would produce a given amplitude at that point to the amplitude, provided the part of the system to the left of the point is removed.

If levers or linkages exist in a mechanical system it is in general possible to reduce the system to an equivalent simple system by multiplying the impedance of all elements attached to the ends of the levers by factors depending on the square of the ratio of the amplitudes at the two ends. For example if two equal masses m were joined together directly their mechanical impedance would be $-2m\omega^2$ whilst if they were attached to a weightless link in such a way that one was twice as far from the fulcrum as the other the mechanical impedance at the mass nearest the fulcrum would be $-5m\omega^2$ and at the mass farthest from the fulcrum would be $-5m\omega^2/4$. If instead of simple masses there are attached to the ends of the linkage mechanical systems of impedances Z_1 and Z_2 respectively then the total impedance at the point of attachment of Z_1 is

$$Z = Z_1 + \left(\frac{l_2}{l_1}\right)^2 Z_2$$

where l_1 and l_2 are the respective distances from the fulcrum of the masses of impedances Z_1 and Z_2.

The general procedure in using the method of mechanical impedance is therefore as follows:

(1) lay out the mechanical system with the element on which the driving force acts to the left if possible.

(2) write down the impedance of each individual element, the impedance of a mass being $-m\omega^2$, of a spring being k, and of a dashpot being $jc\omega$.

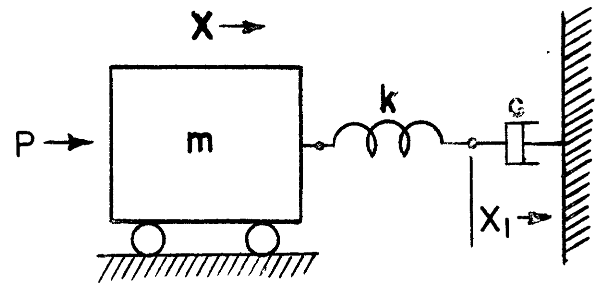
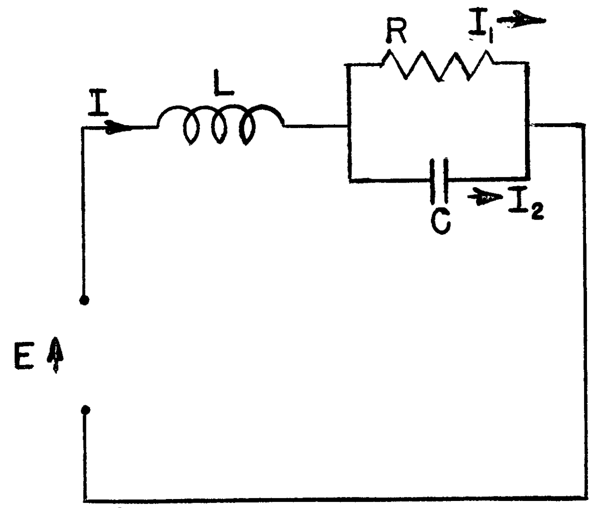
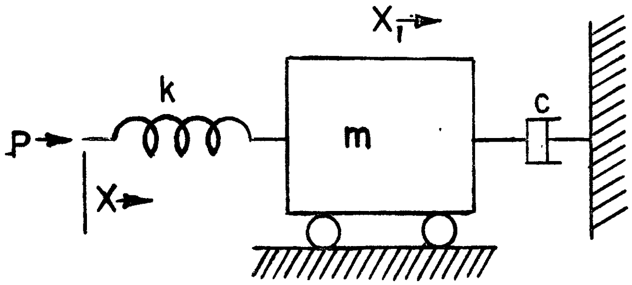
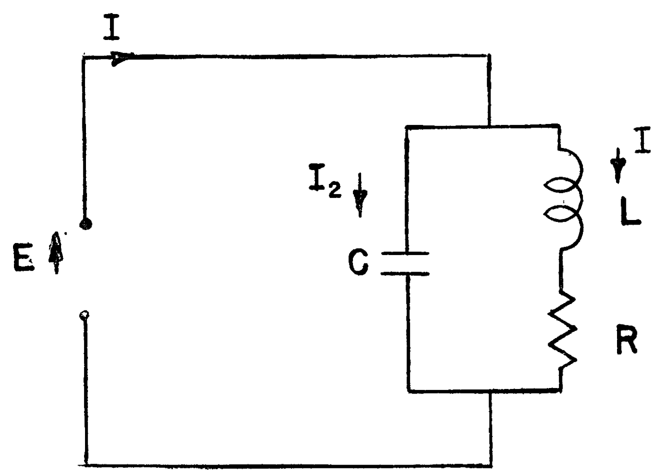
(3) combine the impedances of the individual elements working from left to right using the rules for series or parallel combinations as the case may be.

(4) Resonant frequencies may be found from the impedance formulas by setting up the expression for the square of the absolute value of the impedance, differentiating this with respect to ω, setting the derivative equal to zero, and solving for ω.

The application of the impedance method so far discussed is limited. Mass and spring combinations can be found which cannot be arranged in series and parallel groups. Cases in which the electrical analogy involves mutual impedances fall here. While the method of mechanical impedance can be expanded to cover such cases its advantage over the direct solution from the differential equations is much less clear.

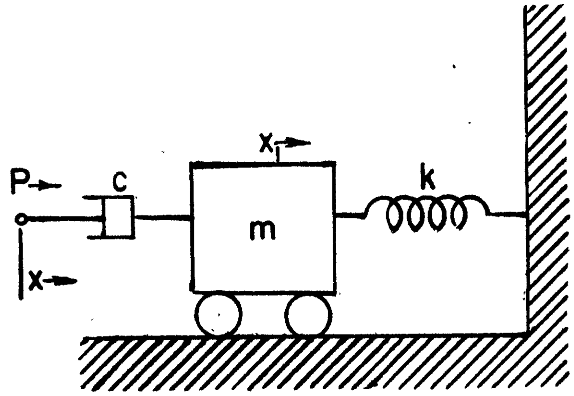
A few examples are believed essential to clarifying the method so far discussed. In all cases it is assumed that it is the solution of the mechanical problem that is primarily required. It is emphasized that it is not essential to consider the electrical analogy in order to use the method of mechanical impedance.

In these examples a dashpot indicates that viscous damping exists between the points where the dashpot is attached. Where Coulomb damping is present the equivalent viscous damping constant must be calculated on the basis of the energy dissipated per cycle as previously shown. The phase angle by which the force leads the displacement can readily be obtained after the numerical values of the real and imaginary parts of the impedance have been found and hence formulas for the phase angles have been omitted. The impedance formulas are given only in the form in which they may be written down directly by inspection. Rationalization of the formulas in their algebraic form is not essential as the process is much simpler after numerical values have been substituted which will always be the case in using them.

Mechanical	Electrical
One degree of freedom	
 let $X_2 = X - X_1$ $Z = -m\omega^2 + \dfrac{1}{\dfrac{1}{k} + \dfrac{1}{jc\omega}}$	 $Z = j\omega L + \dfrac{1}{\dfrac{1}{R} + \dfrac{1}{\dfrac{-j}{\omega C}}}$
 let $X_2 = X - X_1$ $Z = \dfrac{1}{\dfrac{1}{k} + \dfrac{1}{-m\omega^2 + jc\omega}}$	 $Z = \dfrac{1}{\dfrac{1}{\dfrac{-j}{\omega C}} + \dfrac{1}{R + j\omega L}}$

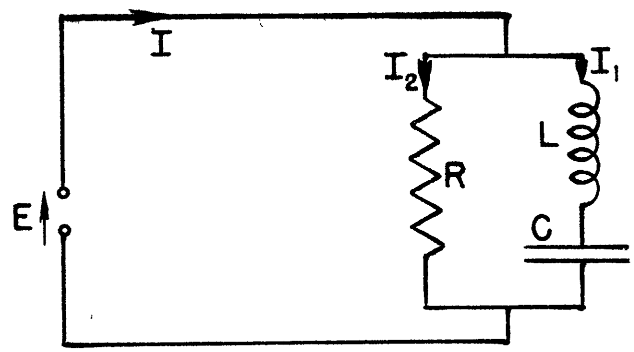
Mechanical	Electrical

One degree of freedom

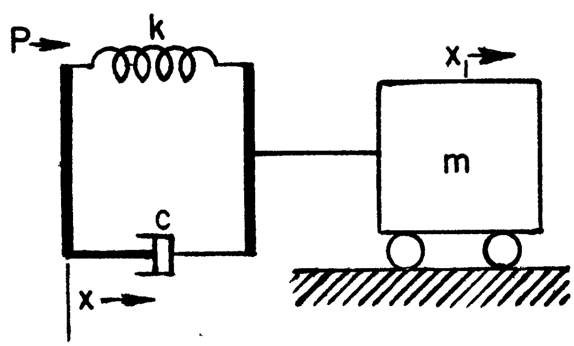


let $x_2 = x - x_1$

$$Z = \frac{1}{\frac{1}{jc\omega} + \frac{1}{-m\omega^2 + k}}$$

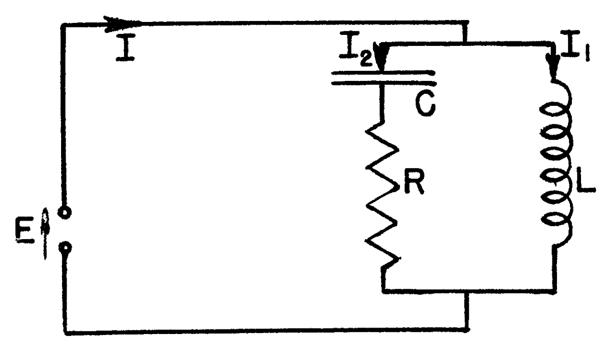


$$Z = \frac{1}{\frac{1}{R} + \frac{1}{j\left(\omega L - \frac{1}{\omega C}\right)}}$$



let $x_2 = x - x_1$

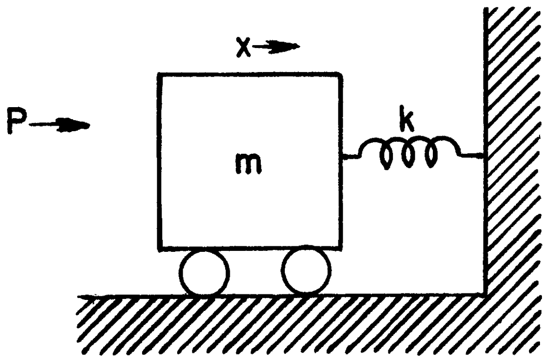
$$Z = \frac{1}{\frac{1}{k + jc\omega} + \frac{1}{-m\omega^2}}$$



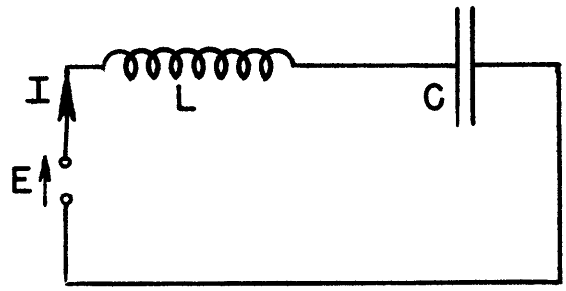
$$Z = \frac{1}{\frac{1}{R - \frac{j}{\omega C}} + \frac{1}{j\omega L}}$$

Mechanical	Electrical

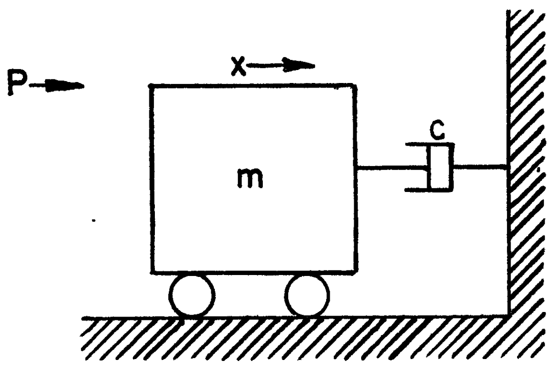
One degree of freedom.



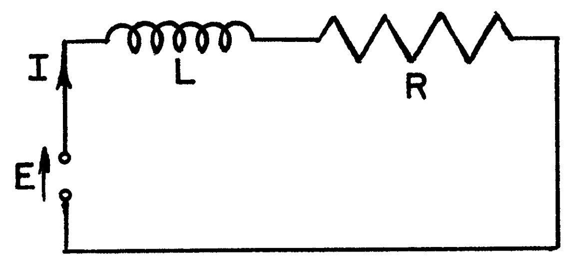
$Z = -m\omega^2 + k$



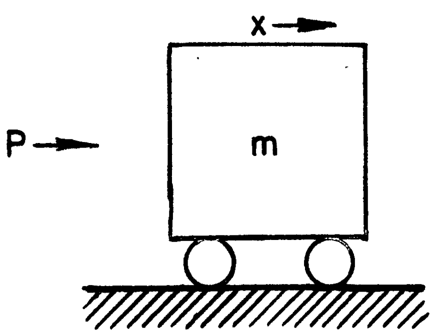
$Z = j\left(\omega L - \frac{1}{\omega C}\right)$



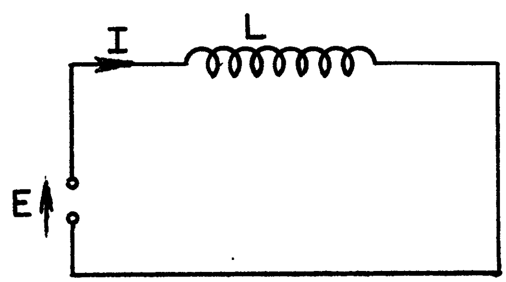
$Z = -m\omega^2 + jc\omega$



$Z = R + j\omega L$



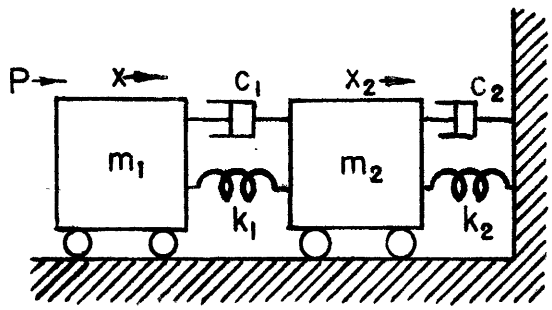
$Z = -m\omega^2$



$Z = j\omega L$

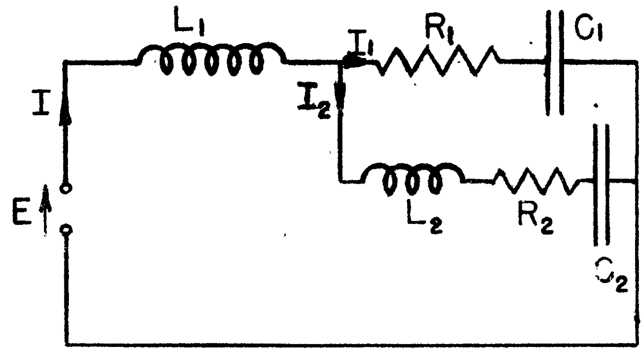
Mechanical	Electrical

Two degrees of freedom

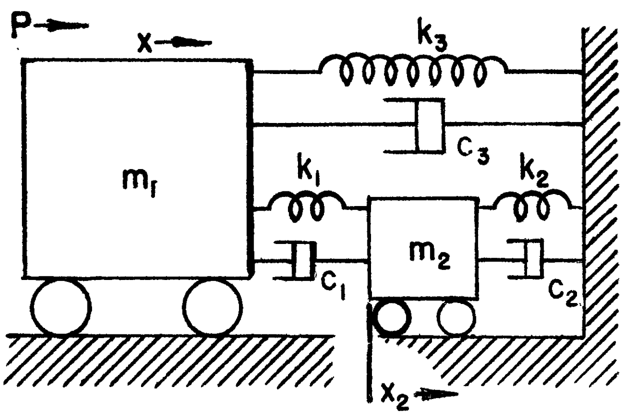


let $x_1 = x - x_2$

$$Z = -m_1\omega^2 + \cfrac{1}{\cfrac{1}{k_1 + jc_1\omega} + \cfrac{1}{-m_2\omega^2 + k_2 + jc_2\omega}}$$



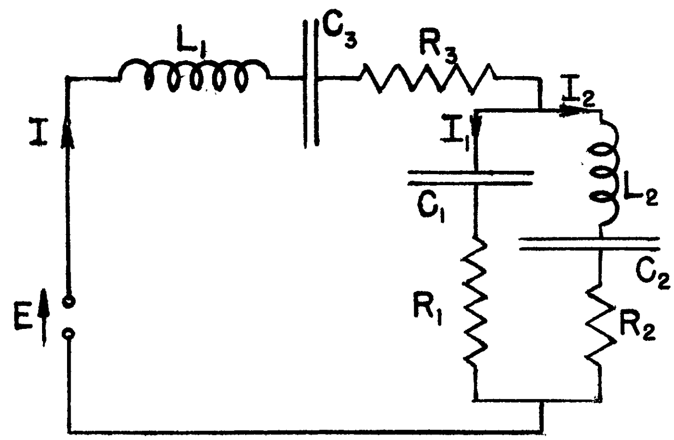
$$Z = j\omega L_1 + \cfrac{1}{\cfrac{1}{R_1 - \cfrac{j}{\omega C_1}} + \cfrac{1}{R_2 + j\left(\omega L_2 - \cfrac{1}{\omega C_2}\right)}}$$



let $x_1 = x - x_2$

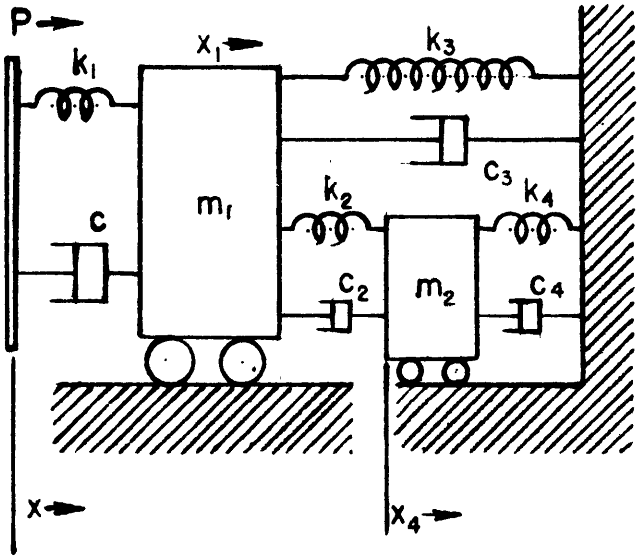
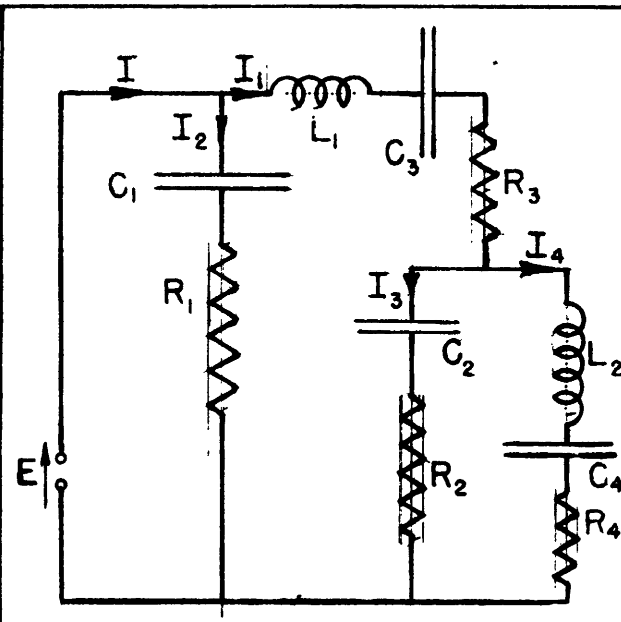
$$Z = k_3 - m_1\omega^2 + jc_3\omega +$$

$$+ \cfrac{1}{\cfrac{1}{k_1 + jc_1\omega} + \cfrac{1}{-m_2\omega^2 + k_2 + jc_2\omega}}$$



$$Z = R_3 + j\left(\omega L_1 - \frac{1}{\omega C_3}\right) +$$

$$+ \cfrac{1}{\cfrac{1}{R_1 - \cfrac{j}{\omega C_1}} + \cfrac{1}{R_2 + j\left(\omega L_2 - \cfrac{1}{\omega C_2}\right)}}$$

Mechanical	Electrical
Two degrees of freedom	
	

let $x_2 = x - x_1$

$x_3 = x_1 - x_4$

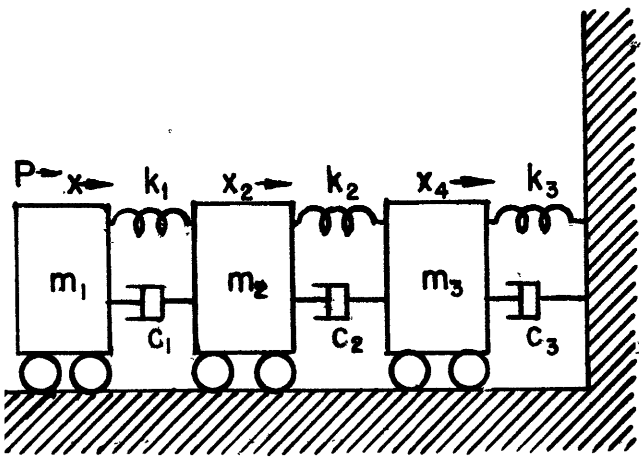
Mechanical Formula

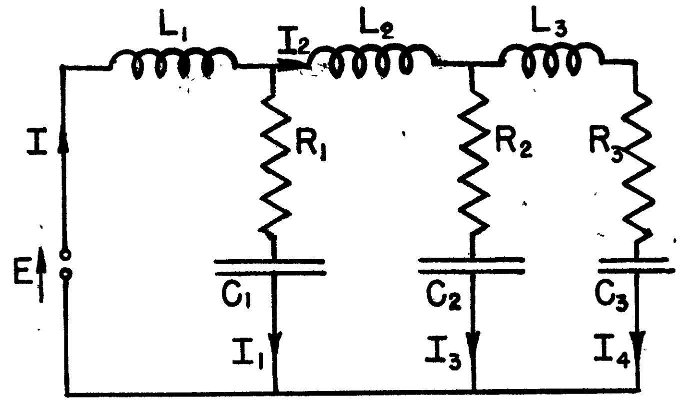
$$Z = \cfrac{1}{\cfrac{1}{k_1 + jc_1\omega} + \cfrac{1}{k_3 - m_1\omega^2 + jc_3\omega + \cfrac{1}{\cfrac{1}{k_2 + jc_2\omega} + \cfrac{1}{k_4 - m_2\omega^2 + jc_4\omega}}}}$$

Electrical Formula

$$Z = \cfrac{1}{\cfrac{1}{R_1 - \cfrac{j}{\omega C_1}} + \cfrac{1}{R_3 + j\left(\omega L_1 - \cfrac{1}{\omega C_3}\right) + \cfrac{1}{\cfrac{1}{R_2 - \cfrac{j}{\omega C_2}} + \cfrac{1}{R_4 + j\left(\omega L_2 - \cfrac{1}{\omega C_4}\right)}}}}$$

Mechanical	Electrical
Three degrees of freedom	





let $x_1 = x - x_2$

$x_3 = x_2 - x_4$

Mechanical Formula

$$Z = -m_1\omega^2 + \cfrac{1}{\cfrac{1}{k_1 + jc_1\omega} + \cfrac{1}{-m_2\omega^2 + \cfrac{1}{\cfrac{1}{k_2 + jc_2\omega} + \cfrac{1}{k_3 - m_3\omega^2 + jc_3\omega}}}}$$

Electrical Formula

$$Z = j\omega L_1 + \cfrac{1}{\cfrac{1}{R_1 - \cfrac{j}{\omega C_1}} + \cfrac{1}{j\omega L_2 + \cfrac{1}{\cfrac{1}{R_2 - \cfrac{j}{\omega C_2}} + \cfrac{1}{R_3 + j\left(\omega L_3 - \cfrac{1}{\omega C_3}\right)}}}}$$