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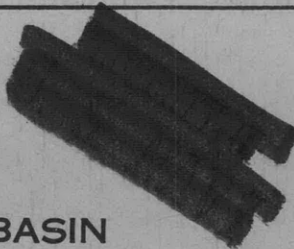
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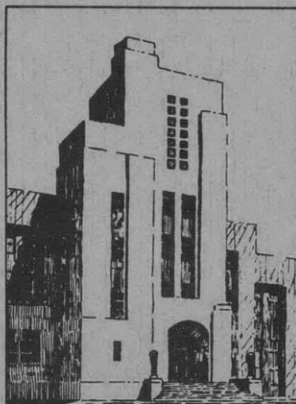
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 THE DAVID W. TAYLOR MODEL BASIN
 WASHINGTON 7, D.C.

THE MOMENT ACTING ON A RANKINE OVOID
 MOVING UNDER A FREE SURFACE

by

Hartley L. Pond

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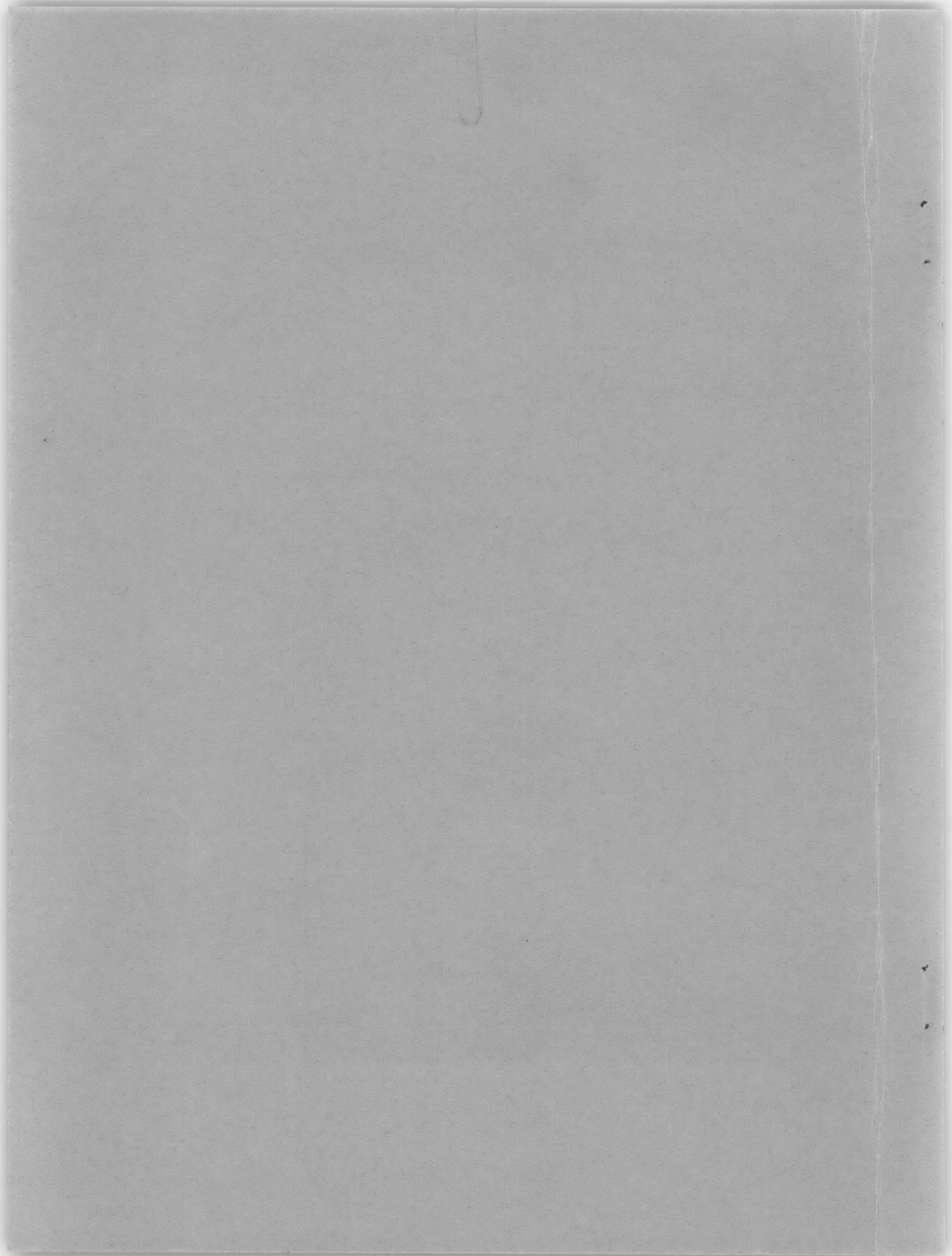
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Errata - October 1951

"The Moment Acting on a Rankine Ovoid Moving Under a Free Surface," by Hartley L. Pond, TMB CONFIDENTIAL Report C-429, September 1951.

List of Symbols - Under K_0, K_1 reads: (Reference 18, p. 19)
Should read: (Reference 18, p. 78)

Page 2, Figure 1 - Coordinates of source are $(a, 0, -f)$
Should be: $(0, 0, -f)$

Coordinates of image sink are $(a, 0, +f)$
Should be: $(0, 0, +f)$

Page 7, Equation [11] reads $-2fm[x+a]^2 + \text{etc.}$
Should read: $-2fm[(x+a)^2 + \text{etc.}$

Page 17, Equation [24] reads:

$$C_M \approx -\frac{\beta \epsilon^2}{64 \alpha^3} \omega e^{-\frac{\omega}{2}} \left[\dots \right]$$

Should read:

$$C_M \approx -\frac{\beta \epsilon^2}{64 \alpha^3} \omega^2 e^{-\frac{\omega}{2}} \left[\dots \right]$$

Page 19, Reference 13 reads: Colluquium
Should read: Colloquium

(Corrections made 11/30/51)

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THE MOMENT ACTING ON A RANKINE OVOID
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by

Hartley L. Pond

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LIST OF SYMBOLS

a	One half the distance between the source and sink of the Rankine Ovoid
b	Maximum radius of body
C_M	Moment coefficient = $\frac{M}{\frac{1}{2} \rho c^2 \pi b^2 l}$
	(Note that this coefficient is designated by M' in Reference 3)
c	Constant uniform stream velocity
d	Maximum diameter of body
F	Froude number based on length = $\frac{c}{\sqrt{g l}}$
\bar{F}	Force vector
f	Distance of source or sink below the undisturbed fluid surface
g	Acceleration of gravity
K_0, K_1	Modified Bessel functions of the second kind (Reference 18, p. 19) 78
k_0	$\frac{g}{c^2}$
L_{2n+1}	A function defined on page 16
l	Over-all length of body
M, M_1, M_2	Moments
m	The strength of a source (a source of strength m emits a volume $4\pi m$ per unit time)
P	Placed before an integral sign means that the Cauchy principal value of the integral is to be taken
\bar{q}	Resultant fluid velocity vector at the location of a source due to all other sources
q_x, q_y, q_z	Magnitude of components of \bar{q} in x, y, z directions, respectively
r	Radial distance
u, v, w	Magnitude of components of local velocity in x, y, z directions, respectively
x, y, z	Rectangular coordinates
α	$\frac{\text{depth of centerline of body}}{\text{diameter of body}} = \frac{f}{d}$

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β	$\frac{\text{length of body}}{\text{diameter of body}} = \frac{l}{d}$
ϵ	$\frac{\text{distance between source and sink}}{\text{length of body}} = \frac{2a}{l}$
η	A parameter
λ_i	Christoffel number used in numerical quadrature formula
μ	Strength of doublet
ρ	Mass density of fluid
ϕ	Velocity potential ($u = -\frac{\partial \phi}{\partial x}$, $v = -\frac{\partial \phi}{\partial y}$, $w = -\frac{\partial \phi}{\partial z}$)
ω	$\frac{2\alpha}{\beta F^2}$

ABSTRACT

The moment acting on a Rankine Ovoid moving under the free surface of a fluid of infinite depth is calculated. The solution is carried through two approximations. The results of the second approximation are shown to be in excellent agreement with available experimental data.

INTRODUCTION

The David Taylor Model Basin is conducting an investigation of the forces and moments acting on submerged bodies moving near a free surface. As a part of this investigation the hydrodynamical theory of the problems involved is being studied. Calculations of resistance^{1,2} and vertical force^{3,4} have been made which show reasonable (though by no means entirely satisfactory) agreement with experimental results. However, calculations of moments have shown little or no agreement with experimental results (for example, see discussion on page 6 of Reference 5). In the present report the calculation for the moment acting on a Rankine Ovoid moving under the free surface of a fluid of infinite depth is carried through two approximations, and the results of the second approximation are found to be in excellent agreement with available experimental data.

THE FIRST APPROXIMATION

With the usual assumptions of small wave slope, and that the velocity (due to the wave motion) of the fluid particles is sufficiently small so that the square of this velocity can be neglected in Bernoulli's equation (Reference 6, p. 1), the velocity potential of fluid motion (for an incompressible, nonviscous fluid) due to a source (Reference 6, p. 404) located below the free surface of a uniform stream of infinite depth is (Reference 7, p. 3)

$$\begin{aligned} \phi(x,y,z) = & cx + \frac{m}{r_1} - \frac{m}{r_2} \\ & - \frac{4k_0 m}{\pi} P \int_0^{\frac{\pi}{2}} \sec^2 \theta d\theta \int_0^{\infty} \frac{e^{-k(f-z)} \cos(kx \cos \theta) \cos(ky \sin \theta)}{k - k_0 \sec^2 \theta} dk \\ & - 4k_0 m \int_0^{\frac{\pi}{2}} \sec^2 \theta e^{-k_0(f-z)\sec^2 \theta} \sin(k_0 x \sec \theta) \cos(k_0 y \sin \theta \sec^2 \theta) d\theta \end{aligned} \quad [1]$$

¹References are listed on page 18.

where x, y, z are rectangular coordinates, z positive upwards (the undisturbed free surface is the xy -plane of Figure 1),

f is the depth of the source below the undisturbed free surface,

$$r_1^2 \text{ is equal to } x^2 + y^2 + (z + f)^2,$$

$$r_2^2 \text{ is equal to } x^2 + y^2 + (z - f)^2,$$

$$\phi(x, y, z) \text{ is the velocity potential } (u = -\frac{\partial \phi}{\partial x}, v = -\frac{\partial \phi}{\partial y}, w = -\frac{\partial \phi}{\partial z}),$$

u, v, w are velocities in positive x, y, z directions,

$-c$ is the uniform stream velocity,

$$k_0 \text{ is equal to } \frac{g}{c^2},$$

g is the acceleration of gravity,

m is the strength of the source (a source of strength m emits a volume $4\pi m$ per unit time), and

P placed before an integral sign means that the Cauchy principal value is to be taken (Reference 8, p. 128).

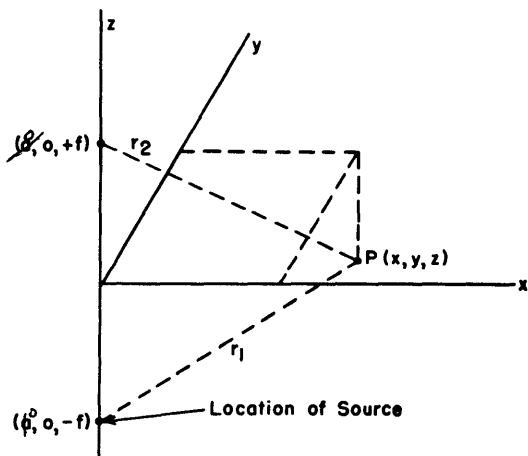


Figure 1

In the notation used the source is considered to be held stationary in a uniform stream. This is, of course, equivalent to having the source move with a uniform velocity through a stationary fluid. It is of interest to note that the first two terms of Equation [1] give the velocity potential of a source in an unbounded uniform stream.

If a source and equal sink are placed a given distance apart on a line parallel to a uniform stream in an infinite fluid, the resulting fluid

motion is that for the flow of a uniform stream about an oval-shaped body called the Rankine Ovoid (Reference 6, p. 411). If the source and sink are near the free surface of a uniform stream the motion is that for the flow of a uniform stream (with a free surface) about a "distorted" Rankine Ovoid. The distortion is small when the source and sink are sufficiently far from the free surface of an infinitely deep stream. Using Equation [1], the velocity potential of the fluid motion about the distorted Rankine Ovoid with source at $(a, 0, -f)$ and sink at $(-a, 0, -f)$ is

$$\begin{aligned}
\phi(x,y,z) = & cx + \frac{m}{r_1} - \frac{m}{r_3} - \frac{m}{r_2} + \frac{m}{r_4} \\
& - \frac{4k_0 m}{\pi} P \int_0^{\frac{\pi}{2}} \sec^2 \theta d\theta \int_0^{\infty} \frac{e^{-k(f-z)} \cos[k(x-a)\cos\theta] \cos(ky \sin\theta)}{k - k_0 \sec^2 \theta} dk \\
& - 4k_0 m \int_0^{\frac{\pi}{2}} \sec^2 \theta e^{-k_0(f-z)\sec^2 \theta} \sin[k_0(x-a)\sec\theta] \cos(k_0 y \sin\theta \sec^2 \theta) d\theta \quad [2] \\
& + \frac{4k_0 m}{\pi} P \int_0^{\frac{\pi}{2}} \sec^2 \theta d\theta \int_0^{\infty} \frac{e^{-k(f-z)} \cos[k(x+a)\cos\theta] \cos(ky \sin\theta)}{k - k_0 \sec^2 \theta} dk \\
& + 4k_0 m \int_0^{\frac{\pi}{2}} \sec^2 \theta e^{-k_0(f-z)\sec^2 \theta} \sin[k_0(x+a)\sec\theta] \cos(k_0 y \sin\theta \sec^2 \theta) d\theta
\end{aligned}$$

where in addition to terms defined for Equation [1]

$$r_1^2 = (x - a)^2 + y^2 + (z + f)^2$$

$$r_2^2 = (x - a)^2 + y^2 + (z - f)^2$$

$$r_3^2 = (x + a)^2 + y^2 + (z + f)^2$$

$$r_4^2 = (x + a)^2 + y^2 + (z - f)^2$$

The first three terms of Equation [2] give the velocity potential of a source and equal sink in an unbounded uniform stream.

Lagally's theorem^{9,10} may be applied to obtain the moment acting on the above distorted Rankine Ovoid. This theorem states that the forces acting on a body whose surface is a closed stream surface of the fluid motion, are given by the vectors

$$\bar{F} = -4\pi\rho m\bar{q}$$

where m is the strength of a source internal to the stream surface,

\bar{q} is the resultant fluid velocity vector at the location of the source due to all other sources (its components in the x, y, z directions are q_x, q_y, q_z), and

ρ is the mass density of the fluid.

Thus, the force has the direction of $-\bar{q}$ and its line of action passes through the point at which the source is located. Each term of Equations [1] and [2] may be considered as the velocity potential due to a certain source or distribution of sources. Applying Lagally's theorem, the vertical force acting on

the distorted Rankine Ovoid through the point (a,o,-f) is

$$Z(a,o,-f) = -4\pi\rho m q_z(a,o,-f) \quad [3]$$

and the vertical force acting on the body through the point (-a,o,-f) is

$$Z(-a,o,-f) = -4\pi\rho(-m)q_z(-a,o,-f) \quad [4]$$

Since there are no other internal sources for this body, Equations [3] and [4] give the only vertical forces acting on the body.

Carrying out the partial differentiations indicated in Equations [3] and [4], the following expressions are obtained

$$\begin{aligned} Z(a,o,-f) = & -4\pi\rho m \left[\frac{m}{4f^2} - \frac{fm}{4(a^2+f^2)^{\frac{3}{2}}} + \frac{4k_0 m}{\pi} P \int_0^{\frac{\pi}{2}} \sec^2 \theta d\theta \int_0^\infty \frac{ke^{-2fk}}{k - k_0 \sec^2 \theta} dk \right. \\ & - \frac{4k_0 m}{\pi} P \int_0^{\frac{\pi}{2}} \sec^2 \theta d\theta \int_0^\infty \frac{ke^{-2fk} \cos[2ak \cos \theta]}{k - k_0 \sec^2 \theta} dk \\ & \left. - 4k_0^2 m \int_0^{\frac{\pi}{2}} \sec^4 \theta e^{-2fk_0 \sec^2 \theta} \sin[2ak_0 \sec \theta] d\theta \right] \quad [3'] \end{aligned}$$

$$\begin{aligned} Z(-a,o,-f) = & +4\pi\rho m \left[-\frac{m}{4f^2} + \frac{fm}{4(a^2+f^2)^{\frac{3}{2}}} - \frac{4k_0 m}{\pi} P \int_0^{\frac{\pi}{2}} \sec^2 \theta d\theta \int_0^\infty \frac{ke^{-2kf}}{k - k_0 \sec^2 \theta} dk \right. \\ & + \frac{4k_0 m}{\pi} P \int_0^{\frac{\pi}{2}} \sec^2 \theta d\theta \int_0^\infty \frac{ke^{-2kf} \cos[2ak \cos \theta]}{k - k_0 \sec^2 \theta} dk \\ & \left. - 4k_0^2 m \int_0^{\frac{\pi}{2}} \sec^4 \theta e^{-2fk_0 \sec^2 \theta} \sin[2ak_0 \sec \theta] d\theta \right] \quad [4'] \end{aligned}$$

Hence, the moment acting on the body about the center of buoyancy, $M_1 = a[Z(a,o,-f) - Z(-a,o,-f)]$, is

$$M_1 = 32\pi\rho am^2 k_0^2 \int_0^{\frac{\pi}{2}} \sec^4 \theta e^{-2fk_0 \sec^2 \theta} \sin[2ak_0 \sec \theta] d\theta \quad [5]$$

(In the above notation a positive moment acts to raise the nose of the body.) Making the substitution $t = \tan \theta$, this equation may be written as

$$M_1 = 32 \pi \rho a m^2 k_0^2 \int_0^\infty (1+t^2) e^{-2fk_0(1+t^2)} \sin[2ak_0 \sqrt{1+t^2}] dt$$

For Rankine Ovoids with sufficiently large length-diameter ratios (see Equations 3-5 of Reference 3)

$$m = \frac{b^2 c}{4}$$

where b is the maximum radius of the Rankine Ovoid. With this substitution the expression for M_1 becomes

$$M_1 = \pi \rho b^4 c^2 k_0 (2ak_0) e^{-2fk_0} \int_0^\infty e^{-2fk_0 t^2} (1+t^2) \sin[2ak_0 \sqrt{1+t^2}] dt \quad [5']$$

THE SECOND APPROXIMATION

Equation [5] gives the moment acting on a distorted Rankine Ovoid and agrees with the results given on pages 45 and 46 of Reference 3. To obtain a closer approximation to the moment acting on an undistorted Rankine Ovoid placed below the surface of a uniform stream the simple distribution of a source and equal sink on the axis of the closed stream surface representing the Rankine Ovoid must be modified. This might be done by an extension of the method of images as applied by Havelock in the case of a circular cylinder. This method would require finding the image system of sources within the closed stream surface due to the source system above the free surface. However, instead of attempting to obtain this image system exactly, an approximate image system will be sought.

It has been shown by von Kármán¹¹ that for a body of revolution with its axis parallel to a uniform stream the effect of superimposing a flow perpendicular to the axis may be obtained approximately by a suitable distribution of doublets (Reference 11, p. 12) along the axis of the body between the limits of the source-sink distribution which defines the body in the uniform stream. The doublets are oriented so that their axes are opposite in direction to the transverse flow and their strength per unit distance along the axis of the body is

$$\mu = -\frac{1}{2} r^2 w \quad [6]$$

where r is the radius of the body at the position of the doublet, and w is the superimposed transverse velocity.

Therefore, in the case of the Rankine Ovoid moving below a free surface, the effect of the velocity induced by the free surface may be approximately accounted for by a suitable distribution of doublets along the axis of the stream surface between the source and sink. It will be apparent that for the calculation of moments (see below) only the vertical component of the induced velocity need be considered. This vertical velocity is obtained from Equation [2] by evaluating $\frac{\partial \phi}{\partial z}$ at points along the line between the source and sink. Since the vertical velocity does not change very rapidly with depth, this calculation of the vertical velocity is probably satisfactory. For Rankine Ovoids with fairly large length-diameter ratios the diameter of the body is nearly constant in the region between the source and sink. Thus, as a further simplification the radius r in Equation [6] will be considered constant and equal to the maximum radius of the body. The desired doublet distribution is now given by

$$\mu = -\frac{1}{2} b^2 w \quad [7]$$

where b is the maximum radius of the undistorted Rankine Ovoid, and w is the vertical component of the induced velocity calculated from Equation [2].

An application of Lagally's theorem shows that a body whose surface is a closed stream surface of the fluid motion experiences a moment if the axis of any internal doublet is normal to the direction of a superimposed uniform stream. (The doublet, however, causes no resultant force on the body.) This moment is given by (Reference 10, p. 13)

$$M = -4\pi\rho\mu c \quad [8]$$

where ρ is the mass density of the fluid,
 μ is the doublet strength, and
 c is the uniform stream velocity.

Hence, from Equations [7] and [8] the moment per unit length for the Rankine Ovoid (due to the uniform stream c) has the algebraic sign of the vertical velocity and is given by

$$M = 2\pi\rho cb^2w \quad [9]$$

and the total moment* on the body due to the doublet distribution is

*The longitudinal velocity due to the wave system is small compared with the uniform stream velocity c and is not considered in calculating M_2 . In addition, the mutual actions of the sources and doublets within the closed stream surface representing the body give no resultant force or moment (Reference 10, p. 5).

$$M_2 = 2\pi\rho cb^2 \int_{-a}^a w dx \quad [10]$$

where, using Equation [2], the velocity w is

$$\begin{aligned} w = w(x, 0, -f) &= -\frac{\partial\phi(x, y, z)}{\partial z} \Bigg|_{\substack{x=x \\ y=0 \\ z=-f}} \\ &= 2fm[(x-a)^2 + (2f)^2]^{-3/2} - 2fm[(x+a)^2 + (2f)^2]^{-3/2} \\ &+ \frac{4k_0 m}{\pi} P \int_0^{\pi/2} \sec^2\theta d\theta \int_0^\infty \frac{ke^{-2fk} \cos[k(x-a)\cos\theta]}{k - k_0 \sec^2\theta} dk \\ &- \frac{4k_0 m}{\pi} P \int_0^{\pi/2} \sec^2\theta d\theta \int_0^\infty \frac{ke^{-2fk} \cos[k(x+a)\cos\theta]}{k - k_0 \sec^2\theta} dk \\ &+ 4k_0^2 m \int_0^{\pi/2} \sec^4\theta e^{-2fk_0 \sec^2\theta} \sin[k_0(x-a)\sec\theta] d\theta \\ &- 4k_0^2 m \int_0^{\pi/2} \sec^4\theta e^{-2fk_0 \sec^2\theta} \sin[k_0(x+a)\sec\theta] d\theta \end{aligned} \quad [11]$$

The first four terms on the right of Equation [11] give that part of w which is skew symmetric about $x = 0$, and the last two the symmetric part. Since the integrals of the skew symmetric terms vanish, Equation [10] now becomes

$$\begin{aligned} M_2 &= 16\pi\rho cb^2 k_0^2 m \left[\int_0^a \int_0^{\pi/2} \sec^4\theta e^{-2fk_0 \sec^2\theta} \sin[k_0(x-a)\sec\theta] d\theta dx \right. \\ &\quad \left. - \int_0^a \int_0^{\pi/2} \sec^4\theta e^{-2fk_0 \sec^2\theta} \sin[k_0(x+a)\sec\theta] d\theta dx \right] \end{aligned} \quad [12]$$

The double integrals in Equation [12] may be expressed in a more convenient form. Consider, for example, the second of these integrals and let

$$I_2(x) = \int_0^{\pi/2} \sec^4\theta e^{-2fk_0 \sec^2\theta} \sin[k_0(x+a)\sec\theta] d\theta$$

The substitution $t = \tan \theta$ transforms this integral into

$$I_2(x) = \int_0^{\infty} (1+t^2) e^{-2fk_0(1+t^2)} \sin[k_0(x+a) \sqrt{1+t^2}] dt$$

The second double integral of Equation [12] can then be written as

$$J_2 = \int_0^a \int_0^{\infty} (1+t^2) e^{-2fk_0(1+t^2)} \sin[k_0(x+a) \sqrt{1+t^2}] dt dx$$

Interchanging the order of integration (Reference 12, p. 277), the expression for J_2 becomes

$$\begin{aligned} J_2 &= \frac{e^{-2fk_0}}{k_0} \int_0^{\infty} e^{-2fk_0 t^2} \sqrt{1+t^2} \cos[k_0 a \sqrt{1+t^2}] dt \\ &\quad - \frac{e^{-2fk_0}}{k_0} \int_0^{\infty} e^{-2fk_0 t^2} \sqrt{1+t^2} \cos[2k_0 a \sqrt{1+t^2}] dt \end{aligned}$$

Designating the first integral of Equation [12] by J_1 , and proceeding as above, it is found that

$$\begin{aligned} J_1 &= \frac{e^{-2fk_0}}{k_0} \int_0^{\infty} e^{-2fk_0 t^2} \sqrt{1+t^2} \cos[k_0 a \sqrt{1+t^2}] dt \\ &\quad - \frac{e^{-2fk_0}}{k_0} \int_0^{\infty} e^{-2fk_0 t^2} \sqrt{1+t^2} dt \end{aligned}$$

Hence, Equation [12] may be written as (putting $m = \frac{b^2 c}{4}$ as in Equation [5'])

$$M_2 = 4\pi\rho b^4 c^2 k_0 e^{-2fk_0} \int_0^{\infty} e^{-2fk_0 t^2} \sqrt{1+t^2} (\cos[2k_0 a \sqrt{1+t^2}] - 1) dt \quad [13]$$

The second approximation to the total moment acting on the undistorted Rankine Ovoid about its center of buoyancy is now given as the sum of the moments given by Equations [5'] and [13]. This sum may be written as

$$\begin{aligned}
M &= M_1 + M_2 \\
&= \pi \rho b^4 c^2 k_0 (2ak_0) e^{-2fk_0} \int_0^\infty e^{-2fk_0 t^2} (1+t^2) \sin[2ak_0 \sqrt{1+t^2}] dt \\
&\quad + 4\pi \rho b^4 c^2 k_0 e^{-2fk_0} \int_0^\infty e^{-2fk_0 t^2} \sqrt{1+t^2} (\cos[2ak_0 \sqrt{1+t^2}] - 1) dt
\end{aligned} \tag{14}$$

OR

$$\begin{aligned}
M &= \pi \rho b^4 c^2 k_0 e^{-2fk_0} \int_0^\infty e^{-2fk_0 t^2} \sqrt{1+t^2} \left[2ak_0 \sqrt{1+t^2} \sin[2ak_0 \sqrt{1+t^2}] \right. \\
&\quad \left. + 4 \cos [2ak_0 \sqrt{1+t^2}] - 4 \right] dt
\end{aligned} \tag{14'}$$

A moment coefficient is given by

$$C_M = \frac{M}{\frac{1}{2} \rho c^2 \pi b^2 l} \tag{15}$$

where ρ is the mass density of the fluid,
 c is the uniform stream velocity,
 πb^2 is the maximum cross-sectional area, and
 l is the length of the body.

Then, for Rankine Ovoids with sufficiently large length-diameter ratios* the moment coefficient can be written as

$$\begin{aligned}
C_M &= \frac{e^{-\frac{2\alpha}{\beta F^2}}}{2\beta^2 F^2} \int_0^\infty e^{-\frac{2\alpha}{\beta F^2} t^2} \sqrt{1+t^2} \left\{ \frac{\epsilon}{F^2} \sqrt{1+t^2} \sin \left[\frac{\epsilon}{F^2} \sqrt{1+t^2} \right] \right. \\
&\quad \left. + 4 \cos \left[\frac{\epsilon}{F^2} \sqrt{1+t^2} \right] - 4 \right\} dt
\end{aligned} \tag{16}$$

*For the present the term "sufficiently large" means 10 or greater. Rankine Ovoids with other length-diameter ratios will be investigated in a later report.

$$\text{where } \beta = \frac{\text{length of body}}{\text{diameter of body}} = \frac{l}{d}$$

$$\alpha = \frac{\text{depth of centerline of body}}{\text{diameter of body}} = \frac{f}{d}$$

$$\epsilon = \frac{\text{distance between source and sink}}{\text{length of body}} = \frac{2a}{l}$$

$$F = \frac{c}{\sqrt{gt}}$$

From Equation [16] it can be seen that C_M approaches zero as F becomes either very small or very large. The two cases of low and high speeds may also be approximated by considering the free surface as a rigid wall for very low speeds and by neglecting gravity forces for very high speeds. In each case the moment coefficient is zero (Reference 3, p. 19).

Writing Equation [14'] in the form

$$M = d^4 \left[\frac{\pi \rho g e}{16} \right] \int_0^\infty e^{-\frac{2fg}{c^2} t^2} \sqrt{1+t^2} \left\{ \frac{\epsilon l g}{c^2} \sqrt{1+t^2} \sin \left[\frac{\epsilon l g}{c^2} \sqrt{1+t^2} \right] + 4 \cos \left[\frac{\epsilon l g}{c^2} \sqrt{1+t^2} \right] - 4 \right\} dt$$

it is seen that for a given length, depth, and speed the moment M is approximately proportional to the fourth power of the diameter for Rankine Ovoids with length-diameter ratios of 10 or greater. The approximation depends on ϵ . For a length-diameter ratio of 10.5 ϵ is 0.95 and as the length-diameter ratio is increased ϵ approaches 1 as a limit.

The integrals appearing in the above work may be evaluated either by numerical quadrature or by expanding the integrand in an infinite series and integrating term by term. The series expansion is rapidly convergent for $F > 1$ (i.e., for large speeds of advance) while for $F < 1$ it is more convenient to use a Gauss type quadrature formula. The two methods are described in the Appendix.

Figure 3 gives the values of C_M given by Equation [16] for a Rankine Ovoid with a length-diameter ratio of 10.5 (see Figure 2) at submergences of 1.5 and 2.8 diameters. The dotted line shows the values of C_M obtained from the approximate formula for C_M given by Equation [24] of the Appendix.

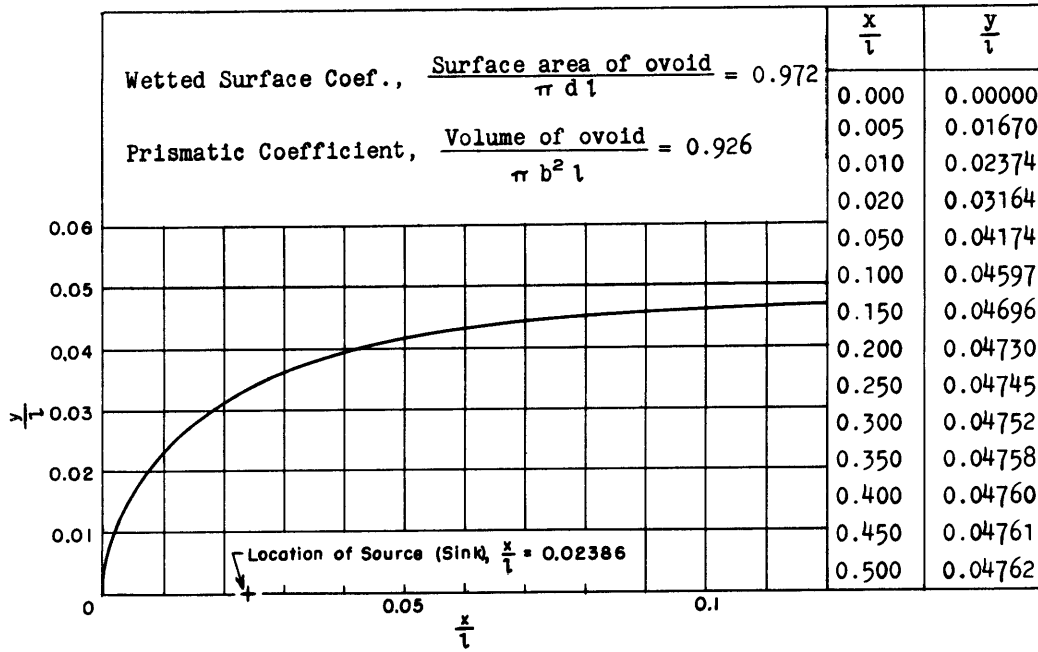


Figure 2 - Offsets of Rankine Ovoid with Length-Diameter Ratio of 10.5

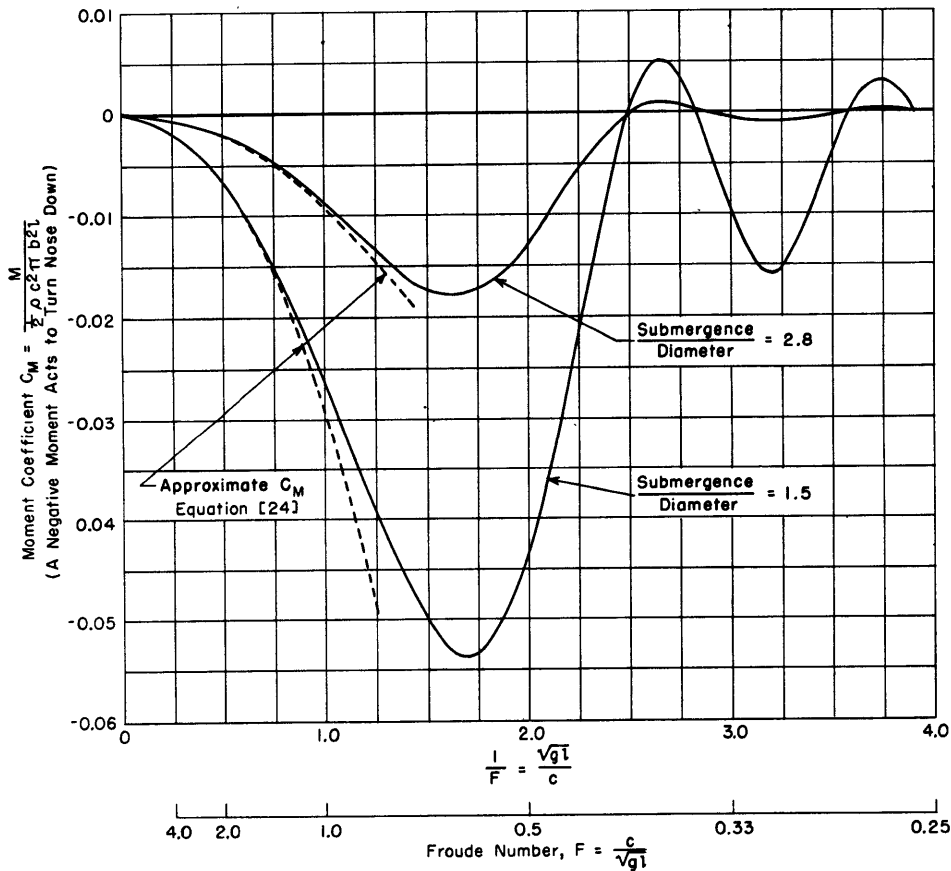


Figure 3 - Moment Coefficient for Rankine Ovoid with Length-Diameter Ratio of 10.5
 Moments are taken about center of buoyancy.

COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS

Figure 33 of Reference 3 gives the moments obtained by experiment and from the first approximation M_1 for a Rankine Ovoid with a length of 48 inches and diameter of 4.572 inches for which the submergence to the model centerline was 1.5 diameters and 2.8 diameters. Figure 4 of the present report reproduces these results and gives in addition the results of the second approximation $M = M_1 + M_2$ as given by Equation [14].

For this particular example, it is seen that the second approximation agrees very well with the experimental results. The application of the method to other forms, and further comparisons with experimental data will be the subject of a later report.

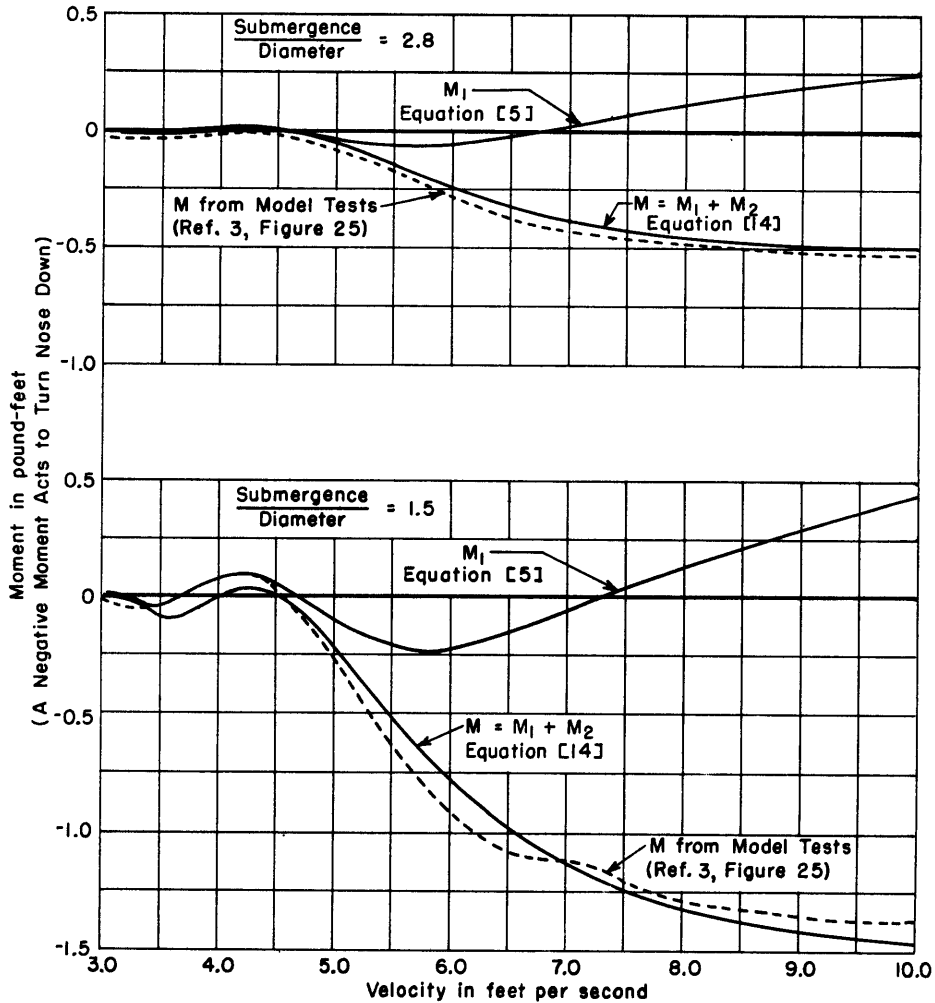


Figure 4 - Comparison of Observed and Predicted Moments About Center of Buoyancy for 4-foot Rankine Ovoid with Length-Diameter Ratio of 10.5

CONCLUSIONS

It may be concluded from the comparison with experimental data that the present theory explains at least the major portion of the moment acting on the Rankine Ovoid model considered. This work will be extended to other forms and further experimental checks will be made.

ACKNOWLEDGMENT

The numerical calculations required for this report were carried out by Mrs. Helen W. Henderson.

APPENDIX

EVALUATION OF INTEGRALS APPEARING IN TEXT

As mentioned in the report the integrals appearing in the expressions for moments can be evaluated either by numerical quadrature or by expanding the integrand in an infinite series and integrating term by term. An efficient method of numerical quadrature for integrals of the type required is given by the Gauss-Christoffel formula

$$\int_{-\infty}^{\infty} e^{-x^2} G(x) dx \approx \sum_{i=0}^n \lambda_i G(x_i) \quad [17]$$

where the x_i 's are roots of the Hermite polynomials of the n-th order and the λ_i 's are called Christoffel numbers. The theory of this method of numerical quadrature is given in References 13 and 14 and tables of the x_i 's and λ_i 's for various values of n are given in References 15 and 16.

Since the x_i 's are symmetrically spaced on either side of $x = 0$ and the λ_i 's are symmetric about $x = 0$, the formula can be used when $G(x)$ is symmetric for the evaluation of integrals of the form

$$\int_0^{\infty} e^{-x^2} G(x) dx$$

In this case only the positive x_i 's of Equation [17] are taken.

As an application of this formula consider Equation [13] for the moment M_2 . Making the transformation $p = \sqrt{2fk_0} t$, the resulting expression for M_2 is

$$M_2 = s \int_0^{\infty} e^{-p^2} G(p) dp \quad [18]$$

where

$$s = \frac{4\pi\rho b^4 c^2 k_0 e^{-2fk_0}}{2fk_0}$$

and

$$G(p) = \sqrt{2fk_0 + p^2} \left(\cos \left[\frac{2ak_0}{\sqrt{2fk_0}} \sqrt{2fk_0 + p^2} \right] - 1 \right)$$

As a particular example consider the Rankine Ovoid described on page 12, and let the speed be $c = 8.02$ feet per second and the depth of

submergence be 1.5 diameters. The constant s is equal to 1.0215, and applying the quadrature formula (with $n = 4$) the result is*

$$\begin{aligned}
 M_2 &= 1.0215[\lambda_0 G(p_0) + \lambda_1 G(p_1) + \dots + \lambda_4 G(p_4)] \\
 &= 1.0215[0.36012G(0) + 0.43265G(0.72355) + 0.088475G(1.4686) \\
 &\quad + 0.0049436G(2.2666) + 0.000039607G(3.1910)] \quad [19] \\
 &= 1.0215[-0.36149 - 0.84890 - 0.22259 - 0.00040587 - 0.00018148] \\
 &= -1.4644 \text{ pound-feet}
 \end{aligned}$$

Repeating this process for $n = 7$ the result is

$$M_2 = -1.4636 \text{ pound-feet}$$

No general expression for the error in this method of quadrature is available. For the case considered here, however, a check can be obtained since the integral can be expressed as an alternating series from which the correct answer may be determined to any desired accuracy. This series expansion is discussed in the next few paragraphs.

The corresponding series expansion for M_2 is obtained by expanding the integrand in an infinite series and then (since the series is uniformly convergent) inverting the order of integration and summation. Thus, substituting the series

$$\sqrt{1+t^2} \left[\cos [2ak_0 \sqrt{1+t^2}] - 1 \right] = \sum_{n=1}^{\infty} (-1)^n \frac{(2ak_0)^{2n}}{(2n)!} (1+t^2)^{\frac{2n+1}{2}}$$

in Equation [13], the result is

$$M_2 = 4\pi\rho b^4 c^2 k_0 e^{-2fk_0} \sum_{n=1}^{\infty} (-1)^n \frac{(2ak_0)^{2n}}{(2n)!} \int_0^{\infty} e^{-2fk_0 t^2} (1+t^2)^{\frac{2n+1}{2}} dt \quad [20]$$

*Values of λ_i and p_i are taken from Reference 15.

Havelock has observed (Reference 17, p. 284) that integrals of the form

$$L_{2n+1} = \int_0^\infty e^{-\eta t^2} (1+t^2)^{\frac{2n+1}{2}} dt$$

where n is an integer, can be expressed in terms of modified Bessel functions of the second kind of orders zero and one. One procedure for doing this depends on the formula*

$$L_{2n+1} = \int_0^\infty e^{-\eta t^2} (1+t^2)^{\frac{2n+1}{2}} dt = \frac{n+\eta}{\eta} \int_0^\infty e^{-\eta t^2} (1+t^2)^{\frac{2n-1}{2}} dt - \frac{2n-1}{2\eta} \int_0^\infty e^{-\eta t^2} (1+t^2)^{\frac{2n-3}{2}} dt \quad [21]$$

For $2n+1 = 3$ this formula gives

$$L_3 = \int_0^\infty e^{-\eta t^2} (1+t^2)^{\frac{3}{2}} dt = \frac{1+\eta}{\eta} \int_0^\infty e^{-\eta t^2} (1+t^2)^{\frac{1}{2}} dt - \frac{1}{2\eta} \int_0^\infty e^{-\eta t^2} (1+t^2)^{\frac{1}{2}} dt = \frac{1+\eta}{\eta} L_1 - \frac{1}{2\eta} L_{-1}$$

and continuing the procedure,

$$L_5 = \frac{2+\eta}{\eta} L_3 - \frac{3}{2\eta} L_1$$

$$\vdots$$

$$L_{2n+1} = \frac{n+\eta}{\eta} L_{2n-1} - \frac{2n-1}{2\eta} L_{2n-3}$$

Hence, from L_1 and L_{-1} all the other L's may be determined. The transformation $t = \sinh \mu$ applied to the integrals for L_1 and L_{-1} transforms them into known expressions for modified Bessel functions of the second kind K_0 and K_1 (Reference 18, p. 181). The result is

*This reduction formula was shown to the author by Dr. J.W. Wrench, Jr. It may be derived by taking the derivative with respect to t of

$$t(1+t^2)^{\frac{2n+1}{2}} e^{-\eta t^2}$$

rearranging terms, and then integrating between the limits 0 and ∞ .

$$L_{-1} = \frac{1}{2} e^{\frac{\eta}{2}} K_0 \left(\frac{\eta}{2} \right)$$

$$L_1 = \frac{1}{4} e^{\frac{\eta}{2}} \left[K_0 \left(\frac{\eta}{2} \right) + K_1 \left(\frac{\eta}{2} \right) \right]$$

Since the functions $K_0(x)$ and $K_1(x)$ have been tabulated,^{18,19} the L-functions can be determined. The expression for M_2 now becomes

$$M_2 = 4\pi\rho b^4 c^2 k_0 e^{-2fk_0} \sum_{n=1}^{\infty} (-1)^n \frac{(2ak_0)^{2n}}{(2n)!} L_{2n+1}(2fk_0) \quad [22]$$

For the same example considered on pages 14 and 15 the sum of the first 10 terms of this series is -1.46373 and the 11th term is -0.00001. Hence, to four significant figures the value of M_2 is -1.464 pound-feet. This result shows that the quadrature formula with $n = 4$ is correct to four significant figures for this particular example. Since three significant figures give sufficient accuracy for any application, the quadrature formula with $n = 4$ was used in calculating the results given in Figures 3 and 4.

The series expansion for the moment coefficient C_M given by Equation [16] can be determined by the same procedure. The result is

$$C_M = \frac{e^{-\frac{2\alpha}{\beta F^2}}}{2\beta^2 F^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-4)}{(2n)!} \left(\frac{\epsilon}{F^2} \right)^{2n} L_{2n+1} \left[\frac{2\alpha}{\beta F^2} \right] \quad [23]$$

Noting that the term for $n = 2$ is zero in this series, it appears that, for sufficiently large F , the first term of the series may be a good estimate of C_M . Figure 3 compares the correct C_M with the values obtained from the first term of Equation [23] for a Rankine Ovoid with a length-diameter ratio of 10.5 submerged 1.5 diameters and 2.8 diameters. The figure shows that for Froude numbers F greater than 1 the first term of Equation [23] is a good approximation to C_M .

Thus for values of the parameters not too different from those in the example, the approximate value of the moment coefficient for Rankine Ovoids is

$$C_M \approx - \frac{\beta \epsilon^2}{64 \alpha^3} \omega^2 e^{\frac{\omega}{2}} \left[\omega K_0 \left(\frac{\omega}{2} \right) + (1 + \omega) K_1 \left(\frac{\omega}{2} \right) \right] \quad [24]$$

where $\omega = \frac{2\alpha}{\beta F^2}$

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