Report 4503

NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

Bethesda, Maryland 20034

DEPARTMENT OF THE NAVY ★ NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER ★

SOLUTION OF A COMPLEX QUADRATIC EIGENVALUE PROBLEM RELATED TO PIPE FLOW

Donald A. Gignac

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

COMPUTATION AND MATHEMATICS DEPARTMENT
RESEARCH AND DEVELOPMENT REPORT

AUGUST 1974

Report 4503

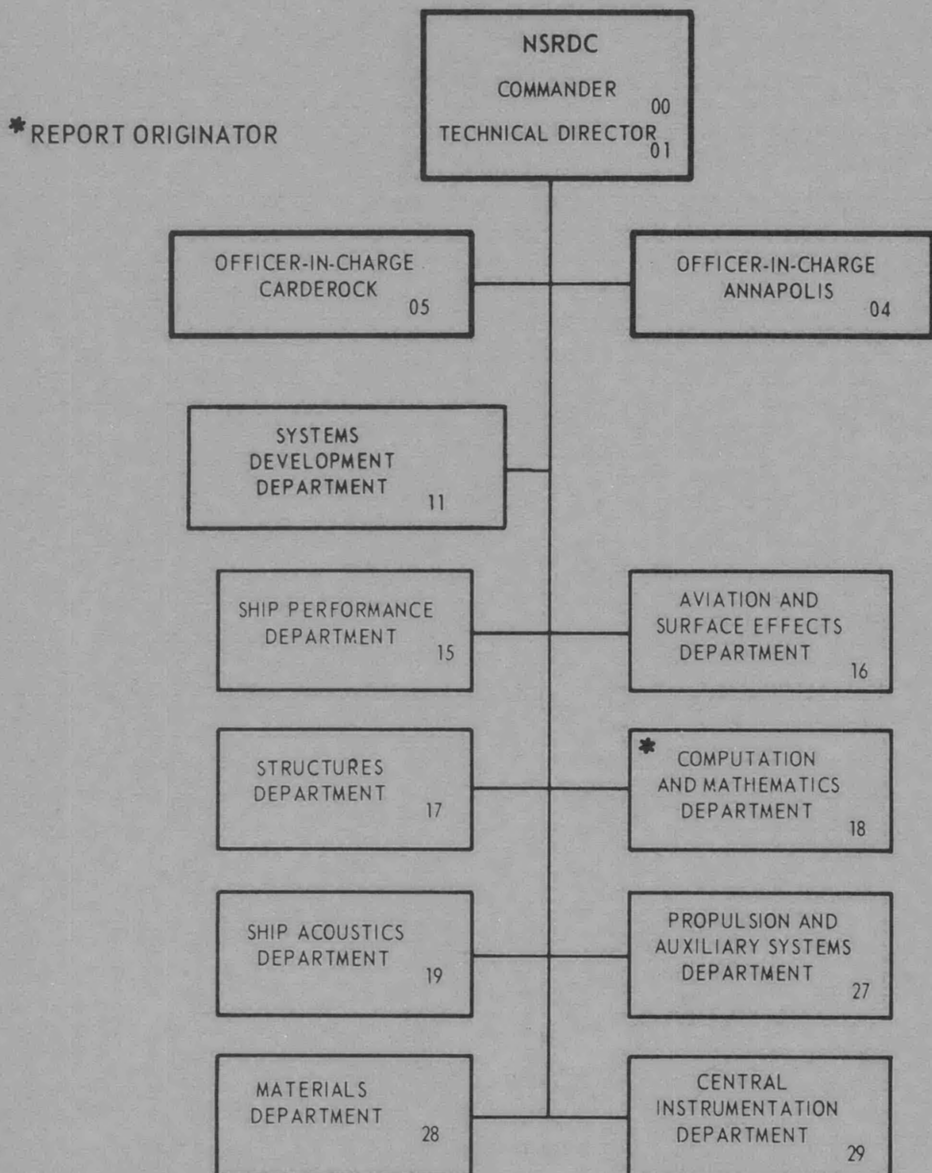
SOLUTION OF A COMPLEX QUADRATIC EIGENVALUE PROBLEM RELATED TO PIPE FLOW

The Naval Ship Research and Development Center is a U. S. Navy center for laboratory effort directed at achieving improved sea and air vehicles. It was formed in March 1967 by merging the David Taylor Model Basin at Carderock, Maryland with the Marine Engineering Laboratory at Annapolis, Maryland.

Naval Ship Research and Development Center

Bethesda, Md. 20034

MAJOR NSRDC ORGANIZATIONAL COMPONENTS



NDW-NSRDC 3960/44 (REV. 8/71)
GPO 917-872

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (*When Data Entered*)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 4503	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (*and Subtitle*) SOLUTION OF A COMPLEX QUADRATIC EIGENVALUE PROBLEM RELATED TO PIPE FLOW		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Donald A. Gignac		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Ship Research & Development Center Bethesda, Maryland 20034		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS SR 014 03 01, Task 15322 Work Unit 1-1844-004
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE August 1974
		13. NUMBER OF PAGES 45
14. MONITORING AGENCY NAME & ADDRESS(*if different from Controlling Office*)		15. SECURITY CLASS. (*of this report*) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (*of this Report*)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (*of the abstract entered in Block 20, if different from Report*)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (*Continue on reverse side if necessary and identify by block number*)

Eigenvalues
Double Precision Arithmetic
Error Analysis
Matrix Analysis

20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

A complex quadratic eigenvalue problem of order n (n = 20, 30, 40, ...) encountered in an investigation of pipe flow was reduced to that of computing the eigenvalues of either a real or a complex matrix of order 2n, depending on the linearization technique used. To get satisfactory results using the CDC 6000 series computer, double precision versions of the appropriate EISPACK subroutines were required. Although the modified subroutines used for the real matrix computations required less core than those used for the

20. continued:

complex matrix, the calculation time was not significantly faster than that for the complex matrix. The use of "balancing" subroutines merits further investigation.

TABLE OF CONTENTS

	Page
INTRODUCTION	1
THE PROBLEM	2
THE SUBROUTINES	4
THE RESULTS	5
OBSERVATIONS AND CONCLUSIONS	11
ACKNOWLEDGMENTS	12
APPENDIX A - PROGRAM LISTINGS	13

LIST OF TABLES

TABLE 1 - Second- and Third-Quadrant Eigenvalues Obtained from Complex Matrix Approach 7

TABLE 2 - Second- and Third-Quadrant Eigenvalues Obtained from Real Matrix Approach 9

INTRODUCTION

The work described in this report was undertaken in response to a request by Dr. R.J. Hansen of the Naval Research Laboratory who was investigating a pipe flow problem in fluid mechanics[1] using a computer program containing an eigenvalue subroutine for complex matrices. The unsatisfactory results previously obtained with this computer program were thought to be due to the complex eigenvalue subroutine then being used, and it was thought that substituting an appropriate sequence of subroutines from the EISPACK subroutine library[2] might solve the problem. However, for certain matrices generated by the program, the EISPACK subroutines did not produce satisfactory results either.

At first it was felt that the matrices had not been generated with sufficient accuracy, but when further effort in that direction proved unproductive, it was conjectured that the source of the difficulty might lie in the fact that the matrix was generated in double precision while its eigenvalues were computed in single precision. At this point the question arises: If it is desirable or necessary to generate these matrices in double precision,* then should not their eigenvalues be computed in double precision also?

This question should be considered in light of the following observations.

---

[1] Hansen, R.J., Little, R.C., Reischman, M.M., and Kelleher, M.D., "Stability and the Laminar-to-Turbulent Transition in Pipe Flows of Drag-Reducing Polymer Solutions," a paper to be presented in the Fall of 1974 at the British Hydromechanics Research Association International Congress on Drag Reduction.

[2] Smith, B.T., Boyle, J.M., Garlow, B.S., Skele, Y., Klema, V.C., and Moler, C.B., "Matrix Eigensystem Subroutines - EISPACK Guide," Vol. 6, "Lecture Notes in Computer Science," New York, Springer-Verlag (1974).

* To accumulate dot products in double precision on the CDC 6400 so as to minimize roundoff error, the whole dot product procedure must be done in double precision, costly though that may be.

First, under certain circumstances even the 48-bit mantissa of the single-precision word of the CDC 6400 may not be sufficient to withstand the accumulation of round-off error in the course of computing the eigenvalues of a complex matrix of rather low order.

Second, if A is the complex matrix generated in double precision and B is the complex matrix obtained by truncating A to single precision, the respective elements of A and B differ by increments of order 10^{-16}. Clearly either matrix can be regarded as having been obtained by perturbing the elements of the other. The crucial question is whether this perturbation significantly changes the eigenvalues, or even alters their very nature?

This report documents an investigation of these difficulties for Hansen's complex eigenvalue problem as well as for an associated real eigenvalue problem.

THE PROBLEM

If A, B, and C are complex matrices of order n, we may define the matrix equation

$$(\lambda^2 A + \lambda B + C)X = 0 \tag{1}$$

as the quadratic eigenvalue problem of order n. If A is invertible, the solution of this problem is equivalent to computing the eigenvalues and eigenvectors of a complex matrix of order 2n.

Two methods can be used to reduce this quadratic eigenvalue problem to a matrix eigenvalue problem. First the substitution of $Y = \lambda X$ into the above equation results in the linear eigenvalue problem of order 2n.

$$\left(\begin{array}{c|c} 0 & I \\ \hline -A^{-1}C & -A^{-1}B \end{array}\right) \left(\begin{array}{c} X \\ \hline Y \end{array}\right) = \lambda \left(\begin{array}{c} X \\ \hline Y \end{array}\right) \quad . \tag{2}$$

The second method involves multiplying the quadratic problem by A^{-1} and completing the square to obtain

$$[(\lambda I + \tfrac{1}{2} A^{-1}B)^2 - (\tfrac{1}{2} A^{-1}B)^2 + A^{-1}C]X = 0 \quad . \tag{3}$$

Next Y is implicitly defined by the relationship

$$(\lambda I + \frac{1}{2} A^{-1}B)X = A^{-1}Y \ . \tag{4}$$

Then this relationship is substituted in Equation (3) and multiplication by A and simplification gives the linear eigenvalue problem

$$\left(\begin{array}{c|c} -\frac{1}{2} A^{-1}B & A^{-1} \\ \hline \frac{1}{4} BA^{-1}B-C & -\frac{1}{2} BA^{-1} \end{array}\right)\left(\begin{array}{c} X \\ \hline Y \end{array}\right) = \lambda \left(\begin{array}{c} X \\ \hline Y \end{array}\right) .$$

The problem at hand is of the form

$$[\lambda^2 P + \lambda(2\theta E + \frac{1}{\theta} Pi) + (-J + Ei)] \ X = 0$$

where P, E, and J are real matrices of order 20 or more, θ is a real parameter, and i is the square root of -1. The first linearization procedure results in the complex eigenvalue problem

$$\left(\begin{array}{c|c} 0 & I \\ \hline P^{-1}J - P^{-1}Ei & 2\theta \ P^{-1}E + \frac{1}{\theta} Ii \end{array}\right)\left(\begin{array}{c} X \\ \hline Y \end{array}\right) = \lambda \left(\begin{array}{c} X \\ \hline Y \end{array}\right)$$

However, the second linearization procedure produces the complex eigenvalue problem

$$\left(\begin{array}{c|c} -\theta P^{-1}E - \frac{1}{2\theta} Ii & P^{-1} \\ \hline \theta^2 \ EP^{-1}E - \frac{1}{4\theta^2} P + J & -\theta EP^{-1} - \frac{1}{2\theta} Ii \end{array}\right)\left(\begin{array}{c} X \\ \hline Y \end{array}\right) = \lambda \left(\begin{array}{c} X \\ \hline Y \end{array}\right)$$

In this instance the coefficient matrix is the difference of a real non-symmetric matrix and a complex scalar multiple of the identity matrix. This coefficient matrix C is similar to a matrix of simpler form. If we let D be the matrix

$$\left(\begin{array}{c|c} I & 0 \\ \hline -\theta E & I \end{array}\right)$$

then the inverse of D is

$$\left(\begin{array}{c|c} I & 0 \\ \hline \theta E & I \end{array}\right)$$

and $D^{-1}CD$ is

$$\left(\begin{array}{c|c} -2\theta P^{-1}E & P^{-1} \\ \hline -\frac{1}{4\theta^2}P + J & 0 \end{array}\right) - \frac{1}{2\theta} Ii$$

Thus we may obtain the eigenvalues of the complex matrix C by computing the eigenvalues of the real non-symmetric matrix

$$\left(\begin{array}{c|c} -2\theta\ P^{-1}E & P^{-1} \\ \hline -\frac{1}{4\theta^2}P + J & 0 \end{array}\right)$$

and subtracting the quantity $\frac{1}{2\theta}$ from their imaginary parts.

THE SUBROUTINES

All the single-precision subroutines (CBAL, COMHES, COMLR, BALANC, ELMHES, ORTHES, HQR) are the standard EISPACK subroutines. They contain instructions pertaining to their calling sequence. The double-precision subroutines (DBAL, DOMHES, DOMLR, DALANC, DLMHES, DRTHES, DQR) were obtained from the corresponding EISPACK subroutines by making the following changes:

- All single-precision variables were changed to double precision.
- The single-precision criterion for the CDC 6400, 2^{-47}, was changed to the double-precision criterion, 2^{-95}.

• The various library subroutines (ABS, SQRT, etc.) were changed to their double-precision counterparts. In the case of DOMLR, it was necessary to supply double-precision subroutines for complex division and complex square-root, since these operations are not available on the CDC 6400. The subroutines provided (CLXDVDE and CLXSQRT) are merely FORTRAN translations of the ALGOL procedures CABS, CDIV, and CSQRT of Martin and Wilkinson.[3]

• The argument lists for these double-precision subroutines differ from the calling sequences for their EISPACK counterparts only in that all non-integer arrays were double precision instead of single precision.

THE RESULTS

The eigenvalues of the complex matrix provided by the first linearization technique were obtained using the following procedure*:

- The complex matrix was balanced.
- The balanced matrix was reduced to Hessenberg form by stabilized elementary similarity transformations.
- The eigenvalues of the Hessenberg matrix were computed using the LR algorithm for complex matrices.

This procedure was carried out in three different ways:

1) The complex matrix generated in double precision was truncated to single precision and the single-precision EISPACK subroutines CBAL, COMHES, COMLR were used (Table 1a).

2) The complex matrix generated in double precision was truncated to single precision and the double-precision versions DBAL, DOMHES, DOMLR

[3] Martin, R.S. and J.H. Wilkinson, "Similarity Reduction of a General Matrix to Hessenberg Form," Contribution II/13 in Wilkinson, J.H. and C. Reinsch, editors "Handbook for Automatic Computation," Vol. II, "Linear Algebra," New York, 1971, Springer-Verlag.

* For information concerning the Hessenberg reduction and the QR and LR algorithms the reader is referred to Wilkinson, J.H., "The Algebraic Eigenvalue Problem," Clarendon Press, Oxford (1965).

of the EISPACK subroutines were used (Table 1b).

3) DBAL, DOMHES, DOMLR were used on the complex matrix generated in double precision (Table 1c).

The considerable simplification of the eigenvalue problem achieved by using the second linearization technique was not anticipated at the outset of this investigation. When a detailed examination of the eigenvalues computed from the matrix of the first linearization technique suggested that a different linearization technique might yield a matrix more amenable to computation, other linearization techniques were investigated.

The eigenvalues of the real matrix provided by the second linearization technique were obtained using the following procedure:

- The real non-symmetric matrix was balanced.

- The balanced matrix was reduced to Hessenberg form using either stabilized elementary similarity transformations or orthogonal transformations.

- The eigenvalues of the Hessenberg matrix were computed using the QR algorithm.

This procedure was carried out in four different ways.

1) The real matrix generated in double precision was truncated to single precision and the single precision EISPACK subroutines BALANC, OR¨HES, and HQR were used (Table 2a).

2) The real matrix generated in double precision was truncated to single precision and the double precision versions DALANC, DRTHES, and DQR of the EISPACK subroutines were used (Table 2b).

3) DALANCE, DRTHES, and DQR were used on the real matrix generated in double precision (Table 2c).

4) DRTHES and DQR were used on the real matrix generated in double precision (Table 2d).

Of the 60 eigenvalues computed, only those that fall in the second and third quadrants are presented in the tables that follow. However, their behavior is representative of the eigenvalues as a whole. The computation times shown are those required to compute all 60 of the eigenvalues.

TABLE 1 - SECOND AND THIRD QUADRANT EIGENVALUES OBTAINED FROM COMPLEX MATRIX APPROACH

Table 1a

CBAL, COMHES, COMLR Subroutines 21.77 Seconds

1	-.166895348951915E+02	.453962332056731E+02
2	-.316749508523681E+02	.248351140020612E+00
3	-.348884093302488E+02	.221199870220845E+02
1	-.816293526043538E+00	-.135811109722657E+03
2	-.816359999160820E+00	-.704598564725760E+01
3	-.262510336877122E+01	-.134477603349308E+03
4	-.262511208539763E+01	-.837954278608919E+01
5	-.570546363332099E+01	-.132807670304536E+03
6	-.570548098766807E+01	-.100494914773950E+02
7	-.112270549054325E+02	-.131454972874669E+03
8	-.112270792516163E+02	-.114021920263506E+02
9	-.166894713808424E+02	-.188253246825858E+03
10	-.205377373902610E+02	-.133027945252527E+03
11	-.205377768261726E+02	-.982922388380672E+01
12	-.316748791446516E+02	-.143105511121843E+03
13	-.348883045381867E+02	-.164977092569118E+03

Table 1b

DBAL, DOMHES, DOMLR Subroutines 96.19 Seconds
(Matrix has been truncated to single precision significance)

1	-.164163526301686D+02	.444157924738040D+02
2	-.309194743856593D+02	.795111187632182D-01
3	-.342104294907155D+02	.214986849517767D+02
1	-.491817227667278D+00	-.136123802804776D+03
2	-.491817235024852D+00	-.673334031666884D+01
3	-.236178882980058D+01	-.817847614309992D+01
4	-.236178886587554D+01	-.134678666891774D+03
5	-.544207538548246D+01	-.983990980218899D+01
6	-.544207544558938D+01	-.133017233233870D+03
7	-.108592821202510D+02	-.111690142642918D+02
8	-.108592821807477D+02	-.131688128825279D+03
9	-.164163521583046D+02	-.187272935438159D+03
10	-.199544704290707D+02	-.133174702838537D+03
11	-.199544704372173D+02	-.968244033591238D+01
12	-.309194742220807D+02	-.142936654328604D+03
13	-.342104291482651D+02	-.164355828892842D+03

TABLE 1 (CONTINUED)

Table 1c

DBAL, DOMHES, DOMLR Subroutines 94.82 Seconds

1	-.161635728329553D+02	.432950045889481D+02
2	-.335577057717484D+02	.207044804956883D+02

1	-.204921408502598D+00	-.685273791773558D+01
2	-.204921408502598D+00	-.136004404939408D+03
3	-.208873186667484D+01	-.134653761423907D+03
4	-.208873186667484D+01	-.820338143323628D+01
5	-.512410079791166D+01	-.133057379553590D+03
6	-.512410079791166D+01	-.979976330355343D+01
7	-.104158704992893D+02	-.131751724309115D+03
8	-.104158704992893D+02	-.111054185480280D+02
9	-.161635728329553D+02	-.186152147446091D+03
10	-.193106276285779D+02	-.133126188924393D+03
11	-.193106276285779D+02	-.973095393274997D+01
12	-.301520647479031D+02	-.142565668129680D+03
13	-.301520647479031D+02	-.291474727463061D+00
14	-.335577057717484D+02	-.163561623352832D+03

TABLE 2 - SECOND AND THIRD QUADRANT EIGENVALUES OBTAINED FROM REAL MATRIX APPROACH

Table 2a

BALANC, ORTHES, HQR Subroutines 24.23 Seconds

1	-.160756150689513E+02	.432634403473735E+02
2	-.335103274930459E+02	.206970748379899E+02
1	-.174021213374999E+00	-.687248959254430E+01
2	-.174021213374999E+00	-.135984653264599E+03
3	-.205661137487414E+01	-.822366618893329E+01
4	-.205661137487414E+01	-.134633476668209E+03
5	-.508934051840720E+01	-.982017385793188E+01
6	-.508934051840720E+01	-.133036968999211E+03
7	-.103794737355472E+02	-.111287013092983E+02
8	-.103794737355472E+02	-.131728441547844E+03
9	-.160756150689513E+02	-.186120583204516E+03
10	-.192775922492228E+02	-.975794068556252E+01
11	-.192775922492228E+02	-.133099202171580E+03
12	-.301234424853915E+02	-.311546785438622E+00
13	-.301234424853915E+02	-.142545596071704E+03
14	-.335103274930459E+02	-.163554217695133E+03

Table 2b

DALANC, DRTHES, DQR Subroutines 84.58 Seconds
(Matrix has been truncated to single precision significance)

1	-.161637993977430D+02	.433345157057497D+02
2	-.335630201800936D+02	.207449518517321D+02
1	-.221363489782061D+00	-.683859559232783D+01
2	-.221363489782061D+00	-.136018547264815D+03
3	-.210845322414368D+01	-.818837272783772D+01
4	-.210845322414368D+01	-.134668770129306D+03
5	-.514736174758617D+01	-.978220853335398D+01
6	-.514736174758617D+01	-.133074934323789D+03
7	-.104416528784868D+02	-.110816366201131D+02
8	-.104416528784868D+02	-.131775506237030D+03
9	-.161637993977430D+02	-.186191658562893D+03
10	-.193337651348488D+02	-.969755942685737D+01
11	-.193337651348488D+02	-.133159583430286D+03
12	-.301660899657195D+02	-.251656928888859D+00
13	-.301660899657195D+02	-.142605485928254D+03
14	-.335630201800936D+02	-.163602094708875D+03

TABLE 2 (CONTINUED)

Table 2c

DALANCE, DRTHES, DQR Subroutines   81.25 Seconds

1	-.161635728329553D+02	.432950045889481D+02
2	-.335577057717484D+02	.207044804956883D+02

1	-.204921408502594D+00	-.685273791773559D+01
2	-.204921408502594D+00	-.136004404939408D+03
3	-.208873186667484D+01	-.820338143323628D+01
4	-.208873186667484D+01	-.134653761423907D+03
5	-.512410079791165D+01	-.979976330355343D+01
6	-.512410079791165D+01	-.133057379553590D+03
7	-.104158704992893D+02	-.111054185480280D+02
8	-.104158704992893D+02	-.131751724309115D+03
9	-.161635728329553D+02	-.186152147446091D+03
10	-.193106276285779D+02	-.973095393274997D+01
11	-.193106276285779D+02	-.133126188924393D+03
12	-.301520647479031D+02	-.291474727463069D+00
13	-.301520647479031D+02	-.142565668129680D+03
14	-.335577057717484D+02	-.163561623352832D+03

Table 2d

DRTHES, DQR Subroutines   76.27 Seconds
(No balancing)

1	-.161635728329026D+02	.432950045888919D+02
2	-.335577057717281D+02	.207044804957628D+02

1	-.204921408518517D+00	-.685273791773917D+01
2	-.204921408518517D+00	-.136004404939404D+03
3	-.208873186669188D+01	-.820338143324120D+01
4	-.208873186669188D+01	-.134653761423902D+03
5	-.512410079793201D+01	-.979976330356033D+01
6	-.512410079793201D+01	-.133057379553583D+03
7	-.104158704993175D+02	-.111054185480356D+02
8	-.104158704993175D+02	-.131751724309108D+03
9	-.161635728329026D+02	-.186152147446035D+03
10	-.193106276286219D+02	-.973095393274980D+01
11	-.193106276286219D+02	-.133126188924393D+03
12	-.301520647479561D+02	-.291474727423287D+00
13	-.301520647479561D+02	-.142565668129720D+03
14	-.335577057717281D+02	-.163561623352906D+03

OBSERVATIONS AND CONCLUSIONS

Study of the preceding tables led to the following observations:

1) The situation as regards the complex matrix requires double-precision eigenvalue computation on the CDC 6400 computer, as indicated by a comparison of the eigenvalues in Tables 1a and 1c, realizing that

i) the real parts of the eigenvalues must be equal in pairs, and

ii) the sequence of double-precision subroutines for the complex LR algorithm obtains eigenvalues satisfying i) by iterating directly for each root and not by some deflation procedure which involves solving a quadratic equation, a procedure which could provide equal real parts.

The situation for the real matrix is similar. The eigenvalues in Tables 2a and 2c agree to only a few decimal places at best, but the eigenvalues in Tables 1c and 2c are virtually identical. However, the eigenvalues in Table 2c, unlike those in Table 1c, appear to be the result of a series of quadratic deflations. Unfortunately, the double-precision computations take more than four times the time required for the single-precision computations.

2) Since the double-precision subroutines are assumed to be more accurate than the corresponding single-precision subroutines, Tables 1b and 1c and Tables 2b and 2c indicate that the original matrix should be generated and input to the eigenvalue subroutines in double precision. Since the 'b' and 'c' eigenvalues are computed using the same double-precision subroutines, the substantial discrepancies between the 'b' and 'c' results can only be explained by the sensitivity of the eigenvalues. Since the 'c' eigenvalues meet the pairwise equality condition on their real parts, the double-precision matrix is the preferred input.

3) The situation with respect to the use of balancing subroutines is not clear. Comparison of Tables 2c and 2d indicates that the use of DALANC, the double-precision balancing subroutine for real matrices, is desirable. However, in one instance the use of DBAL, the double-precision balancing matrix for complex matrices, made no appreciable difference. The use of

balancing subroutines is therefore recommended as a matter of policy, although the value of DBAL appears open to question and to merit further investigation.

4) ELMHES and ORTHES, the Hessenberg reduction subroutines for real matrices, and their double precision counterparts DLMHES and DRTHES give comparable accuracy and require similar computation times respectively.

ACKNOWLEDGMENTS

The author wishes to thank Dr. Elizabeth H. Cuthill (1805) for her invaluable assistance and Dr. R.J. Hansen of the Naval Research Laboratory and Professor Matthew Kelleher of the Naval Postgraduate School for their interest and encouragement.

APPENDIX A

PROGRAM LISTINGS

NOTE: It is to be emphasized that the subroutines listed herein are solely the present author's responsibility and were obtained by the modifications described in this report. The comment cards are those of the original single-precision EISPACK subroutines.

```
C                                                                       71215001
C     ------------------------------------------------------------------71215002
C                                                                       71215003
      SUBROUTINE DBAL(NM,N,AR,AI,LOW,IGH,SCALE)                         71215004
C                                                                       71215005
      INTEGER I,J,K,L,M,N,JJ,NM,IGH,LOW,IEXC                            71215006
*     REAL AR(NM,N),AI(NM,N),SCALE(N)                                   71215007
*     REAL C,F,G,R,S,B2,RADIX                                           71215008
      DOUBLE PRECISION AR(NM,N),AI(NM,N),SCALE(N),C,F,G,R,S,B2,RADIX
C     REAL ABS                                                          71215009
      LOGICAL NOCONV                                                    71215010
C                                                                       71215011
C     THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE           71215012
C     CBALANCE, WHICH IS A COMPLEX VERSION OF BALANCE,                  71215013
C     NUM. MATH. 13, 293-304(1969) BY PARLETT AND REINSCH.              71215014
C     HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 315-326(1971).   71215015
C                                                                       71215016
C     THIS SUBROUTINE BALANCES A COMPLEX MATRIX AND ISOLATES            71215017
C     EIGENVALUES WHENEVER POSSIBLE.                                    71215018
C                                                                       71215019
C     ON INPUT-                                                         71215020
C                                                                       71215021
C        NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL         71215022
C          ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM          71215023
C          DIMENSION STATEMENT,                                         71215024
C                                                                       71215025
C        N IS THE ORDER OF THE MATRIX,                                  71215026
C                                                                       71215027
C        AR AND AI CONTAIN THE REAL AND IMAGINARY PARTS,                71215028
C          RESPECTIVELY, OF THE COMPLEX MATRIX TO BE BALANCED.          71215029
C                                                                       71215030
C     ON OUTPUT-                                                        71215031
C                                                                       71215032
C        AR AND AI CONTAIN THE REAL AND IMAGINARY PARTS,                71215033
C          RESPECTIVELY, OF THE BALANCED MATRIX,                        71215034
C                                                                       71215035
C        LOW AND IGH ARE TWO INTEGERS SUCH THAT AR(I,J) AND AI(I,J)     71215036
C          ARE EQUAL TO ZERO IF                                         71215037
C           (1) I IS GREATER THAN J AND                                 71215038
C           (2) J=1,...,LOW-1 OR I=IGH+1,...,N,                         71215039
C                                                                       71215040
C        SCALE CONTAINS INFORMATION DETERMINING THE                     71215041
C           PERMUTATIONS AND SCALING FACTORS USED.                      71215042
C                                                                       71215043
C     SUPPOSE THAT THE PRINCIPAL SUBMATRIX IN ROWS LOW THROUGH IGH      71215044
C     HAS BEEN BALANCED, THAT P(J) DENOTES THE INDEX INTERCHANGED       71215045
C     WITH J DURING THE PERMUTATION STEP, AND THAT THE ELEMENTS         71215046
C     OF THE DIAGONAL MATRIX USED ARE DENOTED BY D(I,J).  THEN          71215047
C        SCALE(J) = P(J),    FOR J = 1,...,LOW-1                        71215048
C                 = D(J,J)       J = LOW,...,IGH                        71215049
C                 = P(J)         J = IGH+1,...,N.                       71215050
C     THE ORDER IN WHICH THE INTERCHANGES ARE MADE IS N TO IGH+1,       71215051
C     THEN 1 TO LOW-1.                                                  71215052
C                                                                       71215053
C     NOTE THAT 1 IS RETURNED FOR IGH IF IGH IS ZERO FORMALLY.          71215054
```

```
C
C     THE ALGOL PROCEDURE EXC CONTAINED IN CBALANCE APPEARS IN
C     CBAL  IN LINE.  (NOTE THAT THE ALGOL ROLES OF IDENTIFIERS
C     K,L HAVE BEEN REVERSED.)
C
C     ARITHMETIC IS REAL THROUGHOUT.
C
C     QUESTIONS AND COMMENTS SHOULD BE DIRECTED TO B. S. GARBOW,
C     APPLIED MATHEMATICS DIVISION, ARGONNE NATIONAL LABORATORY
C
C     ------------------------------------------------------------------
C
C     ********** RADIX IS A MACHINE DEPENDENT PARAMETER SPECIFYING
C                THE BASE OF THE MACHINE FLOATING POINT REPRESENTATION.
C
C                **********
      RADIX = 2.
C
      B2 = RADIX * RADIX
      K = 1
      L = N
      GO TO 100
C     ********** IN-LINE PROCEDURE FOR ROW AND
C                COLUMN EXCHANGE **********
   20 SCALE(M) = J
      IF (J .EQ. M) GO TO 50
C
      DO 30 I = 1, L
         F = AR(I,J)
         AR(I,J) = AR(I,M)
         AR(I,M) = F
         F = AI(I,J)
         AI(I,J) = AI(I,M)
         AI(I,M) = F
   30 CONTINUE
C
      DO 40 I = K, N
         F = AR(J,I)
         AR(J,I) = AR(M,I)
         AR(M,I) = F
         F = AI(J,I)
         AI(J,I) = AI(M,I)
         AI(M,I) = F
   40 CONTINUE
C
   50 GO TO (80,130), IEXC
C     ********** SEARCH FOR ROWS ISOLATING AN EIGENVALUE
C                AND PUSH THEM DOWN **********
   80 IF (L .EQ. 1) GO TO 280
      L = L - 1
C     ********** FOR J=L STEP -1 UNTIL 1 DO -- **********
  100 DO 120 JJ = 1, L
         J = L + 1 - JJ
C
         DO 110 I = 1, L
```

```
            IF (I .EQ. J) GO TO 110                                      71215110
            IF (AR(J,I) .NE. 0.0 .OR. AI(J,I) .NE. 0.0) GO TO 120        71215111
  110    CONTINUE                                                        71215112
C                                                                        71215113
         M = L                                                           71215114
         IEXC = 1                                                        71215115
         GO TO 20                                                        71215116
  120 CONTINUE                                                           71215117
C                                                                        71215118
      GO TO 140                                                          71215119
C     ********** SEARCH FOR COLUMNS ISOLATING AN EIGENVALUE               71215120
C                AND PUSH THEM LEFT **********                           71215121
  130 K = K + 1                                                          71215122
C                                                                        71215123
  140 DO 170 J = K, L                                                    71215124
C                                                                        71215125
         DO 150 I = K, L                                                 71215126
            IF (I .EQ. J) GO TO 150                                      71215127
            IF (AR(I,J) .NE. 0.0 .OR. AI(I,J) .NE. 0.0) GO TO 170        71215128
  150    CONTINUE                                                        71215129
C                                                                        71215130
         M = K                                                           71215131
         IEXC = 2                                                        71215132
         GO TO 20                                                        71215133
  170 CONTINUE                                                           71215134
C     ********** NOW BALANCE THE SUBMATRIX IN ROWS K TO L **********      71215135
      DO 180 I = K, L                                                    71215136
  180 SCALE(I) = 1.0                                                     71215137
C     ********** ITERATIVE LOOP FOR NORM REDUCTION **********             71215138
  190 NOCONV = .FALSE.                                                   71215139
C                                                                        71215140
      DO 270 I = K, L                                                    71215141
         C = 0.0                                                         71215142
         R = 0.0                                                         71215143
C                                                                        71215144
         DO 200 J = K, L                                                 71215145
            IF (J .EQ. I) GO TO 200                                      71215146
*           C = C + ABS(AR(J,I)) + ABS(AI(J,I))                          71215147
*           R = R + ABS(AR(I,J)) + ABS(AI(I,J))                          71215148
            C = C +DABS(AR(J,I)) +DABS(AI(J,I))
            R = R +DABS(AR(I,J)) +DABS(AI(I,J))
  200    CONTINUE                                                        71215149
C                                                                        71215150
         G = R / RADIX                                                   71215151
         F = 1.0                                                         71215152
         S = C + R                                                       71215153
  210    IF (C .GE. G) GO TO 220                                         71215154
         F = F * RADIX                                                   71215155
         C = C * B2                                                      71215156
         GO TO 210                                                       71215157
  220    G = R * RADIX                                                   71215158
  230    IF (C .LT. G) GO TO 240                                         71215159
         F = F / RADIX                                                   71215160
         C = C / B2                                                      71215161
         GO TO 230                                                       71215162
```

```
C     ********** NOW BALANCE **********                                 71215163
  240    IF ((C + R) / F .GE. 0.95 * S) GO TO 270                       71215164
         G = 1.0 / F                                                    71215165
         SCALE(I) = SCALE(I) * F                                        71215166
         NOCONV = .TRUE.                                                71215167
C                                                                       71215168
         DO 250 J = K, N                                                71215169
            AR(I,J) = AR(I,J) * G                                       71215170
            AI(I,J) = AI(I,J) * G                                       71215171
  250    CONTINUE                                                       71215172
C                                                                       71215173
         DO 260 J = 1, L                                                71215174
            AR(J,I) = AR(J,I) * F                                       71215175
            AI(J,I) = AI(J,I) * F                                       71215176
  260    CONTINUE                                                       71215177
C                                                                       71215178
  270 CONTINUE                                                          71215179
C                                                                       71215180
      IF (NOCONV) GO TO 190                                             71215181
C                                                                       71215182
  280 LOW = K                                                           71215183
      IGH = L                                                           71215184
      RETURN                                                            71215185
C     ********** LAST CARD OF CBAL **********                           71215186
      END                                                               71215187
```

```
C                                                                       82215001
C     ------------------------------------------------------------------82215002
C                                                                       82215003
      SUBROUTINE DOMHES(NM,N,LOW,IGH,AR,AI,INT)                         82215004
C                                                                       82215005
      INTEGER I,J,M,N,LA,NM,IGH,KP1,LOW,MM1,MP1                         82215006
*     REAL AR(NM,N),AI(NM,N)                                            82215007
*     REAL XR,XI,YR,YI                                                  82215008
      DOUBLE PRECISION AR(NM,N),AI(NM,N),XR,XI,YR,YI
C     REAL ABS                                                          82215009
      INTEGER INT(IGH)                                                  82215010
*     COMPLEX X,Y                                                       82215011
*     REAL T1(2),T2(2)                                                  82215012
*     EQUIVALENCE (X,T1(1),XR),(T1(2),XI),(Y,T2(1),YR),(T2(2),YI)       82215013
C                                                                       82215014
C     THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE COMHES,   82215015
C     NUM. MATH. 12, 349-368(1968) BY MARTIN AND WILKINSON.             82215016
C     HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 339-358(1971).   82215017
C                                                                       82215018
C     GIVEN A COMPLEX GENERAL MATRIX, THIS SUBROUTINE                   82215019
C     REDUCES A SUBMATRIX SITUATED IN ROWS AND COLUMNS                  82215020
C     LOW THROUGH IGH TO UPPER HESSENBERG FORM BY                       82215021
C     STABILIZED ELEMENTARY SIMILARITY TRANSFORMATIONS.                 82215022
C                                                                       82215023
C     ON INPUT-                                                         82215024
C                                                                       82215025
C        NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL         82215026
C          ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM          82215027
C          DIMENSION STATEMENT,                                         82215028
C                                                                       82215029
C        N IS THE ORDER OF THE MATRIX,                                  82215030
C                                                                       82215031
C        LOW AND IGH ARE INTEGERS DETERMINED BY THE BALANCING           82215032
C          SUBROUTINE  CBAL.  IF  CBAL  HAS NOT BEEN USED,              82215033
C          SET LOW=1, IGH=N,                                            82215034
C                                                                       82215035
C        AR AND AI CONTAIN THE REAL AND IMAGINARY PARTS,                82215036
C          RESPECTIVELY, OF THE COMPLEX INPUT MATRIX.                   82215037
C                                                                       82215038
C     ON OUTPUT-                                                        82215039
C                                                                       82215040
C        AR AND AI CONTAIN THE REAL AND IMAGINARY PARTS,                82215041
C          RESPECTIVELY, OF THE HESSENBERG MATRIX.  THE                 82215042
C          MULTIPLIERS WHICH WERE USED IN THE REDUCTION                 82215043
C          ARE STORED IN THE REMAINING TRIANGLES UNDER THE              82215044
C          HESSENBERG MATRIX,                                           82215045
C                                                                       82215046
C        INT CONTAINS INFORMATION ON THE ROWS AND COLUMNS               82215047
C          INTERCHANGED IN THE REDUCTION.                               82215048
C          ONLY ELEMENTS LOW THROUGH IGH ARE USED.                      82215049
C                                                                       82215050
C     ARITHMETIC IS REAL EXCEPT FOR THE REPLACEMENT OF THE ALGOL        82215051
C     PROCEDURE CDIV BY COMPLEX DIVISION.                               82215052
C                                                                       82215053
C     QUESTIONS AND COMMENTS SHOULD BE DIRECTED TO B. S. GARBOW,        82215054
```

```
C      APPLIED MATHEMATICS DIVISION, ARGONNE NATIONAL LABORATORY
C
C     ------------------------------------------------------------------
C
      LA = IGH - 1
      KP1 = LOW + 1
      IF (LA .LT. KP1) GO TO 200
C
      DO 180 M = KP1, LA
         MM1 = M - 1
         XR = 0.0
         XI = 0.0
         I = M
C
         DO 100 J = M, IGH
*           IF (ABS(AR(J,MM1)) + ABS(AI(J,MM1))
*    X          .LE. ABS(XR) + ABS(XI)) GO TO 100
            IF(DABS(AR(J,MM1)) +DABS(AI(J,MM1))
     X          .LE.DABS(XR) +DABS(XI)) GO TO 100
            XR = AR(J,MM1)
            XI = AI(J,MM1)
            I = J
  100    CONTINUE
C
         INT(M) = I
         IF (I .EQ. M) GO TO 130
C     ********** INTERCHANGE ROWS AND COLUMNS OF AR AND AI **********
         DO 110 J = MM1, N
            YR = AR(I,J)
            AR(I,J) = AR(M,J)
            AR(M,J) = YR
            YI = AI(I,J)
            AI(I,J) = AI(M,J)
            AI(M,J) = YI
  110    CONTINUE
C
         DO 120 J = 1, IGH
            YR = AR(J,I)
            AR(J,I) = AR(J,M)
            AR(J,M) = YR
            YI = AI(J,I)
            AI(J,I) = AI(J,M)
            AI(J,M) = YI
  120    CONTINUE
C     ********** END INTERCHANGE **********
  130    IF (XR .EQ. 0.0 .AND. XI .EQ. 0.0) GO TO 180
         MP1 = M + 1
C
         DO 160 I = MP1, IGH
            YR = AR(I,MM1)
            YI = AI(I,MM1)
            IF (YR .EQ. 0.0 .AND. YI .EQ. 0.0) GO TO 160
*           Y = Y / X
      CALL CLXDVDE(YR,YI,YR,YI,XR,XI)
            AR(I,MM1) = YR
```

```
            AI(I,MM1) = YI                                              82215107
C                                                                       82215108
            DO 140 J = M, N                                             82215109
               AR(I,J) = AR(I,J) - YR * AR(M,J) + YI * AI(M,J)          82215110
               AI(I,J) = AI(I,J) - YR * AI(M,J) - YI * AR(M,J)          82215111
  140       CONTINUE                                                    82215112
C                                                                       82215113
            DO 150 J = 1, IGH                                           82215114
               AR(J,M) = AR(J,M) + YR * AR(J,I) - YI * AI(J,I)          82215115
               AI(J,M) = AI(J,M) + YR * AI(J,I) + YI * AR(J,I)          82215116
  150       CONTINUE                                                    82215117
C                                                                       82215118
  160    CONTINUE                                                       82215119
C                                                                       82215120
  180 CONTINUE                                                          82215121
C                                                                       82215122
  200 RETURN                                                            82215123
C     ********** LAST CARD OF COMHES **********                         82215124
      END                                                               82215125
```

```
C
C     ------------------------------------------------------------------
C
      SUBROUTINE DOMLR(NM,N,LOW,IGH,HR,HI,WR,WI,IERR)
C
      INTEGER I,J,L,M,N,EN,LL,MM,NM,IGH,IM1,ITS,LOW,MP1,ENM1,IERR
*     REAL HR(NM,N),HI(NM,N),WR(N),WI(N)
*     REAL SI,SR,TI,TR,XI,XR,YI,YR,ZZI,ZZR,MACHEP
      DOUBLE PRECISION HR(NM,N),HI(NM,N),WR(N),WI(N),SI,SR,TI,TR,XI,XR,
     1                 YI,YR,ZZI,ZZR,MACHEP,ZR,ZI
      EQUIVALENCE (ZR,ZZR),(ZI,ZZI)
C     REAL ABS
*     COMPLEX X,Y,Z
C     COMPLEX CSQRT,CMPLX
*     REAL T1(2),T2(2),T3(2)
*     EQUIVALENCE (X,T1(1),XR),(T1(2),XI),(Y,T2(1),YR),(T2(2),YI),
*    X            (Z,T3(1),ZZR),(T3(2),ZZI)
C
C     THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE COMLR,
C     NUM. MATH. 12, 369-376(1968) BY MARTIN AND WILKINSON.
C     HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 396-403(1971).
C
C     THIS SUBROUTINE FINDS THE EIGENVALUES OF A COMPLEX
C     UPPER HESSENBERG MATRIX BY THE MODIFIED LR METHOD.
C
C     ON INPUT-
C
C        NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL
C          ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM
C          DIMENSION STATEMENT,
C
C        N IS THE ORDER OF THE MATRIX,
C
C        LOW AND IGH ARE INTEGERS DETERMINED BY THE BALANCING
C          SUBROUTINE  CBAL.  IF  CBAL  HAS NOT BEEN USED,
C          SET LOW=1, IGH=N,
C
C        HR AND HI CONTAIN THE REAL AND IMAGINARY PARTS,
C          RESPECTIVELY, OF THE COMPLEX UPPER HESSENBERG MATRIX.
C          THEIR LOWER TRIANGLES BELOW THE SUBDIAGONAL CONTAIN THE
C          MULTIPLIERS WHICH WERE USED IN THE REDUCTION BY  COMHES,
C          IF PERFORMED.
C
C     ON OUTPUT-
C
C        THE UPPER HESSENBERG PORTIONS OF HR AND HI HAVE BEEN
C          DESTROYED.  THEREFORE, THEY MUST BE SAVED BEFORE
C          CALLING  COMLR  IF SUBSEQUENT CALCULATION OF
C          EIGENVECTORS IS TO BE PERFORMED,
C
C        WR AND WI CONTAIN THE REAL AND IMAGINARY PARTS,
C          RESPECTIVELY, OF THE EIGENVALUES.  IF AN ERROR
C          EXIT IS MADE, THE EIGENVALUES SHOULD BE CORRECT
C          FOR INDICES IERR+1,...,N,
C
```

```
C        IERR IS SET TO
C          ZERO       FOR NORMAL RETURN,
C          J          IF THE J-TH EIGENVALUE HAS NOT BEEN
C                     DETERMINED AFTER 30 ITERATIONS.
C
C     ARITHMETIC IS REAL EXCEPT FOR THE REPLACEMENT OF THE ALGOL
C     PROCEDURE CDIV BY COMPLEX DIVISION AND USE OF THE SUBROUTINES
C     CSQRT AND CMPLX IN COMPUTING COMPLEX SQUARE ROOTS.
C
C     QUESTIONS AND COMMENTS SHOULD BE DIRECTED TO B. S. GARBOW,
C     APPLIED MATHEMATICS DIVISION, ARGONNE NATIONAL LABORATORY
C
C     ------------------------------------------------------------------
C
C     ********** MACHEP IS A MACHINE DEPENDENT PARAMETER SPECIFYING
C                THE RELATIVE PRECISION OF FLOATING POINT ARITHMETIC.
C
C                **********
      MACHEP = 2.**(-95)
C
      IERR = 0
C     ********** STORE ROOTS ISOLATED BY CBAL **********
  180 DO 200 I = 1, N
         IF (I .GE. LOW .AND. I .LE. IGH) GO TO 200
         WR(I) = HR(I,I)
         WI(I) = HI(I,I)
  200 CONTINUE
C
      EN = IGH
      TR = 0.0
      TI = 0.0
C     ********** SEARCH FOR NEXT EIGENVALUE **********
  220 IF (EN .LT. LOW) GO TO 1001
      ITS = 0
      ENM1 = EN - 1
C     ********** LOOK FOR SINGLE SMALL SUB-DIAGONAL ELEMENT
C                FOR L=EN STEP -1 UNTIL LOW -- **********
  240 DO 260 LL = LOW, EN
         L = EN + LOW - LL
         IF (L .EQ. LOW) GO TO 300
*        IF (ABS(HR(L,L-1)) + ABS(HI(L,L-1)) .LE.
*    X      MACHEP * (ABS(HR(L-1,L-1)) + ABS(HI(L-1,L-1))
*    X              + ABS(HR(L,L)) +ABS(HI(L,L)))) GO TO 300
         IF(DABS(HR(L,L-1)) +DABS(HI(L,L-1)) .LE.
     X      MACHEP *(DABS(HR(L-1,L-1)) +DABS(HI(L-1,L-1))
     X              +DABS(HR(L,L))+DABS(HI(L,L)))) GO TO 300
  260 CONTINUE
C     ********** FORM SHIFT **********
  300 IF (L .EQ. EN) GO TO 660
      IF (ITS .EQ. 30) GO TO 1000
      IF (ITS .EQ. 10 .OR. ITS .EQ. 20) GO TO 320
      SR = HR(EN,EN)
      SI = HI(EN,EN)
      XR = HR(ENM1,EN) * HR(EN,ENM1) - HI(ENM1,EN) * HI(EN,ENM1)
      XI = HR(ENM1,EN) * HI(EN,ENM1) + HI(ENM1,EN) * HR(EN,ENM1)
```

```
      IF (XR .EQ. 0.0 .AND. XI .EQ. 0.0) GO TO 340                        95215105
      YR = (HR(ENM1,ENM1) - SR) / 2.0                                     95215106
      YI = (HI(ENM1,ENM1) - SI) / 2.0                                     95215107
*     Z = CSQRT(CMPLX(YR**2-YI**2+XR,2.0*YR*YI+XI))                       95215108
      CALL CLXSQRT(ZR,ZI,YR*YR-YI*YI+XR,2.D0*YR*YI+XI)
*     IF (YR * ZZR + YI * ZZI .LT. 0.0) Z = -Z                            95215109
      IF (YR * ZZR + YI * ZZI .GE. 0.D0) GO TO 301
      ZR=-ZR
      ZI=-ZI
*     X = X / (Y + Z)                                                     95215110
  301 CALL CLXDVDE(XR,XI,XR,XI,YR+ZR,YI+ZI)
      SR = SR - XR                                                        95215111
      SI = SI - XI                                                        95215112
      GO TO 340                                                           95215113
C     ********** FORM EXCEPTIONAL SHIFT **********                        95215114
* 320 SR = ABS(HR(EN,ENM1)) + ABS(HR(ENM1,EN-2))                          95215115
  320 SR =DABS(HR(EN,ENM1)) +DABS(HR(ENM1,EN-2))
*     SI = ABS(HI(EN,ENM1)) + ABS(HI(ENM1,EN-2))                          95215116
      SI =DABS(HI(EN,ENM1)) +DABS(HI(ENM1,EN-2))
C                                                                         95215117
  340 DO 360 I = LOW, EN                                                  95215118
         HR(I,I) = HR(I,I) - SR                                           95215119
         HI(I,I) = HI(I,I) - SI                                           95215120
  360 CONTINUE                                                            95215121
C                                                                         95215122
      TR = TR + SR                                                        95215123
      TI = TI + SI                                                        95215124
      ITS = ITS + 1                                                       95215125
C     ********** LOOK FOR TWO CONSECUTIVE SMALL                           95215126
C                SUB-DIAGONAL ELEMENTS **********                         95215127
*     XR = ABS(HR(ENM1,ENM1)) + ABS(HI(ENM1,ENM1))                        95215128
      XR =DABS(HR(ENM1,ENM1)) +DABS(HI(ENM1,ENM1))
*     YR = ABS(HR(EN,ENM1)) + ABS(HI(EN,ENM1))                            95215129
      YR =DABS(HR(EN,ENM1)) +DABS(HI(EN,ENM1))
      ZZR =DABS(HR(EN,EN)) +DABS(HI(EN,EN))
C     ********** FOR M=EN-1 STEP -1 UNTIL L DO -- **********              95215131
      DO 380 MM = L, ENM1                                                 95215132
         M = ENM1 + L - MM                                                95215133
         IF (M .EQ. L) GO TO 420                                          95215134
         YI = YR                                                          95215135
*        YR = ABS(HR(M,M-1)) + ABS(HI(M,M-1))                             95215136
         YR =DABS(HR(M,M-1)) +DABS(HI(M,M-1))
         XI = ZZR                                                         95215137
         ZZR = XR                                                         95215138
*        XR = ABS(HR(M-1,M-1)) + ABS(HI(M-1,M-1))                         95215139
         XR =DABS(HR(M-1,M-1)) +DABS(HI(M-1,M-1))
         IF (YR .LE. MACHEP * ZZR / YI * (ZZR + XR + XI)) GO TO 420       95215140
  380 CONTINUE                                                            95215141
C     ********** TRIANGULAR DECOMPOSITION H=L*R **********                95215142
  420 MP1 = M + 1                                                         95215143
C                                                                         95215144
      DO 520 I = MP1, EN                                                  95215145
         IM1 = I - 1                                                      95215146
         XR = HR(IM1,IM1)                                                 95215147
         XI = HI(IM1,IM1)                                                 95215148
```

```
         YR = HR(I,IM1)
         YI = HI(I,IM1)
*        IF (ABS(XR) + ABS(XI) .GE. ABS(YR) + ABS(YI)) GO TO 460
         IF(DABS(XR) +DABS(XI) .GE.DABS(YR) +DABS(YI)) GO TO 460
C     ********** INTERCHANGE ROWS OF HR AND HI **********
         DO 440 J = IM1, N
            ZZR = HR(IM1,J)
            HR(IM1,J) = HR(I,J)
            HR(I,J) = ZZR
            ZZI = HI(IM1,J)
            HI(IM1,J) = HI(I,J)
            HI(I,J) = ZZI
  440    CONTINUE
C
*        Z = X / Y
      CALL CLXDVDE(ZR,ZI,XR,XI,YR,YI)
         WR(I) = 1.0
         GO TO 480
* 460    Z = Y / X
  460 CALL CLXDVDE(ZR,ZI,YR,YI,XR,XI)
         WR(I) = -1.0
  480    HR(I,IM1) = ZZR
         HI(I,IM1) = ZZI
C
         DO 500 J = I, EN
            HR(I,J) = HR(I,J) - ZZR * HR(IM1,J) + ZZI * HI(IM1,J)
            HI(I,J) = HI(I,J) - ZZR * HI(IM1,J) - ZZI * HR(IM1,J)
  500   CONTINUE
C
  520 CONTINUE
C     ********** COMPOSITION R*L=H **********
      DO 640 J = MP1, EN
         XR = HR(J,J-1)
         XI = HI(J,J-1)
         HR(J,J-1) = 0.0
         HI(J,J-1) = 0.0
C     ********** INTERCHANGE COLUMNS OF HR AND HI,
C                IF NECESSARY **********
         IF (WR(J) .LE. 0.0) GO TO 580
C
         DO 540 I = L, J
            ZZR = HR(I,J-1)
            HR(I,J-1) = HR(I,J)
            HR(I,J) = ZZR
            ZZI = HI(I,J-1)
            HI(I,J-1) = HI(I,J)
            HI(I,J) = ZZI
  540    CONTINUE
C
  580    DO 600 I = L, J
            HR(I,J-1) = HR(I,J-1) + XR * HR(I,J) - XI * HI(I,J)
            HI(I,J-1) = HI(I,J-1) + XR * HI(I,J) + XI * HR(I,J)
  600    CONTINUE
C
  640 CONTINUE
```

```
C
      GO TO 240
C     ********** A ROOT FOUND **********
  660 WR(EN) = HR(EN,EN) + TR
      WI(EN) = HI(EN,EN) + TI
      EN = ENM1
      GO TO 220
C     ********** SET ERROR -- NO CONVERGENCE TO AN
C                EIGENVALUE AFTER 30 ITERATIONS **********
 1000 IERR = EN
 1001 RETURN
C     ********** LAST CARD OF COMLR **********
      END
```

```
      SUBROUTINE CLXDVDE(ZR,ZI,XR,XI,YR,YI)
C
      DOUBLE PRECISION ZR,ZI,XR,XI,YR,YI,H,QR,QI,HH
C
      IF (DABS(YR)+DABS(YI)) 3,1,3
C
    1 WRITE(6,2)
    2 FORMAT(1H1/14(1H0/),30X,*CLXDVDE MESSAGE - YOU HAVE JUST ATTEMPTED
     1 TO DIVIDE BY ZERO.*)
      STOP
C
    3 IF (DABS(YR)-DABS(YI)) 5,5,4
C
    4 H=YI/YR
      HH=H*YI+YR
      QR=(XR+H*XI)/HH
      QI=(XI-H*XR)/HH
      ZR=QR
      ZI=QI
      RETURN
C
    5 H=YR/YI
      HH=H*YR+YI
      QR=(H*XR+XI)/HH
      QI=(H*XI-XR)/HH
      ZR=QR
      ZI=QI
      RETURN
      END
```

```
      SUBROUTINE CLXSQRT(YR,YI,XR,XI)
C
      DOUBLE PRECISION YR,YI,XR,XI,XR1,XI1,H,DPCABS
C
C     FIRST COMPUTE DPCABS,THE ABSOLUTE VALUE OF THE RADICAND.
C
      XR1=DABS(XR)
      XI1=DABS(XI)
C
      IF (XI1-XR1) 2,2,1
C
    1 H=XR1
      XR1=XI1
      XI1=H
C
    2 IF (XI1) 4,3,4
C
    3 DPCABS=XR1
      GO TO 5
C
    4 DPCABS=XR1*DSQRT(1.D0+(XI1/XR1)*(XI1/XR1))
C
C
C
    5 H=DSQRT((DABS(XR)+DPCABS)/2.D0)
C
      IF (XI) 6,7,6
C
    6 XI=XI/(2.D0*H)
C
    7 IF (XR) 9,8,8
C
    8 YR=H
      YI=XI
      RETURN
C
    9 IF (XI) 11,10,10
C
   10 YR=XI
      YI=-H
      RETURN
C
   11 YR=-XI
      YI=-H
      RETURN
      END
```

```
C
C     ------------------------------------------------------------------
C
*     SUBROUTINE BALANC(NM,N,A,LOW,IGH,SCALE)
      SUBROUTINE DALANC(NM,N,A,LOW,IGH,SCALE)
C
      INTEGER I,J,K,L,M,N,JJ,NM,IGH,LOW,IEXC
*     REAL A(NM,N),SCALE(N)
*     REAL C,F,G,R,S,B2,RADIX
      DOUBLE PRECISION A(NM,N),SCALE(N),C,F,G,R,S,B2,RADIX
C     REAL ABS
      LOGICAL NOCONV
C
C     THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE BALANCE,
C     NUM. MATH. 13, 293-304(1969) BY PARLETT AND REINSCH.
C     HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 315-326(1971).
C
C     THIS SUBROUTINE BALANCES A REAL MATRIX AND ISOLATES
C     EIGENVALUES WHENEVER POSSIBLE.
C
C     ON INPUT-
C
C        NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL
C          ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM
C          DIMENSION STATEMENT,
C
C        N IS THE ORDER OF THE MATRIX,
C
C        A CONTAINS THE INPUT MATRIX TO BE BALANCED.
C
C     ON OUTPUT-
C
C        A CONTAINS THE BALANCED MATRIX,
C
C        LOW AND IGH ARE TWO INTEGERS SUCH THAT A(I,J)
C          IS EQUAL TO ZERO IF
C           (1) I IS GREATER THAN J AND
C           (2) J=1,...,LOW-1 OR I=IGH+1,...,N,
C
C        SCALE CONTAINS INFORMATION DETERMINING THE
C           PERMUTATIONS AND SCALING FACTORS USED.
C
C     SUPPOSE THAT THE PRINCIPAL SUBMATRIX IN ROWS LOW THROUGH IGH
C     HAS BEEN BALANCED, THAT P(J) DENOTES THE INDEX INTERCHANGED
C     WITH J DURING THE PERMUTATION STEP, AND THAT THE ELEMENTS
C     OF THE DIAGONAL MATRIX USED ARE DENOTED BY D(I,J).  THEN
C        SCALE(J) = P(J),    FOR J = 1,...,LOW-1
C                 = D(J,J),      J = LOW,...,IGH
C                 = P(J)         J = IGH+1,...,N.
C     THE ORDER IN WHICH THE INTERCHANGES ARE MADE IS N TO IGH+1,
C     THEN 1 TO LOW-1.
C
C     NOTE THAT 1 IS RETURNED FOR IGH IF IGH IS ZERO FORMALLY.
C
C     THE ALGOL PROCEDURE EXC CONTAINED IN BALANCE APPEARS IN
```

```
C     BALANC  IN LINE.  (NOTE THAT THE ALGOL ROLES OF IDENTIFIERS
C     K,L HAVE BEEN REVERSED.)
C
C     QUESTIONS AND COMMENTS SHOULD BE DIRECTED TO B. S. GARBOW,
C     APPLIED MATHEMATICS DIVISION, ARGONNE NATIONAL LABORATORY
C
C     ------------------------------------------------------------------
C
C     ********** RADIX IS A MACHINE DEPENDENT PARAMETER SPECIFYING
C                THE BASE OF THE MACHINE FLOATING POINT REPRESENTATION.
C
C                **********
      RADIX = 2.
C
      B2 = RADIX * RADIX
      K = 1
      L = N
      GO TO 100
C     ********** IN-LINE PROCEDURE FOR ROW AND
C                COLUMN EXCHANGE **********
   20 SCALE(M) = J
      IF (J .EQ. M) GO TO 50
C
      DO 30 I = 1, L
         F = A(I,J)
         A(I,J) = A(I,M)
         A(I,M) = F
   30 CONTINUE
C
      DO 40 I = K, N
         F = A(J,I)
         A(J,I) = A(M,I)
         A(M,I) = F
   40 CONTINUE
C
   50 GO TO (80,130), IEXC
C     ********** SEARCH FOR ROWS ISOLATING AN EIGENVALUE
C                AND PUSH THEM DOWN **********
   80 IF (L .EQ. 1) GO TO 280
      L = L - 1
C     ********** FOR J=L STEP -1 UNTIL 1 DO -- **********
  100 DO 120 JJ = 1, L
         J = L + 1 - JJ
C
         DO 110 I = 1, L
            IF (I .EQ. J) GO TO 110
            IF (A(J,I) .NE. 0.0) GO TO 120
  110    CONTINUE
C
         M = L
         IEXC = 1
         GO TO 20
  120 CONTINUE
C
      GO TO 140
```

```
C     ********** SEARCH FOR COLUMNS ISOLATING AN EIGENVALUE           69215109
C                AND PUSH THEM LEFT **********                        69215110
  130 K = K + 1                                                       69215111
C                                                                     69215112
  140 DO 170 J = K, L                                                 69215113
C                                                                     69215114
         DO 150 I = K, L                                              69215115
            IF (I .EQ. J) GO TO 150                                   69215116
            IF (A(I,J) .NE. 0.0) GO TO 170                            69215117
  150    CONTINUE                                                     69215118
C                                                                     69215119
         M = K                                                        69215120
         IEXC = 2                                                     69215121
         GO TO 20                                                     69215122
  170 CONTINUE                                                        69215123
C     ********** NOW BALANCE THE SUBMATRIX IN ROWS K TO L **********  69215124
      DO 180 I = K, L                                                 69215125
  180 SCALE(I) = 1.0                                                  69215126
C     ********** ITERATIVE LOOP FOR NORM REDUCTION **********         69215127
  190 NOCONV = .FALSE.                                                69215128
C                                                                     69215129
      DO 270 I = K, L                                                 69215130
         C = 0.0                                                      69215131
         R = 0.0                                                      69215132
C                                                                     69215133
         DO 200 J = K, L                                              69215134
            IF (J .EQ. I) GO TO 200                                   69215135
*           C = C + ABS(A(J,I))                                       69215136
            C = C +DABS(A(J,I))
*           R = R + ABS(A(I,J))                                       69215137
            R = R +DABS(A(I,J))
  200    CONTINUE                                                     69215138
C                                                                     69215139
         G = R / RADIX                                                69215140
         F = 1.0                                                      69215141
         S = C + R                                                    69215142
  210    IF (C .GE. G) GO TO 220                                      69215143
         F = F * RADIX                                                69215144
         C = C * B2                                                   69215145
         GO TO 210                                                    69215146
  220    G = R * RADIX                                                69215147
  230    IF (C .LT. G) GO TO 240                                      69215148
         F = F / RADIX                                                69215149
         C = C / B2                                                   69215150
         GO TO 230                                                    69215151
C     ********** NOW BALANCE **********                               69215152
  240    IF ((C + R) / F .GE. 0.95 * S) GO TO 270                     69215153
         G = 1.0 / F                                                  69215154
         SCALE(I) = SCALE(I) * F                                      69215155
         NOCONV = .TRUE.                                              69215156
C                                                                     69215157
         DO 250 J = K, N                                              69215158
  250    A(I,J) = A(I,J) * G                                          69215159
C                                                                     69215160
         DO 260 J = 1, L                                              69215161
```

```
  260    A(J,I) = A(J,I) * F
C
  270 CONTINUE
C
      IF (NOCONV) GO TO 190
C
  280 LOW = K
      IGH = L
      RETURN
C     ********** LAST CARD OF BALANC **********
      END
```

```
C                                                                       73215001
C     ------------------------------------------------------------------73215002
C                                                                       73215003
*     SUBROUTINE ELMHES(NM,N,LOW,IGH,A,INT)                             73215004
      SUBROUTINE DLMHES(NM,N,LOW,IGH,A,INT)
C                                                                       73215005
      INTEGER I,J,M,N,LA,NM,IGH,KP1,LOW,MM1,MP1                         73215006
*     REAL A(NM,N)                                                      73215007
*     REAL X,Y                                                          73215008
      DOUBLE PRECISION A(NM,N),X,Y
C     REAL ABS                                                          73215009
      INTEGER INT(IGH)                                                  73215010
C                                                                       73215011
C     THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE ELMHES,   73215012
C     NUM. MATH. 12, 349-368(1968) BY MARTIN AND WILKINSON.             73215013
C     HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 339-358(1971).   73215014
C                                                                       73215015
C     GIVEN A REAL GENERAL MATRIX, THIS SUBROUTINE                      73215016
C     REDUCES A SUBMATRIX SITUATED IN ROWS AND COLUMNS                  73215017
C     LOW THROUGH IGH TO UPPER HESSENBERG FORM BY                       73215018
C     STABILIZED ELEMENTARY SIMILARITY TRANSFORMATIONS.                 73215019
C                                                                       73215020
C     ON INPUT-                                                         73215021
C                                                                       73215022
C        NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL         73215023
C          ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM          73215024
C          DIMENSION STATEMENT,                                         73215025
C                                                                       73215026
C        N IS THE ORDER OF THE MATRIX,                                  73215027
C                                                                       73215028
C        LOW AND IGH ARE INTEGERS DETERMINED BY THE BALANCING           73215029
C          SUBROUTINE  BALANC.  IF  BALANC  HAS NOT BEEN USED,          73215030
C          SET LOW=1, IGH=N,                                            73215031
C                                                                       73215032
C        A CONTAINS THE INPUT MATRIX.                                   73215033
C                                                                       73215034
C     ON OUTPUT-                                                        73215035
C                                                                       73215036
C        A CONTAINS THE HESSENBERG MATRIX.  THE MULTIPLIERS             73215037
C          WHICH WERE USED IN THE REDUCTION ARE STORED IN THE           73215038
C          REMAINING TRIANGLE UNDER THE HESSENBERG MATRIX,              73215039
C                                                                       73215040
C        INT CONTAINS INFORMATION ON THE ROWS AND COLUMNS               73215041
C          INTERCHANGED IN THE REDUCTION.                               73215042
C          ONLY ELEMENTS LOW THROUGH IGH ARE USED.                       73215043
C                                                                       73215044
C     QUESTIONS AND COMMENTS SHOULD BE DIRECTED TO B. S. GARBOW,        73215045
C     APPLIED MATHEMATICS DIVISION, ARGONNE NATIONAL LABORATORY         73215046
C                                                                       73215047
C     ------------------------------------------------------------------73215048
C                                                                       73215049
      LA = IGH - 1                                                      73215050
      KP1 = LOW + 1                                                     73215051
      IF (LA .LT. KP1) GO TO 200                                        73215052
C                                                                       73215053
```

```
      DO 180 M = KP1, LA                                                73215054
         MM1 = M - 1                                                    73215055
         X = 0.0                                                        73215056
         I = M                                                          73215057
C                                                                       73215058
         DO 100 J = M, IGH                                              73215059
*           IF (ABS(A(J,MM1)) .LE. ABS(X)) GO TC 100                    73215060
            IF(DABS(A(J,MM1)) .LE.DABS(X)) GO TO 100
            X = A(J,MM1)                                                73215061
            I = J                                                       73215062
  100    CONTINUE                                                       73215063
C                                                                       73215064
         INT(M) = I                                                     73215065
         IF (I .EQ. M) GO TO 130                                        73215066
C    ********** INTERCHANGE ROWS AND COLUMNS OF A **********            73215067
         DO 110 J = MM1, N                                              73215068
            Y = A(I,J)                                                  73215069
            A(I,J) = A(M,J)                                             73215070
            A(M,J) = Y                                                  73215071
  110    CONTINUE                                                       73215072
C                                                                       73215073
         DO 120 J = 1, IGH                                              73215074
            Y = A(J,I)                                                  73215075
            A(J,I) = A(J,M)                                             73215076
            A(J,M) = Y                                                  73215077
  120    CONTINUE                                                       73215078
C    ********** END INTERCHANGE **********                              73215079
  130    IF (X .EQ. 0.0) GO TO 180                                      73215080
         MP1 = M + 1                                                    73215081
C                                                                       73215082
         DO 160 I = MP1, IGH                                            73215083
            Y = A(I,MM1)                                                73215084
            IF (Y .EQ. 0.0) GO TO 160                                   73215085
            Y = Y / X                                                   73215086
            A(I,MM1) = Y                                                73215087
C                                                                       73215088
            DO 140 J = M, N                                             73215089
  140       A(I,J) = A(I,J) - Y * A(M,J)                                73215090
C                                                                       73215091
            DO 150 J = 1, IGH                                           73215092
  150       A(J,M) = A(J,M) + Y * A(J,I)                                73215093
C                                                                       73215094
  160    CONTINUE                                                       73215095
C                                                                       73215096
  180 CONTINUE                                                          73215097
C                                                                       73215098
  200 RETURN                                                            73215099
C    ********** LAST CARD OF ELMHES **********                          73215100
      END                                                               73215101
```

```
C                                                                       75215001
C     ------------------------------------------------------------------75215002
C                                                                       75215003
*     SUBROUTINE ORTHES(NM,N,LOW,IGH,A,ORT)                             75215004
      SUBROUTINE ORTHES(NM,N,LOW,IGH,A,ORT)
C                                                                       75215005
      INTEGER I,J,M,N,II,JJ,LA,MP,NM,IGH,KP1,LOW                        75215006
*     REAL A(NM,N),ORT(IGH)                                             75215007
*     REAL F,G,H,SCALE                                                  75215008
      DOUBLE PRECISION A(NM,N),ORT(IGH),F,G,H,SCALE
C     REAL SQRT,ABS,SIGN                                                75215009
C                                                                       75215010
C     THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE ORTHES,   75215011
C     NUM. MATH. 12, 349-368(1968) BY MARTIN AND WILKINSON.             75215012
C     HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 339-358(1971).   75215013
C                                                                       75215014
C     GIVEN A REAL GENERAL MATRIX, THIS SUBROUTINE                      75215015
C     REDUCES A SUBMATRIX SITUATED IN ROWS AND COLUMNS                  75215016
C     LOW THROUGH IGH TO UPPER HESSENBERG FORM BY                       75215017
C     ORTHOGONAL SIMILARITY TRANSFORMATIONS.                            75215018
C                                                                       75215019
C     ON INPUT-                                                         75215020
C                                                                       75215021
C        NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL         75215022
C          ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM          75215023
C          DIMENSION STATEMENT,                                         75215024
C                                                                       75215025
C        N IS THE ORDER OF THE MATRIX,                                  75215026
C                                                                       75215027
C        LOW AND IGH ARE INTEGERS DETERMINED BY THE BALANCING           75215028
C          SUBROUTINE  BALANC.  IF  BALANC  HAS NOT BEEN USED,          75215029
C          SET LOW=1, IGH=N,                                            75215030
C                                                                       75215031
C        A CONTAINS THE INPUT MATRIX.                                   75215032
C                                                                       75215033
C     ON OUTPUT-                                                        75215034
C                                                                       75215035
C        A CONTAINS THE HESSENBERG MATRIX.  INFORMATION ABOUT           75215036
C          THE ORTHOGONAL TRANSFORMATIONS USED IN THE REDUCTION         75215037
C          IS STORED IN THE REMAINING TRIANGLE UNDER THE                75215038
C          HESSENBERG MATRIX,                                           75215039
C                                                                       75215040
C        ORT CONTAINS FURTHER INFORMATION ABOUT THE TRANSFORMATIONS.    75215041
C          ONLY ELEMENTS LOW THROUGH IGH ARE USED.                      75215042
C                                                                       75215043
C     QUESTIONS AND COMMENTS SHOULD BE DIRECTED TO B. S. GARBOW,        75215044
C     APPLIED MATHEMATICS DIVISION, ARGONNE NATIONAL LABORATORY         75215045
C                                                                       75215046
C     ------------------------------------------------------------------75215047
C                                                                       75215048
      LA = IGH - 1                                                      75215049
      KP1 = LOW + 1                                                     75215050
      IF (LA .LT. KP1) GO TO 200                                        75215051
C                                                                       75215052
      DO 180 M = KP1, LA                                                75215053
```

```
         H = 0.0                                                        75215054
         ORT(M) = 0.0                                                   75215055
         SCALE = 0.0                                                    75215056
C     ********** SCALE COLUMN (ALGOL TOL THEN NOT NEEDED) **********    75215057
         DO 90 I = M, IGH                                               75215058
*   90   SCALE = SCALE + ABS(A(I,M-1))                                  75215059
    90   SCALE = SCALE +DABS(A(I,M-1))
C                                                                       75215060
         IF (SCALE .EQ. 0.0) GO TO 180                                  75215061
         MP = M + IGH                                                   75215062
C     ********** FOR I=IGH STEP -1 UNTIL M DO -- **********             75215063
         DO 100 II = M, IGH                                             75215064
            I = MP - II                                                 75215065
            ORT(I) = A(I,M-1) / SCALE                                   75215066
            H = H + ORT(I) * ORT(I)                                     75215067
  100    CONTINUE                                                       75215068
C                                                                       75215069
*        G = -SIGN(SQRT(H),ORT(M))                                      75215070
         G = -DSIGN(DSQRT(H),ORT(M))
         H = H - ORT(M) * G                                             75215071
         ORT(M) = ORT(M) - G                                            75215072
C     ********** FORM (I-(U*UT)/H) * A **********                       75215073
         DO 130 J = M, N                                                75215074
            F = 0.0                                                     75215075
C     ********** FOR I=IGH STEP -1 UNTIL M DO -- **********             75215076
            DO 110 II = M, IGH                                          75215077
               I = MP - II                                              75215078
               F = F + ORT(I) * A(I,J)                                  75215079
  110       CONTINUE                                                    75215080
C                                                                       75215081
            F = F / H                                                   75215082
C                                                                       75215083
            DO 120 I = M, IGH                                           75215084
  120       A(I,J) = A(I,J) - F * ORT(I)                                75215085
C                                                                       75215086
  130    CONTINUE                                                       75215087
C     ********** FORM (I-(U*UT)/H)*A*(I-(U*UT)/H) **********            75215088
         DO 160 I = 1, IGH                                              75215089
            F = 0.0                                                     75215090
C     ********** FOR J=IGH STEP -1 UNTIL M DO -- **********             75215091
            DO 140 JJ = M, IGH                                          75215092
               J = MP - JJ                                              75215093
               F = F + ORT(J) * A(I,J)                                  75215094
  140       CONTINUE                                                    75215095
C                                                                       75215096
            F = F / H                                                   75215097
C                                                                       75215098
            DO 150 J = M, IGH                                           75215099
  150       A(I,J) = A(I,J) - F * ORT(J)                                75215100
C                                                                       75215101
  160    CONTINUE                                                       75215102
C                                                                       75215103
         ORT(M) = SCALE * ORT(M)                                        75215104
         A(M,M-1) = SCALE * G                                           75215105
  180 CONTINUE                                                          75215106
```

```
                                                                      75215107
200 RETURN                                                            75215108
    ********** LAST CARD OF ORTHES **********                         75215109
    END                                                               75215110
```

```
C                                                                       86215001
C     ------------------------------------------------------------------86215002
C                                                                       86215003
*     SUBROUTINE HQR(NM,N,LOW,IGH,H,WR,WI,IERR)                         86215004
      SUBROUTINE DQR(NM,N,LOW,IGH,H,WR,WI,IERR)
C                                                                       86215005
      INTEGER I,J,K,L,M,N,EN,LL,MM,NA,NM,IGH,ITS,LOW,MP2,ENM2,IERR      86215006
*     REAL H(NM,N),WR(N),WI(N)                                          86215007
*     REAL P,Q,R,S,T,W,X,Y,ZZ,MACHEP                                    86215008
      DOUBLE PRECISION H(NM,N),WR(N),WI(N),P,Q,R,S,T,W,X,Y,ZZ,MACHEP
C     REAL SQRT,ABS,SIGN                                                86215009
C     INTEGER MIN0                                                      86215010
      LOGICAL NOTLAS                                                    86215011
C                                                                       86215012
C     THIS SUBROUTINE IS A TRANSLATION OF THE ALGOL PROCEDURE HQR,      86215013
C     NUM. MATH. 14, 219-231(1970) BY MARTIN, PETERS, AND WILKINSON.    86215014
C     HANDBOOK FOR AUTO. COMP., VOL.II-LINEAR ALGEBRA, 359-371(1971).   86215015
C                                                                       86215016
C     THIS SUBROUTINE FINDS THE EIGENVALUES OF A REAL                   86215017
C     UPPER HESSENBERG MATRIX BY THE QR METHOD.                         86215018
C                                                                       86215019
C     ON INPUT-                                                         86215020
C                                                                       86215021
C        NM MUST BE SET TO THE ROW DIMENSION OF TWO-DIMENSIONAL         86215022
C          ARRAY PARAMETERS AS DECLARED IN THE CALLING PROGRAM          86215023
C          DIMENSION STATEMENT,                                         86215024
C                                                                       86215025
C        N IS THE ORDER OF THE MATRIX,                                  86215026
C                                                                       86215027
C        LOW AND IGH ARE INTEGERS DETERMINED BY THE BALANCING           86215028
C          SUBROUTINE  BALANC.  IF  BALANC  HAS NOT BEEN USED,          86215029
C          SET LOW=1, IGH=N,                                            86215030
C                                                                       86215031
C        H CONTAINS THE UPPER HESSENBERG MATRIX.  INFORMATION ABOUT     86215032
C          THE TRANSFORMATIONS USED IN THE REDUCTION TO HESSENBERG      86215033
C          FORM BY  ELMHES  OR  ORTHES, IF PERFORMED, IS STORED         86215034
C          IN THE REMAINING TRIANGLE UNDER THE HESSENBERG MATRIX.       86215035
C                                                                       86215036
C     ON OUTPUT-                                                        86215037
C                                                                       86215038
C        H HAS BEEN DESTROYED.  THEREFORE, IT MUST BE SAVED             86215039
C          BEFORE CALLING  HQR  IF SUBSEQUENT CALCULATION AND           86215040
C          BACK TRANSFORMATION OF EIGENVECTORS IS TO BE PERFORMED,      86215041
C                                                                       86215042
C        WR AND WI CONTAIN THE REAL AND IMAGINARY PARTS,                86215043
C          RESPECTIVELY, OF THE EIGENVALUES.  THE EIGENVALUES           86215044
C          ARE UNORDERED EXCEPT THAT COMPLEX CONJUGATE PAIRS            86215045
C          OF VALUES APPEAR CONSECUTIVELY WITH THE EIGENVALUE           86215046
C          HAVING THE POSITIVE IMAGINARY PART FIRST.  IF AN             86215047
C          ERROR EXIT IS MADE, THE EIGENVALUES SHOULD BE CORRECT        86215048
C          FOR INDICES IERR+1,...,N,                                    86215049
C                                                                       86215050
C        IERR IS SET TO                                                 86215051
C          ZERO       FOR NORMAL RETURN,                                86215052
C          J          IF THE J-TH EIGENVALUE HAS NOT BEEN               86215053
```

```
C                      DETERMINED AFTER 30 ITERATIONS.                  86215054
C                                                                       86215055
C     QUESTIONS AND COMMENTS SHOULD BE DIRECTED TO B. S. GARBOW,        86215056
C     APPLIED MATHEMATICS DIVISION, ARGONNE NATIONAL LABORATORY         86215057
C                                                                       86215058
C     ------------------------------------------------------------------86215059
C                                                                       86215060
C     ********** MACHEP IS A MACHINE DEPENDENT PARAMETER SPECIFYING     86215061
C                THE RELATIVE PRECISION OF FLOATING POINT ARITHMETIC.   86215062
C                                                                       86215063
C                **********                                             86215064
*     MACHEP = 2.**(-47)                                                86215065
      MACHEP = 2. D0**(-95)
C                                                                       86215066
      IERR = 0                                                          86215067
C     ********** STORE ROOTS ISOLATED BY BALANC **********              86215068
      DO 50 I = 1, N                                                    86215069
         IF (I .GE. LOW .AND. I .LE. IGH) GO TO 50                      86215070
         WR(I) = H(I,I)                                                 86215071
         WI(I) = 0.0                                                    86215072
   50 CONTINUE                                                          86215073
C                                                                       86215074
      EN = IGH                                                          86215075
      T = 0.0                                                           86215076
C     ********** SEARCH FOR NEXT EIGENVALUES **********                 86215077
   60 IF (EN .LT. LOW) GO TO 1001                                       86215078
      ITS = 0                                                           86215079
      NA = EN - 1                                                       86215080
      ENM2 = NA - 1                                                     86215081
C     ********** LOOK FOR SINGLE SMALL SUB-DIAGONAL ELEMENT             86215082
C                FOR L=EN STEP -1 UNTIL LOW DO -- **********            86215083
   70 DO 80 LL = LOW, EN                                                86215084
         L = EN + LOW - LL                                              86215085
         IF (L .EQ. LOW) GO TO 100                                      86215086
*        IF (ABS(H(L,L-1)) .LE. MACHEP * (ABS(H(L-1,L-1))               86215087
*    X       + ABS(H(L,L)))) GO TO 100                                  86215088
         IF(DABS(H(L,L-1)) .LE. MACHEP *(DABS(H(L-1,L-1))
     X       +DABS(H(L,L)))) GO TO 100
   80 CONTINUE                                                          86215089
C     ********** FORM SHIFT **********                                  86215090
  100 X = H(EN,EN)                                                      86215091
      IF (L .EQ. EN) GO TO 270                                          86215092
      Y = H(NA,NA)                                                      86215093
      W = H(EN,NA) * H(NA,EN)                                           86215094
      IF (L .EQ. NA) GO TO 280                                          86215095
      IF (ITS .EQ. 30) GO TO 1000                                       86215096
      IF (ITS .NE. 10 .AND. ITS .NE. 20) GO TO 130                      86215097
C     ********** FORM EXCEPTIONAL SHIFT **********                      86215098
      T = T + X                                                         86215099
C                                                                       86215100
      DO 120 I = LOW, EN                                                86215101
  120 H(I,I) = H(I,I) - X                                               86215102
C                                                                       86215103
*     S = ABS(H(EN,NA)) + ABS(H(NA,ENM2))                               86215104
      S =DABS(H(EN,NA)) +DABS(H(NA,ENM2))
```

```
      X = 0.75 * S                                                      86215105
      Y = X                                                             86215106
      W = -0.4375 * S * S                                               86215107
  130 ITS = ITS + 1                                                     86215108
C     ********** LOOK FOR TWO CONSECUTIVE SMALL                         86215109
C                SUB-DIAGONAL ELEMENTS.                                 86215110
C                FOR M=EN-2 STEP -1 UNTIL L DO -- **********            86215111
      DO 140 MM = L, ENM2                                               86215112
         M = ENM2 + L - MM                                              86215113
         ZZ = H(M,M)                                                    86215114
         R = X - ZZ                                                     86215115
         S = Y - ZZ                                                     86215116
         P = (R * S - W) / H(M+1,M) + H(M,M+1)                          86215117
         Q = H(M+1,M+1) - ZZ - R - S                                    86215118
         R = H(M+2,M+1)                                                 86215119
*        S = ABS(P) + ABS(Q) + ABS(R)                                   86215120
         S =DABS(P) +DABS(Q) +DABS(R)
         P = P / S                                                      86215121
         Q = Q / S                                                      86215122
         R = R / S                                                      86215123
         IF (M .EQ. L) GO TO 150                                        86215124
*        IF (ABS(H(M,M-1)) * (ABS(Q) + ABS(R)) .LE. MACHEP * ABS(P)     86215125
*    X    * (ABS(H(M-1,M-1)) + ABS(ZZ) + ABS(H(M+1,M+1)))) GO TO 150    86215126
         IF(DABS(H(M,M-1)) *(DABS(Q) +DABS(R)) .LE. MACHEP *DABS(P)
     X    *(DABS(H(M-1,M-1)) +DABS(ZZ) +DABS(H(M+1,M+1)))) GO TO 150
  140 CONTINUE                                                          86215127
C                                                                       86215128
  150 MP2 = M + 2                                                       86215129
C                                                                       86215130
      DO 160 I = MP2, EN                                                86215131
         H(I,I-2) = 0.0                                                 86215132
         IF (I .EQ. MP2) GO TO 160                                      86215133
         H(I,I-3) = 0.0                                                 86215134
  160 CONTINUE                                                          86215135
C     ********** DOUBLE QR STEP INVOLVING ROWS L TO EN AND              86215136
C                COLUMNS M TO EN **********                             86215137
      DO 260 K = M, NA                                                  86215138
         NOTLAS = K .NE. NA                                             86215139
         IF (K .EQ. M) GO TO 170                                        86215140
         P = H(K,K-1)                                                   86215141
         Q = H(K+1,K-1)                                                 86215142
         R = 0.0                                                        86215143
         IF (NOTLAS) R = H(K+2,K-1)                                     86215144
*        X = ABS(P) + ABS(Q) + ABS(R)                                   86215145
         X =DABS(P) +DABS(Q) +DABS(R)
         IF (X .EQ. 0.0) GO TO 260                                      86215146
         P = P / X                                                      86215147
         Q = Q / X                                                      86215148
         R = R / X                                                      86215149
* 170    S = SIGN(SQRT(P*P+Q*Q+R*R),P)                                  86215150
  170    S = DSIGN(DSQRT(P*P+Q*Q+R*R),P)
         IF (K .EQ. M) GO TO 180                                        86215151
         H(K,K-1) = -S * X                                              86215152
         GO TO 190                                                      86215153
  180    IF (L .NE. M) H(K,K-1) = -H(K,K-1)                             86215154
```

```
  190     P = P + S                                              86215155
          X = P / S                                              86215156
          Y = Q / S                                              86215157
          ZZ = R / S                                             86215158
          Q = Q / P                                              86215159
          R = R / P                                              86215160
C     ********** ROW MODIFICATION **********                     86215161
          DO 210 J = K, EN                                       86215162
             P = H(K,J) + Q * H(K+1,J)                           86215163
             IF (.NOT. NOTLAS) GO TO 200                         86215164
             P = P + R * H(K+2,J)                                86215165
             H(K+2,J) = H(K+2,J) - P * ZZ                        86215166
  200        H(K+1,J) = H(K+1,J) - P * Y                         86215167
             H(K,J) = H(K,J) - P * X                             86215168
  210     CONTINUE                                               86215169
C                                                                86215170
          J = MIN0(EN,K+3)                                       86215171
C     ********** COLUMN MODIFICATION **********                  86215172
          DO 230 I = L, J                                        86215173
             P = X * H(I,K) + Y * H(I,K+1)                       86215174
             IF (.NOT. NOTLAS) GO TO 220                         86215175
             P = P + ZZ * H(I,K+2)                               86215176
             H(I,K+2) = H(I,K+2) - P * R                         86215177
  220        H(I,K+1) = H(I,K+1) - P * Q                         86215178
             H(I,K) = H(I,K) - P                                 86215179
  230     CONTINUE                                               86215180
C                                                                86215181
  260 CONTINUE                                                   86215182
C                                                                86215183
      GO TO 70                                                   86215184
C     ********** ONE ROOT FOUND **********                       86215185
  270 WR(EN) = X + T                                             86215186
      WI(EN) = 0.0                                               86215187
      EN = NA                                                    86215188
      GO TO 60                                                   86215189
C     ********** TWO ROOTS FOUND **********                      86215190
  280 P = (Y - X) / 2.0                                          86215191
      Q = P * P + W                                              86215192
*     ZZ = SQRT(ABS(Q))                                          86215193
      ZZ = DSQRT(DABS(Q))
      X = X + T                                                  86215194
      IF (Q .LT. 0.0) GO TO 320                                  86215195
C     ********** REAL PAIR **********                            86215196
*     ZZ = P + SIGN(ZZ,P)                                        86215197
      ZZ = P +DSIGN(ZZ,P)
      WR(NA) = X + ZZ                                            86215198
      WR(EN) = WR(NA)                                            86215199
      IF (ZZ .NE. 0.0) WR(EN) = X - W / ZZ                       86215200
      WI(NA) = 0.0                                               86215201
      WI(EN) = 0.0                                               86215202
      GO TO 330                                                  86215203
C     ********** COMPLEX PAIR **********                         86215204
  320 WR(NA) = X + P                                             86215205
      WR(EN) = X + P                                             86215206
      WI(NA) = ZZ                                                86215207
```

```
      WI(EN) = -ZZ
  330 EN = ENM2
      GO TO 60
C     ********** SET ERROR -- NO CONVERGENCE TO AN
C                EIGENVALUE AFTER 30 ITERATIONS **********
 1000 IERR = EN
 1001 RETURN
C     ********** LAST CARD OF HQR **********
      END
```

INITIAL DISTRIBUTION

Copies:	
1	ONR 430/R. LUNDEGARD
1	ONR 432/L. BRAM
1	NRL/8441/J. HANSEN
1	USNA DEPT MATH
1	USNA LIB
1	NAVPGSCOL/59C1/G. CANTIN
1	NAVPGSCOL/M. KELLEHER
1	NAVPGSCOL/MATH DEPT
1	NAVPGSCOL/LIB
1	NAVWARCOL
1	NROTC & NAVADMINU, MIT
1	NOL 331/E. COHEN

Copies:	
1	NOL 331/M. VANDER VORST
1	NAVSHIPYD BREM/LIB
1	NAVSHIPYD BSN/LIB
1	NAVSHIPYD CHASN/LIB
1	NAVSHIPYD MARE/LIB
1	NAVSHIPYD NORVA/LIB
1	NAVSHIPYD PEARL/LIB
1	NAVSHIPYD PHILA/LIB
1	NAVSHIPYD PTSMH/LIB
1	AIR FORCE AERO RES LABS/P. NIKOLAI

CENTER DISTRIBUTION

Copies:		
1	0000	NELSON PERRY W
1	1725	GIFFORD LEROY N JR
1	1725	JONES REMBERT F JR
1	1725	RODERICK JOAN E
1	1725	ROTH PETER N
1	1745	NG CHRISTOPHER
1	1800	GLEISSNER GENE H
1	1802	SHANKS DANIEL
1	1802	FRENKIEL FRANCOIS N
1	1802	LUGT HANS J
1	1802	THEILHEIMER FEODOR
1	1805	CUTHILL ELIZABETH H
1	1830	ERNST HERBERT M
1	1830	CULPEPPER LINWOOD M
1	1830	WALTON THOMAS S
1	1840	
1	1842	MEALS L KENTON
1	1842	EDDY ROBERT P

Copies:		
1	1843	SCHOT JOANNA WOOD
1	1844	DHIR SURENDRA K
1	1844	EVERSTINE GORDON C
10	1844	GIGNAC DONALD A
1	1844	HENDERSON FRANCIS M
1	1844	MATULA PETRO
1	1850	CORIN THOMAS
1	1860	SULIT ROBERT A
1	1880	CAMARA ABEL W
1	1890	GRAY GILBERT R
1	1890	TAYLOR NORA M
1	1892	GOOD SHARON E
1	1892	RUMSEY JUDITH J
1	1966	CASPAR JOHN R
1	1966	LIU YUAN-NING
30	5614	REPORTS DISTRIBUTION
1	5641	LIBRARY
1	5642	LIBRARY