Report 2008

WASHINGTON, D.C.

DEPARTMENT OF THE NAVY

HYDROMECHANICS

AERODYNAMICS

STRUCTURAL MECHANICS

APPLIED MATHEMATICS

ACOUSTICS AND VIBRATION

THE APPLICATION OF SLENDER BODY THEORY TO STEADY SHIP MOTION

by

E. O. Tuck

HYDROMECHANICS LABORATORY
RESEARCH AND DEVELOPMENT REPORT

June 1965

Report 2008

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Report 2008
S-R001 01 01
Task 0401

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ABSTRACT

Three previously published papers on steady ship motion treated by slender body theory are reprinted, together with an introduction which summarizes the three papers and reviews other efforts in this field up to 1965. Suggestions are made for further work to remove present limitations of the theory.

ADMINISTRATIVE INFORMATION

The work reported herein was conducted at the University of Cambridge from 1960 to 1962 and continued at the University of Manchester from September 1962 until the author joined the staff of the Taylor Model Basin late in 1963. The portion undertaken at the Model Basin represents part of the in-house independent research program of the Hydromechanics Laboratory (Project S-R001 01 01, Task 0401).

INTRODUCTION

Apart from this introduction, which is in the nature of a review, the present report consists of reprints of three papers on the application of slender body theory to the steady motion of a ship. The papers are here placed in the following order:

1. "A Systematic Asymptotic Expansion Procedure for Slender Ships," Journal of Ship Research, Vol 8, No. 1, pp. 15–23 (1964).

2. "On Line Distributions of Kelvin Sources," Journal of Ship Research, Vol 8, No. 2, pp. 45–52 (1964).

3. "On Vossers' Integral," Proceedings of International Seminar on Theoretical Wave Resistance, Ann Arbor, Vol II, pp. 699–710 (1963).

The actual order of publication was (3), (1), (2) whereas the order in which the research was completed was (2), (3), (1). Thus paper (1) contains the latest and most complete treatment of the problem and is the key paper of the set. The other two papers are ancillary to paper (1); paper (2) provides certain specific analytical results which are required in the general theoretical foundation, and paper (3) investigates some of the properties of the resulting formula for the wave resistance. However, both papers (2) and (3) were originally intended to be at least partially self-sufficient, and both, therefore, include a short discussion of the basis for a simplified slender ship theory. The unified general treatment of paper (1) is to be preferred, however.

The introduction to paper (3) contains a brief description of the work that had been done by Vossers and Maruo prior to 1963, but it may be of interest to give a more complete historical survey at this time. It is possible to see in some of Sir Thomas Havelock's work a close approach to slender ship theory; indeed he frequently used distributions of sources chosen to represent a ship but usually related the source strengths to hull shape by Michell's thin ship formula. However, in at least one paper,[1] he used a different method which related the strength at any section to the rate of change of cross-sectional area at that section, as in slender body theory. In this paper he was suggesting a high-speed theory wherein the ship could be approximated by discrete sections, but if he had let his subdivisions become finer and finer, he would have obtained slender ship theory.

Apart from this near thing from Havelock, the first serious attempt at a slender body theory for ships appears to have been made by W.E. Cummins of the Model Basin.[2] Aware of the basic aerodynamic slender body theories of Munk, Ward, and Adams and Sears, he attempted to apply similar arguments to ship problems but was unable to obtain the wave resistance in a form immediately suitable for calculation. The basic property of slender body theory is that the solution can be built up as the sum of a two-dimensional solution worked out for each cross section separately, plus a "non-uniqueness" addition in the form of an interaction between sections which must be calculated by three-dimensional methods. Cummins' method required his two-dimensional potentials to satisfy the "impulse" condition $\phi = 0$ on the free surface; later workers found the "wall" condition $\partial\phi/\partial n = 0$ to be more appropriate since this is the natural limit of the exact free surface condition as the lateral length scale tends to zero.

Effort in slender ship theory really began to get under way in the early 1960's, simultaneously and independently in four different countries. In Holland G. Vossers worked on the subject for his Ph.D. thesis under the supervision of Professor R. Timman of Delft, and in Cambridge, England, the author began his own Ph.D. project under Professor F. Ursell. At the same time, Professor H. Maruo in Japan became interested in the problem; interest continued at the David Taylor Model Basin although there Cummins and Newman were concerned more with unsteady problems. Vossers' thesis appeared in 1962 and was followed closely by a paper by Maruo in 1962 and the author's thesis in 1963; the general features of the Vossers and Maruo approaches are described in paper (3).

In addition to paper (3), several other papers bearing on slender ships were presented at the International Seminar on Theoretical Wave Resistance in August 1963. W.P.A. Joosen and S.I. Ciolkowski also presented versions of slender ship theory, both obtaining the same ("Vossers") wave resistance integral. In particular, Joosen attacked the problem by a method similar to Voosers' original work but without committing Vossers' errors, and furthermore, obtained systematically the special end-effect corrections given by Maruo. In addition,

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[1] Havelock, Sir Thomas, "The Approximate Calculation of Wave Resistance at High Speeds," Trans. North-East Coast Institute of Engineers and Shipbuilders, Vol. 60, pp. 47–58 (1943).

[2] Cummins, W.E., "The Wave Resistance of a Floating Slender Body," Ph.D. Thesis, American University (May 1956).

Joosen also presented some results on nonsteady motions at zero speed; F. Ursell's paper was also in this category. Some experimental results on a sharply pointed body of revolution were presented by G.R.G. Lewison, together with comparisons of his experimental results with calculations from both Michell's and Vossers' integrals. The conclusion was distinctly unfavorable to slender ship theory. J.V. Wehausen presented a new formulation of thin ship theory, with a note indicating how, by allowing the draft to tend to zero at an appropriate point, the results of slender ship theory could be obtained.

Apart from papers (1) and (2), little further work on the steady motion problem has appeared since this meeting. This is partly as a result of the negative conclusions reached at the meeting (and since) on the applicability of the slender ship theory to wave resistance calculations. It appears that in the practical range of speeds, the Vossers integral overestimates the wave resistance by a factor 3 or 4 whereas Michell's integral can be within a factor of 1 1/2. Further discussion of this discrepancy appeared in a paper by J. Kotik and P. Thomsen.[3] They show that slender ship theory cannot give the correct wave resistance at either low or high speeds and that practical speeds are a little on the low side of its range of validity.

Although there seems little hope of salvaging the wave resistance formula except perhaps for simple qualitative deductions such as those for high speed in paper (3), this is not an indication of the uselessness of the theory as a whole. On the contrary, in every respect but wave resistance, the theory continues to show great promise. In particular, several papers have appeared on unsteady motions at zero speed which show excellent agreement with practical expectations, and this application of slender body theory to ships can be said to be on a sound basis. Papers by Ogilvie, Newman and Tuck, and Joosen at the 5th Symposium on Naval Hydrodynamics at Gergen, Norway, in 1964 describe features of the unsteady motion theory. But if we are interested in quantities other than wave resistance, there appears to be hope of good agreement with practice even for the steady motion problem. For instance, recent estimates of sinkage forces (as yet unpublished) made at the Model Basin show a most encouraging agreement with observation. Some of the more successful aspects of the theory, even as it stood in 1963, are mentioned by Lewison in his discussion of paper (3).

Some of the material of the three papers included here appears also in the author's thesis,[4] together with a number of additional results. However, paper (1) in particular is thought to represent an improvement over both the material and presentation of the thesis. This applies especially to the development of the "inner and outer expansions" procedure for constructing a singular perturbation solution to the problem. Since publication of this paper, an excellent textbook[5] has appeared on this technique, and can be usefully read in

[3]Kotik, J. and Thomsen, P., "Various Wave Resistance Theories for Slender Ships," Schiffstechnik, Vol. 10, pp. 178–186 (1963).

[4]Tuck, E.O., "The Steady Motion of Slender Ships," Ph.D. Thesis, University of Cambridge (Jun 1963).

[5]Van Dyke, M., "Perturbation Methods in Fluid Mechanics," Academic Press, New York (1964).

conjunction with paper (1). The technique was originally associated with the names of Kaplun and Lagerstrom who published papers applying it to viscous flows in 1957, as referenced in paper (1). In recent years, a large number of papers have applied the technique in various fields, and it can now be said to have taken its place as one of the most important tools of fluid dynamics.

By use of this technique, it was possible in paper (1) to construct a formal second-order solution to the problem and to give an explicit method for finding the second approximation to wave resistance. It may be worth spelling out in detail at this point the steps required to find this second approximation; the details are, of course, meaningful only in the context of paper (1):

a. Set $a_1(x) = \dfrac{U}{\pi} S'(x)$ (Equation [34]). This is the first approximation to the source strength.
b. Solve a simple Neumann problem in the cross-sectional plane, as illustrated in Figure 1 for Φ_1. The "given outflow" is given by Equation [22], and the boundary condition at infinity is Equation [36].
c. Evaluate the right-hand side of Equation [31], using the potential Φ_1 just determined. This gives the velocity $\partial\Phi_2/\partial z$ at which fluid crosses the free surface in the second approximation, illustrated by Figure 1 for Φ_2.
d. Substitute this value of $\partial\Phi_2/\partial z$ in Equation [40] to find the second approximation $a_2(x)$ to the source strength.
e. Use the corrected source strength $a(x) = a_1(x) + a_2(x)$ in the wave resistance formula (Equation [35]).

There seems no reason in principle why this process could not be carried through numerically for a practical ship, and some progress along this line has been made by the author. It must be emphasized that in this problem, the second approximation arises only from the nonlinearity of the free-surface condition; if we had started with the linearized free-surface condition (Equation [14]) instead of with Equations [9] and [10], we should have found $\Phi_2 \equiv 0$, with the second approximation being only a factor $1 + 0(\epsilon^2)$ as in aerodynamic slender body theory. Further work on related problems has appeared wherein the importance of the free-surface nonlinearity is stressed.[6] J.V. Wehausen in his paper at the International seminar on Theoretical Wave Resistance and B. Yim in unpublished work since then have indicated that when the draft is small, the most important neglected term in thin-ship theory is the well-known line integral around the intersection of the hull with the plane free surface. It is probable that this term is of the same nature as the second approximation described above although this still awaits proof.

[6] Tuck, E.O., "The Effect of Nonlinearity at the Free Surface on Flow past a Submerged Cylinder," Journal of Fluid Mechanics, Vol. 21 (1965).

In addition to computations of second approximations in slender ship theory, there still remains the question of end effects. The theory requires ships to be more sharply pointed than is the case in practice, and when we allow too great a degree of bluntness (specifically nonzero slope of the curve of cross-sectional areas at the bow or stern), there is still confusion concerning the correct formula for the wave resistance. Maruo and Joosen have given some extra terms for the case when the bow (or stern) is a vertical (or horizontal) wedge, but some more general results are needed to cover such things as bulbous bows and transom sterns. For the case of an infinite fluid, a version of slender body theory has been produced which allows blunt ends,[7] but it has not yet been found possible to apply these methods to the ship problem.

Finally it would be desirable to remove the restriction of slender ship theory to finite speeds. In particular, a theory valid for low speed might have more chance of agreement with experiment. Since a ship produces short waves at low speeds, this has turned out to be a difficult undertaking, but work is continuing on this interesting extension of slender ship theory.

[7] Tuck, E.O., "Some Methods for Flows past Blunt Slender Bodies," Journal of Fluid Mechanics, Vol. 18, pp. 619–635 (1964).

A Systematic Asymptotic Expansion Procedure for Slender Ships[1]

By E. O. Tuck[2]

Inner and outer expansions are used to formulate a systematic solution to the problem of the steady translation of a slender ship of arbitrary shape. Careful consideration is given to finding the correct boundary conditions to be satisfied by successive terms in the expansions, and certain of the individual terms are determined partly or completely as functions of hull shape. Some results are given concerning the second approximations to the potential and wave resistance.

THIS paper seeks to lay the foundation of a systematic and self-consistent theory for the motion of a ship in an infinite ocean. We restrict attention to the case where the ship is moving with constant velocity into still water (or, equivalently, where the ship is held fixed and fluid streams past with a uniform velocity at infinity). Viscosity and compressibility are neglected, so that attention is concentrated upon those aspects of steady ship motion which are directly concerned with the fact that the ship produces a system of waves on the surface of the ocean.

To fix ideas, let us set up a system of Cartesian axes, with the x-axis in the direction of the uniform stream at infinity, and with the plane $z = 0$ representing the equilibrium free surface (z is positive *downwards*). The ship's bulk is supposed to be largely concentrated on a line parallel to the x-axis, which is either in the equilibrium free surface $z = 0$ or at some small depth $z = z_0 \geq 0$; this line is taken as the line $y = 0$, $z = z_0$. It is also convenient to define polar coordinates (r, θ) with the foregoing line as the axis $r = 0$, i.e.

$$r \cos \theta = z - z_0 \qquad r \sin \theta = y \tag{1}$$

We now take the ship's hull to be described in these coordinates by the equation

$$r = r_0(x, \theta) \tag{2}$$

for some given function $r_0(x, \theta)$. Since the ship has in general no axis of symmetry, one may ask how the axis $r = 0$ is to be chosen, and in fact this may be done almost at random. The results for the first approximation do not depend on choice of axis, but in practice an unsuitable choice of axis could make the series expansions for higher approximations poorly convergent. Note also that if the maximum value of r_0 is r_{max}, and if $r_{max} < z_0$, then the "ship" is wholly below the equilibrium free surface. Although such a body is more like a submarine, we shall use the word "ship" to denote an arbitrary body with the hull equation (2).

In practice one expects the ship to be of finite extent only, e.g., such that $r_0(x, \theta) \equiv 0$ for $|x| > \frac{1}{2}L$, say, where L is the length of the ship, but one need not confine attention to such a class of ships initially, and we shall in this general treatment suppose r_0 to be a perfectly arbitrary function of x. In any case we shall suppose that a length scale L for changes in x may be defined even if the ship extends all the way to infinity; e.g., L may be taken as a typical value of $r_0/|\partial r_0/\partial x|$.

Now a condition that the ship is slender is that the slenderness ratio $\epsilon = r_{max}/L$ is small, but this is not sufficient, for due to the influence of gravity there is a further length scale involved in the problem. If U is the velocity of the uniform stream at infinity, i.e., in practice the speed of the ship moving into still water, and g is the acceleration of gravity, then U^2/g has the dimensions of a length—it is in fact proportional to the

[1] This paper contains work done between 1960 and 1962 in the Department of Applied Mathematics and Theoretical Physics at the University of Cambridge; between September 1962 and September 1963 in the Department of Mathematics at the University of Manchester; and at the Taylor Model Basin in late 1963.

[2] David Taylor Model Basin, Washington, D. C.

Nomenclature

g = acceleration of gravity
$\mathbf{H}_0$ = Struve function of zero order
L = representative length scale
r, θ = polar coordinates
$r_0(x, \theta)$ = function describing hull shape
r_{max} = greatest value of $r_0(x, \theta)$
R_W = wave resistance
$S(x)$ = immersed cross-sectional area of ship
U = velocity of uniform stream (or speed of ship)
x, y, z = Cartesian coordinates
Y_0 = Bessel function of second kind, zero order
ϵ = slenderness ratio = r_{max}/L
$\zeta(x, y)$ = depression of free surface
$\eta(x)$ = hull waterline curve
κ = g/U^2
ρ = water density
ϕ = total velocity potential
ϕ_n = terms in outer expansion
Φ_n = terms in inner expansion

wave length at infinity of the transverse wave system produced by the ship. Thus there are three basic lengths in our problem (further length scales could be important if viscosity, compressibility, surface tension, and so on, were to be taken into account; there is also the depth of submersion z_0 which is discussed later), namely r_{max}, L, and U^2/g, and depending upon the relative sizes of these quantities, quite different "slender-body" theories may be formulated. The problem to be studied here is that obtained by assuming that r_{max}/L is small but that U^2/gL is finite (the last ratio is the square of the Froude number). It is most important to bear in mind that the theory which follows is not valid in the first instance if the Froude number is either large or small, or, what is the same thing, if the waves produced by the ship are either long or short compared with the length scale L.

This is clearly illustrated if the problem is made nondimensional with respect to L by putting $x' = x/L$, and so on, so that the ship becomes of unit length and has maximum radius ϵ in the dashed coordinates. Then the velocity potential, for instance, is a function

$$\phi = \phi\left(x', y', z'; \epsilon, \frac{U^2}{gL}\right)$$

i.e., a function of the nondimensional space coordinates and of the two dimensionless parameters concerned. For an asymptotic slender-body theory we might suppose that ϕ has an expansion of the form

$$\phi = \sum_{n=0}^{\infty} \epsilon^n \phi_n \left(x', y', z'; \frac{U^2}{gL}\right)$$

But now if $U^2/gL \neq 0(1)$ with respect to ϵ, each ϕ_n must itself be expanded as a series in ϵ, and individual terms of the reordered series will not in general satisfy the same equations and boundary conditions as did the original ϕ_n obtained on the assumption that $U^2/gL = 0(1)$.

In fact we shall not in the text use any such nondimensional formulation. The author would like at this point to make a small plea that complicated boundary-value problems, especially those involving asymptotic expansions and small parameters, be not handled in nondimensional formulations. There are at least three good reasons for this:

1 The algebra is nearly always more difficult to follow.

2 Dimensions have no meaning in the midst of a mathematical calculation. Thus the symbol x can be taken to represent merely a real number during an algebraic manipulation and its dimensions discussed only at the end, if necessary. There is no need to make x dimensionless first explicitly; one may consider this to have been done implicitly with respect to some suitable length scale.

3 Final answers in dimensional form are usually more readily understood, interpreted and checked (e.g., dimensionally) than when put into dimensionless form, although this may in the final analysis be necessary for practical application.

With this philosophy in mind, we shall freely make statements such as "U^2/g = finite" or "$r = 0(\epsilon)$"; the purist may interpret such statements as referring to the size of these dimensional quantities with respect to the length scale L.

The velocity potential for flow with a free surface present must satisfy two conditions at the free surface. The first is the condition that the pressure is constant over the whole surface, while the second is a kinematic condition that fluid particles do not enter or leave the free surface. Both of these boundary conditions are nonlinear when written in terms of the velocity potential (for instance the pressure condition introduces the nonlinearity of the Bernoulli equation) and, further, they are to be applied on a surface which is *a priori* unknown and must be determined as part of the problem. The usual answer to these difficulties is to linearize the boundary conditions on the assumption that the departure of the free surface from a plane and of the fluid velocity from a uniform stream is small. The resulting "linearized free-surface condition" is of the form (for steady flows)

$$g\frac{\partial\phi}{\partial z} - U^2\frac{\partial^2\phi}{\partial x^2} = 0 \qquad \text{on } z = 0 \tag{3}$$

(note that it is now applied on a known surface $z = 0$, which is consistent with the approximation that the free surface departs little from this plane).

A great deal of classical water-wave theory is based on potential theory with this type of linear (but mixed) boundary condition, and the linear problems involved are of great mathematical and practical interest. However, in the present case of steady motion of a ship, it is clear that it is only because the ship itself is "small," e.g., slender, that the free surface is nearly plane. A finite ship would create a finite disturbance and the boundary condition could not be linearized. The fact that the free surface is nearly plane should therefore be derived as part of the solution to the problem and not used as an assumption by means of which the free surface conditions are linearized; this approach is possible if asymptotic expansions are used from the outset, and at the same time higher terms in the expansions, satisfying boundary conditions in general different from (3), are obtained systematically.

The foregoing argument does not hold for some unsteady and some other free-surface flows. Thus if a body is made to undergo small oscillations about an equilibrium fixed position, then the smallness of the disturbance at the free surface is ensured by the smallness of the oscillation alone, and the body itself need not necessarily be small. Hence providing only that oscillations are of small amplitude, the linearized free surface condition, which is similar to (3), may be taken as exact, and slenderness is only required as a means of enabling the hull boundary condition to be satisfied analytically within the context of a wholly linear problem. Ursell [1][3] has investigated ship motions at zero speed from this point of view. As soon as the body has a finite velocity of translation, however, slenderness or some other restriction on the size of the body is necessary for linearization.

There is another case in which a finite body may produce a small disturbance of the free surface; namely, when the body is deeply submerged. This is again a subject that has been treated by others, notably by Havelock (for a bibliography, see [2]), and it is not proposed to discuss this case in the present paper. Notice however that for a slender body, "deeply submerged" clearly means submerged to a depth large compared with r_{max}. Hence, there is no need to consider finite values of the depth of submersion z_0 and we shall in fact make the assumption throughout that $z_0 = 0(\epsilon)$; i.e., that the depth of submersion is of the same order as the lateral dimensions of the body.

In the present paper we discuss the first and second approximations to the velocity potential without specifying any particular ship form; i.e., we retain a quite general function $r_0(x, \theta)$. The first approximation can be carried through almost completely for a general $r_0(x, \theta)$ and, for instance, we are able to find a formula for the first approximation to the wave resistance. In order to obtain second approximations it is necessary to specify the shape of the cross section of the ship, and hence we are only able to discuss the second approximation in general terms at the present time; however it is hoped to present shortly some results of calculations of both first and second approximations for a simple assumed cross section.

Exact Equations and Boundary Conditions

The velocity potential ϕ satisfies Laplace's equation

$$\nabla^2\phi = 0 \tag{4}$$

where

$$\begin{aligned}\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ &= \frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \\ &= \frac{\partial^2}{\partial x^2} + \nabla_{(2)}{}^2\end{aligned}$$

with $\nabla_{(2)}{}^2$ as the two-dimensional Laplacian in planes normal to the axis of x.

At infinity the flow becomes the uniform stream

$$\phi \rightarrow Ux \qquad \text{at } \infty$$

This is not quite sufficient, and it is generally necessary to add a so-called radiation condition which specifies that ϕ shall approach Ux in a physically acceptable manner; this will be incorporated into the solution when required (see reference [3]). At infinity one also requires that the free surface becomes plane; that is; if the height (more accurately "depression," since z is positive downwards) of the free surface is described by

$$z = \zeta(x, y) \tag{5}$$

then we require that

$$\zeta \rightarrow 0 \qquad \text{at } \infty$$

giving $z = 0$ as the equilibrium free surface. This says nothing, of course, about ζ being small at any finite value of the space coordinates.

On the ship's hull surface the boundary condition of zero normal velocity states

$$\frac{D}{Dt}[r - r_0(x, \theta)] = 0$$

i.e.

$$\frac{\partial\phi}{\partial r} - \frac{1}{r_0{}^2}\frac{\partial r_0}{\partial\theta}\frac{\partial\phi}{\partial\theta} - \frac{\partial r_0}{\partial x}\frac{\partial\phi}{\partial x} = 0 \qquad \text{on } r = r_0(x, \theta) \tag{6}$$

This may be written in the form

$$\frac{\partial\phi}{\partial N} = \frac{(\partial r_0/\partial x)(\partial\phi/\partial x)}{[1 + (1/r_0{}^2)(\partial r_0/\partial\theta)^2]^{1/2}} \qquad \text{on } r = r_0(x, \theta) \tag{7}$$

where $\partial/\partial N$ denotes differentiation normal to the curve $r = r_0(x, \theta)$ in planes of constant x, as distinct from the derivative normal to the hull *surface* (which is zero). Equation (7) is seen from the fact that the vector $\mathbf{dN}$ with axial component zero, radial component $r_0 d\theta$ and tangential component $-dr_0 = -(\partial r_0/\partial\theta)d\theta$ is orthogonal to the tangent vector $(0, dr_0, r_0 d\theta)$ and has length

$$|\mathbf{dN}| = r_0 d\theta \left[1 + \frac{1}{r_0{}^2}\left(\frac{\partial r_0}{\partial\theta}\right)^2\right]^{1/2}$$

whence

$$\frac{\partial\phi}{\partial N} = \frac{\mathbf{dN}}{|\mathbf{dN}|}\cdot\boldsymbol{\nabla}\phi$$

is as given by (7).

Finally we must specify the two boundary conditions on the unknown free surface $z = \zeta(x, y)$. Now Bernoulli's equation can be written in the form

$$p/\rho = gz - \tfrac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \tfrac{1}{2}U^2 \tag{8}$$

where the constant $\frac{1}{2}U^2$ is chosen so that the pressure p will vanish on the free surface at infinity where $|\boldsymbol{\nabla}\phi| = U$ and $z = 0$; thus we have chosen to measure p with atmospheric pressure as zero. The "dynamic" free-surface condition is now the statement that $p \doteq 0$ everywhere on the free surface; i.e., that

$$g\zeta = \tfrac{1}{2}|\boldsymbol{\nabla}\phi|^2 - \tfrac{1}{2}U^2 \qquad \text{on } z = \zeta(x, y) \tag{9}$$

The "kinematic" free-surface condition is the condition that fluid particles remain in the surface; viz.

$$\frac{D}{Dt}[z - \zeta(x, y)] = 0$$

which gives

[3] Numbers in brackets designate References at end of paper.

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \zeta}{\partial y} \qquad \text{on } z = \zeta(x, y) \qquad (10)$$

This completes the specification of the boundary-value problem, which is to determine the potential ϕ for a given hull function $r_0(x, \theta)$, subject to the foregoing equations and boundary conditions. The following theory provides a systematic analytic method for finding ϕ for arbitrary but small $r_0(x, \theta)$.

Inner and Outer Expansions

A natural way of approaching an asymptotic expansion for small slenderness ϵ is to write

$$\phi = \sum_{n=0}^{\infty} \phi_n(x, r, \theta;\ \epsilon) \qquad (11)$$

where the terms ϕ_n are in descending order of magnitude with respect to ϵ, i.e., $\phi_{n+1} = o(\phi_n)$ as $\epsilon \to 0$, where each ϕ_n is homogeneous in order with respect to ϵ, and where x, r, and θ take values of $0(1)$ with respect to ϵ. Note that it is not necessary or desirable to assume that the expansion is a power series, and indeed we shall find that it is not. The use of this expansion leads to the usual "linearized" results [4] and is in some cases quite satisfactory by itself. However, for a slender body there is a little difficulty with the hull boundary condition, for as $\epsilon \to 0$ the ship must necessarily shrink down to its limiting line. Since each term ϕ_n must satisfy boundary conditions on the limit of the hull as $\epsilon \to 0$, we are faced with solving a series of three-dimensional boundary-value problems with "boundary conditions" specified on a line.

One way to circumvent this difficulty is to introduce a second, so-called "inner" expansion, and instead of requiring the "outer" expansion (11) to satisfy the hull boundary condition, to merely require it to agree with the inner expansion. The inner expansion is formally defined by

$$\phi = \sum_{n=0}^{\infty} \Phi_n\left(x,\ \frac{r}{\epsilon},\ \theta;\ \epsilon\right) \qquad (12)$$

where the Φ_n are ordered with respect to ϵ as before, but now x, r/ϵ, and θ are assumed finite with respect to ϵ; i.e., $r = 0(\epsilon)$. Clearly now as $\epsilon \to 0$, the body defined by $r/\epsilon = r_0(x, \theta)/\epsilon$ does *not* shrink to the limiting line $r = 0$, and we will expect no difficulty in formulating the hull boundary condition. However, since $r = 0(\epsilon)$ we can not apply the outer boundary condition that ϕ behaves like a uniform stream at infinity.

Thus both the inner and outer expansions are by themselves non-unique since the outer expansion has no inner boundary condition while the inner expansion has no outer boundary condition. This non-uniqueness is then removed by "matching" the expansions together, matching being a process of comparison between the inner and outer expansions carried out in an "intermediate" limiting region which is such that $r = 0(\epsilon^\alpha)$ for some α, $0 < \alpha < 1$. In other words, if we take the outer expansion and look at its behavior for *small* $r = 0(\epsilon^\alpha)$, this must agree term by term with the limit of the inner expansion for *large* $r/\epsilon = 0(\epsilon^{-1+\alpha})$. For a careful rigorous discussion of this matching process, reference may be made to work by Kaplun and Lagerstrom [5], where the method of inner and outer expansions is applied to problems in viscous flow at low Reynolds numbers.

Sometimes it is helpful in looking at the inner expansion to write $R = r/\epsilon$ and imagine the inner expansion as valid in the (x, R, θ) space obtained by stretching radial distances in the real (x, r, θ) space. Thus in the (x, R, θ) or inner region the ship remains of finite size as $\epsilon \to 0$. This also explains why we have taken $z_0 = 0(\epsilon)$, for if z_0 were finite, then in the inner region the ship would be infinitely submerged as $\epsilon \to 0$ and the free surface would not be present. It is in order to concentrate on the situation when free-surface effects are dominant that we restrict z_0 to be of order ϵ.

However, the validity of this picture should not be pushed too far, and in particular it should be remembered that the statements made are always limiting statements. Thus it is not helpful to picture the inner expansion as holding only in some physical region of space close to the ship and the outer expansion as valid far from the ship, with an actual physical bounding surface on which they agree. Both complete expansions are asymptotic expansions valid throughout the whole of space, except that the inner expansion cannot be valid at $r = \infty$ while the outer expansion cannot be valid at $r = 0$; the matching process is a limiting operation and there is no physical intermediate region between the two expansions.

We must assume that the asymptotic expansions (11) and (12) can be differentiated as often as we need with respect to their respective space variables (although sometimes it turns out that this cannot be done *uniformly* with respect to the remaining space variables, as when end effects occur). Thus for the inner expansion,

$$\frac{\partial \phi}{\partial r} = \frac{1}{\epsilon} \sum_{n=0}^{\infty} \frac{\partial}{\partial R} \Phi_n(x, R, \theta;\ \epsilon)$$

so that for the inner expansion where $r = 0(\epsilon)$ we must also require $\partial/\partial r = 0(\epsilon^{-1})$. This has immediate consequences, since it means that in the Laplace equation (4) and in all the boundary conditions, axial derivatives will be dropped compared with radial derivatives. In particular, all Φ_n will satisfy two-dimensional Laplace or Poisson equations in cross-sectional planes. The details of the linearization for the inner expansion will be worked out in a later section.

Determination of Outer Expansion

We now formally substitute the asymptotic expansion (11) into equations (4), (9) and (10), and perform the limit as $\epsilon \to 0$ successively to give the limiting versions of the equations and boundary conditions satisfied by individual ϕ_n. First we should notice that as $\epsilon \to 0$ in real space the ship disappears entirely to leave an undisturbed uniform stream, so that the first term must be taken as

$$\phi_0 = Ux \tag{13}$$

while as a consequence of the ordering process we must have

$$\phi_1 = o(\phi_0) = o(1)$$

Now for $n = 1, 2 \ldots$ the Laplace equation gives trivially

$$\nabla^2 \phi_n = 0$$

while the condition at infinity gives

$$\phi_n \to 0 \quad \text{at } \infty$$

Since we must not attempt to force individual ϕ_n to satisfy the inner boundary condition (6), the free-surface conditions (9) and (10) are the only remaining boundary conditions to be satisfied, in the correct limiting forms as $\epsilon \to 0$, by the functions ϕ_n.

The formal result of substituting the outer expansion into (9) and (10) and letting $\epsilon \to 0$ is the following sequence of "linearized" boundary conditions

$$\left.\begin{aligned} g\frac{\partial \phi_1}{\partial z} - U^2 \frac{\partial^2 \phi_1}{\partial x^2} &= 0 \\ g\frac{\partial \phi_2}{\partial z} - U^2 \frac{\partial^2 \phi_2}{\partial x^2} &= 0 \end{aligned}\right\} \tag{14}$$

$$g\frac{\partial \phi_3}{\partial z} - U^2 \frac{\partial^2 \phi_3}{\partial x^2} = U\frac{\partial}{\partial x}|\nabla \phi_1|^2 - U\frac{\partial \phi_1}{\partial x}\frac{\partial^2 \phi_1}{\partial z^2} + \frac{U^3}{g}\frac{\partial \phi_1}{\partial x}\frac{\partial^3 \phi_1}{\partial x^2 \partial z} \tag{15}$$

and so on, all satisfied on the equilibrium free surface $z = 0$; the analysis leading to these results will not be reproduced here as it has been given many times by Wehausen [4] and others, and is similar to the analysis used in the next section for the inner expansion.

While there appears at the moment to be no justification for writing the boundary conditions for ϕ_1 and ϕ_2 as identical, it should be noted that there is no loss of generality in doing so, since the distinction is only one of nomenclature; all we are saying is that it is convenient to split the dominant term in the outer expansion into two parts, ϕ_1 and ϕ_2. In fact the reason for this is that we shall show by use of the inner expansion that $\phi_1 = 0(\epsilon^2)$ so that the forcing terms on the right of equation (15) are of order ϵ^4. However we shall also see that the outer expansion needs to have a term ϕ_2 of order $\epsilon^3 \log \epsilon$, so that these forcing terms cannot affect ϕ_2 and are therefore written into the boundary condition for ϕ_3. Physically the right-hand side of (15) represents the pressure distribution due to the presence of the first-order wave system associated with ϕ_1, and (15) indicates that ϕ_3 is a flow due to this imposed pressure distribution.

Since the forcing terms of (15) involve ϕ_1 quadratically, the task of determining a potential function ϕ_3 satisfying (15) is formidable, and the calculation may be assumed to have terminated once this stage has been reached. The main task is therefore that of finding potential functions ϕ_1 and ϕ_2 satisfying the homogeneous free-surface condition (14) and, as this is a well-known type of potential problem, the determination of ϕ_1 (for instance; the following applies equally well to ϕ_2 with suffix "1" replaced by "2" throughout) follows a well-trodden path. Note that as $\epsilon \to 0$ the ship shrinks down to the axis $r = 0$ and that there are no other disturbing agents in the fluid for $z > 0$. Thus ϕ_1 must be analytic everywhere in $z > 0$ except for singularities *on* this line, and the appropriate behavior for ϕ_1 is then that of a distribution of sources of unknown strength along this line.

This problem was discussed in reference [3], where the potential for a line distribution of sources of arbitrary strength $-\frac{1}{4}a_1(x)$ along the line $r = 0$ was obtained and investigated. In particular, for use in matching we require the *intermediate* limit of the potential ϕ_1 for small $r = 0(\epsilon^\alpha)$ with α arbitrary, $0 < \alpha < 1$. This can be obtained from the results given in [3] in the form

$$\phi_1 = a_1(x) \log r + b_1(x) + 0(r \log r) \tag{16}$$

where

$$b_1(x) = -\tfrac{1}{2}\int_{-\infty}^{\infty} da_1(\xi)\Big\{\operatorname{sgn}(x-\xi)\log 2|x-\xi| + \frac{\pi}{2}\mathbf{H}_0(\kappa(x-\xi)) + (2+\operatorname{sgn}(x-\xi))\frac{\pi}{2}Y_0(\kappa|x-\xi|)\Big\} \tag{17}$$

($\kappa = g/U^2$; $\mathbf{H}_0$, Y_0 are Struve, Bessel functions, respectively [6]).

Equation (16) states that ϕ_1 behaves for small r like a two-dimensional source of unknown strength $a_1(x)$ at $r = 0$ together with a "constant" term $b_1(x)$ which is determined directly and uniquely from $a_1(x)$ and which introduces the free-surface character of the problem. The arbitrariness of the function $a_1(x)$ can only be removed by matching with the inner expansion, and we must therefore proceed to determine some terms of this expansion.

Determination of Inner Expansion

We now expand formally both the potential ϕ and the wave height ζ, putting

$$\phi(x, r, \theta; \epsilon) = \sum_{n=0}^{\infty} \Phi_n(x, r/\epsilon, \theta; \epsilon)$$

or

$$\phi(x, y, z; \epsilon) = \sum_{n=0}^{\infty} \Phi_n(x, y/\epsilon, z/\epsilon; \epsilon)$$

and

$$\zeta(x, y; \epsilon) = \sum_{n=0}^{\infty} \zeta_n(x, y/\epsilon; \epsilon) \tag{18}$$

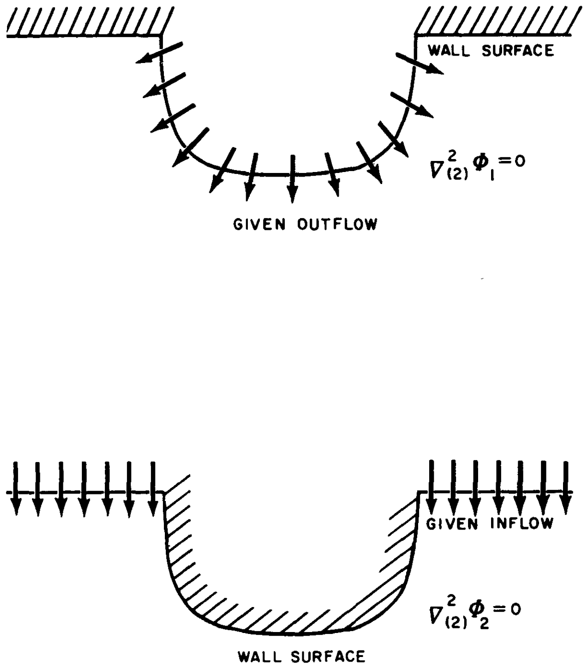

Fig. 1

In order to save time at this stage we anticipate the fact that in order to match the outer expansion, the inner expansion must start with the term

$$\Phi_0 = Ux \tag{19}$$

although we could easily have proceeded a little way farther with an unknown Φ_0 until we were ready for the full matching process.

Now the Laplace equation gives

$$\nabla_{(2)}{}^2\Phi_1 + \nabla_{(2)}{}^2\Phi_2 + \ldots = -\frac{\partial^2\Phi_1}{\partial x^2} - \frac{\partial^2\Phi_2}{\partial x^2} - \ldots$$

Hence on taking the limit as $\epsilon \to 0$ and remembering that $\nabla_{(2)}{}^2 = 0(\epsilon^{-2})$, we have successively that

$$\nabla_{(2)}{}^2\Phi_1 = 0 \tag{20}$$

$$\nabla_{(2)}{}^2\Phi_2 = -\Phi_{1xx} \tag{21}$$

and so on. That is, successive terms satisfy two-dimensional Poisson equations, with the "charge density" on the right hand side dependent on the preceding term in the series.

Similarly, from the hull boundary condition in the form (7), we have

$$\frac{\partial\Phi_1}{\partial N} + \frac{\partial\Phi_2}{\partial N} + \ldots = \left[1 + r_0^{-2}\left(\frac{\partial r_0}{\partial\theta}\right)^2\right]^{1/2} \cdot \frac{\partial r_0}{\partial x}\left[U + \frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_2}{\partial x} + \ldots\right]$$

so that

$$\frac{\partial\Phi_1}{\partial N} = \frac{U\dfrac{\partial r_0}{\partial x}}{\left[1 + \dfrac{1}{r_0^2}\left(\dfrac{\partial r_0}{\partial\theta}\right)^2\right]^{1/2}} \tag{22}$$

$$\frac{\partial\Phi_2}{\partial N} = \frac{\dfrac{\partial\Phi_1}{\partial x}\dfrac{\partial r_0}{\partial x}}{\left[1 + \dfrac{1}{r_0^2}\left(\dfrac{\partial r_0}{\partial\theta}\right)^2\right]^{1/2}} \tag{23}$$

and so on, all applied on the curve $r = r_0(x, \theta)$ in planes of constant x. Thus each term Φ_n satisfies a Neumann boundary condition specifying the velocity at which fluid crosses the curve $r = r_0$ in terms of the previously determined Φ_{n-1}. In particular, we see that the outflow for Φ_1 is completely determined when r_0 is given, and also obtain the important result that $\Phi_1 = 0(\epsilon^2)$.

It remains to find the free-surface condition satisfied by Φ_n. The dynamic condition gives

$$\begin{aligned} g\zeta_0 + g\zeta_1 + \ldots = {} & \tfrac{1}{2}(U + \Phi_{1x} + \Phi_{2x} + \ldots)^2 \\ & + \tfrac{1}{2}(\Phi_{1y} + \Phi_{2y} + \ldots)^2 \\ & + \tfrac{1}{2}(\Phi_{1z} + \Phi_{2z} + \ldots)^2 \\ & - \tfrac{1}{2}U^2 \end{aligned}$$

i.e.

$$\begin{aligned} g\zeta_0 + \text{smaller terms} = {} & U\Phi_{1x} + \tfrac{1}{2}\Phi_{1y}{}^2 + \tfrac{1}{2}\Phi_{1z}{}^2 \\ & + \text{smaller terms} \\ & \text{on } z = \zeta_0 + \text{smaller terms} \end{aligned} \tag{24}$$

Thus as $\epsilon \to 0$, remembering that $\Phi_1 = 0(\epsilon^2)$ and $\partial/\partial y = 0(\epsilon^{-1})$ etc., we have

$$g\zeta_0 = U\Phi_{1x} + \tfrac{1}{2}\Phi_{1y}{}^2 + \tfrac{1}{2}\Phi_{1z}{}^2 \qquad \text{on } z = 0 \tag{25}$$

Similarly, the kinematic free-surface condition states

$$\begin{aligned} \Phi_{1x} + \Phi_{2x} + \ldots = {} & (U + \Phi_{1x} + \Phi_{2x} + \ldots) \\ & \cdot(\zeta_{0x} + \zeta_{1x} + \ldots) \\ & + (\Phi_{1y} + \Phi_{2y} + \ldots) \\ & \cdot(\zeta_{0y} + \zeta_{1y} + \ldots) \\ & \text{on } z = \zeta_0 + \zeta_1 + \ldots \end{aligned} \tag{26}$$

i.e.

$$\begin{aligned} \Phi_{1z} + \text{smaller terms} = {} & U\zeta_{0x} + \Phi_{1y}\zeta_{0y} \\ & + \text{smaller terms} \\ & \text{on } z = \zeta_0 + \text{smaller terms} \end{aligned} \tag{27}$$

But (25) implies that $\zeta_0 = 0(\epsilon^2)$ so that the right-hand side of (27) is of order ϵ^2 whereas the left-hand side is of order ϵ. Hence, in the limit as $\epsilon \to 0$, we have as the free-surface condition for the first approximation Φ_1 that

$$\Phi_{1z} = 0 \qquad \text{on } z = 0 \tag{28}$$

Thus Φ_1 satisfies a "wall" free-surface condition on $z = 0$; this together with the given outflow (22) across the curve $r = r_0$ is all the information we have to determine the two-dimensional potential function Φ_1. The solution is non-unique since we can add any potential satisfying

(28) and giving zero outflow over $r = r_0$ but with arbitrary behavior at infinity (e.g., a constant). The matching process to be described in the next section is effectively a method for supplying a boundary condition at infinity which makes the solution for Φ_1 unique.

Now from (25), once Φ_1 has been determined, the first approximation to the wave height ζ_0 in the inner expansion is given by

$$g\zeta_0 = U\Phi_{1\,x} + \tfrac{1}{2}\Phi_{1\,y}{}^2 \qquad \text{on } z = 0 \tag{29}$$

and, by further consideration of the kinematic free-surface condition (26), the boundary condition for Φ_2 is obtained in the form

$$\Phi_{2\,z} = U\zeta_{0\,x} + \Phi_{1\,y}\zeta_{0\,y} - \Phi_{1\,zz}\zeta_0 \qquad \text{on } z = 0, \tag{30}$$

(the last term arises from expanding $\Phi_{1\,z}(x, y, \zeta_0)$ in a Taylor series), so that on substituting for ζ_0 from (29) we have

$$g\Phi_{2\,z} = U^2\Phi_{1\,xx} + 2U\Phi_{1\,y}\Phi_{1\,xy} + U\Phi_{1\,x}\Phi_{1\,yy} + \tfrac{3}{2}\,\Phi_{1y}{}^2\,\Phi_{1\,yy}\,, \text{on } z = 0 \tag{31}$$

Equation (31) is the free-surface condition for Φ_2. But since $\Phi_1 = 0(\epsilon^2)$, all terms on the right-hand side of (31) are of order ϵ^2 and hence Φ_2 is of order ϵ^3. But now in equation (21) the right-hand side is of order ϵ^2 whereas the left-hand side is of order $\Phi_2/\epsilon^2 \doteq 0(\epsilon)$. Hence, Φ_2 must satisfy the homogeneous Laplace equation $\nabla_{(2)}{}^2\Phi_2 = 0$. Similarly in equation (23) the right-hand side is $0(\epsilon^3)$ whereas the left-hand side is $0(\epsilon^2)$. Hence

$$\frac{\partial\Phi_2}{\partial N} = 0 \qquad \text{on } r = r_0 \tag{32}$$

so that there is no outflow over the hull curve $r = r_0(x, \theta)$ for the second-order flow, which is determined solely by the given "outflow" (31) over the line $z = 0$. Once again the solution for Φ_2 is initially non-unique owing to the lack of a boundary condition at infinity, but the matching process will provide such a condition for us. The potential problems to be solved in finding Φ_1, Φ_2 are illustrated diagrammatically in Fig. 1.

Matching the First Approximation

The matching principle asserts that the intermediate limit of the inner expansion must agree with the intermediate limit of the outer expansion term by term. But now we have given in equation (16) the first few terms of the latter in the form

$$\phi = Ux + a_1(x)\log r + b_1(x) + \ldots$$

for some as yet unknown $a_1(x)$. Hence, the intermediate limit of Φ_1 must be

$$\Phi_1 = a_1(x)\log r + b_1(x)$$

This limit can serve as an extra boundary condition for Φ_1 and will render Φ_1 unique. That is, of all the possible solutions for Φ_1 satisfying (22) and (28) we select the solution that behaves at infinity like $a_1 \log r + b_1$ for any a_1 but with b_1 determined directly from a_1 by (17).

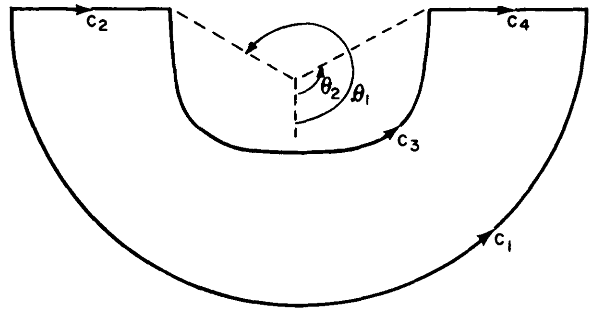

Fig. 2

However, we can go a little further in the case of this first term Φ_1 and determine the function $a_1(x)$ directly for an arbitrary $r_0(x, \theta)$ by conservation of mass without actually solving for Φ_1. First let us note that if C_1 is any curve lying wholly in $z \geq 0$ and wholly in the intermediate region, i.e., with $r = 0(\epsilon^\alpha)$, then the flux of Φ_1 across C_1 is $\pi a_1(x)$. This is exactly one half of the total flux produced by the source, since in the intermediate region, $z_0 \to 0$ and the source appears to lie in the plane $z = 0$. Now by Gauss's theorem (i.e., conservation of mass), the total flux of Φ_1 out of the closed curve $C = -C_1 + C_2 + C_3 + C_4$ of Fig. 2 is zero. Here C_1 is as defined in the foregoing, C_2 and C_4 are portions of the line $z = 0$, while C_3 is the hull contour $r = r_0(x, \theta)$. But the flux across C_2 and C_4 vanishes by (28) so that the flux of Φ_1 across C_1 must equal that across C_3, and can be written as

$$\pi a_1(x) = \int_{C_3} \frac{\partial\Phi_1}{\partial N}\,[(r_0 d\theta)^2 + (dr_0)^2]^{1/2} = U\int_{\theta_1(x)}^{\theta_2(x)} r_0\frac{\partial r_0}{\partial x}\,d\theta$$

from the boundary condition (22). Here θ_1, θ_2 are the waterline values of θ as shown in Fig. 2, which are roots of the transcendental equation

$$z_0 = -r_0(x, \theta)\cos\theta \tag{33}$$

But now the *immersed* cross-sectional area of the ship is

$$S(x) = \int_{\theta_1}^{\theta_2} \tfrac{1}{2}r_0{}^2(x, \theta)d\theta + \tfrac{1}{2}z_0{}^2\tan\theta_1 - \tfrac{1}{2}z_0{}^2\tan\theta_2$$

the last two terms of which comprise the area of the triangle bounded by $z = 0$, $\theta = \theta_1$, and $\theta = \theta_2$; note that θ_1 is in the third and θ_2 in the second quadrant. Hence

$$S'(x) = \int_{\theta_1}^{\theta_2} r_0\frac{\partial r_0}{\partial x}\,d\theta + \theta_2'(x)\,[\tfrac{1}{2}r_0{}^2 - \tfrac{1}{2}z_0{}^2\sec^2\theta_2] - \theta_1'(x)\,[\tfrac{1}{2}r_0{}^2 - \tfrac{1}{2}z_0{}^2\sec^2\theta_2]$$

$$= \int_{\theta_1}^{\theta_2} r_0\frac{\partial r_0}{\partial x}\,d\theta$$

by use of (33). Thus we have shown that

$$a_1(x) = \frac{U}{\pi} S'(x) \tag{34}$$

a most important result.

There are a number of ways of viewing the result (34). First it is clear that the solution for the first term ϕ_1 in the outer expansion is now complete and unique, for the only unknown quantity in ϕ_1 was the source strength $a_1(x)$. Hence first approximations are now available for any physical quantity which is determined from the behavior of the flow in the outer region; e.g., from the behavior of the wave pattern at infinity. The most important such quantity is of course the wave resistance, which may be calculated by methods of Havelock [7] from the energy left behind the ship, yielding a formula of the type

$$R_W = -\tfrac{1}{2}\rho\pi^2 \iint da(x)da(\xi)Y_0(\kappa|x-\xi|) \tag{35}$$

for a line distribution of sources of strength $-\frac{1}{4}a(x)$. Putting

$$a(x) = a_1(x) = \frac{U}{\pi} S'(x)$$

from (34) gives immediately the first approximation to the wave resistance of a slender ship, as obtained by Vossers [8] and others (see [9]). From the outer expansion it should also be possible to calculate such things as trim forces and the asymptotic form of the wave pattern.

However, there are many details of the flow for which ϕ_1 is not adequate even for a first approximation. As an example, suppose we have a ship which is not symmetric with respect to y (e.g. a ship at a small angle of yaw), and we wish to find the side forces or moments on the ship. Then ϕ_1 is unable to provide this since it is essentially symmetric with respect to y whatever are the symmetry properties of the ship—the only information fed into ϕ_1 from the ship form is the area curve $S(x)$. For quantities such as side forces we must look either to the inner expansion or else to further terms $\phi_2\ldots$ in the outer expansion; since it is first necessary to calculate Φ_1 before we can find ϕ_2 these two alternatives are really equivalent.

The result (34) also provides a definite outer boundary condition for the plane potential problem to find Φ_1, namely

$$\Phi_1 \rightarrow \frac{U}{\pi} S'(x) \log r + b_1(x) \qquad \text{as } r \rightarrow \infty \tag{36}$$

with $b_1(x)$ known. Although formally (36) contains redundant information since the solution for Φ_1 is rendered unique merely by specifying

$$\Phi_1 \rightarrow a_1(x) \log r + b_1(x)$$

even if $a_1(x)$ were not known, the fact that we now have a well-defined boundary condition at infinity renders the task of solving for Φ_1 less difficult. It should even be possible now to solve for Φ_1 numerically if the hull data were given in numerical or polynomial form.

Some Second-Order Considerations

One cannot expect to be able to say very much about the second approximation for a general hull cross section, since the boundary condition (31) indicates that the behavior of Φ_2 even in the intermediate region is intimately related to the details of the behavior of Φ_1. Thus a full description must await the solution of the plane potential problem to determine Φ_1 for a given shape of cross section. However, there are a few general statements we can make, some of which must be taken on trust at the present time.

First we shall require further terms in the intermediate expansion of ϕ_1 begun by equation (16). These were found in reference [10], and we have

$$\begin{aligned}\phi_1 = a_1(x) \log r &+ b_1(x) \\ &+ \kappa^{-1}a_1''(x)r \log r \cos\theta \\ &- \kappa^{-1}(a_1''(x) - b_1''(x))r\cos\theta \\ &- \kappa^{-1}a_1''(x)r\theta\sin\theta + 0(r^2\log r)\end{aligned} \tag{37}$$

Then the intermediate expansion of the outer expansion must begin

$$\begin{aligned}\phi = {} & [Ux] \\ &+ [a_1 \log r + b_1] \\ &+ [\kappa^{-1}a_1''r\log r\cos\theta \\ &- \kappa^{-1}(a_1'' - b_1'')r\cos\theta \\ &- \kappa a_1''r\theta\sin\theta + a_2\log r + b_2] + \ldots\end{aligned} \tag{38}$$

where $\frac{1}{4}a_2(x)$ is the strength of the source distribution associated with ϕ_2. Since the contents of the first two square brackets are already matched by the intermediate expansions of Φ_0 and Φ_1, the third group of terms must match Φ_2. Thus Φ_2 must behave at infinity like (i.e., have an intermediate limit of)

$$\begin{aligned}\Phi_2 \rightarrow \kappa^{-1}a_1''r\log r\cos\theta &- \kappa^{-1}(a_1'' - b_1'')r\cos\theta \\ &- \kappa^{-1}a_1''r\theta\sin\theta \\ &+ a_2\log r + b_2\end{aligned} \tag{39}$$

as $r \rightarrow \infty$.

Now if we repeat the conservation-of-mass argument of the preceding section for Φ_2, we find that the flux of Φ_2 across C_1 must equal

$$\int_{C_2+C_4} \frac{\partial\Phi_2}{\partial z}\,dy$$

with $\partial\Phi_2/\partial z$ prescribed by the boundary condition (31), since there is no flux of Φ_2 over the hull C_3 by (32). Let us specialize C_1 to be a semi-circle of radius R and suppose for the sake of definiteness that the ship is symmetrical with respect to y and has a waterline curve

$$y = \pm\eta(x)$$

Then conservation of mass requires

$$2\int_\eta^R \frac{\partial\Phi_2}{\partial z}\,dy = 2\kappa^{-1}a_1''R\log R + 2\kappa^{-1}b_1''R - 2\kappa^{-1}a_1''R + \pi a_2$$

We may now let the radius R of the semi-circle tend to infinity (within the intermediate limiting region), obtaining

$$a_2(x) = \frac{2}{\pi\kappa} \lim_{R\to\infty} \left[\kappa \int_{\eta(x)}^{R} \frac{\partial \Phi_2}{\partial z}\, dy - a_1''(x)R \log R + a_1''(x)R - b_1''(x)R \right] \quad (40)$$

This limit must exist since Φ_2 is required to behave at infinity like (39); note however that if the integral with respect to y is to produce a term to cancel $a_1''(x)R \log R$ at its upper limit, it will also produce a term of the form $-a_1''(x)\eta \log \eta$ at the lower limit. This suggests that $a_2(x)$ (and hence ϕ_2) is of order $\epsilon^3 \log \epsilon$ confirming that the asymptotic expansions used are not strictly power series in ϵ, and also that the splitting up of the dominant term of the outer expansion into ϕ_1 and ϕ_2 both satisfying (14) was correct.

Once Φ_1 has been determined for any given shape of cross section, the second approximation $a_2(x)$ to the source strength may be evaluated immediately from (40) and hence ϕ_2 is determined completely and uniquely. A second approximation to the wave resistance also follows directly by substituting $a(x) = a_1(x) + a_2(x)$ in the formula (35), and other information (including some first approximations not available from ϕ_1) should be accessible. Even more information can be obtained by then solving for Φ_2 using the boundary condition (39) at infinity; for the special case where the cross section is a circle centered in the free surface the complete solution to the problem has been carried this far [10].

References

1 F. Ursell, "Slender Oscillating Ships at Zero Forward Speed," *Journal of Fluid Mechanics*, vol. 14, 1962, pp. 496–516.

2 T. H. Havelock, "The Effect of Speed of Advance upon the Damping of Heave and Pitch," *Transactions of the Institution of Naval Architects*, vol. 100, 1958, pp. 131–135.

3 E. O. Tuck, to be published.

4 J. V. Wehausen and E. V. Laitone, "Surface Waves," vol. 9, *Handbuch der Physik*, Springer, Berlin, Germany, 1960.

5 S. Kaplun and P. A. Lagerstrom, "Asymptotic Expansions of Navier-Stokes Solutions for Small Reynolds Numbers," *Journal of Rational Mechanics and Analysis*, vol. 6, 1957, pp. 585–593.

6 A. Erdelyi, editor, *Higher Transcendental Functions*, vol. 2, McGraw-Hill Book Company, Inc., New York, N. Y., 1953.

7 T. H. Havelock, "The Calculation of Wave Resistance," *Proceedings of The Royal Society*, Series A, vol. 144, 1934, pp. 514–521.

8 G. Vossers, "Some Applications of the Slender Body Theory in Ship Hydrodynamics," PhD thesis, Delft, 1962.

9 *Proceedings of the International Seminar on Theoretical Wave Resistance*, Ann Arbor, Mich., 1963.

10 E. O. Tuck, "The Steady Motion of a Slender Ship," PhD thesis, Cambridge, 1963.

On Line Distributions of Kelvin Sources

By E. O. Tuck[1]

The velocity potential for the flow due to point sources distributed arbitrarily along a straight line near to or at a linearized gravitational free surface is obtained in a new form by use of Fourier transforms. Such a method of representing the potential facilitates the determination of its behavior near to the line of singularities; this behavior is derived formally and its physical properties discussed. A brief illustration is given of a method of using this result in a theory for the motion of a slender ship.

THE main purpose of this paper is to demonstrate that a slender-body theory for ships can be constructed in a manner highly analogous to the usual slender-body theory of aerodynamics [1,2].[2] One formulation of the latter [2] starts by assuming that the slender body can be represented by a suitable distribution of sources along a suitable line, and then shows that near this line the flow is approximately equivalent to that due to a two-dimensional source in planes normal to the line. We shall show that the same statement is true also for a ship in steady motion; Ursell [3] has obtained a similar conclusion for the case of a ship heaving and pitching at zero speed.

Once this approximation has been made it is possible to apply an approximate boundary condition on the hull, thus determining the density of the source distribution as a function of hull shape. Such quantities as wave resistance, the pressure distribution on the hull, trim forces and moments, and the wave pattern are now also determinate as a function of hull shape. However, since there are many other questions to be answered in a satisfactory treatment of the boundary-value problem for a ship, we give at this stage only a brief illustration of the use of our results for a simple type of hull form. A complete treatment of slender ships analogous to the work of Peters and Stoker [4] or of Newman [5] on thin ships is in preparation. Notice the distinction between "thinness" and "slenderness"; thin ships are approximated by a plane surface while slender ships are approximated by a line. A similar distinction is made in aerodynamics between thin aerofoil theory and slender-body theory.

Even without the hull boundary condition in detail, one can consider distributed sources as representing a ship at least qualitatively, provided the boundary condition at the free surface can be linearized. This is, for the most part, the approach of Havelock (in a long series of papers; for a bibliography see [6]), though he does approximate the hull boundary condition in the particular case of bodies submerged to a depth large compared with their own dimensions. Thus the problem to be studied is also of interest in its own right, since a qualitative and quantitative understanding of the flow due to idealized models of ships is a useful guide to the behavior of real ships.

In mathematical terms, then, we investigate the steady irrotational flow of an inviscid incompressible fluid past a fixed distribution of sources situated on a line at or near a gravitational free surface, on which the velocity potential satisfies the usual linearized free-surface condition. In the particular case in which the distribution constitutes a dipole at the surface, the flow is proportional to that produced by an isolated pressure point, as studied by Kelvin [7], and more recently by Peters [8] and Ursell [9]. Hence the term "Kelvin source" is used to indicate an ordinary source in the presence of a linearized free surface.

In view of the known irregular behavior near the source of the potential due to a Kelvin source, it is at first sight surprising that any analogy with aerodynamic slender-body theory is possible. For instance, Ursell [9] has found that the wave height near the track of a pressure point is highly oscillatory, with ever-increasing amplitude and ever decreasing wavelength. However,

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[2] Numbers in brackets designate References at end of paper.

Nomenclature

$a(x)$ = arbitrary function describing source strength
$b(x)$ = function calculated from $a(x)$ by equation (22)
g = acceleration of gravity
$\mathbf{H}_0$ = Struve function of zero order
k = x-wise Fourier transform variable
K_0 = modified Bessel function of third kind, zero order
L = representative length scale
r, θ = polar coordinates
r', θ' = image polar coordinates
$S(x)$ = immersed cross-sectional area of slender ship
U = velocity of uniform stream (or speed of ship)
R = wave resistance
x, y, z = Cartesian coordinates
Y_0 = Bessel function of second kind, zero order
κ = g/U^2
λ = y-wise Fourier transform variable
ρ = water density
ϕ = disturbance velocity potential

it appears that distributing Kelvin sources over a line smooths out this irregular behavior, provided that the density function describing the distribution is itself sufficiently smooth. Indeed it is clear that in regions of very small wavelength the smoothing process would be most efficient, due to interference effects (or in other words, by the principle of stationary phase).

Statement of Problem

We shall use Cartesian coordinates x, y, z, with origin in the free surface, z vertically downwards, and x in the direction of the free stream at infinity (or alternatively, the sources may be considered to move in the negative x-direction in a fluid otherwise at rest). There is a distribution of point sources along the line $y = 0$, $z = z_0 \geq 0$, of source strength $-\frac{1}{4} a(x)$ per unit length. The fluid fills the half-space $z \geq 0$, in which it possesses a velocity potential $Ux + \phi$ satisfying Laplace's equation (except at the source-points).

The disturbance potential ϕ satisfies a linearized free surface boundary condition on the plane $z = 0$, in the form:

$$\kappa \frac{\partial \phi}{\partial z} - \frac{\partial^2 \phi}{\partial x^2} = 0 \qquad \text{on } z = 0 \tag{1}$$

where $\kappa = g/U^2$, g being the acceleration due to gravity and U the free-stream velocity.[3] The parameter κ has dimensions (length)$^{-1}$, and if L is a representative length (e.g., the linear extent of the sources or a typical value of $|a(x)|/|a'(x)|$), then $(\kappa L)^{-1}$ is the square of the Froude number for the problem. κL is assumed finite, of $0(1)$, throughout; it is important to note that all approximate work needs to be carefully reexamined if κL is either small (high Froude number) or large (low Froude number), since under these circumstances there is no *a priori* reason to believe that the linearization process is justified.

Although we usually have in mind situations where the source distribution is only of limited extent, we shall nevertheless take it to extend formally from $x = -\infty$ to $x = +\infty$; it only requires specifying $a(x)$ to vanish identically outside a finite range of x to include the case of limited extent. The factor "$\frac{1}{4}$" is merely for convenience in later formulas; the point is that $a(x)$ or equally $-\frac{1}{4}a(x)$ is quite arbitrary throughout the bulk of this paper.

It would appear that, with the addition of the far-field boundary condition:

$$\phi \text{ and its derivatives} \rightarrow 0 \text{ at infinity}$$

we have a well-formulated potential problem. However, it is well known that solutions of Laplace's equation with boundary conditions like (1) are not unique; the usual uniqueness proofs do not work for "mixed" boundary conditions. Physically, one expects waves behind a disturbance to a free surface, but not in front of it, whereas there is nothing in the problem so far to suggest that symmetrical source distributions would not produce symmetrical flow patterns. It is necessary to add a "radiation condition" at infinity, which is essentially a requirement that the waves lie behind the disturbance. The mathematical formulation of this boundary condition will be described when it is required in the analysis.

Determination of the Velocity Potential

The velocity potential may be obtained by extending to distributed sources the work of Peters [8] on isolated pressure points, or by specializing to line distributions the work of Havelock, e.g., [12], on general source distributions in space. However the results in both these cases are not in a form suitable for our purpose, and as the labor involved in manipulating the Havelock or Peters' formulas is comparable with that in obtaining ϕ from scratch, we give a new formal solution for ϕ by means of Fourier transforms.

Now the potential of an isolated unit source satisfies

$$\nabla^2 \phi = -4\pi\delta(\mathbf{r})$$

where $\mathbf{r}$ is the vector from an arbitrary point to the source point, and δ is the Dirac function. Hence the relevant "Laplace" equation in our case is

$$\nabla^2 \phi = \pi a(x) \delta(y) \delta(z - z_0) \tag{2}$$

Let us Fourier transform the potential with respect to both x and y, putting

$$\phi^{**}(k, \lambda; z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{ikx + i\lambda y} \phi(x, y, z) \tag{3}$$

Then equation (2) becomes

$$\frac{\partial^2 \phi^{**}}{\partial z^2} - (k^2 + \lambda^2)\phi^{**} = \tfrac{1}{2} a^*(k) \delta(z - z_0) \tag{4}$$

where

$$a^*(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ikx} a(x) \tag{5}$$

is the Fourier transform of $a(x)$, while the free-surface condition (1) transforms to

$$\kappa \frac{\partial \phi^{**}}{\partial z} + k^2 \phi^{**} = 0 \qquad \text{at } z = 0 \tag{6}$$

Equation (4) is a simple second-order ordinary differential equation in z with impulsive forcing at the point $z = z_0$. Its solution subject to the boundary condition (6) together with $\phi^{**} \rightarrow 0$ as $z \rightarrow +\infty$, is best obtained by Laplace transforming with respect to z. In any case, the relevant solution is found to be

[3] This linearized equation is derived naturally [10] as the first approximation in the course of a thoroughgoing expansion procedure for the slender-ship theory. However, it is also the usual steady condition [11] resulting from the *additional* assumption that the waves are small in a certain sense, a fact which may be considered physically plausible.

$$\phi^{**} = -\frac{a^*(k)}{4(k^2+\lambda^2)^{1/2}}\left\{e^{-|z-z_0|(k^2+\lambda^2)^{1/2}} + e^{-(z-z_0)(k^2+\lambda^2)^{1/2}} + \frac{2k^2}{\kappa(k^2+\lambda^2)^{1/2}-k^2}\, e^{-(z+z_0)(k^2+\lambda^2)^{1/2}}\right\}$$

Now let us take the inverse transform with respect to λ, recovering the y-dependence. Thus

$$\begin{aligned}\phi^*(k;y,z) = \\ &-\tfrac{1}{4}\, a^*(k) \int_{-\infty}^{\infty} \frac{d\lambda}{(k^2+\lambda^2)^{1/2}}\, e^{-i\lambda y-|z-z_0|(k^2+\lambda^2)^{1/2}} \\ &-\tfrac{1}{4}\, a^*(k) \int_{-\infty}^{\infty} \frac{d\lambda}{(k^2+\lambda^2)^{1/2}}\, e^{-i\lambda y-(z+z_0)(k^2+\lambda^2)^{1/2}} \\ &+\tfrac{1}{2}\, a^*(k)k^2 \int_{-\infty}^{\infty} \frac{d\lambda}{(k^2+\lambda^2)^{1/2}}\, \frac{e^{-i\lambda y-(z+z_0)(k^2+\lambda^2)^{1/2}}}{\kappa(k^2+\lambda^2)^{1/2}-k^2} \end{aligned} \tag{7}$$

Note that in the particular case $z_0 = 0$ these three terms can be combined to give

$$\phi^* = -\tfrac{1}{2}\,\kappa\, a^*(k) \int_{-\infty}^{\infty} \frac{d\lambda}{\kappa(k^2+\lambda^2)^{1/2}-k^2}\, e^{-i\lambda y-z(k^2+\lambda^2)^{1/2}} \tag{8}$$

If $z_0 \neq 0$, let us define two sets of polar coordinates, (r, θ) and (r', θ'), such that

$$\begin{aligned} r\cos\theta &= z - z_0 \\ r'\cos\theta' &= z + z_0 \end{aligned} \qquad r'\sin\theta' = r\sin\theta = y$$

i.e., r and r' are distances in planes normal to the x-axis from the line of sources and its image in the plane $z = 0$, respectively.

Now the first two integrals in equation (7) can be shown to be representations of Bessel K-functions,[4] so that

$$\begin{aligned}\phi^* = &-\tfrac{1}{2}\, a^*(k)K_0(|k|r) - \tfrac{1}{2}a^*(k)K_0(|k|r') \\ &-\tfrac{1}{2}\, a^*(k)\,\frac{|k|}{\kappa} \int_{-\infty}^{\infty} d\gamma\, \frac{e^{-|k|r'\cosh(\gamma - i|\theta'|)}}{\cosh\gamma - |k|/\kappa}\end{aligned} \tag{9}$$

where we have made the substitution $\lambda = |k| \sinh \gamma$.

Notice that, by the Fourier convolution theorem and the known Fourier transform of K_0-functions, the contribution to ϕ from the first two terms of (9) is

$$\phi_1 = \int_{-\infty}^{\infty} d\xi\, \left(-\tfrac{1}{4}\, a(x-\xi)\right) \left[\frac{1}{(\xi^2+r^2)^{1/2}} + \frac{1}{(\xi^2+r'^2)^{1/2}}\right] \tag{10}$$

i.e., a distribution $-\frac{1}{4}\, a(x)$ of simple sources in an unbounded fluid along the line $r = 0$ and also along its image line $r' = 0$. But this should be expected, since as $\kappa \to \infty$ these terms clearly dominate equation (9), while the free-surface condition (1) when $\kappa \to \infty$ becomes the "wall" condition

$$\frac{\partial\phi_1}{\partial z} = 0 \qquad \text{on} \qquad z = 0$$

and the potential satisfying this condition is precisely the "source + image source" potential ϕ_1.

But it is the third term of (9) which contains the true free-surface character of the problem, and this term does not have such a simple description. The integral involved needs to be interpreted correctly, for if $|k| \geq \kappa$ there are poles on the line of integration; a Cauchy principal-value interpretation would not be sufficient since it can be shown that the main wave pattern is due to the residue at these poles. The correct interpretation is obtained by replacing $|k|$ by $|k| + ip \operatorname{sgn} k$ and later letting $p \to 0$ through positive real values. This method of supplying the radiation condition is equivalent to the artificial friction device of Rayleigh [11], although the interpretation of p as a "coefficient of friction" is misleading; Stoker [14] has supplied the correct justification of this trick (p is essentially a Laplace transform variable, and $p \downarrow 0$ corresponds to $t \to +\infty$ in an unsteady problem).

Finally, the solution for the velocity potential can be obtained by again taking the Fourier transform of equation (7), in the form

$$\phi = \phi_1 - \frac{1}{2}\int_{-\infty}^{\infty} dk e^{-ikx} a^*(k) \cdot \frac{|k|}{\kappa} \int_{-\infty}^{\infty} d\gamma\, \frac{e^{-|k|r'\cosh(\gamma - i|\theta'|)}}{\cosh\gamma - |k|/\kappa} \tag{11}$$

Flow in Far Field and Wave Resistance

Expressed as it is in terms of a Fourier integral in x, equation (11) has certain advantages and disadvantages as compared to previous methods of writing down this type of potential. The advantages are that the effect of a distribution of sources rather than an isolated source or pressure point is clearly shown through the functions $a^*(k)$ or $a(x)$, and that, as will be shown in the next section, the behavior near the disturbance is almost trivially determined from this expression.

However, if information is required about flow in the far field, the result is better expressed in forms such as that of Peters [8] where the potential is split into two terms, one of which vanishes for points in front of the disturbance and dominates for points far to the rear of the disturbance. But we can show immediately that our result is equivalent to Peters' in the particular case of a dipole (i.e., $a^*(k) \sim k$) at the surface (i.e., $z_0 = 0$). For, making the change of variable

$$t = (k^2+\lambda^2)^{1/2}, \quad tdt = kdk$$

in the Fourier transform of equation (8), with $a^*(k) = k$, we have

$$\begin{aligned}\phi &= \tfrac{1}{2}\kappa \int_{-\infty}^{\infty} d\lambda e^{-i\lambda y} \int_{-\infty}^{\infty} \frac{kdke^{-ikx-z(k^2+\lambda^2)^{1/2}}}{k^2-\kappa(k^2+\lambda^2)^{1/2}} \\ &= 2\kappa \operatorname{Im} \int_0^{\infty} d\lambda \cos\lambda y \int_{\lambda}^{\infty} \frac{tdte^{-ix(t^2-\lambda^2)^{1/2}-zt}}{t^2-\kappa t-\lambda^2},\end{aligned}$$

[4] Reference [13], p. 83.

which, with a suitable distortion of the path of t-integration to avoid the poles of the integrand is proportional to the potential given by Peters' equation (3.8).

Now by writing down the convolution integral giving the generalization of the dominant term (Peters' "φ^{**}") to distributed sources, one could use this form of the potential to calculate the wave pattern at a great distance behind the disturbance. This would seem a formidable undertaking, since, for instance, Peters', and Ursell's expressions for asymptotic wave height are complicated enough even for the case of an isolated disturbance. Since we are also more concerned in this paper with behavior *close* to the disturbance, we shall not pursue this interesting problem further.

However, the behavior in the far field also serves to determine the wave resistance, which would of course be infinite for isolated point disturbances. To every distribution function $a(x)$ there corresponds a certain wave resistance, associated with the rate at which energy is radiated to infinity in the waves. For a ship, this provides the (inviscid) drag opposing its motion, and in a complete theory of ship motion the latter can be calculated by a pressure integration over the hull, which may be a most difficult piece of analysis. But if a unique relationship can be established between hull form and source distribution, the wave resistance is determined immediately.

The wave resistance may be found from the behavior of the potential in the far field using methods of Havelock; e.g., as in [15]. The result can be expressed in several familiar ways. Provided $a(x)$ is continuous and piecewise differentiable, so that the Fourier transform of $a'(x)$ exists and is equal to $-ika^*(k)$

$$R = \pi\rho \int_0^\infty d\gamma e^{-2\kappa z_0 \cosh^2\gamma}[P^2(\kappa\cosh\gamma) + Q^2(\kappa\cosh\gamma)] \tag{12}$$

where

$$P(k) + iQ(k) = \int_{-\infty}^{\infty} dx e^{-ikx} a'(x) \tag{13}$$

In the particular case $z_0 = 0$, or as an approximation when z_0 is small (as it is for a ship near or at the surface), we have on interchanging the orders of x and γ integrations that

$$R = -\tfrac{1}{2}\pi^2\rho \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx d\xi a'(x)a'(\xi)Y_0(\kappa|x-\xi|) \tag{14}$$

which is a quadratic functional of $a'(x)$ with a Bessel Y_0-function,[5] as kernel; this is the most useful form for analytic treatments of the resistance. Finally, by making the substitutions

$$x - \xi = 2v \qquad x + \xi = 2u$$

we obtain the "hull function" form

$$R = -2\pi^2\rho \int_0^\infty dv Y_0(2\kappa v) \int_{-\infty}^{\infty} du\ a'(u+v)a'(u-v) \tag{15}$$

[5] Ibid., p. 82.

For numerical work either (15) or (12) (with $z_0 = 0$) would probably be preferred to (14) since one of the two integrations is performed once and for all independently of κ, for any given $a(x)$.

The First Slender-Body Approximation

We want to look at the potential as given by equation (11) for points close to the line on which the disturbance takes place. For use in slender-ship problems it happens that we are also only interested in disturbances at or *near* the surface, which means that the depth z_0 of the line of disturbance must be chosen as zero or small. We could investigate the case when z_0 is finite, but this implies for ship problems a body submerged to a depth large compared with its breadth, and this type of problem has been well treated by Havelock.

Thus the approximation that is of interest here is one in which r and r' *both* become small. But now the behavior of ϕ in this region is easily obtained from equation (11), since the integral expression for $\phi - \phi_1$ is bounded as $r' \to 0$. Thus without more ado, the slender-body approximation to $\phi - \phi_1$ is given by the expression

$$\phi - \phi_1 = -\frac{1}{2}\int_{-\infty}^{\infty} dk e^{-ikx} a^*(k) \frac{|k|}{\kappa}\int_{-\infty}^{\infty} \frac{d\gamma}{\cosh\gamma - |k|/\kappa} + o(1) \tag{16}$$

where o(1) implies that the error tends to zero as $r' \to 0$.

The expression (16) is a function of x (and the parameter κ) alone, which we now reduce to an explicit single integral involving Bessel-type functions. Now the γ-integration can be performed immediately since the integrand is a rational function of $\cosh\gamma$, and we have by elementary methods that

$$\tfrac{1}{2}\cosh\beta \int_{-\infty}^{\infty} \frac{d\gamma}{\cosh\gamma + \cosh\beta} = \beta\coth\beta$$

provided that $\mathrm{Re}\beta \geq 0$, $|\mathrm{Im}\beta| < \pi$. In the present case

$$\cosh\beta = -\left(\frac{|k|}{\kappa} + ip\frac{\operatorname{sgn} k}{\kappa}\right)$$

for a small positive number p which we immediately let tend to zero. Thus if $\beta = \beta_1 + i\beta_2$, then

$$\cosh\beta_1\cos\beta_2 = -|k|/\kappa$$

$$\sinh\beta_1\sin\beta_2 = -\frac{p}{\kappa}\operatorname{sgn} k = \text{small}$$

so that as $p \downarrow 0$, either $\beta_1 \to 0$ or $\beta_2 \to 0, \pm\pi$. Of these possibilities, only the following combinations are seen to be permissible:

$$\begin{aligned}
&\text{(i)} && \beta = i(\pi - \operatorname{arc\,cos}|k|/\kappa) && \text{if } |k| \leqslant \kappa \\
&\text{(ii)} && \beta = -i\pi + \operatorname{ar\,cosh}|k|/\kappa && \text{if } k \geqslant \kappa \\
&\text{(iii)} && \beta = +i\pi + \operatorname{ar\,cosh}|k|/\kappa && \text{if } k \leqslant -\kappa
\end{aligned} \tag{17}$$

Thus (16) becomes

$$\phi - \phi_1 = \int_{-\infty}^{\infty} dk e^{-ikx}\, a^*(k)\, \beta \coth \beta + o(1)$$

where $\beta = \beta(k)$ takes the values (17) in different ranges of k. Now by use of the Fourier transform convolution theorem

$$\phi - \phi_1 = -\int_{-\infty}^{\infty} d\xi a'(\xi) g(x - \xi) + o(1) \quad (18)$$

where $g(x)$ has the Fourier transform

$$\left(\frac{\beta \coth \beta}{2\pi i k}\right)$$

(since $a'(x)$ has the Fourier transform $-ika^*(k)$). Explicitly

$$\begin{aligned} g(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-ikx}\left(\frac{\beta \coth \beta}{ik}\right) \\ &= \frac{1}{\pi}\int_0^{\kappa} dk \frac{\sin kx}{k}\left(\pi - \operatorname{arc\,cos}\frac{k}{\kappa}\right)\cot\left(\operatorname{arc\,cos}\frac{k}{\kappa}\right) \\ &\quad + \frac{1}{\pi}\int_{\kappa}^{\infty} dk \frac{\sin kx}{k} \operatorname{ar\,cosh}\frac{k}{\kappa}\coth\left(\operatorname{ar\,cosh}\frac{k}{\kappa}\right) \\ &\qquad - \int_{\kappa}^{\infty} dk \frac{\cos kx}{k}\coth\left(\operatorname{ar\,cosh}\frac{k}{\kappa}\right) \\ &= \int_0^{\pi/2} d\alpha \sin(\kappa x \cos\alpha) \\ &\quad - \frac{1}{\pi}\int_0^{\pi/2} d\alpha\, \alpha \sin(\kappa x\cos\alpha) \\ &\quad - \frac{1}{\pi}\int_0^{\infty} d\beta\, \beta \sin(\kappa x \cosh\beta) \\ &\qquad - \int_0^{\infty} d\beta \cos(\kappa x\cosh\beta) \quad (19) \end{aligned}$$

The first and last of the integrals in (19) are recognizable as representations of Struve and Bessel Y_0-functions, respectively,[6] and the second and third integrals are shown in the Appendix to be also combinations of these functions, so that we have finally from (18) and (33) that

$$\begin{aligned} \phi - \phi_1 = -\frac{\pi}{4}\int_{-\infty}^{\infty} d\xi a'(\xi)[\mathbf{H}_0(\kappa(x-\xi)) \\ + (2 + \operatorname{sgn}(x-\xi))Y_0(\kappa|x-\xi|)] + o(1) \quad (20) \end{aligned}$$

where $\mathbf{H}_0$ is a Struve function of zero order (an *odd* function of its argument). Notice that since this expression is a function of x and κ alone and the function ϕ_1 does not depend on κ, the free-surface character of the problem affects the slender-body approximation only through this term.

Of course, ϕ_1 itself depends on r and r', but its behavior when these are small is well known from similar work for slender-body theory in an unbounded fluid.[7] We have

$$\begin{aligned} \phi_1 = \tfrac{1}{2}a(x)\log r + \tfrac{1}{2}a(x)\log r' \\ - \frac{1}{2}\int_{-\infty}^{\infty} d\xi a'(\xi)\operatorname{sgn}(x-\xi)\log 2|x-\xi| + o(1) \end{aligned} \quad (21)$$

where by writing the error as o(1) we now imply that it tends to zero as both r and r' tend to zero. This result can be derived easily via equation (9), using known expansions for the Bessel K_0-functions.

Thus finally, adding the "free-surface" contribution (18) to the "fixed-surface" contribution (21), we have

$$\phi = \tfrac{1}{2}a(x)\log r + \tfrac{1}{2}a(x)\log r' + b(x) + o(1) \quad (22)$$

where

$$\begin{aligned} b(x) = -\frac{1}{2}\int_{-\infty}^{\infty} d\xi a'(\xi)\Big[\operatorname{sgn}(x-\xi)\log 2|x-\xi| \\ + \frac{\pi}{2}\mathbf{H}_0(\kappa(x-\xi)) \\ + (2 + \operatorname{sgn}(x-\xi))\frac{\pi}{2}Y_0(\kappa|x-\xi|)\Big] \quad (23) \end{aligned}$$

While this result as presented is asymptotic only, in the sense that the error vanishes as r, r' vanish, it can be shown [10] to be a true approximation to the exact potential, such that the error is uniformly bounded by a function of r, r' alone that is small when r, r' are small, provided:

(i) $a'(x)$ exists, is continuous and piecewise differentiable over the whole real line.

(ii) $a(x)$ is absolutely integrable over the whole real line.

For the case when the distribution is only over a finite portion of the axis, condition (i) specifies how smoothly it should fall to zero at its ends, while for the case in which the distribution extends over the whole axis, condition (ii) shows that $a(x)$ must converge rapidly to zero as $x \to \pm\infty$. These are sufficient, but probably not necessary conditions for the validity of equation (22) as an approximate, rather than a limiting, result; in particular, discontinuities in $a'(x)$ could probably be tolerated.

The form of equation (22) is that of a two-dimensional potential function in cross-sectional planes of constant x, with equal two-dimensional sources at $z = z_0$ and its image $z = -z_0$, the strength of which are dependent only on the local value of the three-dimensional source strength $a(x)$. To this two-dimensional potential is added an "arbitrary constant" $b(x)$ (i.e., a constant in planes of constant x, which is necessarily a two-dimensional potential), which provides the coupling between one cross section and another. This behavior is familiar in slender-body theory for an unbounded medium; it is essentially neglected in some "strip" theories of ship motion.

The expression (23) for $b(x)$ has an interesting physical description. Let us separate the regions of integration $\xi < x$ and $\xi > x$, giving

[6] Ibid., pp. 37 and 82.
[7] E.g., [1], p. 387.

$$
\begin{aligned}
b(x) = &-\frac{1}{2}\int_{-\infty}^{x} d\xi a'(\xi) \\
&\left[\log 2(x-\xi) + \frac{\pi}{2}\,\mathbf{H}_0(\kappa(x-\xi)) + \frac{3\pi}{2}\,Y_0(\kappa(x-\xi))\right] \\
&+\frac{1}{2}\int_{x}^{\infty} d\xi a'(\xi) \qquad\qquad (24) \\
&\left[\log 2(\xi-x) + \frac{\pi}{2}\,\mathbf{H}_0(\kappa(\xi-x)) - \frac{\pi}{2}\,Y_0(\kappa(\xi-x))\right]
\end{aligned}
$$

Now the combination, $\mathbf{H}_0 - Y_0$, occurring in the second of these integrals is well known to be a monotone decreasing function of its (positive) argument, namely[8]

$$
\begin{aligned}
\frac{\pi}{2}\,\mathbf{H}_0(u) - \frac{\pi}{2}\,Y_0(u) &= \int_0^{\infty} dt\,\frac{e^{-ut}}{(1+t^2)^{1/2}} \\
&= u^{-1} + 0(u^{-3}) \quad \text{as } u \to +\infty
\end{aligned}
$$

Hence the second integral in (24) does not contribute to the wave-like character of the problem, which is what we should expect, since it is a contribution from disturbance points ξ behind the point x of observation. Further, the kernel of the second integral is bounded, since the "log" term precisely cancels the logarithmic behavior of the Y_0-function as $\xi \to x+$.

On the other hand, the kernel of the first integral contains a different combination of log, $\mathbf{H}_0$, and Y_0, retaining all the oscillatory nature of the Bessel functions, and behaving logarithmically as $\xi \to x-$. Thus this term, which corresponds to disturbances *ahead* of the observer, contributes most to both the wave-like and the source-like behavior of the flow near the disturbance. By using the known asymptotic behavior of the $\mathbf{H}_0$ and Y_0 functions, it can be shown [10] that for large positive x, i.e., at a large distance to the rear of the bulk of the disturbance, the function $b(x)$ can be approximated in the form

$$
b(x) = -\left(\frac{2\pi}{\kappa x}\right)^{1/2} \Lambda \sin\left(\kappa x - \frac{\pi}{4} - \chi\right) + 0(x^{-1})
$$

where

$$
\Lambda e^{i\chi} = P(\kappa) + iQ(\kappa)
$$

as defined by equation (13). This represents a wave motion in x, with wave number κ, with amplitude decreasing with distance as the inverse square root, and with initial amplitude and phase determined by the shape of the function $a(x)$. This is a description of the "transverse" waves of Kelvin's ship-wave theory; clearly the "diverging" waves would not be obtained in regions close to the track of the disturbance.

Slender Ships

Let us illustrate the way in which the results of the last section may be used in a comprehensive study of the motion of a slender ship, by the example of a body with circular cross section situated at the free surface. Of course the validity of the linearized free surface boundary condition (1) needs to be justified within the context of such a theory, but we shall gloss over this point since (1) is the sort of boundary condition one expects in this type of problem.

Now if the body is defined by the equation $r = r_0(x)$, where r_0 is small signifying slenderness, and is situated with its centerline in the free surface, so that we may take $z_0 = 0$ in the previous results, the boundary condition of zero normal velocity on the ship's hull can be approximated in the form

$$
\frac{\partial\phi}{\partial r} = U r_0'(x) \qquad \text{on } r = r_0(x)
$$

i.e., the component of the free-stream velocity normal to the hull is largely balanced by the radial component of the disturbance velocity; the approximation consists of omitting the effect of the longitudinal component of the disturbance velocity.

But now if we assume that the ship is equivalent to some line distribution of sources at the surface, we have for small r

$$
\phi = a(x) \log r + b(x) \qquad (25)
$$

approximately, by putting $r' = r$ in equation (22). Hence

$$
\frac{a(x)}{r_0(x)} = U r_0'(x)
$$

or

$$
a(x) = \frac{U}{\pi} S'(x) \qquad (26)
$$

where $S(x) = \frac{1}{2}\pi[r_0(x)]^2$ is the *immersed* cross-sectional area. In the last form, it can be shown by analysis similar to that given by Ward [16] that the result also applies for noncircular ships, whether at the surface, or partly or wholly submerged (see also [17]).

The foregoing formula for $a(x)$ may now be used to give the wave resistance via equations (12), (14) or (15). Alternatively, the pressure on the hull may be calculated in the form[9]

$$
\frac{p}{\rho} = gz - Ua' \log r_0 - \tfrac{1}{2}U^2(r_0')^2 - Ub' \qquad (27)
$$

and on multiplying by the surface area element in the x-direction, namely $S'(x)dx = (\pi/U)a(x)\,dx$, and integrating, all terms except that involving $b'(x)$ give zero contribution, so that (after an integration by parts)

$$
R = \pi\rho \int_{-\infty}^{\infty} dx\, a'(x) b(x) \qquad (28)
$$

[8] Reference [13], pp. 38 and 39.

[9] It may seem surprising that the hydrodynamic part of this pressure is independent of θ; that is, the problem is effectively axisymmetric to the present approximation in spite of the presence of a free surface. However, a second approximation has been obtained [10] which is strongly dependent on θ even for a circular ship, including the multiple-valuedness characteristic of all water-wave problems (e.g. [18]).

This further simplifies to give (14) since only the part,

$$-\frac{\pi}{2} Y_0(\kappa|x - \xi|)$$

of the kernel for $b(x)$ as given by (23) which is *even* in its argument contributes to the foregoing integral. Thus we have obtained the same wave resistance by two methods which are entirely different both in their physical basis and their mathematical detail.

Notice that (26) applies even if the "ship" extends all the way to infinity, provided its area curve converges sufficiently rapidly to a constant (small) value, so that $a(x)$ can be integrable. Thus one can with little extra difficulty consider the effect of adding a displacement thickness boundary layer to the ship, or even a deadwater wake extending to infinity if required; it is only necessary to compute the area derivative for this bigger ship and substitute the corrected value of $a(x)$ in the wave-resistance integral. This process, suggested by Havelock [19], may provide a first step in allowing for frictional effects.

On the other hand, if it is assumed that $S(x)$ vanishes outside a finite interval, then equations (14) and (26) together give a wave-resistance formula for a slender ship of finite length. When nondimensionalized in some convenient manner (and interpreted in the Stieltjes sense if $S'(x)$ does not vanish at the ends of the ship) this formula is equivalent to the result first obtained for a slender ship by Vossers [20] and since derived also by Maruo [17] and others (e.g. [10], [21]); the formula is also derivable as an approximate version of the well-known Michell integral [22] for thin ships of small draft.

References

1 B. Thwaites, editor, *Incompressible Aerodynamics*, Oxford University Press, Oxford, England, 1960.

2 S. Goldstein, *Lectures on Fluid Mechanics*, Interscience Publishers, New York, N. Y., 1960.

3 F. Ursell, "Slender Oscillating Ships at Zero Forward Speed," *Journal of Fluid Mechanics*, vol. 14, 1962, p. 496.

4 A. S. Peters and J. J. Stoker, "The Motion of a Ship, as a Floating Rigid Body, in a Seaway," *Communications on Pure and Applied Mathematics*, vol. 10, 1957, p. 399.

5 J. N. Newman, "A Linearized Theory for the Motion of a Thin Ship in Regular Waves," Journal of Ship Research, vol. 5, 1961, p. 34.

6 T. H. Havelock, "The Effect of Speed of Advance upon the Damping of Heave and Pitch," *Transactions*, Institution of Naval Architects, vol. 100, 1958, p. 131.

7 Lord Kelvin, "On Ship Waves," *Proceedings*, The Institution of Mechanical Engineers, 1887.

8 A. S. Peters, "A New Treatment of the Ship Wave Problem," *Communications on Pure and Applied Mathematics*, vol. 2, p. 123.

9 F. Ursell, "On Kelvin's Ship Wave Pattern," *Journal of Fluid Mechanics*, vol. 8, 1960, p. 418.

10 E. O. Tuck, "The Steady Motion of a Slender Ship," PhD thesis, Cambridge, 1963.

11 H. Lamb, *Hydrodynamics*, Cambridge University Press, Cambridge, England, 1932, sixth edition.

12 T. H. Havelock, "The Theory of Wave Resistance," *Proceedings* of *The Royal Society*, series A, vol. 138, 1932, p. 339.

13 A. Erdelyi, editor, *Higher Transcendental Functions*, vol. 2, McGraw-Hill Book Company, Inc., New York, N. Y., 1953.

14 J. J. Stoker, "On Radiation Conditions," *Communications on Pure and Applied Mathematics*, vol. 9, 1956, p. 577.

15 T. H. Havelock, "The Calculation of Wave Resistance," *Proceedings of The Royal Society*, series A, vol. 144, 1934, p. 514.

16 G. N. Ward, "Supersonic Flow Past Slender Pointed Bodies," *Quarterly Journal of Mechanics and Applied Mathematics*, vol. 2, 1949, p. 75.

17 H. Maruo, "Calculation of the Wave Resistance of Ships the Draught of Which is as Small as the Beam," *Journal of the Zosen Kiokai of Japan*, vol. 112, 1962, p. 161.

18 F. Ursell, "On the Heaving Motion of a Circular Cylinder on the Surface of a Fluid," *Quarterly Journal of Mechanics and Applied Mathematics*, vol. 2, 1949, p. 218.

19 T. H. Havelock, "Calculations Illustrating the Effect of Boundary Layer on Wave Resistance," *Transactions*, Institution of Naval Architects, vol. 90, 1948, p. 259.

20 G. Vossers, "Some Applications of the Slender Body Theory in Ship Hydrodynamics," PhD thesis, Delft, 1962.

21 W. P. A. Joosen, "The Velocity Potential and Wave Resistance Arising from the Motion of a Slender Ship," *Proceedings of the International Seminar on Theoretical Wave Resistance*, Ann Arbor, Mich., 1963.

22 J. V. Wehausen, "Wave Resistance of Thin Ships," *Symposium on Naval Hydrodynamics*, Washington, D. C., 1957.

Appendix

In this appendix we reduce the function $g(x)$ defined by equation (19) to a combination of Struve and Bessel Y_0-functions. First let us observe that for $x > 0$

$$g(x) + g(-x) = -2\int_0^\infty d\beta \cos\,(\kappa x \cosh \beta)$$

$$= \pi Y_0(\kappa x) \tag{29}$$

since all other terms are odd in x. Hence we need only consider positive values of x.

Now for $X > 0$, let us evaluate

$$\int ze^{iX\cosh z}\, dz$$

where C is a rectangle with vertices at $0, +\infty, +\infty + i\pi, i\pi$. Since the integrand is regular inside C, the value of the foregoing integral is zero, as is also the contribution from the side of the rectangle joining $+\infty$ and $+\infty + i\pi$ since the integrand is exponentially small there. Thus

$$\int_0^{\infty} = \int_{i\pi}^{+\infty+i\pi} + \int_0^{i\pi}$$

or

$$\int_0^{\infty} d\beta\beta e^{iX\cosh\beta} = \int_0^{\infty} d\beta(i\pi + \beta)e^{-iX\cosh\beta} - \int_0^{\pi} d\alpha\alpha e^{iX\cos\alpha} \quad (30)$$

On taking the imaginary part of equation (30), we have

$$\int_0^{\infty} d\beta\beta \sin(X\cosh\beta) = \pi\int_0^{\infty} d\beta \cos(X\cosh\beta) - \int_0^{\infty} d\beta\beta\sin(X\cosh\beta) - \int_0^{\pi} d\alpha\alpha\cos(X\cos\alpha)$$

or

$$2\int_0^{\infty} d\beta\beta\sin(X\cosh\beta) + 2\int_0^{\pi/2} d\alpha\, \alpha\sin(X\cos\alpha) = \pi\int_0^{\infty} d\beta\cos(X\cosh\beta) + \pi\int_0^{\pi/2} d\alpha\sin(X\cos\alpha) = \frac{\pi^2}{2}[\mathbf{H}_0(X) - Y_0(x)] \quad (31)$$[10]

Equation (31) gives a representation of the second and third integrals in equation (19) in terms of $\mathbf{H}_0$ and Y_0 functions. Since the first and fourth integrals are already well-known representations of these functions, we have for $x > 0$ that

$$g(x) = \frac{\pi}{4}[\mathbf{H}_0(\kappa x) + 3Y_0(\kappa,x)] \quad (32)$$

Using (29) this result can be extended to the case $x < 0$, so that for all x we have

$$g(x) = \frac{\pi}{4}[\mathbf{H}_0(\kappa x) + (2 + \operatorname{sgn} x)Y_0(\kappa|x|)] \quad (33)$$

[10] Reference [13], pp. 37 and 82.

ON VOSSERS' INTEGRAL

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Slender Ship Theories

Let us consider the uniform translation of a rigid body (ship) situated at or near the gravitational free surface of an infinite inviscid incompressible ocean which is at rest apart from the disturbance produced by the passage of the ship. This is a well formulated boundary value problem in hydrodynamics, but as it is non-linear, in order to approximate it by a mathematically tractable linearized problem it is generally necessary to assume that the disturbance produced by the ship at the free surface is small. In practice this means either that the ship is deeply submerged or the ship itself "small" in some sense.

For many years the only successful and mathematically sound approach to problems of the latter type has been that usually associated with the name of Michell(1898), and the theory of Michell, or "thin", ships has reached its highest stage of development in the work of Peters and Stoker (1957). Since thin ship theory is based upon the same type of approximation as is used in the thin aerofoil theory of aerodynamics, one is tempted to try another technique used by aerodynamicists for linearizing their problems, namely the theory of slender bodies. Recently several authors working independently (Vossers 1962, Maruo 1962, Tuck 1963) have been successful in establishing slender ship theories for the case of steady motion which are broadly on the same basic as aerodynamic slender body theory.

Vossers' approach is perhaps the least like the traditional aerodynamic method instituted by Ward (1949). Vossers formulates the linearized problem as an integral equation using Green's functions, and by constructing the approximate form of the Green's function in the slender region (i.e. in a region near to a straight line) occupied by the ship obtains a first approximation to the potential in this region. Although Vossers' analysis is unnecessarily complicated and contains numerous serious errors, his method has much to be said for it from the point of view of mathematical rigour, and it would seem worthwhile applying it to the original aerodynamic problem as well. In spite of errors Vossers obtains a wave resistance formula by integrating hull pressure which is only wrong by a factor of four.

Maruo represents the ship by a volume distribution of sources and then, by non-dimensionalising his problem with distances in cross-sectional planes stretched compared with axial distances, reduces the volume distribution to a line distribution (except at the ends of the ship). The strength of this line distribution may now be shown by arguments similar to those of Ward to be proportional to the axial derivative of immersed cross-sectional area. The wave resistance follows from results of Havelock (1932), and Maruo obtains essentially the same resistance formula as Vossers, together with some new end-effect terms.

Perhaps a reason why the development of slender ship theories has lagged so long behind that of thin ship theories is that in the latter case the hull boundary condition may be approximately applied on the plane surface to which the ship shrinks as its beam tends to zero, whereas it is not mathematically feasible to apply boundary conditions on the line in three dimensions to which a slender body shrinks. This is why it is useful to consider the problem with a distorted co-ordinate system such as Maruo used, since in the new co-ordinate system the body does not shrink down to a line. However, in this "inner" region we can say little about how the flow behaves at infinity and to complete the solution it is necessary to also consider the original physical space with the ship shrunk down to a line. In this "outer" region the only possible disturbance is on the limiting line, so that the flow behaves like a line distribution of (Kelvin) sources of unknown density. The solution to the problem may now be completed by matching the behavior of the inner and outer solutions in a supposed common domain of validity.*

Thus in the inner region the disturbance potential becomes two dimensional in cross-sectional planes, and behaves at large distances r from the ship in such planes like a 2D line source of known strength; in fact

$$\phi \sim \frac{U}{\pi} S'(x) \log r + b(x) \tag{1}$$

where U is the speed of the ship, and $S(x)$ is the immersed cross-sectional area as a function of the axial co-ordinate x. The function "$b(x)$" is a constant in planes of constant x, and is arbitrary until matching takes place.

On the other hand (Tuck, 1963) the outer solution behaves for <u>small</u> r like

$$\begin{aligned}\phi \sim\ & a(x) \log r \\ & - \frac{1}{2} \int_{-\infty}^{\infty} da(\xi) \left\{ \operatorname{sgn}(x-\xi) \log 2 |x-\xi| + \frac{\pi}{2} \underset{\sim}{H}_0 (\varkappa(x-\xi)) \right. \\ & \left. \qquad + (2 + \operatorname{sgn}(x-\xi)) \frac{\pi}{2} Y_0(\varkappa|x-\xi'|) \right\}\end{aligned} \tag{2}$$

*This is a brief and inadequate symmary of a singular perturbation technique of wide applicability in applied mathematics. The method of inner and outer expansions has been mainly used till now for problems involving viscous flow at low Reynolds' number and a very careful rigorous statement of the principles involved is given by Lagerstrom and Cole (1955).

where $\mathrm{sgn} = \pm 1$ according to the sign of its argument and $\underset{\sim}{H}_0$, Y_0 are Struve and Bessel Y functions respectively (Erdelyi, 1953, pp. 8, 37). That is, the outer solution behaves like a 2D line source of unknown strength $a(x)$ together with a function of x (and of the constant $\kappa = g/U^2$) which is determined uniquely from $a(x)$. Now matching the inner and outer solutions together gives

$$a(x) = \frac{U}{\pi} S'(x)$$

$$b(x) = -\frac{U}{2\pi} \int_{-\infty}^{\infty} d\, S'(\xi) \left\{ \mathrm{sgn}(x-\xi) \log 2 |x-\xi| + \frac{\pi}{2} \underset{\sim}{H}_0 (k(x-\xi)) + (2 + \mathrm{sgn}(x-\xi)) \frac{\pi}{2} Y_0 (k|x-\xi|) \right\} \qquad (4)$$

so that both solutions are now rendered complete and unique. Vossers has given a result similar to our Equation (1) but his formula for the function corresponding to $b(x)$ as given by (4) is incorrect.

The formula (4) has an interesting physical description. Let us consider separately the ranges of integration $\xi > x$ and $\xi < x$. In the former case the Struve and Y functions occur only in the particular combination "$\underset{\sim}{H}_0 - Y_0$" which is well known (Erdelyi, p. 38) to be a monotone decreasing function of its (positive) argument. Hence this portion of the range of integration does not contribute to the wave-like character of the disturbance, which is what we should expect since it is a contribution from disturbance points behind the point x of observation (we have chosen axes such that the ship moves in the negative x direction). Further, the kernel of the integral in (4) is bounded as $\xi \to x+$, since the "log" term precisely cancels the logarithmic behavior of the Y_0 function.

On the other hand, when $\xi < x$ the kernel contains a different combination of log, $\underset{\sim}{H}_0$, and Y_0, retaining all the oscillatory nature of the Bessel and Struve functions, and behaving logarithmically as $\xi \to x-$. Thus this portion of the range of integration, which corresponds to disturbance ahead of the observer, contributes most to both the wave-like and source-like nature of the flow near the ship. This description of the behavior of the function $b(x)$ has more than qualitative importance, since it can be shown (Tuck, 1963) that the amplitude of the transverse wave either in front of or behind the ship is proportional to $b'(x)$. In front of the ship we must have $\xi > x$ so that no waves are predicted, whereas far behind the ship we can predict from the asymptotic behavior of the H_0 and Y_0 functions that there is a wave system with wave number κ and with amplitude decreasing like $x^{-\frac{1}{2}}$.

Vossers' Integral

The wave resistance may be found either from the inner potential by pressure integration along the hull, or from the outer potential by use of results of Havelock on source distributions, and the same answer is obtained in both cases. Thus by a pressure integration we have the wave resistance as

$$
\begin{aligned}
R &= - \rho \int_{-\infty}^{\infty} dS(x) \; (U \, b'(x)) \\
&= \rho \, U \int_{-\infty}^{\infty} dS'(x) \; b(x) \\
&= - \frac{1}{2} \rho \, U^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dS'(x) \, dS'(\xi) \; Y_0 \; (K|x-\xi|)
\end{aligned}
\tag{5}
$$

since only that part (involving Y_0) of the kernel in (4) which is an even function of its argument will contribute to the above integral. (The reason why Vossers is able to obtain the correct form of the resistance integral is that the error in his expression for $b(x)$ involves omission of some odd terms only from the kernel.) If it is assumed that $S(x)$ vanishes identically outside a finite interval, the length of the ship, then the above Stieltjes integral may be written as a finite Riemann integral involving the second derivative $S''(x)$; however, if $S'(x)$ is not continuous over the whole real axis, then extra terms will be introduced at the points of discontinuity, and Vossers supplies these explicitly for the case when $S'(x)$ does not vanish at the ends of the ship.

The formula (5) may be called the Vossers integral in dimensional form. Non-dimensional formulations are sometimes (but by no means always) more useful. There are innumerable methods of carrying out non-dimensionalisation, but in the present context the most meaningful is that based on displacement, as used by Maruo. For definiteness let us assume that the ship is of finite length L and that $S'(\pm \frac{L}{2}) = 0$, so that the end-effect terms do not occur. Then we put

$$
\sigma(t) = \frac{L}{\Delta} S(\frac{Lt}{2})
\tag{6}
$$

where Δ is the displacement,

$$
\Delta = \int_{-L/2}^{L/2} S(x) \, dx
$$

Hence we have

$$\int_{-1}^{1} dt\, \sigma(t) = 2 \tag{7}$$

In terms of $\sigma(t)$, Vossers' integral becomes

$$\frac{R}{\frac{1}{2}\rho U^2 L^2} = \frac{8}{\pi} \left(\frac{\Delta}{L^3}\right)^2 f(F) \tag{8}$$

where we have defined a resistance coefficient $f(F)$ of the form

$$f(F) = -\frac{\pi}{2} \int_{-1}^{1} \int_{-1}^{1} dt_1\, dt_2\, \sigma''(t_1)\, \sigma''(t_2)\, Y_o \left(\frac{|t_1 - t_2|}{2F^2}\right) \tag{9}$$

with $F = \frac{U}{\sqrt{gL}} = \frac{1}{\sqrt{\kappa L}}$ as the Froude number. There are other useful forms for the resistance; e.g. a "p's and q's" form is

$$f(F) = \int_{o}^{\infty} d\gamma\, p^2\left(\frac{\cosh\gamma}{2F^2}\right) + q^2\left(\frac{\cosh\gamma}{2F^2}\right) \tag{10}$$

where

$$p(k) + i\, q(k) = \int_{-1}^{1} d\, t\, e^{i\, k\, t}\, \sigma''(t) \tag{11}$$

is the Fourier Transform of $\sigma''(t)$. Also it is sometimes convenient to use a "hull function" form, with

$$f(F) = \frac{-\pi}{2} \int_{o}^{1} dv\, Y_o \left(\frac{v}{F^2}\right) H(v) \tag{12}$$

where

$$H(v) = 4 \int_{-1+v}^{1-v} du\, \sigma''(u + v)\, \sigma''(u - v) \tag{13}$$

High Speed Approximations

The integrals (9) or (12) can easily be approximated for large values of F using known behavior of the Y_o function at small values of its argument (Erdelyi, p. 8). The result from (12) is

$$f(F) = f_o + f_1 F^{-4} + \frac{3}{16} F^{-8} \log F + f_2 F^{-8} + O(F^{-12} \log F) + \ldots \tag{14}$$

where

$$f_o = -\int_o^1 dv \left[\log_e v \quad H(v) \right] \tag{15}$$

$$f_1 = \frac{1}{4} \int_o^1 dv \left[\log_e v \quad v^2 H(v) \right] \tag{16}$$

$$f_2 = -\frac{1}{64} \int_o^1 dv \left[\log_e v \quad v^4 H(v) \right] + \frac{3}{32} \left(\frac{3}{2} + \log_e 2 - \gamma \right) \tag{17}$$

etc. It may easily be demonstrated (Tuck, 1963) that f_o and f_1 are non-negative, and that the contribution of the integral to f_2 and further terms is of ever diminishing importance compared with the constant terms involving Euler's constant γ, which occur in all f_n, $n \geq 2$.

Now the terms actually written down in the above series (14) already give information about the highest peak on the resistance curve, for (14) clearly breaks down at a value of $F(< 1)$ such that the term "$\frac{3}{16} F^{-8} \log F$" sends f negative, whereas as F tends to indinity, f tends to the non-negative value f_o from above (since f_1 is non-negative). In fact there are plausible grounds for estimating the position of the last peak as that value of F for which the "F^{-8}" terms cancel each other, i.e. at $F = F_o$, where

$$F_o \simeq e^{-\frac{16}{3} f_2} \tag{18}$$

Further, we may gain an even rougher, but universal, estimate of this peak by putting $f_2 \simeq \frac{3}{32} (\frac{3}{2} + \log 2 - \gamma) = 0.1515$ .. in this formula, i.e. by neglecting the contribution of the integral to (17). This gives $F_o = 0.466$..; if exact values of f_2 are used in any particular case, somewhat higher values of F_o are obtained, in the range 0.45 to 0.5. This appears to agree to the accuracy expected with experimental observations, and the fact that F_0 does not appear in experiments to depend strongly on the shape of the hull is a reflection of the fact that f_2 is not much different from 0.1515 for practical ship shapes.

If further terms in the series (14) were used, the lower limit of F for which the formula is valid might well be pushed down to the practical operating range of ships; as it stands the formula (14) ceases to give sensible results for F below about 0.4. It is possible (Tuck, 1963) to make low speed approximations similar to those obtained by Kotik (Wehausen, 1957) for Michell's integral, but these seem to have no quantitative use; in any case the full Vossers integral itself is not expected to be valid for small F.

Minimisation at High Speed

Maruo (1963) has given a method for finding the shape $\sigma(t)$ which gives least drag in terms of Mathieu functions. It may be worth noticing that at high speed a very simple result (Tuck, 1963) is obtained for the minimisation of the coefficient f_0. The full kernel Y_0 has now reduced to a log function, and the integral equation for the minimisation problem can be solved by use of Hilbert transforms, giving

$$\sigma(t) = \frac{8}{3\pi} (1-t^2)^{\frac{3}{2}} \tag{19}$$

(normalised by (7)), with a corresponding resistance at infinite Froude number of $f_0 = 32$. This is the least possible value of f_0; the same ship would give a resistance of

$$f(F) = 64 \int_0^\infty d\gamma \ [J_2 (\frac{\cosh\gamma}{2F^2})]^2 \tag{20}$$

at finite F, but the latter is of course not by any means the least possible value.

REFERENCES

Erdelyi, A., Bateman Manuscript Project, "Higher Transcendental Functions", Vol. 2, McGraw-Hill, 1953.

Havelock, T. H., Proc. Roy. Soc. A 138 (1932) 339.

Langerstrom, P. A., and Cole, J. D., J. Rational Mech. Anal. 4 (1955) 817.

Maruo, H., J. Zosen Kiokai Japan 112 (1962) 161.

Michell, J. H., Phil. Mag. 45 (1898) 106.

Peters, A. S. and Stoker, J. J., Commun. P. Appl. Math. 2 (1957) 379.

Tuck, E. O., Ph.D. Thesis, Cambridge, 1963.

Vossers, G., Ph.D. Thesis, Delft, 1962.

Ward, G. N., Q. J. Mech. Appl. Math. 1 (1949) 75.

Wehausen, J. V., Symposium on Naval Hydrodynamics, Washington, 1957.

DISCUSSION

by G. R. G. Lewison

The experimental confirmation of Mr. Tuck's theory as to wave resistance is a little distant, as indicated by results described yesterday, but I should like to mention some other results which I have computed and may be of interest.

Firstly, the function b, of Equation (4), or rather its derivative (b') with respect to x, has been calculated for a range of speeds. The general behavior is much as described by Mr. Tuck, and b' is very small and non-oscillatory ahead, and oscillates with decreasing amplitude astern There are discontinuities of slope at the ends, $x = \pm 1$; within the body [the function resembles the wave profile as experimentally observed,* when corrected by two small terms dependant upon the section area derivative]. This resemblance is more noticable as speed increases, and is fair at $f \sim .3$.

Secondly, the position of the last hump, for my choice of model, is given quite well by the calculations of page 10; in fact $f \simeq .49$ theoretically, in close agreement with the experimental value for one model of $F = .47$.

Thirdly, the calculated wave resistance coefficient, proportional to Tuck's f(F) of Equation (14), seems to behave very much as indicated by this equation, at high speeds, and to approach a minimum value, which remains quite large, as the speed increases further.

* As mentioned yesterday by Mr. Tuck in his discussion of Professor Takahei's paper.

by J. N. Newman

I should like to outline an alternative approach to the wave resistance of a slender body. This is based upon the relatively simple analysis of the far-field potential through Green's theorem. For a point in the interior of the fluid we have

$$\phi = \frac{1}{4\Pi} \iint (G\phi_n - \phi G_n)\, dS + O\,(\varepsilon^3 \log \varepsilon)$$

where the integral is over the submerged surface of the body, and the Green's function G represents the potential of a source satisfying the linearized free surface condition. Now we can apply the above relation to a point in the far-field, whence G and G_n are non-singular and $O(1)$. On the other hand, from the solution of the first order problem in the near field, $\phi = O(\varepsilon^2 \log \varepsilon)$ while $\phi_n = O(\varepsilon)$. Thus for points in the far-field the above equation reduces to the simple relation

$$\phi = \frac{1}{4\Pi} \iint G\phi_n\, dS + O\,(\varepsilon^3 \log \varepsilon)$$

and the first-order far-field potential consists simply of a source distribution of known strength. Since the far-field potential is sufficient for the determination of the wave resistance, the above relation may be used for this purpose. Moreover it suggests an explanation of the fact that the first order slender body wave resistance is so closely related to Michell's integral, for the Mitchell potential for a thin ship is also a source distribution of strength ϕ_n .

The above approach can also be carried out to second order, providing a relation for the far-field second-order potential in terms of the first order potential in the near field.

by H. Maruo

I appreciate Mr. Tuck and Mr. Joosen who obtained independently and simultaneously the right expression for the velocity potential of a slender ship. When Dr. Vossers and I got the wave resistance formula for the slender ship independently last year, there was a difference between our results unfortunately. However, the results presented here show a thorough agreement, in spite of the methods by which the formula is derived are different. Therefore we can conclude that the slender ship theory can claim its right as a consistent theory. I wish to point out some of the difficulties when the slender ship theory is considered. That is the higher singularity which appear in the kernel of the integral. In the case of the ship form which has finite angle of entrance, a special treatment is needed for the end singularities in order to secure a converging integral. For a ship form with pointed nose, this difficulty does not appear in the calculation of the wave resistance, but if we need to discuss the surface profile, singularities occur at both ends. Though the slender ship theory looks nicely as a linearized theory, it does not necessarily so if a practical application is considered, as we can see in Mr. Lewison's result. One of the reasons is the fact that the theory assumes free surface as a rigid plane. The basic idea of the slender ship is to regard the fluid flow near the body surface as two-dimensional, and this assumption is so strong as to be satisfied by the second approximation as well. The assumption of the rigid free surface however is much weaker than it. It does not hold in the second approximation. When the second approximation is considered in order to obtain a better agreement with the actual phenomenon, a difficulty appears again from the higher singularity. Converging form can be obtained only for a very sharp pointed form with cusps. This case is by no means practical. This fact suggests that the perturbation scheme is not regular. Some special techniques are needed to handle the end singularities. Multipole expansion cannot be applied to the slender ship with finite forward velocity. Anyway, the theory presented here is the only self-consistant linearized theory for a slender ship, and it should be highly evaluated.

AUTHOR'S REPLY

Professor Maruo's remarks are very pertinent. Time did not allow discussion of them in my presentation, but in my thesis (Tuck, 1963) I investigated in detail all the questions raised by Professor Maruo. In particular I obtained a second approximation with the properties described by him, viz. it is still derived from a two-dimensional potential function but not from a rigid wall boundary condition and it is indeed very badly behaved at the ends except for cusped ships. These end-effects, which are analogous to those experienced in thin aerofoil theory, clearly need special treatment in some sort of singular perturbation scheme.

Dr. Newman's method of constructing a slender ship theory is very useful, especially insofar as it indicates the essential similarities (and differences) between thin and slender ship theories. The application of his method to finding a second order potential would, however, be very difficult since the second order potential depends essentially on non-linear effects from the free surface condition near the ship. Thus a naive approach which assumes the usual linearized outer free surface condition to be valid everywhere does not work (in spite of the fact that the linearized condition <u>is</u> valid even for the second approximation, except in the inner region near the ship).

The fact that the qualitative and asymptotic descriptions given in my paper seem to agree well with calculations for a specific hull form, as reported by Mr. Lewison, suggests that simple approximations to the full slender ship results (such as the high speed asymptotics of Equation (14)) may be as useful as detailed computations. One should not in any case expect much quantitative agreement with experiment from a theory which itself requires severe approximations in its derivation, and the chief merit of slender ship theory may well be the simplicity of some of the final results which may be utilized in qualtative deductions.

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DOCUMENT CONTROL DATA - R&D			
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)			
1 ORIGINATING ACTIVITY (Corporate author) David Taylor Model Basin		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		2b. GROUP	
3 REPORT TITLE THE APPLICATION OF SLENDER BODY THEORY TO STEADY SHIP MOTION			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates)			
5. AUTHOR(S) (Last name, first name, initial) Tuck, Ernest O.			
6. REPORT DATE June 1965	7a. TOTAL NO. OF PAGES 37		7b. NO. OF REFS 36
8a. CONTRACT OR GRANT NO. b. PROJECT NO. S-R011 01 01 c. Task 0401 d.	9a. ORIGINATOR'S REPORT NUMBER(S) 2008		
	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)		
10. AVAILABILITY/LIMITATION NOTICES (No restriction on distribution)			
11 SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Bureau of Ships		

13 ABSTRACT

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Three previously published papers on steady ship motion treated by slender body theory are reprinted, together with an introduction which summarized the three papers and reviews other efforts in this field up to 1965. Suggestions are made for further work to remove present limitations of the theory.

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