THE INFLUENCE OF A MASS ON THE FREE FLEXURAL VIBRATIONS OF A CIRCULAR RING

by

E. W. Palmer

This document has been approved for public release and sale; its distribution is unlimited.

August 1967

Report No. 2536
The Naval Ship Research and Development Center is a U. S. Navy center for laboratory effort directed at achieving improved sea and air vehicles. It was formed in March 1967 by merging the David Taylor Model Basin at Carderock, Maryland and the Marine Engineering Laboratory at Annapolis, Maryland.

Naval Ship Research and Development Center
Washington, D. C. 20007
THE INFLUENCE OF A MASS ON THE FREE FLEXURAL VIBRATIONS OF A CIRCULAR RING

by

E. W. Palmer

This document has been approved for public release and sale; its distribution is unlimited.

Sub Project S-F013 10 05,
Task 11656

August 1967

Report No. 2536
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>1</td>
</tr>
<tr>
<td>ADMINISTRATIVE INFORMATION</td>
<td>1</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>BACKGROUND</td>
<td>1</td>
</tr>
<tr>
<td>OBJECTIVE</td>
<td>2</td>
</tr>
<tr>
<td>SCOPE OF REPORT</td>
<td>2</td>
</tr>
<tr>
<td>THEORETICAL ANALYSIS</td>
<td>2</td>
</tr>
<tr>
<td>STATEMENT OF THE PROBLEM</td>
<td>2</td>
</tr>
<tr>
<td>THE DIFFERENTIAL EQUATION OF MOTION</td>
<td>3</td>
</tr>
<tr>
<td>SOLUTION OF THE DIFFERENTIAL EQUATION</td>
<td>5</td>
</tr>
<tr>
<td>CONSIDERATION OF THE POINT MASS</td>
<td>6</td>
</tr>
<tr>
<td>FREQUENCY EQUATIONS</td>
<td>8</td>
</tr>
<tr>
<td>NORMAL MODE SHAPES</td>
<td>11</td>
</tr>
<tr>
<td>ORTHOGONALITY</td>
<td>13</td>
</tr>
<tr>
<td>NUMERICAL SOLUTIONS</td>
<td>15</td>
</tr>
<tr>
<td>VARIATION OF FREQUENCY WITH MASS</td>
<td>15</td>
</tr>
<tr>
<td>VARIATION OF MODE SHAPES WITH MASS</td>
<td>16</td>
</tr>
<tr>
<td>DISCUSSION</td>
<td>35</td>
</tr>
<tr>
<td>INFLUENCE OF MASS ON FREQUENCY</td>
<td>35</td>
</tr>
<tr>
<td>INFLUENCE OF MASS ON MODE SHAPES</td>
<td>35</td>
</tr>
<tr>
<td>SUMMARY AND CONCLUSIONS</td>
<td>36</td>
</tr>
<tr>
<td>RING WITHOUT A POINT MASS</td>
<td>36</td>
</tr>
<tr>
<td>RING WITH A POINT MASS</td>
<td>36</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>37</td>
</tr>
<tr>
<td>APPENDIX A, Computer Programs by L. F. Parker</td>
<td>39</td>
</tr>
<tr>
<td>INITIAL DISTRIBUTION</td>
<td>46</td>
</tr>
<tr>
<td>DD FORM 1473</td>
<td>47</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

Figure 1 - Ring Coordinate System ........................................... 3
Figure 2 - Dynamical Equilibrium of a Ring Element ....................... 3
Figure 3 - Dynamical Equilibrium of the Point Mass ......................... 7
Figure 4 - Variation of Frequency with Mass, First Mode .................. 18
Figure 5 - Variation of Frequency with Mass, Second Mode ................. 19
Figure 6 - $C_A$ versus $\lambda$, First Mode ................................ 20
Figure 7 - $C_B$ versus $\lambda$, First Mode .................................. 21
Figure 8 - $C_A$ versus $\lambda$, Second Mode ................................ 22
Figure 9 - $C_B$ versus $\lambda$, Second Mode ................................ 22
Figure 10 - Radial Displacements, Symmetrical Branch of First Mode .... 23
Figure 11 - Tangential Displacements, Symmetrical Branch of First Mode 24
Figure 12 - Radial Displacements, Antisymmetrical Branch of First Mode 25
Figure 13 - Tangential Displacements, Antisymmetrical Branch of First Mode 26
Figure 14 - Radial Displacements, Symmetrical Branch of Second Mode .... 27
Figure 15 - Tangential Displacements, Symmetrical Branch of Second Mode 28
Figure 16 - Radial Displacements, Antisymmetrical Branch of Second Mode 29
Figure 17 - Tangential Displacements, Antisymmetrical Branch of Second Mode 30
Figure 18 - Mode Shapes, Symmetrical Branch of First Mode ............... 31
Figure 19 - Mode Shapes, Antisymmetrical Branch of First Mode ......... 32
Figure 20 - Mode Shapes, Symmetrical Branch of Second Mode .......... 33
Figure 21 - Mode Shapes, Antisymmetrical Branch of Second Mode .... 34

LIST OF TABLES

Table 1 - Natural Frequencies .................................................. 35
NOTATION

\( A_k \) Mode constants for antisymmetrical branch.

\( a \) Real part of mode parameters \( n_a \) and \( n_b \) in first mode.

\( B_k \) Mode constants for symmetrical branch.

\( b \) Imaginary part of mode parameters \( n_a \) and \( n_b \) in first mode.

\( C \) Frequency function (see Equation (25)).

\( C_A \) Frequency function for antisymmetrical branch (see Equation (36)).

\( C_B \) Frequency function for symmetrical branch (see Equation (38)).

\( D \) Determinant of coefficients of mode constants \( A_k \) and \( B_k \) (see Equation (27)).

\( D_A \) Determinant of coefficients of mode constants \( A_k \) (see Equation (35)).

\( D_B \) Determinant of coefficients of mode constants \( B_k \) (see Equation (37)).

\( E \) Young's modulus.

\( e \) Base of natural logarithms.

\( G \) In-plane bending moment.

\( H \) Amplitude constant.

\( I \) Moment of inertia of ring cross section about the centroidal axis normal to the plane of the ring.

\( \sqrt{-1} \), or integer subscript.

\( j \) An integer subscript.

\( k \) An integer subscript relating the three mode constants \( A_k \) or \( B_k \) to the three mode parameters \( n_k \).

\( M \) Point mass.

\( m \) Mass of ring per-unit-length.

\( N \) Shear on ring cross section in radial direction.

\( n \) An integer greater than unity defining the mode of vibration of a ring without a point mass.

\( n_k \) Mode parameters.

\( r \) Radius of ring neutral axis.

\( T \) Normal force on ring cross section.

\( t \) Time variable.

\( U \) Radial displacement variable, time independent.

\( u \) Radial displacement variable, time dependent.

\( V \) Tangential displacement variable, time independent.

\( v \) Tangential displacement variable, time dependent.

\( \omega_k \) Circular argument \( n_k \pi \) for coefficients of mode constants \( A_k \) and \( B_k \).

\( \Gamma \) Time function (see Equation (11)).

\( \delta \) Phase angle.

\( \Theta \) Angular coordinate.

\( \lambda \) Eigenvalue (see Equation (10)).

\( \omega \) Circular frequency of vibration.
ABSTRACT

The general solution is obtained for the free flexural vibrations of a thin circular ring containing a point mass. The solution for a uniform ring alone is derived by taking the point mass to be zero. Numerical calculations of the frequencies of the first and second flexural modes are presented for values of the point mass in the range from zero to infinity. Mode shapes are presented in graphical form.

The predominant feature of the investigation is the difference in frequency and mode shape found in the symmetrical and antisymmetrical branches of each mode. It is noted that similar phenomena have been observed experimentally for vibrations of imperfect bodies of revolution.

ADMINISTRATIVE INFORMATION

This report is related to the program entitled Hull Penetration Design Criteria, Sub Project S-F013 10 05, Task 11656, NSRDC Problem Number 791-038, which was authorized by Bureau of Ships letter Serial 423-227 of 6 January 1959.

INTRODUCTION

BACKGROUND

The problem of the vibration of a ring containing a mass is of interest in many engineering situations (e.g., in the dynamic behavior of cylindrical shells used in hydro and aerospace vehicles having attached equipment). With the mass considered as an imperfection, it should be particularly suited also to the study of vibrations of imperfect bodies of revolution inasmuch as the effect of such imperfections can be quite distinct as has been shown by several experimental investigations.¹,²,³

¹ References are listed on page 37.
The equation of motion and its general solution for the inextensional flexural vibrations of thin circular rings are given in the literature.\textsuperscript{4, 5} A comparison of theoretical and experimental results for an elastically supported ring is also available.\textsuperscript{6} Some theoretical work\textsuperscript{7} has been done in deriving approximate fundamental frequencies by the Rayleigh method for complete and partial rings with attached masses.

Starting with the general solution, free vibration frequencies and mode shapes may be found by describing suitable continuity and equilibrium conditions at the mass. The influence of the mass may be ascertained by comparison with the vibrations of a ring without a point mass, the solution of which may be obtained from the general solution by taking the point mass to be zero.

OBJECTIVE

The objective of this investigation is to determine the influence of a concentrated mass on the free flexural vibrations in the plane of a complete thin circular ring with attention to the problem of imperfections in the vibrations of bodies of revolution.

SCOPE OF REPORT

First a statement of the problem is given and then the differential equation of motion and its solution for a ring with and without a point mass, and the orthogonality condition. Next, numerical solutions for the first two modes with values of the point mass ranging from zero to infinity are obtained, and the influence of the mass on the normal mode frequencies and shapes is discussed. The results are summarized along with the conclusions that are drawn. Computer Programs developed and used in this study are given in Appendix A.

THEORETICAL ANALYSIS

STATEMENT OF THE PROBLEM

This investigation pertains to the free flexural vibrations of a naturally curved, thin circular ring of constant cross section containing a point mass. One of the principal axes of the cross section is in the plane of the ring with the vibrations occurring in that plane.

The assumptions employed are:
(1) flexure without extension of the neutral axis,  
(2) strain varies linearly over a cross section,  
(3) Hookes' law applies, and  
(4) shear distortion and rotatory inertia effects are neglected.

The mass is taken as rigid and integral with the ring and is assumed to be concentrated at a point on the ring neutral axis.

THE DIFFERENTIAL EQUATION OF MOTION

The orientation of the polar coordinate system of the ring with respect to the point mass is shown in Figure 1 together with the positive directions of the radial and tangential displacements, \( u \) and \( v \), of a point on the ring neutral axis.

![Figure 1 - Ring Coordinate System](image1)

To establish the equations of equilibrium consider an element of the ring as shown in Figure 2.

![Figure 2 - Dynamical Equilibrium of a Ring Element](image2)
The variation of the internal shear and normal forces $N$ and $T$, and flexural moment $G$ along the element was obtained by Taylor's series expansions retaining only the first-order term in each case.

The sum of forces in the radial direction gives an equation of equilibrium,

$$\frac{\partial N}{\partial \theta} + T = mr \frac{\partial^2 u}{\partial t^2}$$

(1)

and in the tangential direction an equilibrium equation,

$$\frac{\partial T}{\partial \theta} - N = mr \frac{\partial^2 v}{\partial t^2}$$

(2)

Neglecting rotatory inertia, the sum of the moments gives an equation of equilibrium,

$$\frac{\partial G}{\partial \theta} + Nr = 0.$$  

(3)

A single differential equation of motion is derived by eliminating the $N$, $T$ and $G$ terms of Equations (1), (2) and (3) with the aid of the relationship between bending moment and displacement. For thin circular bars, the distribution of bending stresses approaches a linear one and the neutral axis is assumed to pass through the centroid of the cross section. The bending moment can then be related to the change in curvature of the bar as is done for straight bars and the following bending moment-displacement expression is obtained.\(^8\)

$$G = \frac{EI}{r^2} \left( \frac{\partial^2 u}{\partial \theta^2} + u \right).$$

(4)

For flexural vibrations without extension the following geometrical condition must be satisfied\(^4,\!8\)

$$u = \frac{\partial v}{\partial \theta}.$$  

(5)
By use of the relationships (4) and (5), Equations (1), (2) and (3) may be reduced to obtain the equation of motion in terms of the tangential displacements:

\[
\frac{EI}{r^4} \left( \frac{\partial^6 v}{\partial \theta^6} + 2 \frac{\partial^4 v}{\partial \theta^4} + \frac{\partial^2 v}{\partial \theta^2} \right) = m \frac{\partial^2}{\partial t^2} \left( v - \frac{\partial^2 v}{\partial \theta^2} \right). \tag{6}
\]

SOLUTION OF THE DIFFERENTIAL EQUATION

Substituting an assumed product solution of the form

\[
v(\theta, t) = V(\theta)\Gamma(t) \tag{7}
\]

into (6), the process of separation of variables leads to the equations

\[
\frac{d^2 \Gamma}{dt^2} + \omega^2 \Gamma = 0 \tag{8}
\]

and

\[
\frac{d^6 V}{d\theta^6} + 2 \frac{d^4 V}{d\theta^4} + \frac{d^2 V}{d\theta^2} \left( 1 - \lambda \right) + \lambda V = 0 \tag{9}
\]

where

\[
\lambda = \frac{mr^4 \omega^2}{EI}. \tag{10}
\]

Solving (8) gives the time function

\[
\Gamma = H \cos(\omega t + \delta) \tag{11}
\]

where \( \omega \) is the circular frequency, and \( H \) (amplitude) and \( \delta \) (phase angle) are constants which can be determined from the initial conditions.

For frequencies and mode shapes of free vibration the solution of (9) for the shape function is of primary interest. The complete solution of (9) is of the form

\[
\text{For frequencies and mode shapes of free vibration the solution of (9) for the shape function is of primary interest. The complete solution of (9) is of the form}^6
\]
\[ V = \sum_{k=1}^{k=3} \left( A_k \cos n_k \theta + B_k \sin n_k \theta \right) \]  

(12)

where \( n_1, n_2 \) and \( n_3 \) are roots of the equation

\[ n_k^2 \left( n_k^2 - 1 \right)^2 = \left( n_k^2 + 1 \right) \lambda. \]  

(13)

Equation (13) results from the substitution of (12) in (9).

**CONSIDERATION OF THE POINT MASS**

As the solution of (9) contains six constants of integration, \( A_k \) and \( B_k \) (in general at least one of these always remains arbitrary as (9) is homogeneous), and since the eigenvalue \( \lambda \) in (9) must also be determined, then six conditions are needed at the mass. Continuity conditions at sides \( \theta = \pi \), and \( \theta = -\pi \) (see Figure 1) of the mass require that the radial and tangential deflections, \( u \) and \( v \), and the slope of the radial deflection, \( \partial u / \partial \theta \), be equal. Equilibrium conditions at the mass require that the sum of the forces in the radial and tangential directions and the sum of the moments be zero.

From the continuity conditions the following equations can be written

\[ [u]_{-\pi}^\pi = [u]_\pi - [u]_{-\pi} = 0 \]  

(14)

\[ [v]_{-\pi}^\pi = 0 \]  

(15)

and

\[ \left( \frac{\partial u}{\partial \theta} \right)_{-\pi}^\pi = 0 \]  

(16)

From Figure 3, summing forces in the radial direction gives

\[ M \left[ \frac{\partial^2 u}{\partial t^2} \right]_\pi + \left[ N \right]_{-\pi}^\pi = 0 \]  

(17)
and in the tangential direction gives

\[ M \left[ \frac{3^2 v}{\partial t^2} \right]_\pi + \left[ T \right]_\pi = 0 \]  \hspace{1cm} (18)

and summing the moments (disregarding rotatory inertia since the mass is assumed to be concentrated at a point) gives

\[ [G]_\pi = 0. \]  \hspace{1cm} (19)

Evaluating (14) through (19) with expressions (1) through (5) and (11) and (12) gives six linear homogeneous algebraic equations arranged as follows:

from (14)

\[ \sum_{k=3}^{k=3} A_k n_k \sin \alpha_k = 0 \]  \hspace{1cm} (20)

from (19)

\[ \sum_{k=1}^{k=3} A_k n_k^3 \sin \alpha_k = 0 \]  \hspace{1cm} (21)

from (18)

\[ \sum_{k=1}^{k=3} A_k \left( \cos \alpha_k + Cn_k^6 \sin \alpha_k \right) = 0 \]  \hspace{1cm} (22)
from (15)

\[ \sum_{k=1}^{3} B_k \sin a_k = 0 \]  

(23)

from (16)

\[ \sum_{k=1}^{3} B_k n_k^2 \sin a_k = 0 \]  

(24)

and from (17)

\[ \sum_{k=1}^{3} B_k n_k \left( \cos a_k + C_k \sin a_k \right) = 0 \]  

(25)

where

\[ a_k = n_k \pi, \ (k = 1, 2 \text{ or } 3) \]

and

\[ C = \frac{2EI}{Mr^3\omega^2}. \]  

(26)

FREQUENCY EQUATIONS

Since (20) through (25) are homogeneous, the determinant of the coefficients of the \( A_k \) and \( B_k \) terms must vanish in order for nontrivial values of these constants to exist. Evaluating this determinant establishes the frequency equation from which the natural frequencies may be obtained.

The sixth-order determinant from (20) through (25) is

\[
\begin{vmatrix}
n_1 \sin \alpha_1 & n_2 \sin \alpha_2 & n_3 \sin \alpha_3 & 0 & 0 & 0 \\
n_1^2 \sin \alpha_1 & n_2^2 \sin \alpha_2 & n_3^2 \sin \alpha_3 & 0 & 0 & 0 \\
\cos \alpha_1 + C_1 \sin \alpha_1 & \cos \alpha_2 + C_2 \sin \alpha_2 & \cos \alpha_3 + C_3 \sin \alpha_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \sin \alpha_1 & \sin \alpha_2 & \sin \alpha_3 \\
0 & 0 & 0 & n_1^2 \sin \alpha_1 & n_2^2 \sin \alpha_2 & n_3^2 \sin \alpha_3 \\
0 & 0 & 0 & n_1 \cos \alpha_1 & n_2 \cos \alpha_2 & n_3 \cos \alpha_3 + C_1 \sin \alpha_1 \\
\end{vmatrix} = 0
\]  

(27)
Equation (27) may be expressed as a product of two third-order determinants, one containing only $A_k$ coefficients and the other containing only $B_k$ coefficients. Thus

$$D = D_A \cdot D_B = 0,$$  \hspace{1cm} (28)

where

$$D_A = \begin{vmatrix} n_1 \sin \alpha_1 & n_2 \sin \alpha_2 & n_3 \sin \alpha_3 \\ n_1' \sin \alpha_1 & n_2' \sin \alpha_2 & n_3' \sin \alpha_3 \\ \cos \alpha_1 + Cn_1' \sin \alpha_1 & \cos \alpha_2 + Cn_2' \sin \alpha_2 & \cos \alpha_3 + Cn_3' \sin \alpha_3 \end{vmatrix}$$  \hspace{1cm} (29)

and

$$D_B = \begin{vmatrix} \sin \alpha_1 & \sin \alpha_2 & \sin \alpha_3 \\ n_1' \sin \alpha_1 & n_2' \sin \alpha_2 & n_3' \sin \alpha_3 \\ n_1 \cos \alpha_1 + Cn_1' \sin \alpha_1 & n_2 \cos \alpha_2 + Cn_2' \sin \alpha_2 & n_3 \cos \alpha_3 + Cn_3' \sin \alpha_3 \end{vmatrix}$$  \hspace{1cm} (30)

There are three ways to satisfy (27):

$$D_A = 0$$  \hspace{1cm} (31)

$$D_B = 0$$  \hspace{1cm} (32)

$$D_A = D_B = 0$$  \hspace{1cm} (33)

In the first of these, (31), nontrivial values are possible for the $A_k$ constants since $D_A = 0$. But if $D_A = 0$ and $D_B \neq 0$, then all the $B_k$ constants will be zero, since by Cramer's rule nontrivial values cannot exist. The reverse is true for the second of these, (32). From the expression for the radial displacement function $U$ obtained from (12) by virtue of (5),

$$U = \sum_{k=1}^{3} \left( -A_k n_k \sin n_k \theta + B_k n_k \cos n_k \theta \right)$$  \hspace{1cm} (34)

we find that the $A_k$ constants are associated with sine functions which are antisymmetrical, whereas the $B_k$ constants are associated with symmetrical cosine functions. Condition (31) then would result in a solution containing only antisymmetrical terms and the solution from (32) would contain only symmetrical terms.
In either case, (31) or (32), both the governing equation of motion and the continuity and equilibrium conditions at the mass will be satisfied.

The frequency equations may be found by evaluating $D_A$ and $D_B$. From (29)

$$D_A = n_2 n_3 (n_2^2 - n_3^2) \cos \alpha_1 \sin \alpha_2 \sin \alpha_3 + n_1 n_3 (n_3^3 - n_1^3) \sin \alpha_1 \cos \alpha_2 \sin \alpha_3 + n_1 n_2 (n_1^2 - n_2^2) \sin \alpha_1 \sin \alpha_2 \cos \alpha_3 - C n_1 n_2 n_3 (n_1^2 - n_2^2) (n_2^2 - n_3^2) (n_3^2 - n_1^2) \sin \alpha_1 \sin \alpha_2 \sin \alpha_3. \quad (35)$$

Solving (35) for $C$ (let $C = C_A$ for $D_A = 0$) the frequency equation for antisymmetrical vibrations is

$$C_A = \frac{\cos \alpha_1}{n_1 (n_1^2 - n_2^2) (n_3^3 - n_1^3) \sin \alpha_1} + \frac{\cos \alpha_2}{n_2 (n_2^2 - n_3^2) (n_2^2 - n_3^2) \sin \alpha_2} + \frac{\cos \alpha_3}{n_3 (n_3^2 - n_1^2) (n_3^3 - n_2^2) \sin \alpha_3}. \quad (36)$$

By the same procedure we find from (30)

$$D_B = n_1 (n_2^2 - n_3^2) \cos \alpha_1 \sin \alpha_2 \sin \alpha_3 + n_2 (n_3^2 - n_1^2) \sin \alpha_1 \cos \alpha_2 \sin \alpha_3 + n_3 (n_1^2 - n_2^2) \sin \alpha_1 \sin \alpha_2 \cos \alpha_3 - C (n_2^2 - n_3^2) (n_2^2 - n_3^2) (n_3^2 - n_1^2) \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 \quad (37)$$

from which the frequency equation for symmetrical vibrations is (let $C = C_B$ for $D_B = 0$)

$$C_B = \frac{n_1 \cos \alpha_1}{(n_1^2 - n_2^2) (n_3^2 - n_1^2) \sin \alpha_1} + \frac{n_2 \cos \alpha_2}{(n_2^2 - n_3^2) (n_2^2 - n_3^2) \sin \alpha_2} + \frac{n_3 \cos \alpha_3}{(n_2^2 - n_3^2) (n_3^2 - n_1^2) \sin \alpha_3}. \quad (38)$$

In general, both $D_A$ and $D_B$ will not vanish for the same $n_k$ values since frequency expressions (36) and (38) for the two branches are not the same; $C_A$ would equal $C_B$ only if all the $n_k$ were unity since (36) would equal (38).
only if \( l/n_k = n_k \). Such a situation cannot exist for flexural vibrations (see (13)). Now referring to the previous discussion of conditions (31) and (32), it is apparent that for free vibrations of a ring with a point mass, two solutions are possible; one consisting of symmetrical vibrations and another of antisymmetrical vibrations, each of these having different natural frequencies. An illustration of this is provided in the next section, where numerical solutions are obtained for both the symmetrical and antisymmetrical branches of the first two modes.

The frequency equation for the case of a ring without a point mass may be obtained by dividing (22) and (25) by \( C \) and letting the point mass become zero. Then from (29) and (30) the following expressions are obtained:

from (29)

\[
D_A = n_1 n_2 n_3 (n_1^2 - n_2^2)(n_2^2 - n_3^2)(n_3^2 - n_1^2) \sin a_1 \sin a_2 \sin a_3
\]  
(39)

from (30)

\[
D_B = (n_1^2 - n_2^2)(n_2^2 - n_3^2)(n_3^2 - n_1^2) \sin a_1 \sin a_2 \sin a_3
\]  
(40)

If one of the \( n_k \) is an integer, then both (39) and (40) are identically zero, which is condition (33), and the frequency equation from (13) is

\[
n^2(n^2 - 1)^2 = (n^2 + 1) \lambda.
\]  
(41)

NORMAL MODE SHAPES

The antisymmetrical branch mode constants \( A_k \) may be found by solving (20) and (21) simultaneously; the solution is

\[
A_2 = \frac{n_1 (n_1^2 - n_2^2) \sin a_1}{n_2 (n_2^2 - n_3^2) \sin a_2}
\]  
(42)

and

\[
A_3 = \frac{n_1 (n_1^2 - n_2^2) \sin a_1}{n_3 (n_2^2 - n_3^2) \sin a_3}
\]  
(43)

and the symmetrical constants \( B_k \) from (23) and (24) are
\[
\frac{B_2}{B_1} = \frac{(n_3^2 - n_1^2) \sin \alpha_1}{(n_2^2 - n_3^2) \sin \alpha_2}
\] (44)

and

\[
\frac{B_3}{B_1} = \frac{(n_1^2 - n_3^2) \sin \alpha_1}{(n_2^2 - n_3^2) \sin \alpha_3}
\] (45)

Since \(A_1\) and \(B_1\) were taken as arbitrary, they may be set equal to unity. Then from (12) and (34) the tangential and radial displacements for antisymmetrical vibrations are

\[
V = \cos(n_1 \theta) + A_2 \cos(n_2 \theta) + A_3 \cos(n_3 \theta)
\] (46)

and

\[
U = -n_1 \sin(n_1 \theta) - A_2 n_2 \sin(n_2 \theta) - A_3 n_3 \sin(n_3 \theta)
\] (47)

and for symmetrical vibrations are

\[
V = \sin(n_1 \theta) + B_2 \sin(n_2 \theta) + B_3 \sin(n_3 \theta)
\] (48)

and

\[
U = n_1 \cos(n_1 \theta) + B_2 n_2 \cos(n_2 \theta) + B_3 n_3 \cos(n_3 \theta)
\] (49)

The mode shapes for a ring without a point mass may be found by solving the six simultaneous equations, (20) through (25), with one of the \(n_k\) taken as an integer and the point mass equal to zero. Two of the mode constants, one \(A_k\) and one \(B_k\), are found to be arbitrary and the other four zero. Taking the arbitrary constants \(A_1\) and \(B_1\) as unity, the tangential and radial displacements for antisymmetrical vibrations are

\[
V = \cos(n_1 \theta)
\] (50)

and

\[
U = -n_1 \sin(n_1 \theta)
\] (51)

and for symmetrical vibrations are

\[
V = \sin(n_1 \theta)
\] (52)
and

\[ U = n_1 \cos n_1 \theta. \] (53)

ORTHOGONALITY

To obtain the orthogonality relation between the normal modes, let \( \lambda_i \) and \( \lambda_j \) be any two different eigenvalues with \( V_i \) and \( V_j \) the associated eigenvectors or mode shapes. From (9) then follows

\[
\frac{d^6 V_i}{d\theta^6} + 2 \frac{d^4 V_i}{d\theta^4} + \frac{d^2 V_i}{d\theta^2} - \lambda_i \left( \frac{d^2 V_i}{d\theta^2} - V_i \right) = 0
\] (54)

and

\[
\frac{d^6 V_j}{d\theta^6} + 2 \frac{d^4 V_j}{d\theta^4} + \frac{d^2 V_j}{d\theta^2} - \lambda_j \left( \frac{d^2 V_j}{d\theta^2} - V_j \right) = 0.
\] (55)

Multiplying Equation (54) by \( V_j d\theta \) and Equation (55) by \( V_i d\theta \), then integrating each over the ring and subtracting (55) from (54), we obtain

\[
\int_{-\pi}^{\pi} V_j \left( \frac{d^6 V_i}{d\theta^6} + 2 \frac{d^4 V_i}{d\theta^4} + \frac{d^2 V_i}{d\theta^2} \right) d\theta - \int_{-\pi}^{\pi} V_i \left( \frac{d^6 V_j}{d\theta^6} + 2 \frac{d^4 V_j}{d\theta^4} + \frac{d^2 V_j}{d\theta^2} \right) d\theta
\]

\[ + \lambda_j \int_{-\pi}^{\pi} V_i \left( \frac{d^2 V_j}{d\theta^2} - V_j \right) d\theta - \lambda_i \int_{-\pi}^{\pi} V_j \left( \frac{d^2 V_i}{d\theta^2} - V_i \right) d\theta = 0. \] (56)

Integrated by parts, the first term of (56) becomes

\[
\int_{-\pi}^{\pi} V_j \left( \frac{d^6 V_i}{d\theta^6} + 2 \frac{d^4 V_i}{d\theta^4} + \frac{d^2 V_i}{d\theta^2} \right) d\theta = \int_{-\pi}^{\pi} V_i \left( \frac{d^6 V_j}{d\theta^6} + 2 \frac{d^4 V_j}{d\theta^4} + \frac{d^2 V_j}{d\theta^2} \right) d\theta
\]

\[ + \left[ V_j \left( \frac{d^5 V_i}{d\theta^5} + 2 \frac{d^3 V_i}{d\theta^3} + \frac{dV_i}{d\theta} \right) - \frac{dV_j}{d\theta} \left( \frac{d^4 V_i}{d\theta^4} + 2 \frac{d^2 V_i}{d\theta^2} + V_i \right) + \frac{d^2 V_j \left( \frac{d^3 V_i}{d\theta^3} + 2 \frac{dV_i}{d\theta} \right)}{d\theta} \right]
\]

\[ - \frac{d^3 V_j}{d\theta^3} \left( \frac{d^2 V_i}{d\theta^2} + 2V_i \right) + \frac{d^4 V_j}{d\theta^4} \frac{dV_i}{d\theta} - \frac{d^5 V_j}{d\theta^5} V_i \right]_{-\pi}^{\pi}. \] (57)
The bracketed part of (57) may be evaluated from the boundary conditions (14) through (19) at the point mass, which are recast in the following time-independent form:

from (15)

\[ [V]_{\theta=\pi} = 0 \]  \hspace{1cm} (58)

from (14)

\[ \left[ \frac{dV}{d\theta} \right]_{\theta=\pi} = 0 \]  \hspace{1cm} (59)

from (16)

\[ \left[ \frac{d^2V}{d\theta^2} \right]_{\theta=\pi} = 0 \]  \hspace{1cm} (60)

from (19)

\[ \left[ \frac{d^3V}{d\theta^3} \right]_{\theta=\pi} = 0 \]  \hspace{1cm} (61)

from (17)

\[ \frac{M}{m r} \lambda \left[ \frac{dV}{d\theta} \right]_{\theta=\pi} + \left[ \frac{d^4V}{d\theta^4} \right]_{\theta=\pi} = 0 \]  \hspace{1cm} (62)

and from (18)

\[ \frac{M}{m r} \lambda \left[ V \right]_{\theta=\pi} - \left[ \frac{d^5V}{d\theta^5} \right]_{\theta=\pi} = 0 \]  \hspace{1cm} (63)

After (58) through (63) are applied, Equation (57) becomes

\[
\int_{\pi}^{\pi} V_j \left( \frac{d^6V_i}{d\theta^6} + 2 \frac{d^4V_i}{d\theta^4} + \frac{d^2V_i}{d\theta^2} \right) d\theta = \int_{\pi}^{\pi} V_i \left( \frac{d^6V_j}{d\theta^6} + 2 \frac{d^4V_j}{d\theta^4} + \frac{d^2V_j}{d\theta^2} \right) d\theta \\
+ \left( \lambda_i - \lambda_j \right) \frac{M}{m r} \left[ V_i V_j + \left( \frac{dV_i}{d\theta} - \frac{dV_j}{d\theta} \right) \right] \]  \hspace{1cm} (64)
Now integrating the fourth term of (56) by parts and applying (58) and (59), we obtain

\[
\lambda_1 \int_{-\pi}^{\pi} V_j \left( \frac{d^2 V_i}{d\theta^2} - V_j \right) d\theta = \lambda_i \int_{-\pi}^{\pi} V_i \left( \frac{d^2 V_j}{d\theta^2} - V_j \right) d\theta
\]

(65)

With the use of (64) and (65), Equation (56) becomes

\[
\left( \lambda_j - \lambda_i \right) \int_{-\pi}^{\pi} V_i \left( \frac{d^2 V_j}{d\theta^2} - V_j \right) d\theta - \left( \lambda_j - \lambda_i \right) \frac{M}{mr} \left[ V_i V_j + \left( \frac{dV_i}{d\theta} \frac{dV_j}{d\theta} \right) \right]_{\pi} = 0.
\]

(66)

Since \((\lambda_j - \lambda_i) \neq 0\), the orthogonality relation from (66) is

\[
\int_{-\pi}^{\pi} V_i \left( \frac{d^2 V_j}{d\theta^2} - V_j \right) d\theta - \frac{M}{mr} \left[ V_i V_j + \left( \frac{dV_i}{d\theta} \frac{dV_j}{d\theta} \right) \right]_{\pi} = 0,
\]

(67)

with the point mass located at \(\theta = \pi\), as shown in Figure 1.

NUMERICAL SOLUTIONS

To illustrate the influence of the point mass on the free vibrations of the ring, frequencies and mode shapes were calculated* for the first and second flexural modes by the following procedure:

VARIATION OF FREQUENCY WITH MASS

1. Values of \(\lambda\) were chosen and (13) was solved for \(n_k\). These values are listed in Table 1.

2. The \(n_k\) values were then inserted into the frequency expressions (36) and (38) to obtain values for \(C_A\) and \(C_B\), which are listed in Table 1. From inspection of (26) it is noted that \(C = 0\) is the case of an infinite point mass, and \(C = \infty\) is the case of zero point mass.

3. Frequency ratios of the ring with a point mass to that of the ring alone may be obtained from (10):

*All numerical computations were performed by electronic digital computers. The programs are given in the Appendix.
For the ring with a point mass \( M \), (10) is

\[
\omega^2 = \frac{EI}{mr^4} \lambda^*_M
\]

(68)

and for the ring alone, (10) is

\[
\omega_m^2 = \frac{EI}{mr^4} \lambda_m .
\]

(69)

Then from (68) and (69) the following is obtained

\[
\frac{\omega_M}{\omega_m} = \left( \frac{\lambda_M}{\lambda_m} \right)^{\frac{1}{2}}
\]

(70)

Mass and frequency ratios are related by eliminating \( \omega^2 \) from (10) and (26) which gives

\[
\frac{M}{2\pi rm} = \frac{1}{\pi C^*} .
\]

(71)

Frequency ratios, \( \omega_M/\omega_m \), and corresponding mass ratios, \( M/(2\pi rm) \), are given in Table 1. The relationship between frequency and mass is shown graphically in Figures 4 and 5.

Notice that in Table 1 some negative values of the frequency functions \( C_A \) and \( C_B \) are included but that corresponding mass ratios are not, as they also would be negative and therefore meaningless: this can be seen by inspection of (71). Complete curves of \( C_A \) and \( C_B \) versus \( \lambda \) for the first and second modes are shown in Figures 6 through 9.

VARIATION OF MODE SHAPES WITH MASS

Equations (46) through (49) were used to determine the mode shapes for four values of the mass ratio: \( M/(2\pi rm) \) = 0, 0.25, 1.0 and \( \infty \). The \( n_k \) values that correspond to these mass ratios were found by choosing a \( \lambda \) and solving (13), together with either (36) or (38), depending on the branch of interest, until (71) was satisfied. These values are included in Table 1.

The displacement functions \( U \) and \( V \) of both branches of the first two modes are shown in Figures 10 through 17 for \( \theta \) between 0 degree and 180 degrees. Complete shapes for mass ratios of 0 and \( \infty \) are shown in Figures 18 through 21.
<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$n_1$</th>
<th>$n_2 = a + bi$</th>
<th>$n_3 = a - bi$</th>
<th>$\frac{\omega_M}{\omega_m}$</th>
<th>antisymmetrical branch</th>
<th>symmetrical branch</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$a$</td>
<td>$b$</td>
<td></td>
<td>$C_A$ $2\pi \nu m$</td>
<td>$C_B$ $2\pi \nu m$</td>
</tr>
<tr>
<td>7.20000</td>
<td>2.00000</td>
<td>0.413304</td>
<td>1.082045</td>
<td>1.00000</td>
<td>$\infty$</td>
<td>$0.00000$</td>
</tr>
<tr>
<td>7.15000</td>
<td>1.997572</td>
<td>0.414399</td>
<td>1.082199</td>
<td>0.996522</td>
<td>2.458990</td>
<td>0.018105</td>
</tr>
<tr>
<td>7.00000</td>
<td>1.990215</td>
<td>0.417674</td>
<td>1.074676</td>
<td>0.986013</td>
<td>0.547188</td>
<td>0.083103</td>
</tr>
<tr>
<td>6.747744</td>
<td>1.977590</td>
<td>0.423147</td>
<td>1.065122</td>
<td>0.968084</td>
<td>0.186891</td>
<td>0.250000</td>
</tr>
<tr>
<td>6.50000</td>
<td>1.964688</td>
<td>0.428483</td>
<td>1.055438</td>
<td>0.950146</td>
<td>0.085449</td>
<td>0.573102</td>
</tr>
<tr>
<td>6.3454518</td>
<td>1.956759</td>
<td>0.431793</td>
<td>1.049237</td>
<td>0.938783</td>
<td>0.050163</td>
<td>1.000000</td>
</tr>
<tr>
<td>6.10000</td>
<td>1.943593</td>
<td>0.437019</td>
<td>1.039116</td>
<td>0.920447</td>
<td>0.012874</td>
<td>4.053380</td>
</tr>
<tr>
<td>6.00000</td>
<td>1.938123</td>
<td>0.439139</td>
<td>1.034893</td>
<td>0.912871</td>
<td>0.853848</td>
<td>0.387804</td>
</tr>
<tr>
<td>5.958868</td>
<td>1.937345</td>
<td>0.439438</td>
<td>1.034291</td>
<td>0.911795</td>
<td>0.000000</td>
<td>$\infty$</td>
</tr>
<tr>
<td>5.50000</td>
<td>1.909780</td>
<td>0.449650</td>
<td>1.012825</td>
<td>0.874007</td>
<td>$-0.039379$</td>
<td>0.258039</td>
</tr>
<tr>
<td>5.378353</td>
<td>1.902626</td>
<td>0.452182</td>
<td>1.007205</td>
<td>0.864302</td>
<td>$-0.046990$</td>
<td>0.236726</td>
</tr>
<tr>
<td>5.00000</td>
<td>1.879586</td>
<td>0.460021</td>
<td>0.988959</td>
<td>0.833333</td>
<td>$-0.067988$</td>
<td>0.184051</td>
</tr>
<tr>
<td>4.00000</td>
<td>1.812258</td>
<td>0.480342</td>
<td>0.934274</td>
<td>0.745356</td>
<td>$-0.117928$</td>
<td>0.095572</td>
</tr>
<tr>
<td>3.802196</td>
<td>1.798950</td>
<td>0.483931</td>
<td>0.922301</td>
<td>0.728413</td>
<td>$-0.127393$</td>
<td>0.083322</td>
</tr>
<tr>
<td>3.00000</td>
<td>1.720531</td>
<td>0.500000</td>
<td>0.866025</td>
<td>0.645947</td>
<td>$-0.180476$</td>
<td>0.030940</td>
</tr>
<tr>
<td>2.60000</td>
<td>1.694764</td>
<td>0.507602</td>
<td>0.832930</td>
<td>0.600925</td>
<td>$-0.215768$</td>
<td>0.003389</td>
</tr>
<tr>
<td>2.546680</td>
<td>1.689304</td>
<td>0.508657</td>
<td>0.828001</td>
<td>0.594498</td>
<td>$-0.221421$</td>
<td>$0.000000$</td>
</tr>
<tr>
<td>2.00000</td>
<td>1.630634</td>
<td>0.518553</td>
<td>0.773551</td>
<td>$-0.232431$</td>
<td>$\infty$</td>
<td>$0.45636$</td>
</tr>
<tr>
<td>1.997572</td>
<td>2.00000</td>
<td>0.414399</td>
<td>1.082199</td>
<td>0.996522</td>
<td>$\infty$</td>
<td>$0.000000$</td>
</tr>
<tr>
<td>1.977590</td>
<td>2.00000</td>
<td>0.417674</td>
<td>1.074676</td>
<td>0.986013</td>
<td>$\infty$</td>
<td>$0.000000$</td>
</tr>
</tbody>
</table>

**Note:** All values are rounded-off to the sixth decimal place, except for the $\lambda$ values which are exact.
Figure 4 - Variation of Frequency with Mass, First Mode
Figure 5 - Variation of Frequency with Mass, Second Mode

- Symmetrical Branch
- Antisymmetrical Branch
- Asymptote $\frac{M}{2\pi r m} = \infty$
Asymptote, $\lambda = 7.2$

Asymptote, $\lambda = 0$

$C_A = \infty$

$C_A = -\infty$

Figure 6 - $C_A$ versus $\lambda$, First Mode
Figure 7 - $C_B$ versus $X$, First Mode

Asymptote, $\lambda = 7.2$

$C_B = \infty$

Asymptote, $\lambda = 0$

$C_B = -\infty$
Figure 8 - $C_A$ versus $\lambda$, Second Mode

Figure 9 - $C_B$ versus $\lambda$, Second Mode
Figure 10 - Radial Displacements, Symmetrical Branch of First Mode
Figure 11 - Tangential Displacements, Symmetrical Branch of First Mode
Figure 12 - Radial Displacements. Antisymmetrical Branch of First Mode.
Figure 13 - Tangential Displacements, Antisymmetrical Branch of First Mode
Figure 14 - Radial Displacements, Symmetrical Branch of Second Mode
Figure 15 - Tangential Displacements, Symmetrical Branch of Second Mode
Figure 16 - Radial Displacements, Antisymmetrical Branch of Second Mode
Figure 17 - Tangential Displacements, Antisymmetrical Branch of Second Mode
Figure 18 - Modes Shapes, Symmetrical Branch of First Mode
Figure 19 - Mode Shapes, Antisymmetrical Branch of First Mode
Figure 20 - Mode Shapes, Symmetrical Branch of Second Mode
Figure 21 - Mode Shapes, Antisymmetrical Branch of Second Mode
DISCUSSION

INFLUENCE OF MASS ON FREQUENCY

The most noticeable feature here is the difference in frequency between the symmetrical and antisymmetrical branches of a normal mode, as shown in Figures 4 and 5. For a ring with zero point mass both branches have the same frequency, but as the point mass increases the difference in frequency becomes greater and is greatest for the limiting case of a ring with an infinite point mass.

For the first two flexural modes, as the mass ratio increases the reduction in frequency is greater in the symmetrical branch than in the antisymmetrical branch. In other words, for a given value of the mass ratio the higher frequency occurs in the antisymmetrical branch while in the symmetrical branch it is lower.

Comparing Figures 4 and 5 it is noted that for the same mass ratio the reduction in frequency is less for both branches in the second mode than in the first. That is, the mass has a lesser effect in the second mode than in the first.

INFLUENCE OF MASS ON MODE SHAPES

From Figures 10 through 21 it is noted that the displacement of the point mass decreases as the mass ratio increases. In the case of an infinite point mass there is no movement of the mass and it becomes a stationary point. Note also that the mass has appreciably less influence on the radial displacement of the antisymmetrical shapes than on the symmetrical as is illustrated by comparing Figures 10 and 12, and 14 and 16. The major influence of the mass for antisymmetrical vibrations occurs in the tangential displacements, since for these shapes the mass is located at a radial node.

For a ring without a point mass it is interesting to note from Equations (51) and (53), and from Figures 19 and 21, that the axis of symmetry of the antisymmetrical branch is rotated $\pi/(2n)$ radians or $90/n$ degrees from the axis of symmetry of the symmetrical branch, where $n$ is an integer greater than unity denoting the number of wavelengths to the circumference of the ring. Such a relationship does not exist when the point mass is present. Not only are the antisymmetrical branch mode shapes different from the symmetrical branch mode shapes, but as expected, they are completely antisymmetrical.
SUMMARY AND CONCLUSIONS

RING WITHOUT A POINT MASS

1. The solution was obtained from the general equations for a ring with a point mass by taking the point mass to be zero.

2. The solution was found to be a degenerate case of the general solution of a ring with a point mass since for a given mode the frequencies and mode shapes of both branches are the same.

RING WITH A POINT MASS

1. With the addition of a small point mass to the ring, the natural frequencies of each branch were reduced to a different degree. Symmetrical branch frequencies were reduced to a greater extent than those of the antisymmetrical branch. It was found that a given mass exerted less influence on the frequency in the second mode than in the first.

2. The normal mode shapes for each branch were found to be different. The presence of the point mass influenced the symmetrical branch shapes to a greater degree than the antisymmetrical. This can be briefly explained by the fact that the mass was located at a radial antinode for symmetrical vibrations, whereas for antisymmetrical vibrations it was located at a radial node.

Experimental investigations\textsuperscript{1,2,3} into the vibrations of imperfect bodies of revolution have shown that the presence of a small asymmetry tended to resolve a single natural frequency into two nearby values. The larger the asymmetry or irregularity, the farther apart and more distinct the frequencies. Resonance tests\textsuperscript{1} have revealed so-called preferential planes; that is, if the body was excited in one of these planes only one resonance peak appeared, but if excited at any other position around the body two peaks occurred for the same mode number. For a body of revolution with a mass irregularity the results of this investigation tend to support such experimental phenomena. A qualitative illustration is provided by Figures 18 and 19. Consider the ring with mass shown in Figure 18 excited radially at $\theta = 0$ degree, opposite the mass. In this case only one resonance peak would appear for the first mode and symmetrical vibrations would occur. If the ring with mass shown in Figure 19 was excited radially at $\theta \approx 50$ degrees then antisymmetrical vibrations would occur and again only one resonance peak would appear, but at a different frequency (see Figure 4). From this it can be expected that for a ring with a mass irregularity excited arbitrarily, two resonance peaks would occur in a given mode, one at the natural frequency of the symmetrical branch and the other at the natural frequency of the antisymmetrical branch.


APPENDIX A

Computer Programs

by

L. F. Parker
Program 1.

Title: ROOT

Language: FORTRAN IV

Computer Used: IBM 1130, located at Norfolk Naval Shipyard, Portsmouth, Va.

Purpose: Computes \( n_k \) from Equation (13) for given values of \( \lambda \).

Sub routines called: POLRT (an IBM SSP routine)

Legend: 
\( n_k \) (real part) = \text{ROOTR} \\
\( n_k \) (imaginary part) = \text{ROOTI} \\
Number of data sets = \text{NUMP} \\
Initial value of \( \lambda \) for a data set = \( Q \) \\
Final value of \( \lambda \) for a data set = \( R \) \\
Increment size of \( \lambda \) between \( Q \) and \( R \) = \( S \)

```
C PROGRAM - ROOT

DIMENSION XCOF(7), COF(7), ROOTR(6), ROOTI(6)
DIMENSION Q(50), R(50), S(50)

1 FORMAT(6F12.7)
2 FORMAT(/,1H )
3 FORMAT(2F14.10)
4 FORMAT(/,4HIER=,I3)
5 FORMAT(IF10.6,5H 1)
6 FORMAT(3F15.7)
7 FORMAT(15)
8 FORMAT(/)
9 XCOF(7)=1.
XCOF(6)=0.
XCOF(5)=-2.
XCOF(4)=0.
XCOF(2)=0.
M=6

READ(2,7)NUMP
READ(2,6) (Q(I), R(I), S(I), I=1, NUMP)
READ(2,8)
DO 40 I=1, NUMP
10 XCOF(1)=-Q(I)
XCOF(3)=1.-Q(I)
WRITE(1,2)
WRITE(1,1)(XCOF(N), N=1,7)
CALL POLRT(XCOF, COF, M, ROOTR, ROOTI, IER)
WRITE(1,4)IER
WRITE(2,5) Q(I)
DO 20 J=1,6
WRITE(2,3)ROOTR(J), ROOTI(J)
20 CONTINUE
Q(I)=Q(I)+S(I)
IF(Q(I)-R(I))10,10,40
40 CONTINUE
CALL EXIT
END
```
Program 2.

Title: CACB

Language: FORTRAN IV

Computer Used: IBM 7090, located at NSRDC, Washington, D.C.

Purpose: Computes $C_A$ from Equation (36) and $C_B$ from Equation (38) for given $n_k$.

Computes $M/(2\pi r_m)$ from Equation (71).

Computes $\omega_M/\omega_m$ from Equation (70).

Sub routines called: None

Legend: $\lambda = Q$

\[ n_1 = N1 \text{ (In print out)} = ZN1 \text{ (In input)} \]
\[ n_2 = N2 \text{ (In print out)} = ZN2 \text{ (In input)} \]
\[ n_3 = N3 \text{ (In print out)} = ZN3 \text{ (In input)} \]
\[ C_A = CA \text{ (In print out)} \]
\[ C_B = CB \text{ (In print out)} \]
\[ \frac{M}{2\pi r_m} \quad \text{(for } C_A \text{)} = MA \text{ (In print out)} \]
\[ \frac{M}{2\pi r_m} \quad \text{(for } C_B \text{)} = MB \text{ (In print out)} \]
\[ \frac{\omega_M}{\omega_m} = P \]
C PROGRAM CACB
COMPLEX ZN1,ZN2,ZN3,CA,CB
COMPLEX A,B,C,D,E,F
COMPLEX WA,WB
1 FORMAT(F10.6,15)
2 FORMAT(2F14.10)
3 FORMAT(1H1)
4 FORMAT(15X,1H0,19X,2HCA,13X,2HCB,13X,2HMA,13X,2HMB,13X,1HF
5 FORMAT(15X,2HN1/15X,2HN2/15X,2HN3///)
6 FORMAT(1H0)
7 FORMAT(5X,F10.6,15X,F10.6,4(5X,F10.6))
8 FORMAT(2F10.6)
WRITE(6,3)
WRITE(6,4)
WRITE(6,5)
10 READ(5,1)0,LAST
READ(5,2)ZN1
READ(5,2)ZN2
READ(5,2)ZN3
PI=3.1415927
A=CCOS(ZN1*PI)/CSIN(ZN1*PI)
B=CCOS(ZN2*PI)/CSIN(ZN2*PI)
C=CCOS(ZN3*PI)/CSIN(ZN3*PI)
D=ZN1**2-ZN2**2
E=ZN3**2-ZN1**2
F=ZN2**2-ZN3**2
CA=A/(ZN1*CE)+B/(ZN2*F)+C/(ZN3*E)
CB=(ZN1*A)/(D*E)+(ZN2*8)/(D*F)+(ZN3*C)/(F*E)
WA=1./(PI*CA*O)
WB=1./(PI*CB*O)
IF(Q-10.)20,20.30
20 Q0=7.2
GO TO 40
30 Q0=57.6
40 P=SQR(T(0/Q0))
ACA=REAL(CA)
ACB=REAL(CB)
AMA=REAL(WA)
AMB=REAL(WB)
WRITE(6,7)Q,ACA,ACB,AMA,AMB,P
WRITE(6,8)ZN1
WRITE(6,8)ZN2
WRITE(6,8)ZN3
WRITE(6,6)
IF(LAST)50,50,10
50 CONTINUE
STOP
END
Program 3.

Title: RING

Language: FORTRAN IV

Computer Used: IBM 7090, located at NSRDC, Washington, D. C.

Purpose: Computes U and V from Equations (46), (47), (48), and (49) for given nk.

Sub Routines called: None

Legend: λ = Q

\n
n_1 = N1 (In print out) = ZN1 (In input) 

n_2 = N2 (In print out) = ZN2 (In input) 

n_3 = N3 (In print out) = ZN3 (In input) 

θ = THETA (In print out) 

U (for Equation (47)) = UA 

U (for Equation (49)) = UB 

V (for Equation (46)) = VA 

V (for Equation (48)) = VB
C PROGRAM - RING
C MODE SHAPES OF A RING WITH ATTACHED MASS
1 FORMAT(2F10.7)
2 FORMAT(F11.7,15)
3 FORMAT(///,10X,6HFOR Q=,F11.7//)
4 FORMAT(10X,3HN1=,2F10.6,5X,3HN2=,2F10.6,5X,3HN3=,2F10.6///)
5 FORMAT(10X,5HTHETA,15X,2HVA,23X,2HUA,23X,2HV8,23X,2HUB//)
6 FORMAT(1H1)
7 FORMAT(10X,114,6X,2F10.6,6F10.6,2F10.6,6F10.6,6F10.6)

COMPLEX ZN1,ZN2,ZN3,VA,UA,VB,UB
COMPLEX A2,A3,B2,B3
COMPLEX C,D,E,F,G,H
PI=3.1415927
R=0.01745329

10 READ(5,2)Q,LAST
READ(5,1)ZN1
READ(5,1)ZN2
READ(5,1)ZN3
WRITE(6,6)
WRITE(6,3)
WRITE(6,4)ZN1,ZN2,ZN3
WRITE(6,5)
C=ZN3**2-ZN1**2
D=ZN2**2-ZN3**2
E=ZN1**2-ZN2**2
F=CSIN(ZN1*PI)/CSIN(ZN2*PI)
G=CSIN(ZN1*PI)/CSIN(ZN3*PI)
B2=C*F/D
A2=ZN1*B2/ZN2
B3=E*G/D
A3=ZN1*A3/ZN3

J=0

40 TH=R*DEL
VA=CCOS(ZN1*TH)+A2*CCOS(ZN2*TH)+A3*CCOS(ZN3*TH)
UA=-ZN1*CSIN(ZN1*TH)-A2*ZN2*CSIN(ZN2*TH)-A3*ZN3*CSIN(ZN3*TH)
VB=CSIN(ZN1*TH)+B2*CSIN(ZN2*TH)+B3*CSIN(ZN3*TH)
UB=ZN1*CCOS(ZN1*TH)+B2*ZN2*CCOS(ZN2*TH)+B3*ZN3*CCOS(ZN3*TH)
WRITE(6,7)J,VA,UA,VB,UB
J=J+15
DEL=DEL+15.

IF(DDEL-180.,)40,40.20
20 IF(LAST)30,30,10
30 CONTINUE
STOP
END
INITIAL DISTRIBUTION

Copies
6 NAVSHIPSYSCOM
   1 Ship 031
   1 Ship 033
   1 Ship 034
   1 Ship 052
   3 Ship 2052
   1 Ship 525

5 NAVSEC
   3 Sec 6115
   1 Sec 6120
   1 Sec 6132

3 CHONR
   1 Code 439
   1 Code 468

CNO
   1 DP 07T

NAVMAT
   1 Mat 033

2 NRL, Code 2027
   1 CO & DIR, USCIVENGLAB, Code L31
   1 CO & DIR, USNUSL

1 CO, NAVAIRDEVGEN, Johnsville

1 NAVSHIPYD, PTSMH
   Code 260D
   1 CO, PGSCOL, Monterey
   1 CO, PGSCOL, Webb
   1 CO, USNROTC & NAVADMINU, MIT

20 DIR, DDC
   1 DIR, DEF R & E
      Attn: Tech Lib
   1 Science & Tech Division, Library of Congress

1 National Research Council
   National Academy of Sciences
   Ship Hull Res Committee
   2101 Constitution Ave,
   Washington, D. C. 20418

1 National Science Foundation, Engr. Div.
   1951 Constitution Ave, N.W.
   Washington, D. C. 20550

1 National Aeronautics & Space Admin
   Assoc Admin for Advances Res & Tech
   Washington, D. C. 20546

1 Structures Research Div
   National Aeronautics & Space Admin
   Langley Research Center
   Langley Sta, Hampton, Va. 23665

1 Sandia Corporation
   Attn: Dr. M. L. Merritt
   Sandia Base
   Albuquerque, N. M. 87115

1 Cambridge Acoustical Associates
   Attn: Dr. M. C. Junger
   129 Mount Auburn St
   Cambridge, Mass 02138

1 Electric Boat Division, Gen Dyn Corp.
   Attn: Dr. L. H. Chen
   Groton, Conn 06340

1 Bolt, Beranek & Newman, Inc.
   Attn: Dr. E. M. Kerwin
   50 Moulton St
   Cambridge, Mass 02138

1 Newport News Shipbuilding & Dry Dock Co.
   Attn: Tech Library
   Newport News, Va. 23607

1 Southwest Research Institute
   Attn: Dr. H. N. Abramson; Dr. R. C. DeHart
   8500 Culebra Road
   San Antonio, Texas 78206

1 J. G. Engineering Research Associates
   Attn: Dr. Joshua E. Greenspon
   3831 Menlo Drive
   Baltimore, Md 21215

1 Paul Weidlinger Consulting Engineers
   Attn: Dr. M. L. Baron
   5515 Randolph Road
   Rockville, Md 20852

1 Engineering Physics Co.
   Attn: Dr. Vincent Cushing
   5515 Randolph Road
   Rockville, Md 20852

1 Technical Library
   Battelle Memorial Institute
   Columbus, Ohio 43201

1 Dept. of Mechanics, Illinois Inst of Tech
   Chicago, Ill. 60616

1 University of Illinois, Attn: Prof N.M. Newmark
   Urbana, Ill. 61803

2 University of Michigan, Dept of NAME;
   Dept of Engineering Mechanics
   Ann Arbor, Mich 48106

1 Massachusetts Institute of Technology
   Attn: Dr. A. H. Keil
   Cambridge, Mass 02139

1 Virginia Polytechnic Institute
   Engr Mech Dept
   Blacksburg, Va. 24061

1 Brown University
   Attn: Prof P. S. Symonds
   Providence, R.I. 02912

1 ODC
   Civil Engr Dept
   Norfolk, Va. 23508
THE INFLUENCE OF A MASS ON THE FREE FLEXURAL VIBRATIONS OF A CIRCULAR RING

The general solution is obtained for the free flexural vibrations of a thin circular ring containing a point mass. The solution for a uniform ring alone is derived by taking the point mass to be zero. Numerical calculations of the frequencies of the first and second flexural modes are presented for values of the point mass in the range from zero to infinity. Mode shapes are presented in graphical form.

The predominant feature of the investigation is the difference in frequency and mode shape found in the symmetrical and anti-symmetrical branches of each mode. It is noted that similar phenomena have been observed experimentally for vibrations of imperfect bodies of revolution.
<table>
<thead>
<tr>
<th>KEY WORDS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamics (mechanics)</td>
</tr>
<tr>
<td>Rings, thin circular with point mass</td>
</tr>
<tr>
<td>Bodies of Revolution, imperfect</td>
</tr>
<tr>
<td>Vibrations, flexural</td>
</tr>
<tr>
<td>Numerical analysis</td>
</tr>
<tr>
<td>Experimental data</td>
</tr>
<tr>
<td>Equations of motion</td>
</tr>
</tbody>
</table>

UNCLASSIFIED
Security Classification

DD FORM 1473 (BACK)
(PAGE 2)

48
The general solution is obtained for the free flexural vibrations of a thin circular ring containing a point mass. The solution for a uniform ring alone is derived by taking the point mass to be zero. Numerical calculations of the frequencies of the first and second flexural modes are presented for values of the point mass in the range from zero to infinity. Mode shapes are presented in graphical form.

The predominant feature of the investigation is the difference in frequency and mode shape found in the symmetrical and antisymmetrical branches of each mode. It is noted that similar phenomena have been observed experimentally for vibrations of imperfect bodies of revolution.

This document has been approved for public release and sale; its distribution is unlimited.